极限必做 150 解答

刘

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17 级

刘言：我的个人解答，经过老司机（柯）检查，如果还有错误，欢迎大佬联系我及时改正，我的 QQ：198924030

$$1.\lim_{x\to 0}\frac{1}{\mathrm{x}}(\frac{1}{\sin x}-\frac{1}{\tan x})$$

$$=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\tan x(1-\cos x)}{x^3}=\frac{1}{2}$$

$$2.\lim_{x\to 0}\frac{\ln(a+x)+\ln(a-x)-2\ln a}{x^2}$$

$$=\lim_{x\to 0}\frac{\frac{1}{a+x}-\frac{1}{a-x}}{2x}=\lim_{x\to 0}\frac{-2x}{2x(a+x)(a-x)}=-\frac{1}{a^2}$$

$$3.\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}x^4}}{\frac{1}{2}x^2}=\sqrt{2}$$

$$4.\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$$

$$=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}+\lim_{x\to a}\frac{\sqrt{x-a}}{\sqrt{(x+a)(x-a)}}=\lim_{x\to a}\frac{x-a}{\sqrt{x^2-a^2}(\sqrt{x}+\sqrt{a})}+\frac{1}{\sqrt{2a}}=\frac{1}{\sqrt{2a}}$$

$$5.\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1+x}-1}$$

$$=\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt{1+x}-1}-\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{\sqrt{1+x}-1}=1$$

$$6.\lim_{x\to 0}\frac{\tan mx}{\sin nx}(m\text{、n为正整数})$$

$$=\lim_{x\to 0}\frac{mx}{nx}=\frac{m}{n}$$

$$7.\lim_{x\to 0}\frac{\ln(1+x+x^2)+\ln(1-\mathrm{x}+\mathrm{x}^2)}{\mathrm{secx}-\mathrm{cosx}}$$

$$=\lim_{x\to 0}\frac{\ln\left[(x^2+1)^2-x^2\right]}{\sec x(1-\cos^2 x)}=\lim_{x\to 0}\frac{x^4+x^2}{x^2}=1$$

$$8.\lim_{x\to 0}\frac{1}{x}\ln(\frac{e^x+e^{2x}+..+e^{nx}}{n})$$

$$=\lim_{x\to 0}\frac{1}{x}\left(\frac{e^x-1}{n}+\frac{e^{2x}-1}{n}+..+\frac{e^{nx}-1}{n}\right)=\frac{1}{n}+\frac{2}{n}+..+\frac{n}{n}=\frac{n+1}{2}$$

$$9.\lim_{n\to\infty}\sin\left(\sqrt{n^2+a^2}\pi\right)$$

$$=\lim_{n\to\infty}(-1)^n\sin(\sqrt{n^2+a^2}\pi-n\pi)=\lim_{n\to\infty}(-1)^n\sin\frac{a^2}{\sqrt{n^2+a^2}}\pi=0$$

$$10.\lim_{n\to\infty}\left(\frac{3n^2-2}{3n^2+4}\right)^{n(n+1)}$$

$$=\lim_{n\to\infty}\left(1-\frac{6}{3n^2+1}\right)^{n(n+1)}=e^{\lim\limits_{n\to\infty}\frac{-6n(n+1)}{3n^2+1}}=e^{-2}$$

$$11.\lim_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n$$

$$=\lim_{n\to\infty}\left(1+\frac{2}{2n-1}\right)^n=e^{\lim\limits_{n\to\infty}\frac{2n}{2n-1}}=e$$

$$12.\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$$

$$=e^{\lim\limits_{n\to\infty}n\left(\frac{\sqrt[n]{a}-1}{2}+\frac{\sqrt[n]{b}-1}{2}\right)}=e^{\lim\limits_{n\to\infty}n(\frac{\ln a}{n}+\frac{\ln b}{n})}=e^{\frac{a}{2}+\frac{b}{2}}=\sqrt{ab}$$

$$13.\lim_{n\to\infty}n^2\left[e^{(2+\frac{1}{n})}+e^{(2-\frac{1}{n})}-2e^2\right]$$

令$\dfrac{1}{n}=t$

$$=\lim_{t\to0}\frac{e^{(2+t)}+e^{(2-t)}-2e^2}{t^2}=\lim_{t\to0}\frac{e^{(2+t)}-e^{(2-t)}}{2t}=\lim_{t\to0}\frac{e^{(2+t)}+e^{(2-t)}}{2}=e^2$$

$$14.\lim_{n\to\infty}n\left[a^{\frac{1}{n}}-1\right]$$ (a为整数)

$$=\lim_{n\to\infty}n\frac{\ln a}{n}=\ln a$$

$$15.\lim_{n\to\infty}\left(\frac{\sqrt{n^2+1}}{n+1}\right)^n$$

$$=e^{\lim\limits_{n\to\infty}n\left(\frac{\sqrt{n^2+1}}{n+1}-1\right)}=e^{\lim\limits_{n\to\infty}n\left(\frac{\sqrt{n^2+1}-n-1}{n+1}\right)}=e^{\lim\limits_{n\to\infty}\frac{-2n}{\sqrt{n^2+1}+n+1}}=e^{-1}$$

$$16.\lim_{n\to\infty} n^2\left[\ln(a+\frac{1}{n})+\ln(a-\frac{1}{n})-2\ln a\right]$$

令$\frac{1}{\text{n}}=t$,同第二题

$$17.\lim_{n\to\infty} n\left(e^{\frac{a}{n}}-e^{\frac{b}{n}}\right)$$

$$=\lim_{n\to\infty} n(e^{\frac{a}{n}}-1)-\lim_{n\to\infty} n\left(e^{\frac{b}{n}}-1\right)=a-b$$

$$18.\lim_{n\to\infty}\left(\frac{1}{n}+e^{\frac{1}{n}}\right)^n$$

$$=e^{\lim\limits_{n\to\infty}}\frac{n}{n}+n(e^{\frac{1}{n}}-1)=e^2$$

$$19.\lim_{n\to\infty} n\left[\ln(n+1)-\ln n\right]$$

$$=\lim_{n\to\infty} n\ln\left(\frac{n+1}{n}\right)=\lim_{n\to\infty} n\ln(1+\frac{1}{n})=1$$

$$20.\lim_{x\to-1}\frac{x^2-1}{\ln|x|}$$

$$=\lim_{x\to-1}\frac{(x+1)(x-1)}{\ln(-x)}=\lim_{x\to-1}\frac{(x+1)(x-1)}{-(x+1)}=2$$

$$21.\lim_{x\to+\infty}\left[\ln(1+x)-\ln(x-1)\right]x$$

$$=\lim_{x\to+\infty}\ln(\frac{1+x}{x-1})x=\lim_{x\to+\infty}\ln(1+\frac{2}{x-1})x=\lim_{x\to+\infty}\frac{2x}{x-1}=2$$

$$22.\lim_{x\to0}\frac{\ln\cos x}{x^2}$$

$$=\lim_{x\to0}\frac{\ln(1+\cos\mathrm{x}-1)}{x^2}=\lim_{x\to0}\frac{\cos x-1}{x^2}=-\frac{1}{2}$$

$$23.\lim_{x\to+\infty}\left[(x+2)\ln(x+2)-2(x+1)\ln(x+1)+x\ln x\right]x$$

$$=\lim_{x\to+\infty}\left[x\ln(x+2)-x\ln(x+1)+x\ln x-x\ln(x+1)+2\ln(\frac{x+2}{x+1})\right]x$$

$$=\lim_{x\to+\infty}\left[x\ln\frac{x+2}{x+1}+x\ln\frac{x}{x+1}\right]x+2=\lim_{x\to+\infty}x^2\ln(1-\frac{1}{(x+1)^2})+2=2-1=1$$

$$24.\lim_{x\to 0}\left(\sqrt{1+x^2}+x\right)^{\frac{1}{x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}+1}=e$$

$$25.\lim_{x\to 0}(\cos\sqrt{x})^{\frac{1}{x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\cos\sqrt{x}-1}{x}}=e^{\lim\limits_{x\to 0}\frac{-\frac{1}{2}x}{x}}=e^{-\frac{1}{2}}$$

$$26.\lim_{x\to 0}\left[\tan(\frac{\pi}{4}-x)\right]^{\cot x}$$

$$=e^{\lim\limits_{x\to 0}\frac{\sin(\frac{\pi}{4}-x)-\cos(\frac{\pi}{4}-x)}{\cos(\frac{\pi}{4}-x)\tan x}}=e^{\sqrt{2}\lim\limits_{x\to 0}-\cos(\frac{\pi}{4}-x)-\sin(\frac{\pi}{4}-x)}=e^{-2}$$

$$27.\lim_{x\to 0}\left(\sin x+\cos x\right)^{\frac{1}{x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\sin x+\cos x-1}{x}}=e^{\lim\limits_{x\to 0}\frac{\cos x-1}{x}+1}=e$$

$$28.\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan^2 x}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos^2 x}}=e^{\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{-2\cos x}}=e^{-\frac{1}{2}}$$

$$29.\lim_{x\to\infty}\left(\frac{2x^2-x+1}{2x^2+x-1}\right)^{x}$$

$$=e^{\lim\limits_{x\to\infty}x\left(\frac{2x^2-x+1}{2x^2+x-1}-1\right)}=e^{\lim\limits_{x\to\infty}x\left(\frac{-2x+2}{2x^2+x-1}\right)}=e^{-1}$$

$$30.\lim_{x\to\infty}\left(\frac{2x+1}{2x-1}\right)^{3x}$$

$$=\lim_{x\to\infty}(1+\frac{2}{2x+1})e^{\lim\limits_{x\to\infty}3x\left(\frac{2}{2x+1}\right)}=e^{6}$$

$$31.\lim_{x\to 0}(1-2x)^{\frac{1}{x}}=e^{-2}$$

32. $\lim\limits_{x\to+\infty}\cos^x(\frac{\pi}{\sqrt{x}})$

$=e^{\lim\limits_{x\to+\infty}x\left[\cos\left(\frac{\pi}{\sqrt{x}}-1\right)\right]}=e^{\lim\limits_{x\to+\infty}x(-\frac{\pi^2}{2x})}=e^{-\frac{\pi^2}{2}}$

33. $\lim\limits_{x\to a}\left(\frac{\cos x}{\cos a}\right)^{\frac{1}{x-a}}$

$=e^{\lim\limits_{x\to a}\frac{\cos x-\cos a}{\cos a(x-a)}}=e^{\lim\limits_{x\to a}\frac{-\sin x}{\cos a}}=e^{-\tan a}$

34. $\lim\limits_{x\to0}\frac{\ln(x_0+x)+\ln(x_0-x)-2\ln x_0}{+x^2}$同第二题$-\frac{1}{x_0^2}$

35. $\lim\limits_{x\to+\infty}\ln(1+e^{ax})\ln(1+\frac{b}{x})$

$=\lim\limits_{x\to+\infty}\ln(1+e^{ax})\frac{b}{x}=\lim\limits_{x\to+\infty}\frac{b\ln(1+e^{ax})}{x}=\lim\limits_{x\to+\infty}\frac{abe^{ax}}{1+e^{ax}}=ab$

36. $\lim\limits_{x\to0}\frac{\ln(\sec x+\tan x)}{\sin x}$

$=\lim\limits_{x\to0}\frac{\ln\left[(1+\sin x)\cos x\right]}{x}=\lim\limits_{x\to0}\frac{\ln(1+\sin x)}{x}+\lim\limits_{x\to0}\frac{\ln\cos x}{x}=1$

37. $\lim\limits_{x\to+\infty}x^2(a^{\frac{1}{x}}-a^{\frac{1}{x+1}})$

$=\lim\limits_{x\to+\infty}x^2a^{\frac{1}{x+1}}(a^{\frac{1}{x}-\frac{1}{x+1}}-1)=\lim\limits_{x\to+\infty}x^2(\frac{1}{x}-\frac{1}{x+1})\ln a=\lim\limits_{x\to+\infty}\frac{x^2}{x(x+1)}\ln a=\ln a$

38. $\lim\limits_{x\to0}\left(\frac{1+xa^x}{1+xb^x}\right)^{\frac{1}{x^2}}$

$=\lim\limits_{x\to0}\left(\frac{xa^x-xb^x}{1+xb^x}\right)^{\frac{1}{x^2}}=\exp\lim\limits_{x\to0}\frac{a^x-b^x}{x(1+xb^x)}=\exp\lim\limits_{x\to0}\frac{a^x-1}{x}-\frac{b^x-1}{x}=e^{(\ln a-\ln b)}=\frac{a}{b}$

39. $\lim\limits_{x\to0}\frac{e^{5x}-1}{x}=5$

$$40.\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$$

$$=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x-1}{2x}-\lim_{x\to 0}\frac{e^{-x}-1}{2x}=\frac{1}{2}+\frac{1}{2}=1$$

$$41.\lim_{x\to 0}\frac{e^{\tan x}-e^{3x}}{\sin x}$$

$$=\lim_{x\to 0}\frac{e^{\tan x}-1}{x}-\lim_{x\to 0}\frac{e^{3x}-1}{x}=1-3=-2$$

$$42.\lim_{x\to 0}\frac{a^{3x}-1}{x}$$

$$=\lim_{x\to 0}\frac{3x\ln a}{x}=3\ln a$$

$$43.\lim_{x\to a}\frac{a^x-a^a}{x-a}$$

$$=\lim_{x\to a}\frac{a^a(a^{x-a})}{x-a}=\lim_{x\to a}\frac{a^a(x-a)\ln a}{x-a}=a^a\ln a$$

$$44.\lim_{x\to x_0}\frac{\ln x-\ln x_0}{x-x_0}=\frac{1}{x_0}$$

$$45.\lim_{x\to 1}\frac{x^n-1}{x-1}$$

令$x-1=t$

$$=\lim_{t\to 0}\frac{(1+t)^n-1}{t}=\lim_{t\to 0}\frac{nt}{t}=n$$

$$46.\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$$

令$\frac{1}{\mathrm{x}}=t$，如12题

$$47.\lim_{x\to 0}(ax+e^{bx})^{\frac{1}{x}}$$

$$=\exp\lim_{x\to 0}\frac{ax+e^{bx}-1}{x}=\exp\lim_{x\to 0}\frac{e^{bx}-1}{x}+a=e^{a+b}$$

48.证明不等式:$\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}$ 其中n为正整数

解：令$f(x)=\ln(1+x)-x$

$f'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}\le 0$,当$x\in[0,+\infty)$

所以$f(x)$在$[0,+\infty)$递减 所以$f(x)< f(0), x\in[0,+\infty)$

即$\ln(1+x)-x<0\Rightarrow \ln(1+x)<x\Rightarrow \ln(1+\frac{1}{n})<\frac{1}{n}$

证毕

49. 设$\alpha(x)=x^3-3x+2, \beta(x)=c(\mathrm{x}-1)^n$,确定$c$及n, 使当$\mathrm{x}\to 1$时，$\alpha(x)\sim\beta(x)$

解:$\lim\limits_{x\to 1}\frac{\alpha(x)}{\beta(x)}=1\Rightarrow \lim\limits_{x\to 1}\frac{x^3-3x+2}{c(x-1)^n}=1\Rightarrow \lim\limits_{x\to 1}\frac{3x^2-3}{cn(x-1)^{n-1}}=1$

$\Rightarrow \lim\limits_{x\to 1}\frac{3(x+1)(\mathrm{x}-1)}{cn(x-1)^{n-1}}=1\Rightarrow \lim\limits_{x\to 1}\frac{3(x+1)}{cn(x-1)^{n-2}}=1$

所以n−2=0,$\frac{6}{cn}=1\Rightarrow n=2, \mathrm{c}=3$

50.设$f(x)=\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}, g(x)=\frac{A}{x^k}$,确定K及A，使当$\mathrm{x}\to+\infty, f(x)\sim g(x)$

解:$\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)}=1\Rightarrow \lim\limits_{x\to+\infty}\frac{\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}}{Ax^{-k}}=1$

$\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}=\frac{1}{\sqrt{x+2}+\sqrt{x+1}}-\frac{1}{\sqrt{x+1}+\sqrt{x}}$

$=\frac{\sqrt{x}-\sqrt{x+2}}{\left(\sqrt{x+2}+\sqrt{x+1}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}\sim\frac{-x^{\frac{1}{2}}}{\left(\sqrt{x+2}+\sqrt{x+1}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}, x\to\infty$

$\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)}=1\Rightarrow \lim\limits_{x\to+\infty}\frac{\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}}{Ax^{-k}}=1$

$\Rightarrow \lim\limits_{x\to+\infty}\frac{-x^{\frac{1}{2}}x^k}{A\left(\sqrt{x+2}+\sqrt{x+1}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}=1$

所以k+$\frac{1}{2}=1, k=-\frac{1}{2}, -\frac{1}{4A}=1, A=-4$