Week 2

August 18, 2021

Linear Regression

The previously described home price data example is an example of a supervised learning regression problem.

Notation (used throughout the course)

- m = number of training examples.
- $x's$ = input variables / features.
- $y's$ = output variable "target" variables.
- (x, y) - single training example.
- (x^i, y^j) - specific example (ith training example).

Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
...	...

$m = 47$

With our training set defined - how do we used it?

- Take training set.
- Pass into a learning algorithm.
- Algorithm outputs a function (h = hypothesis).
- This function takes an input (e.g. size of new house).
- Tries to output the estimated value of Y.

$$h_\theta(x) = \theta_0 + \theta_1 x$$

- Y is a linear function of x!
- θ_0 is zero condition.
- θ_1 is gradient.

Cost function

- We may use a cost function to determine how to fit the best straight line to our data.
- We want to want to solve a minimization problem. Minimize $(h_\theta(x) - y)^2$.
- Sum this over the training set.

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{m}^{i=1} (h_\theta(x^{(i)}) - y^{(i)})^2$$

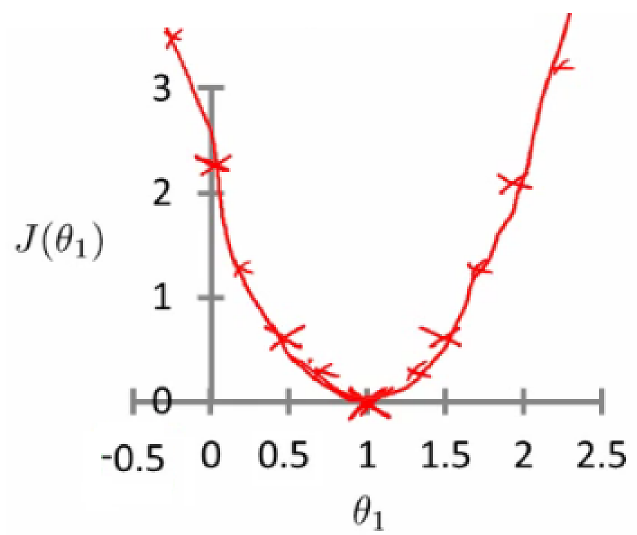
Example

For $\theta_0 = 0$ we have:

$$h_\theta(x) = \theta_1 x \quad and \quad J(\theta_1) = \frac{1}{2m} \sum_{m}^{i=1} (h_\theta(x^{(i)}) - y^{(i)})^2$$

Data:

- $\theta_1 = 1$ and $J(\theta_1) = 0$.
- $\theta_1 = 0.5$ and $J(\theta_1) = 0.58$.
- $\theta_1 = 0$ and $J(\theta_1) = 2.3$.



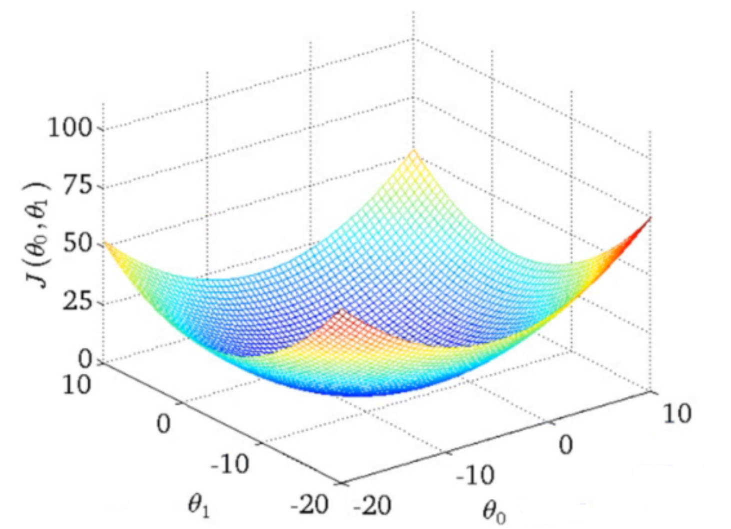
The optimization objective for the learning algorithm is find the value of θ_1 which minimizes $J(\theta_1)$. So, here $\theta_1 = 1$ is the best value for θ_1.

A deeper insight into the cost function - simplified cost function

The real cost function takes two variables as parameters! $J(\theta_0, \theta_1)$.

We can now generates a 3D surface plot where axis are:

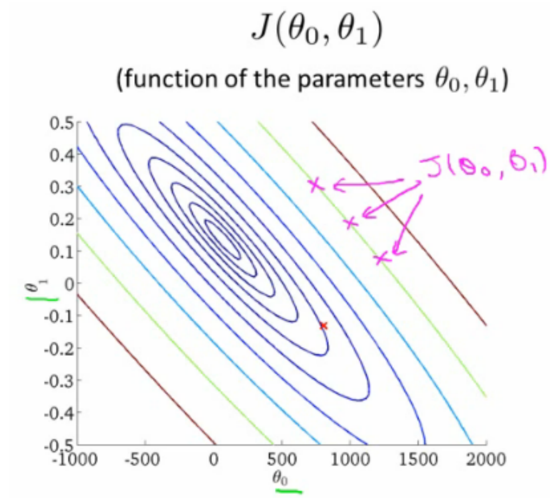
- $X = \theta_1$.
- $Z = \theta_0$.
- $Y = J(\theta_0, \theta_1)$.



The best hypothesis is at the bottom of the bowl.

Instead of a surface plot we can use a contour figures/plots.

- Set of ellipses in different colors.
- Each colour is the same value of $J(\theta_0, \theta_1)$, but obviously plot to different locations because θ_1 and θ_0 will vary.
- Imagine a bowl shape function coming out of the screen so the middle is the concentric circles.

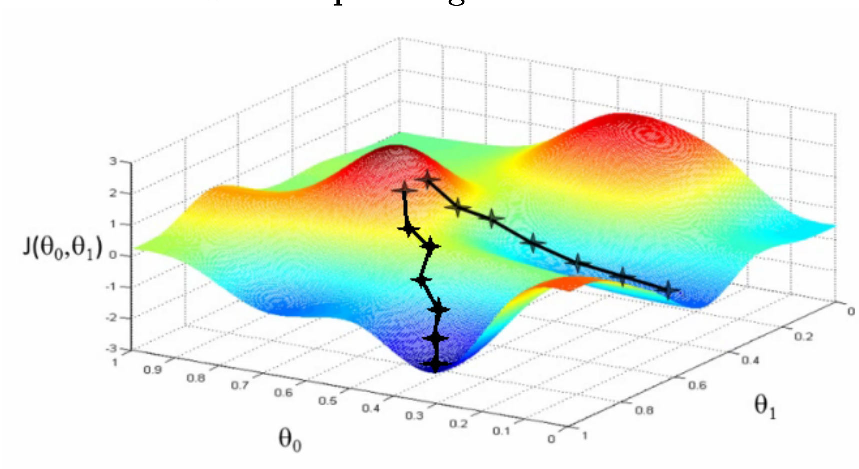


The best hypothesis is located in the center of the contour plot.

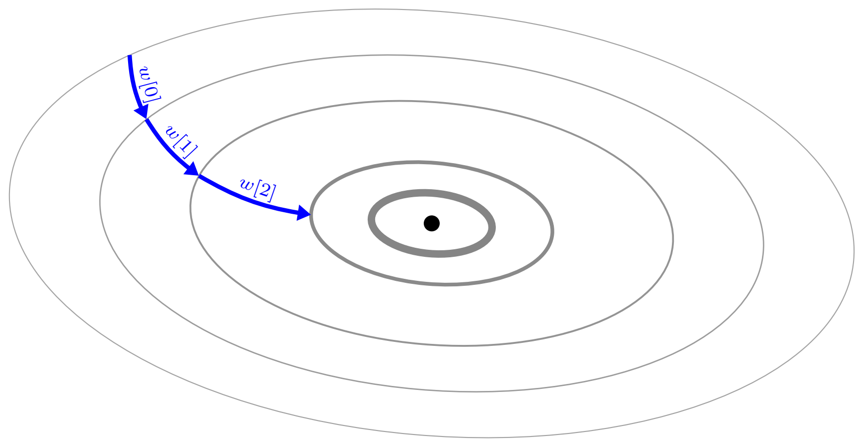
Gradient descent algorithm

- Begin with initial guesses, could be (0,0) or any other value.
- Continually change values of θ_0 and $theta_1$ to try to reduce $J(\theta_0, \theta_1)$.
- Continue until you reach a local minimum.
- Reached minimum is determined by the starting point.

Surface plot of gradient descent.



Contour plot of gradient descent.



Algorithm 1 Gradient Descent

while not converged **do**
$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1).$$
$$(for\ j = 1\ and\ j = 0)$$

Two key terms in the algorithm:

- α term
- Derivative term

Partial derivative vs. derivative

- Use partial derivative when we have multiple variables but only derive with respect to one.
- Use derivative when we are deriving with respect to all the variables.

Derivative says:

- Let's look at the slope of the line by taking the tangent at the point.
- As a result, going towards the minimum (down) will result in a negative derivative; nevertheless, alpha is always positive, thus $J(\theta_1)$ will be updated to a lower value.
- Similarly, as we progress up a slope, we increase the value of $J(\theta_1)$.

α term

- If it's too small, it takes too long to converge.
- If it is too large, it may exceed the minimum and fail to converge.

When you get to a local minimum

- Gradient of tangent/derivative is 0
- So derivative term = 0
- $\alpha \cdot 0 = 0$
- So $\theta_1 = \theta_1 - 0$.
- So θ_1 remains the same.

Linear regression with gradient descent

Apply gradient descent to minimize the squared error cost function $J(\theta_0, \theta_1)$.

$$\begin{aligned}\frac{\partial}{\partial\theta_j}J(\theta_0,\theta_1) &= \frac{\partial}{\partial\theta_j}\frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 \\ &= \frac{\partial}{\partial\theta_j}\frac{1}{2m}\sum_{i=1}^{m}(\theta_0 + \theta_1 x^{(i)} - y^{(i)})^2\end{aligned}$$

For each case, we must determine the partial derivative:

$$j = 0 : \frac{\partial}{\partial\theta_0}J(\theta_0,\theta_1) = \frac{\partial}{\partial\theta_j}\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})$$

$$j = 1 : \frac{\partial}{\partial\theta_1}J(\theta_0,\theta_1) = \frac{\partial}{\partial\theta_j}\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

The linear regression cost function is always a convex function - always has a single minimum. So gradient descent will always converge to global optima.

Two extension to the algorithm

Normal equation for numeric solution

- To solve the minimization problem we can solve it $[minJ(\theta_0, \theta_1)]$ exactly using a numeric method which avoids the iterative approach used by gradient descent.

- Normal equations method.
- Can be much faster for some problems, but it is much more complicated (will be covered in detail later).

We can learn with a larger number of features

- e.g. with houses: Size, Age, Number bedrooms, Number floors...
- Can't really plot in more than 3 dimensions.
- Best way to get around with this is the notation of linear algebra (matrices and vectors).