Exercise 7, problem set 6

Dawid Żywczak, 19.04.2020.

We have to find distribution of random variable $Z = \sum_{k=1}^{n} X_k$. Where variables X_k are independent. We will use MGFs to do it. Let X_k has $Gamma(b, p_k)$ distribution for $k = 1, \ldots, n$.

First let's find MGF of gamma distribution

$$M(t) = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx} \frac{b^p}{\Gamma(p)} x^{p-1} e^{-bx} dx = \int_0^\infty \frac{b^p}{\Gamma(p)} x^{p-1} e^{(t-b)x} dx$$

Now we will use substitution

$$u = (b-t)x \qquad x = \frac{u}{b-t}$$
$$dx = \frac{1}{b-t} du$$

Then

$$\int_0^\infty \frac{b^p}{\Gamma(p)} x^{p-1} e^{-u} \frac{1}{b-t} uy = \int_0^\infty \frac{b^p}{\Gamma(p)} (\frac{u}{b-t})^{p-1} e^{-u} \frac{1}{b-t} du =$$
$$\frac{b^p}{\Gamma(p)(b-t)^p} \int_0^\infty u^{p-1} e^{-u} du = \frac{b^p}{\Gamma(p)(b-t)^p} \Gamma(p) = (\frac{b}{b-t})^p$$

Now when we have MGF of gamma distribution we can calculate MGF of variable Z. Using fact that X_k are independent we can write:

$$M_Z(t) = M_{X_1}(t) M_{X_2}(t) \ldots M_{X_n}(t) = (\frac{b}{b-t})^{p_1} (\frac{b}{b-t})^{p_2} \ldots (\frac{b}{b-t})^{p_3} =$$
$$(\frac{b}{b-t})^{\sum_{k=1}^{n} p_k}$$

Which is MGF of $Gamma(b, \sum_{k=1}^{n} p_k)$