EECS 16A Midterm 1 Review Session

Presented by <NAMES >(HKN)

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If you want to follow along: head to hkn.mu/16Aslides

Slides are also posted at @# on Piazza.

HKN Drop-In Tutoring

- HKN has office hours every weekday from **11 AM - 3 PM** and **8 PM - 10 PM** on hkn.mu/ohqueue
- The schedule of tutors can be found at hkn.mu/tutor
- Name 1:

Time 1:

Matrices and Linear Transformations

Matrices

- Matrices are **collections of vectors**.
- Typically represent **systems of equations**, where each row is an equation and each column is a variable.
- Notable matrices:
  - Identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - Rotation matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Augmented Matrices

Augmented matrices are a way of represening both sides of a system of equations using one matrix:

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ -2x + 2y - 2z &= -10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -2 & 2 & -2 & -10 \end{array}\right]$$

Gaussian Elimination

Gaussian Elimination

- **Gaussian elimination** is a methof for solving systems of linear equations.
- Use row operations to reduce matrix to **row echelon form** (a matrix that is all zero below the diagonal)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The row operations are:

1. **Row exchange**: reordering rows
2. **Row scaling**: scaling a row by a real number
3. **Superposition**: replace a row with the sum of itself and a scalar multiple of another row

Practice: Gaussian Elimination

Let's use Gaussian elimination to solve this system of equations!

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

Practice: Gaussian Elimination [Solution]

First, write out the system of equations into matrix-vector form:

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

Practice: Gaussian Elimination [Solution]

Now, use row operations to get the system into row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 0 & \boxed{6} & -1 & 7 \\ 0 & 6 & -1 & 8 \end{array}\right]$$

The values on the diagonals (boxed in the matrix above) are known as **pivots** or **leading coefficients**.

Practice: Gaussian Elimination [Solution]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row says $0 = -1$, so there is **no solution** to this system of equations!

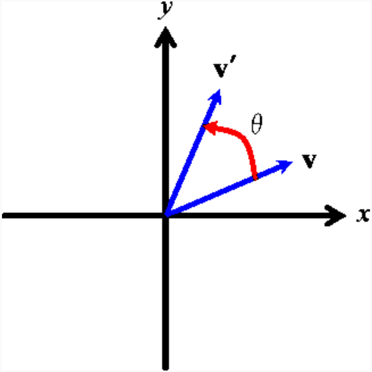
Possible results of Gaussian Elimination on $A\vec{x} = \vec{b}$

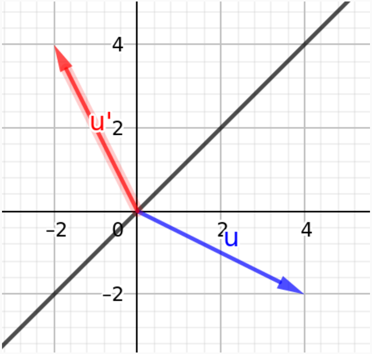
Result	Row picture	Column picture	Properties of A
Unique solution	Equations intersect at exactly one point	$\vec{b}$ is **uniquely represented** by a linear combination of the columns of A	A is **invertible**
Infinite solutions	Equations intersect along an **infinite space** (eg. line, plane, volume)	There are **multiple ways** of representing $\vec{b}$ in terms of the linear combinations of the columns of A	A has **linearly dependent columns**
No solution	Equations **do not intersect**	$\vec{b}$ is not in the span of the columns (columnspace) of A	Columnspace of A does not include $\vec{b}$. Columns of A are **linearly dependent**

Linear Transformations

Linear Transformations

- **Linear transformations** are operations that can be performed by applying a matrix to a vector.
- Some common transformations include **rotation** and **reflection**.





Common Linear Transformations: Rotation

The **rotation matrix** rotates points by a specific angle, θ:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Use this matrix by **plugging in the desired rotation angle**, then multiply it to a vector.

$$R(\theta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation matrices also **preserve the length** of a vector. Check it! (*Think about the eigenvalues! Real? Complex? How about the magnitude of these eigenvalues?*)

Common Linear Transformations: Reflection

- The reflection matrix **reflects vectors** across a line. (Notice that such matrix also *preserves the length of a vector*.)
- Notable reflection matrices:
  - Reflection across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Reflection across y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Reflection across line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by 45 deg
2. Reflect across $y = x$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by 45 deg

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix}$$

2. Reflect across $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear Independence

Linear Combinations

A **linear combination** of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a sum of the vectors, scaled by scalars $\{a_1, \ldots, a_n\}$:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

- If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are the *columns of a matrix* B, the set of all linear combinations is called the **columnspace** or **range** of B.
- Note: the **columnspace** of B is a **vector space**.
    - To check a subset is a vector space, it must be **closed under addition and scalar multiplication**.

Linear Independence

- Informally speaking, a set of vectors is linearly independent if *no vector in the set can be represented as a linear combination of other vectors*.
- Formal definition:
  - Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, if some scalars $\{c_1, c_2, \ldots, c_m\}\ neq \{0, \ldots, 0\}$ exist such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$, then the vectors are **linearly dependent**.
  - If no such scalars exist, then the vectors are **linearly independent**.
- In other words, for a set of linearly independent vectors, $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$ implies that all $c_1 = 0, c_2 = 0, \ldots, c_m = 0$ (*useful when doing proofs*).

Practice: Linear Independence

Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Row reduce the matrix. If any of the **pivots** (numbers on the diagonal) are 0, then the columns are **linerly dependent**.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Let's look at the row-reduced matrix as the system of equations:

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = 0$$

Remember, that if there exist nonzero c_1, c_2, c_3 that satisfy that equation, the columns are **linearly dependent**.

$$\begin{aligned} c_2 &= -3c_3 \\ 2c_1 = -3c_2 - 5c_3 = 9c_3 - 5c_3 &= 4c_3 \\ c_1 &= 2c_3 \end{aligned}$$

There are infinitely many solutions, so the columns of A are **linearly dependent**.

Span and Rank

- **Definition**: The **span** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots\}$ is the set of *all possible linear combinations* of the vectors.
- The span of a collection of vectors is always a **vector space**
- Since the column space of a matrix A is the *span of its columns*, it is a vector space.

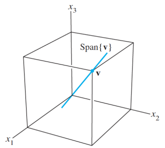


FIGURE 10 Span{**v**} as a line through the origin.

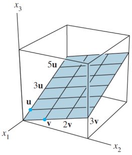
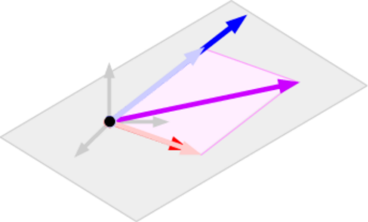


FIGURE 11 Span{**u**, **v**} as a plane through the origin.



Practice: Span

Do the following sets of vectors span $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Practice: Span [Solution]

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{No}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{Yes}$$

How do you know if a set of vectors spans $\mathbb{R}^3$?

- Are there **n pivots**, i.e. **n linearly independent vectors**?
- Use *Gaussian Elimination*!

Rank

- Definition: The **rank** of a matrix A is the *dimension of the column space of A*.
- Altenatively:
  - The number of rows with *nonzero leading coefficients*
  - The number of *linearly independent columns*.
- A **pivot** is the first nonzero element of a row for a matrix in *row echelon form*, and a pivot column is a column that contains a pivot.
- In fact, pivots are shared by both the columns and the rows, so $\mathbf{dim}(\mathbf{colspace}(A)) = \mathbf{dim}\ (\mathbf{rowspace}(A))$.
- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$

Practice: Rank

Find the rank of B:

$$B = \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

Practice: Rank [Solution]

Do Gaussian Elimination!

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 0 & 2 & 0 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_3]{R_3/2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} \boxed{1} & 2 & 1 & -4 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

There are *3 pivot columns*, so $\mathbf{rank}(B) = \mathbf{3}$.

Matrix Inverses

Inverses

A matrix, A, is **invertible** if there exists a matrix B, such that

$$AB = BA = I_n \implies B = A^{-1}$$

Conditions for inverse to exist:

- The matrix must be **square** (n x n)
- The columns must be **linearly independent** (injective) and they must **span** $\mathbb{R}^n$ (surjective).

Inverse Properties

Here are some useful properties of matrix inverses:

- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $kA^{-1} = k^{-1}A^{-1}$ for scalar k
- $(AB)^{-1} = B^{-1}A^{-1}$ (similar to transpose)
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

Invertibility

If A is a $n \times n$ **invertible** matrix, then the following are also true:

- A has n pivot positions.
- A has a trivial nullspace ($A\vec{x} = \vec{0}$ only if $\vec{x} = \vec{0}$).
- The columns and rows of A are **linearly independent**, and **span** $\mathbb{R}^n$. As a result, they form a **basis** for $\mathbb{R}^n$.
- The columnspace of A is $\mathbb{R}^n$, and is $n-$dimensional. So, A has a **rank** of n.
- For every $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a *unique solution*.
- The determinant of A is not 0.
- A does not have an eigenvalue of 0.

For a full description of the **invertible matrix theorem** (warning: parts are out of scope), look **here**.

Computing Inverses

Use Gaussian elimination! (Who would've guessed...)

- Construct an **augmented matrix** consisting of A and the identity matrix:

$$\left[A \mid I\right]$$

- Row reduce this augmented matrix until the *left side becomes the identity*, and the *right side becomes A^{-1}*:

$$\left[A \mid I\right] \longrightarrow \left[I \mid A^{-1}\right]$$

Computing Inverses: 2×2 Matrices

For a 2×2 matrix, you can **find the inverse quickly** using the following formula:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can derive this from the general method of finding inverses, but this can be convenient.

Side Note: In real numerical computation, we generally use gaussian elimination to find the inverse of a large matrix even through there is a theoretical formula deduced from Cramer's rule which has a terrible runtime.

Practice: Computing Inverses

Find the **inverse** of

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Practice: Computing Inverses [Solution]

Make an **augmented matrix** with A and the identity, and then perform **Gaussian Elimination**:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2\to R_2]{-R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1-3R_2\to R_1]{R_1-3R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Vector Spaces

A set of elements closed under vector addition and scalar multiplication.

- *"No escape properties"* for addition and scalar multiplication
- Must also contain the **0 vector** (special case of closed scalar multiplication).

Let $\mathbb{V}$ be a vector space:

- If $\vec{u}, \vec{v}$ are vectors in $\mathbb{V}$, then $\vec{u} + \vec{v}$ must also be in $\mathbb{V}$
- If $\vec{u} \in \mathbb{V}$ and k is a real number, then $k\vec{u}$ must be in $\mathbb{V}$

Thus, any **linear combination** of vectors in $\mathbb{V}$ is also in $\mathbb{V}$.

Practice: Is it a Vector Space?

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$)
- 5D space (spanning $\mathbb{R}^5$)
- n-D space (spanning $\mathbb{R}^n$)
- Line in $\mathbb{R}^2$ intersecting origin
- Line in $\mathbb{R}^2$ *not* intersecting origin
- First and third quadrant of $\mathbb{R}^2$
- Plane in $\mathbb{R}^3$ intersecting origin
- $\{\vec{0}\}$ (just the zero vector)
- $\{\vec{v}, \vec{v} \neq 0\}$
- $\text{span}(\vec{v})$

Practice: Is it a Vector Space? [Solution]

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$) [yes]
- 5D space (spanning $\mathbb{R}^5$) [yes]
- n-D space (spanning $\mathbb{R}^n$) [yes]
- Line in $\mathbb{R}^2$ intersecting origin [yes]
- Line in $\mathbb{R}^2$ *not* intersecting origin [no]
- First and third quadrant of $\mathbb{R}^2$ [no]
- Plane in $\mathbb{R}^3$ intersecting origin [yes]
- $\{\vec{0}\}$ (just the zero vector) [yes]
- $\{\vec{v}, \vec{v} \neq 0\}$ [no]
- $\text{span}(\vec{v})$ [yes]

Subspaces

- Definition: **A subspace is a subset of a vector space that is itself a vector space**.
- Suppose we have a vector space $\mathbb{V}$. A subset $\mathbb{S}$ of $\mathbb{V}$ is *only a subspace if the following three properties are met*:
    1. The **zero vector** of $\mathbb{V}$ is in $\mathbb{S}$
    2. $\mathbb{S}$ is closed under **vector addition**
    3. $\mathbb{S}$ is closed under **scalar multiplication**

(Same rules as before!)

Basis

Definition: A **basis** of a vector space is a **linearly independent** set of vectors that **spans** the vector space.

- **Linearly independent** (*not too big*): No vectors in a basis can be written as a linear combination of the other vectors.
- **Spanning** (*not too small*): All vectors in the vector space can be represented as a linear combination of the basis vectors.

Basis

A basis does not necessarily have to span $\mathbb{R}^n$ - it can span any vector space.

- A basis is a **minimum set of vectors** required to completely span a vector space.
- eg. Any basis of $\mathbb{R}^n$ contains **exactly n vectors**. In fact, an m-dimensional vector space must have m vectors in its basis.

Bases are **not unique**! Why? How many are there?

Practice: Basis

Are the following bases for $\mathbb{R}^n$?

$$\mathbb{V}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{V}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Practice: Basis [Solution]

Are the following bases for $\mathbb{R}^n$?

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Yes! The set is linearly independent and spanning.

$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ No! The vectors are negatives of each other!

Null Space

Null Space

- **Definition**: The **null space** of a matrix (transformation) is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.
- It is a *subspace of* $\mathbb{R}^n$.
- Solving a null space:
    1. Reduce to **reduced row echelon form**.
    2. Find solution to the system of equations.
    3. Represent the solutions in *matrix form*.

Practice: Null Space

Find the **null space** of A:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix}$$

Practice: Null Space [Solution]

First, **row reduce**.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Practice: Null Space [Solution]

- We saw that in the row-reduced matrix, there were **3 pivot columns**.
- Additionally, we know that there are 5 total “variables”
- Thus, we can say that there are **2 free variables**, and *obtain a basis for our null space in terms of these free variables*!
- The **pivot columns** occur at x_1, x_3, and x_5, so we can set $x_2 = r$ and $x_4 = s$
- Let's find our basis in terms of r, s!

Practice: Null Space [Solution]

Let's find our basis in terms of r, s!

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2r + s, \;\; x_2 = r$$

$$x_3 = -2s, \;\; x_4 = s, \;\; x_5 = 0$$

In matrix form:

$$\vec{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors

****The most important concept in linear algebra**

An scalar-vector pair $\lambda, \vec{v}$ are an **eigenvalue-eigenvector pair** of matrix A if applying A to $\vec{v}$ produces a version of $\vec{v}$ scaled by λ.

$$A\vec{v} = \lambda\vec{v}$$

Properties:

- Eigenvectors w/ distinct eigenvalues are **linearly independent**
- A matrix is **non invertible** iff 0 is an eigenvalue
- A scalar times an eigenvector is still an eigenvector
- Eigenvalues remain the **same across transposes** - not necessarily true for eigenvectors!

$$A^{-1}\vec{v} = \lambda^{-1}\vec{v} \text{ and } A^n\vec{v} = \lambda^n\vec{v}$$

Finding Eigenvalues and Eigenvectors

To find eigenvalues and eigenvectors of A:

- Find solutions to the **characteristic polynomial**, which is:

$$\det(A - \lambda I) = 0$$

Aside: the determinant of a 2×2 *matrix is*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- The solutions for λ give the **eigenvalues** of A.
- For each eigenvalue λ, calculate the null space of

$$A - \lambda I$$

- The basis for the null space will be your **eigenvectors**.

Practice: Eigenvalues and Eigenvectors

Find the **eigenvalues and eigenvectors** of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Practice: Eigenvalues and Eigenvectors [Solution]

First, find the solutions to the **characteristic polynomial**.

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
(3-\lambda)(4-\lambda) - 2 \cdot 1 &= 0 \\
\lambda^2 - 7\lambda + 10 &= 0 \\
\lambda_1 = 2,\ \lambda_2 &= 5
\end{aligned}$$

Practice: Eigenvalues and Eigenvectors [Solution]

Then, find the **null space** of $A - \lambda I$ for each lambda:

$$\left[\begin{array}{cc|c} 3-\lambda_1 & 1 & 0 \\ 2 & 4-\lambda_1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3-\lambda_2 & 1 & 0 \\ 2 & 4-\lambda_2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

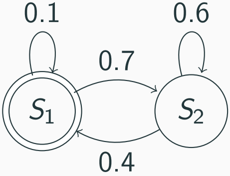
PageRank

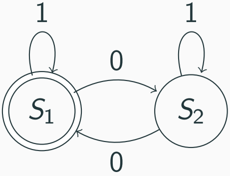
Graphs, Flow, and Transition Matrices

A **transition matrix** represents a directed graph of states and transitions.

- Edges of the graph represent *what fraction of one state moves to the next*.
- Entry **ij** in matrix means *fraction of water from node j entering node i*.
- Columns sum to ≤ 1 (*What would it mean of the sum of a column was greater than 1?*).
- Examples: Social networks, PageRank

Here are some examples of **graphs** and their corresponding **transition matrices**. Notice that element ij represents the *fractional transition* from state j to state i.





$$T = \begin{bmatrix} 0.1 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Practice: Transition Matrices

- What do we know about the system if all *columns* each **sum to 1**?
- Less than 1? (Think about what physically happens if the system was water flows)
- Greater than 1?

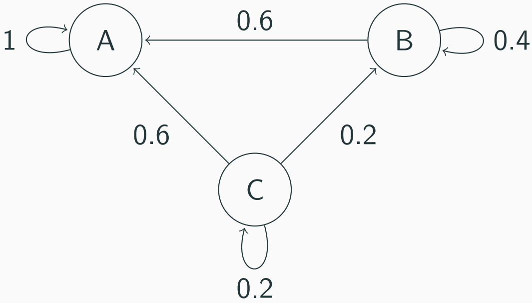
Practice: Transition Matrices [Solution]

- What do we know about the system if all *columns* each **sum to 1**? **Flow is conserved: no "water" is added or lost.**
- Less than 1? (Think about what physically happens if the system was water flows) **Flow is lost.**
- Greater than 1? **Flow increases.**

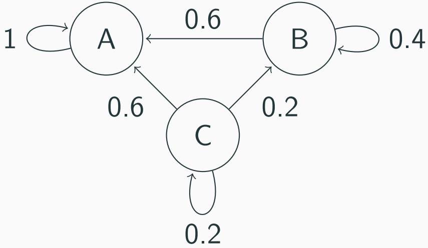
- Given a flow graph, we wish to **rank the nodes by their "importance"**, i.e. which state will hold the most "water" when the system reaches a steady state.
- Procedure:
  1. Write A, the **transition matrix** of the flow diagram.
  2. Find the eigenvector(s) of A, whose **eigenvalue is 1**. If $A\vec{v} = \vec{v}$, $\vec{v}$ is a **steady state vector**.
  3. $\vec{v}$ contains the *values the nodes will stabilize at*, and ranks the importance of each node

Practice: PageRank

Find the **transition matrix** for this flow diagram.



Practice: PageRank [Solution]



Assuming the first column represents *flow from state A*, the second represents *flow from state B*, etc. the **transition matrix** is:

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank

Find the **steady state values** of the system.

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank [Solution]

To find the **steady state values**, find the *eigenvectors corresponding to an eigenvalue of 1*.

$$A\vec{v} = \vec{v}$$

$$(A - I)\vec{v} = \vec{0}$$

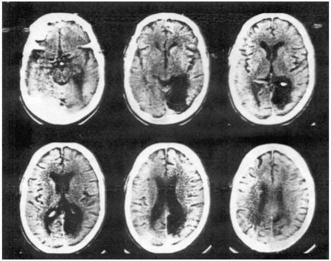
$$\left[\begin{array}{ccc|c} 0 & 0.6 & 0.6 & 0 \\ 0 & -0.6 & 0.2 & 0 \\ 0 & 0 & -0.8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Practice Problems

Tomography

- **Tomography** refers to *imaging by section*
- Idea: use a penetrating wave to image sections of an object
- Then process those section to *recreate a 3D image*
- Examples of tomography: X-ray, CAT scan, MRI, Ultrasound



Tomography

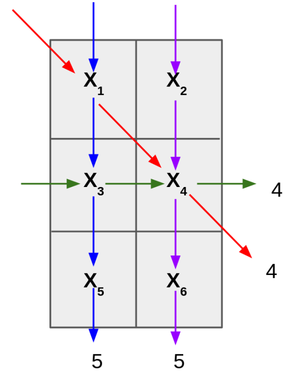
Basic Tomography Example: Suppose we have 3 types of bottles, each absorbing some amount of light:

Problem: Given an *grid of bottles*, and a straight-line laser, how can we *determine what bottles we have*?

Idea: Shine a light through different rows and columns and record the light absorbed.

Material	Milk (M)	Juice(J)	Empty (O)
Light absorbed	3	2	1

Tomography

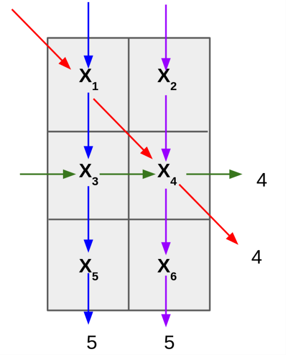


Suppose **each bottle absorbs X_i light**, and we *record the light absorbed from these four light rays*.

Write the **system of equations** describing the light absorbed by each bottle.

Is this system *solvable*? Why or why not? What if we know each X_i must be *either 1, 2, or 3*?

Tomography [Solution]



$$X_1 + X_4 = 4$$
$$X_1 + X_3 + X_5 = 5$$
$$X_2 + X_4 + X_6 = 5$$
$$X_3 + X_4 = 5$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 0 & 0 & 5 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0.5 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0.5 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 3 \\ 0 & 0 & 0 & 1 & -0.5 & 0 & 2 \end{array}\right]$$

Tomography [Solution]

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0.5 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0.5 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 3 \\ 0 & 0 & 0 & 1 & -0.5 & 0 & 2 \end{array}\right]$$

This system would have **infinitely many solutions**, but we can solve the system because we know all X_i must be 1, 2, or 3!

$$X_1 + 0.5X_5 = 2 \implies X_1 = 1 \text{ and } X_5 = 2.$$

$$X_3 + 0.5X_5 = 3 \implies X_3 = 2$$

$$X_4 - 0.5X_5 = 2 \implies X_4 = 3$$

$$X_2 + 0.5X_5 + X_6 = 3 \implies X_2 + X_6 = 2 \implies X_2 = X_6 = 1$$

Bases in $\mathbb{R}^3$

1. **True/False**: There exists a scalar x such that the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$.

2. **True/False**: There exists a scalar x such that the vectors $\left\{ \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}, \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 1 \\ 1+x \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$.

Bases in $\mathbb{R}^3$ [Solution]

1. Can *two vectors* ever be a basis for a **3 dimensional vector space**?
    - No!
    - The two vectors considered in the question can **never** be a basis for the three dimensional vector space
2. To form a basis for a vector space, the vectors must: **span the space** and be **independent**.
    - Are the three vectors **linearly independent**?
    - No! The first two vectors sum to give the third.
    - Therefore, no matter what the value of x is, these three vectors can **never** form a basis for the vector space $\mathbb{R}^3$.

Spring 2015 Midterm 1: Alice's Photos

At the beginning of each day, Alice puts \$1 of her money into the stock market. At the end of the day she sells her stocks and looks at how much money she got back. She lets a Berkeley-based mutual fund manage the \$1 she invests. This fund invests in three companies from the area: Friendly Faces Inc., Search and Spend Inc., and Delicious Devices Inc.

Each company has a growth $g_f(i), g_(i)$, and $g_d(i)$ on dat i. What that means is that if Alice invests \$1 in Friendly Faces on day i, then she gets back $g_f(i)$ dollars at the end of day i. The mutual fund invests fractions α_f, α_s, and α_d of Alice's money in each of the three companies. Note that these fractions are always the same (i.e. they do not change from one day to the next), and that **all of her money is always invested in the market**.

Part 1: Using the terms defined above, write an expression using an **inner product** that would *compute how many dollars Alice gets back after the first day*.

Spring 2015 Midterm 1: Alice's Photos [Solution]

The model: Let us first think about how to model the situation, before the math:

- One thing that we notice is that we have **three growth functions**, each of which represent *some sort of component of Alice's investments*.
- Additionally, we have **investment fractions** corresponding to each of these components.
- Finally, we want to find the *total amount of money Alice would get after one day*. The operations needed to do this: **summing over the products of components sound awfully like an inner product**. Additionally, it seems natural to *represent the growth and the investment fraction as vectors*.

Spring 2015 Midterm 1: Alice's Photos [Solution]

Growth Vector:

$$\vec{g}(1) = \begin{bmatrix} g_f(1) \\ g_s(1) \\ g_d(1) \end{bmatrix}$$

Investment Vector:

$$\vec{\alpha} = \begin{bmatrix} \alpha_f \\ \alpha_s \\ \alpha_d \end{bmatrix}$$

Dollars returned after Day 1:

$$\vec{g}(1)^T \vec{\alpha} = g_f(1)\alpha_f + g_s(1)\alpha_s + g_d(1)\alpha_d$$

Spring 2015 Midterm 1: Alice's Photos

Part 2: Assume that Alice gets \$1.7 back from her \$1 investment on the first day (i.e. the result of the expression you wrote in part 1 is \$1.7).

On the second day, Alice again invests \$1, but gets \$0 back! This makes Alice angry, and she wants to find out the investment strategy of the mutual fund.

She does some research and finds that $g_f(1) = 2, g_s(1) = 1$, and $g_d(1) = 2$, abd on te second dat $g_f(2) = 2, g_s(2) = 2$, but $g_d(2) = -3$.

Write a matrix equation Alice could use to find our what the mutual fund's investment strategy was (i.e. find the values of α_f, α_s, and α_d).

Spring 2015 Midterm 1: Alice's Photos [Solution]

- First thing we need to note is that we are **solving for the alpha-vector**
- We get **two equations** from noticing the following:

$$\vec{g}(1)^T \vec{\alpha} = 1.7, \vec{g}^T \vec{\alpha} = 0$$

- We have three unknowns... *how do we get the third equation*?

Spring 2015 Midterm 1: Alice's Photos [Solution]

- Key idea: *all of Alice's money is invested*, so:

$$\alpha_f + \alpha_s + \alpha_d = 1$$

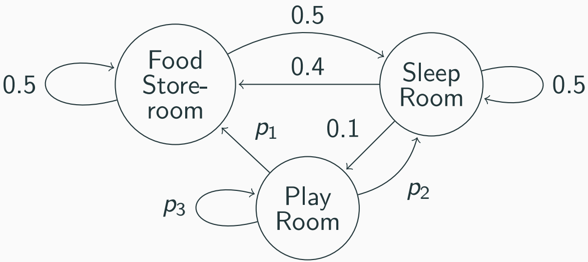
- If we write our equations in **matrix form**:

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_f \\ \alpha_s \\ \alpha_d \end{bmatrix} = \begin{bmatrix} 1.7 \\ 0 \\ 1 \end{bmatrix}$$

- Assuming the matrix is invertible (it is)

$$\begin{bmatrix} \alpha_f \\ \alpha_s \\ \alpha_d \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1.7 \\ 0 \\ 1 \end{bmatrix}$$

Page Rank



(a) Let the number of furballs in each room (Food Storeroom, Sleep Room, and Play Room) at time n be $x_f[n]$, $x_s[n]$, and $x_p[n]$, respectively. We would like to find the **transition matrix** A such that

$$\begin{bmatrix} x_f[n+1] \\ x_s[n+1] \\ x_p[n+1] \end{bmatrix} = A \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix}$$

Write A using the numbers and variables in the diagram.

Page Rank [Solution]

The **transition matrix** is

$$A = \begin{bmatrix} 0.5 & 0.4 & p_1 \\ 0.5 & 0.5 & p_2 \\ 0 & 0.1 & p_3 \end{bmatrix}$$

Remember that the element at row i, column j represents the number of furballs going from room j to room i.

(b) We know that *no furballs enter or leave* the configuration of tunnels shown above and that during the time you're observing the behavior, *no furballs die or are born*. What **constraint** does this place on the values of p_1, p_2, p_3? Write your answer in *equation form*.

Page Rank [Solution]

(b) We know that *no furballs enter or leave* the configuration of tunnels shown above and that during the time you're observing the behavior, *no furballs die or are born*. What **constraint** does this place on the values of p_1, p_2, p_3? Write your answer in *equation form*.

No furballs enter or leave the system, so the **sum of each column must be 1**. So,

$$p_1 + p_2 + p_3 = 1$$

(c) Suppose we let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\vec{x}[n] = \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix}$, and that we are sure that $x_p[n]$ is nonzero.

Express $\vec{p}$ as a function of the numbers in the diagram. $\vec{x}[n]$, and $\vec{x}[n+1]$. (*Hint: what is the relationship between* $\vec{x}[n+1]$ *and* $\vec{x}[n]$*?*)

Page Rank [Solution]

(c) Suppose we let $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\vec{x}[n] = \begin{bmatrix} x_f[n] \\ x_s[n] \\ x_p[n] \end{bmatrix}$, and that we are sure that $x_p[n]$ is nonzero.

Express $\vec{p}$ as a function of the numbers in the diagram. $\vec{x}[n]$, and $\vec{x}[n+1]$. (*Hint: what is the relationship between* $\vec{x}[n+1]$ *and* $\vec{x}[n]$*?*)

$$\vec{p} = \frac{1}{x_p[n]}\left(\vec{x}[n+1] - x_f[n]\begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} - x_s[n]\begin{bmatrix} 0.4 \\ 0.5 \\ 0.1 \end{bmatrix}\right)$$

Steady State

(a) Consider a 2x2 matrix A, where max(eigenvalues(A)) > 1. Is it possible for the system defined by $x(t+1) = Ax(t)$ to be **stable** (i.e. non-exploding) for some initial x?

Steady State [Solution]

(a) Consider a 2x2 matrix A, where $\max(\text{eigenvalues}(A)) > 1$. Is it possible for the system defined by $x(t+1) = Ax(t)$ to be **stable** (i.e. non-exploding) for some initial x?

Yes, but only as long as there is **another eigenvalue such that abs(eigenvalue) $\leq$ 1** and only for an **appropriate initial state x_0**

Steady State

(b) Given a **conservative** state transition matrix A, is it possible for A to have **multiple eigenvectors** corresponding to the eigenvalue 1? *What does it say about the system?*

Steady State [Solution]

(b) Given a **conservative** state transition matrix A, is it possible for A to have **multiple eigenvectors** corresponding to the eigenvalue 1? *What does it say about the system?*

Yes. There is a set of **linearly independent steady-state**s, and *any linear combination of those states is stable*.