Principles of Functional Analysis

SECOND EDITION

Martin Schechter

Graduate Studies in Mathematics

Volume 36

American Mathematical Society

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Abstract. The book is intended for a one-year course for beginning graduate or senior undergraduate students. However, it can be used at any level where the students have the prerequisites mentioned below. Because of the crucial role played by functional analysis in the applied sciences as well as in mathematics, the author attempted to make this book accessible to as wide a spectrum of beginning students as possible. Much of the book can be understood by a student having taken a course in advanced calculus. However, in several chapters an elementary knowledge of functions of a complex variable is required. These include Chapters 6, 9, and 11. Only rudimentary topological or algebraic concepts are used. They are introduced and proved as needed. No measure theory is employed or mentioned.

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$BS''D$

To my wife, children, and grandchildren.

May they enjoy many happy years.

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PREFACE TO THE REVISED EDITION

The first edition of *Principles of Functional Analysis* enjoyed a successful run of 28 years. In revising the text, we were confronted with a dilemma. On the one hand, we wanted to incorporate many new developments, but on the other, we did not want to smother the original flavor of the book. As one usually does under such circumstances, we settled for a compromise. We considered only new material related to the original topics or material that can be developed by means of techniques existing within the original framework. In particular, we restricted ourselves to normed vector spaces and linear operators acting between them. (Other topics will have to wait for further volumes.) Moreover, we have chosen topics not readily available in other texts.

We added sections to Chapters 3, 5, 7, 9, and 13 and inserted a new chapter – Chapter 14. (The old Chapter 14 now becomes Chapter 15.) Added topics include products of operators (Sections 5.7 and 7.7), a more general theory of semi-Fredholm operators (Sections 5.6 and 7.6), Riesz operators (Section 9.6), Fredholm and semi-Fredholm perturbations (Sections 9.6 and 9.7), spectral theory for unbounded selfadjoint operators (Section 13.6), and measures of operators and perturbation functions (Chapter 14).

We attempted to strengthen those areas in the book that demonstrate its unique character. In particular, new material introduced concerning Fredholm and semi-Fredholm operators requires minimal effort since the required machinery is already in place. By these means we were able to provide very useful information while keeping within our guidelines.

The new chapter (Chapter 14) deserves some additional remarks. It is designed to show the student how methods that were already mastered can be used to attack new problems. We gathered several topics which are new, but relate only to those concepts and methods emanating from other parts of the book. These topics include perturbation classes, measures of noncompactness, strictly singular operators and operator constants. This last topic illustrates in a very surprising way how a constant associated with an operator can reveal a great deal of information concerning the operator. No new methods of proof are needed, and, again, most of this material cannot be readily found in other books.

We went through the entire text with a fine toothed comb. The presentation was clarified and simplified whenever necessary, and misprints were corrected. Existing lemmas, theorems, corollaries and proofs were expanded when more elaboration was deemed beneficial. New lemmas, theorems and corollaries (with proofs) were introduced as well. Many new problems were added.

We have included two appendices. The first gives the definitions of important terms and symbols used throughout the book. The second lists major theorems and indicates the pages on which they can be found.

The author would like to thank Richard Jasiewicz for installing LaTeX 2_ε into his computer. He would also like to thank the editors and staff of the AMS for helpful suggestions.

Irvine, California

March, 2001

TVSLB″O

The following are a few excerpts from a review of the original edition by Einar Hille in the *American Scientist.*[1]

" 'Charming' is a word that seldom comes to the mind of a science reviewer, but if he is charmed by a treatise, why not say so? I am charmed by this book."

"Professor Schechter has written an elegant introduction to functional analysis including related parts of the theory of integral equations. It is easy to read and is full of important applications. He presupposes very little background beyond advanced calculus; in particular, the treatment is not burdened by topological 'refinements' which nowadays have a tendency of dominating the picture."

"The book can be warmly recommended to any reader who wants to learn about this subject without being deterred by less relevant introductory matter or scared away by heavy prerequisites."

[1]From Hille, Einar, Review of *Principles of Functional Analysis,* American Scientist {Vol. 60}, No. 3, 1972, 390.

FROM THE PREFACE TO THE FIRST EDITION

Because of the crucial role played by functional analysis in the applied sciences as well as in mathematics, I have attempted to make this book accessible to as wide a spectrum of beginning students as possible. Much of the book can be understood by a student having taken a course in advanced calculus. However, in several chapters an elementary knowledge of functions of a complex variable is required. These include Chapters 6, 9, and 11. Only rudimentary topological or algebraic concepts are used. They are introduced and proved as needed. No measure theory is employed or mentioned.

The book is intended for a one-year course for beginning graduate or senior undergraduate students. However, it can be used at any level where the students have the prerequisites mentioned above.

I have restricted my attention to normed vector spaces and their important examples, Banach and Hilbert spaces. These are venerable institutions upon which every scientist can rely throughout his or her career. They are presently the more important spaces met in daily life. Another consideration is the fact that an abundance of types of spaces can be an extremely confusing situation to a beginner.

I have also included some topics which are not usually found in textbooks on functional analysis. A fairly comprehensive treatment of Fredholm operators is given in Chapters 5 and 7. I consider their study elementary. Moreover, they are natural extensions of operators of the form $I - K$, K compact. They also blend naturally with other topics. Additional topics include unbounded semi-Fredholm operators and the essential spectrum considered in Chapter 7. Hyponormal and seminormal operators are treated in Chapter 11, and the numerical range of an unbounded operator is studied in Chapter 12. The last chapter is devoted to the study of three types of operators on the space $L^2(-\infty, \infty)$.

One will notice that there are few applications given in the book other than those treated in the last chapter. In general, I used as many illustrations as I could without assuming more mathematical knowledge than is needed to follow the text. Moreover, one of the basic philosophies of the book is that the theory of functional analysis is a beautiful subject which can be motivated and studied for its own sake. On the other hand, I have devoted a full chapter to applications that use a minimum of additional knowledge.

The approach of this book differs substantially from that of most other mathematics books. In general one uses a "tree" or "catalog" structure, in which all foundations are developed in the beginning chapters, with later chapters branching out in different directions. Moreover, each topic is introduced in a logical and indexed place, and all the material concerning that topic is discussed there complete with examples, applications, references to the literature and descriptions of related topics not covered. Then one proceeds to the next topic in a carefully planned program. A descriptive introduction to each chapter tells the reader exactly what will be done there. In addition, we are warned when an important theorem is approaching. We are even told which results are of "fundamental importance." There is much to be said for this approach. However, I have embarked upon a different path. After introducing the first topic, I try to follow a trend of thought wherever it may lead without stopping to fill in details. I do not try to describe a subject fully at the place it is introduced. Instead, I continue with my trend of thought until further information is needed. Then I introduce the required concept or theorem and continue with the discussion.

This approach results in a few topics being covered in several places in the book. Thus, the Hahn-Banach theorem is discussed in Chapters 2 and

9, with a complex form given in Chapter 6, and a geometric form in Chapter 7. Another result is that complex Banach spaces are not introduced until Chapter 6, the first place that their advantage is clear to the reader.

This approach has further resulted in a somewhat unique structure for the book. The first three chapters are devoted to normed vector spaces, and the next four to arbitrary Banach spaces. Chapter 8 deals with reflexive Banach spaces, and Chapters 11 – 13 cover Hilbert spaces. Chapters 9 and 10 discuss special topics.

Chapter 1

BASIC NOTIONS

1.1. A problem from differential equations

Suppose we are given the problem of finding a solution of

$$f''(x) + f(x) = g(x) \tag{1.1}$$

in an interval $a \le x \le b$ with the solution satisfying

$$f(a) = 1, f'(a) = 0. \tag{1.2}$$

(We shall not enter into a discussion as to why anyone would want to solve this problem, but content ourselves with the statement that such equations do arise in applications.) From your course in differential equations you will recall that when $g = 0$, equation (1.1) has a general solution of the form

$$f(x) = A \sin x + B \cos x, \tag{1.3}$$

where A and B are arbitrary constants. However, if we are interested in solving (1.1) for $g(x)$ an arbitrary function continuous in the closed interval, not many of the methods developed in the typical course in differential equations will be of any help. A method which does work is the least popular and would rather be forgotten by most students. It is the method of *variation of parameters* which states, roughly, that one can obtain a solution of (1.1) if one allows A and B to be functions of x instead of just constants. Since we are only interested in a solution of (1.1), we shall not go into any justification of the method, but merely apply it and then check to see if what we get is actually a solution. So we differentiate (1.3) twice, substitute into (1.1) and see what happens. Before proceeding, we note that we shall get one equation with two unknown functions. Since we were brought up from childhood to believe that one should have two equations to determine two functions, we shall feel free to impose a further restriction on A and B, especially if such action will save labor on our part. So here we go:

$$f' = A\cos x - B\sin x + A'\sin x + B'\cos x.$$

Now it becomes clear to us that further differentiation will yield eight terms, a circumstance that should be avoided if possible. Moreover, the prospect of obtaining higher order derivatives of A and B does not appeal to us. So we make the perfectly natural assumption

$$A'\sin x + B'\cos x = 0. \tag{1.4}$$

Thus,

$$f'' = A'\cos x - B'\sin x - f,$$

showing that we must have

$$g = A'\cos x - B'\sin x.$$

Combining this with (1.4), we get

$$A' = g\cos x, \quad B' = -g\sin x.$$

From the initial conditions (1.2) we see that $B(a) = \cos a$, $A(a) = \sin a$. Thus,

$$A(x) = \sin a + \int_a^x g(t)\cos t\, dt, \quad B(x) = \cos a - \int_a^x g(t)\sin t\, dt$$

and

$$\begin{aligned} f(x) &= \cos(x-a) + \int_a^x [\sin x\cos t - \cos x \sin t] g(t)\, dt \\ &= \cos(x-a) + \int_a^x \sin(x-t) g(t)\, dt. \qquad (1.5) \end{aligned}$$

So far so good. Since we made no claims concerning the method, we really should verify that (1.5) truly is a solution of (1.1) and (1.2). Differentiating twice, we have

$$f'(x) = -\sin(x-a) + \int_a^x \cos(x-t)g(t)\, dt$$

$$\begin{aligned} f''(x) &= -\cos(x-a) - \int_a^x \sin(x-t)g(t)\, dt + g(x) \\ &= -f(x) + g(x). \end{aligned}$$

(Make sure to check back in your advanced calculus text about differentiating an integral with respect to a parameter appearing in the integrand and in the limits of integration.)

Encouraged by our success, we generalize the problem. Suppose in place of (1.1) we want to solve

$$f''(x) + f(x) = \sigma(x)f(x), \quad a \le x \le b, \tag{1.6}$$

under the initial conditions (1.2). Here $\sigma(x)$ is a given function continuous in the closed interval. One thing is certain. Any solution of (1.6) satisfies

$$f(x) = \cos(x-a) + \int_a^x \sin(x-t)\sigma(t)f(t)\,dt. \tag{1.7}$$

Thus, by what was just shown, a function $f(x)$ is a solution of (1.6), (1.2) if, and only if, it is a solution of (1.7). Do not believe that this has improved the situation much. It has merely transformed a differential problem into one of solving the integral equation (1.7).

A more disturbing fact is that equation (1.7) is rather complicated, and matters appear to be getting worse unless taken in hand. We must introduce some shorthand. If we write

$$Kh = \int_a^x \sin(x-t)\sigma(t)h(t)\,dt, \quad u(x) = \cos(x-a), \tag{1.8}$$

then (1.7) takes on the more manageable form

$$f = u + Kf. \tag{1.9}$$

The "object" K is called an "operator" or "transformation," since it acts on continuous functions and transforms them into other continuous functions. (We will give a more precise definition in Chapter 3.) Thus, we are looking for a continuous function f such that Kf added to u gives back f.

Now that (1.7) has been simplified to (1.9), we can think about it a bit more clearly. It really does seem like a difficult equation to solve. To be sure, if one takes an arbitrary function f_0 and plugs it into the right hand side of (1.9), one would have to be extremely lucky if $u + Kf_0$ turned out to be f_0. In general, we would only get some other function f_1, which is probably no closer to an actual solution (should one exist) than f_0. On second thought, perhaps it is, in some way. After all, it is obtained by means of the right hand side of (1.9). Well, if this is the case, let us plug in f_1. This gives another function $f_2 = u + Kf_1$. Continuation of the procedure leads to a sequence $\{f_n\}$ of continuous functions defined by

$$f_n = u + Kf_{n-1}, \tag{1.10}$$

where it is hoped that f_n is "closer" to a solution than f_{n-1}. This suggests that the sequence $\{f_n\}$ might even converge to some limit function f. Would such an f be a solution? Well, if f is continuous and Kf_n converges to Kf, we then have that (1.9) holds, showing that f is indeed a solution. Thus, the big question: is our operator K such that these things will happen? Now before we go further, we must consider what type of convergence we want. Since we want the limit function to be continuous, it is quite natural to consider uniform convergence. (Here you are expected to know that the uniform limit of continuous functions is continuous, and to suspect that it may not be so if the convergence is otherwise.) Now you may recall that a continuous function has a maximum on a closed interval and that a sequence $\{f_n(x)\}$ of continuous functions converges uniformly if and only if

$$\max_{a \le x \le b} |f_n(x) - f_m(x)|$$

can be made as small as we like by taking m and n large enough. Again I am compelled to pause a moment. The expression above is too tedious to write often, and since we expect it to occur frequently, more shorthand is in order. One idea is to put

$$\|h\| = \max_{a \le x \le b} |h(x)|. \tag{1.11}$$

This is the most appealing to me, so we shall use it. So we now want

$$\|f_n - f_m\| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty. \tag{1.12}$$

Now let us carry out our program. By (1.10),

$$\begin{aligned} f_1 &= u + Kf_0 \\ f_2 &= u + Kf_1 \\ &= u + K(u + Kf_0) \\ &= u + Ku + K^2 f_0, \end{aligned}$$

where $K^2 h = K(Kh)$, and we have used the property

$$K(v + w) = Kv + Kw. \tag{1.13}$$

If we define

$$K^n h = K(K^{n-1} h)$$

by induction, we have

$$f_3 = u + Kf_2 = u + Ku + K^2 u + K^3 f_0,$$

$$f_n = u + Ku + \cdots + K^{n-1} u + K^n f_0. \tag{1.14}$$

Thus, for say $n > m$,

$$f_n - f_m = K^m u + \cdots + K^{n-1} u + K^n f_0 - K^m f_0.$$

Now we want (1.12) to hold. Since

$$\|f_n - f_m\| \le \|K^m u\| + \cdots + \|K^{n-1} u\| + \|K^n f_0\| + \|K^m f_0\|, \tag{1.15}$$

the limit (1.12) would be guaranteed if the right hand side of (1.15) went to zero as $m, n \to \infty$. Note that we have used the properties

$$\|v + w\| \le \|v\| + \|w\|, \tag{1.16}$$

$$\| - v\| = \|v\|, \tag{1.17}$$

which are obvious consequences of (1.11). Now the right hand side of (1.15) converges to zero as $m, n \to \infty$ if for each continuous function v

$$\sum_0^\infty \|K^n v\| < \infty, \tag{1.18}$$

where we defined $K^0 v = v$. This follows from the fact that

$$\|K^m u\| + \cdots + \|K^{n-1} u\|$$

is contained in the tail end of such a series, and $\|K^n f_0\|$ is the n-th term of another such series (perhaps a review of convergent series is in order).

Now before attempting to verify (1.18), we note that it implies that $\{f_n(x)\}$ is a uniformly convergent Cauchy sequence and hence has a continuous limit $f(x)$. Thus, all we need is

$$\|Kf_n - Kf\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \tag{1.19}$$

Now an examination of (1.8) reveals that there is a constant M such that

$$\|Kh\| \le M\|h\| \tag{1.20}$$

for all continuous functions h. In fact, we can take $M = (b-a)\|\sigma\|$. This coupled with the further property

$$K(-h) = -Kh, \tag{1.21}$$

shows that

$$\|Kf_n - Kf\| = \|K(f_n - f)\| \le M\|f_n - f\| \longrightarrow 0$$

as $n \to \infty$. This gives (1.19). Thus, we are in business if we can verify (1.18).

As it happens, (1.18) is an easy consequence of (1.8). In fact, we have

$$|Kv| \le \|\sigma\| \int_a^x |v(t)|\,dt \le \|\sigma\|\,\|v\|(x-a)$$

for all x in the interval $[a,b]$. Thus,

$$\begin{aligned} |K^2 v| &\le \|\sigma\| \int_a^x |Kv|\,dt \le \|\sigma\|^2\,\|v\| \int_a^x (t-a)\,dt \\ &= \|\sigma\|^2\|v\|(x-a)^2/2, \\ |K^3 v| &\le \|\sigma\| \int_a^x |K^2 v|\,dt \le \frac{1}{2}\|\sigma\|^3\|v\| \int_a^x (t-a)^2\,dt \\ &= \frac{1}{3!}\|\sigma\|^3\|v\|(x-a)^3, \end{aligned}$$

and by induction

$$|K^n v| \le \frac{1}{n!}\|\sigma\|^n\|v\|(x-a)^n.$$

Thus,

$$\|K^n v\| \le \frac{1}{n!}\|\sigma\|^n\|v\|(b-a)^n,$$

and

$$\sum_0^\infty \|K^n v\| \le \|v\| \sum_0^\infty \frac{1}{n!}\|\sigma\|^n (b-a)^n = \|v\| e^{\|\sigma\|(b-a)},$$

giving (1.18).

Now we tie the loose ends together. From property (1.18), we see by (1.15) that the sequence $\{f_n\}$ defined by (1.10) is a Cauchy sequence converging uniformly. Thus, the limit f is continuous in $[a,b]$. Moreover, by (1.20) and (1.21) we see that (1.19) holds, showing that f is a solution of (1.9), i.e., of (1.7). Differentiation now shows that f is a solution of (1.6), (1.2). Note that the choice of f_0 did not play a role, and we did not have to know anything about the functions σ or u other than the fact that they are continuous.

Those of you who are familiar with the Picard iteration method that is used to solve differential equations will notice a similarity to the presentation given here.

Let $k(x,t)$ be a function continuous in the triangle $a \le t \le x \le b$. Then the method above gives a solution of the integral equation

$$f(x) = u(x) + \int_a^x k(x,t) f(t)\, dt.$$

This is known as a *Volterra* equation.

1.2. An examination of the results

We now sit back and contemplate what we have done. We have solved a problem. We have shown that for any function $\sigma(x)$ continuous in $[a,b]$ we can find a solution of (1.6), (1.2). But I claim we have done more. We have also shown that if K is any operator which maps continuous functions into continuous functions and satisfies (1.13), (1.18), (1.20) and (1.21), then the equation (1.9) has a solution. Other properties of K were not needed. Now one may ask: why did the method work? Could it work in other situations? What properties of continuous functions were used?

To get some further insight, let us examine some of the well-known properties of continuous functions. Let $C \equiv C[a,b]$ be the set of functions continuous on the closed interval $[a,b]$. We note the following properties:

(1) They can be added. If f and g are in C, so is $f+g$.
(2) $f+(g+h) = (f+g)+h, \quad f,\ g,\ h \in C.$
(3) There is an element $0 \in C$ such that $h+0=h$ for all $h \in C$.
(4) For each $h \in C$ there is an element $-h \in C$ such that $h+(-h)=0$.
(5) $g+h = h+g, \quad g,\ h \in C.$

(6) For each real number α, $\alpha h \in C$.

(7) $\alpha(g+h) = \alpha g + \alpha h$.

(8) $(\alpha+\beta)h = \alpha h + \beta h$.

(9) $\alpha(\beta h) = (\alpha\beta)h$.

(10) To each $h \in C$ there corresponds a real number $\|h\|$ (called a *norm* with the following properties:

(11) $\|\alpha h\| = |\alpha|\,\|h\|$.

(12) $\|h\| = 0$ if, and only if, $h = 0$.

(13) $\|g+h\| \le \|g\| + \|h\|$.

(14) If $\{h_n\}$ is a sequence of elements of C such that $\|h_n - h_m\| \to 0$ as $m, n \to \infty$, then there is an element $h \in C$ such that $\|h_n - h\| \to 0$ as $n \to \infty$.

Note that we haven't used all of the properties (1)–(14) in solving our problem, but all of the properties used are among them.

Now for some definitions. A collection of objects which satisfies statements (1)–(9) and the additional statement

(15) $1h = h$

is called a *vector space* (VS) or *linear space.* We will be using real scalars until we reach a point where we will be forced to allow complex scalars. A set of objects satisfying statements (1)–(13) is called a *normed vector space* (NVS), and the number $\|h\|$ is called the *norm* of h. Although statement (15) is not implied by statements (1)–(9), it is implied by statements (1)–(13). A sequence satisfying

$$\|h_n - h_m\| \to 0 \text{ as } m, n \to \infty,$$

is called a *Cauchy sequence.* Property (14) states that every Cauchy sequence converges in norm to a limit. Property (14) is called *completeness,* and a normed vector space satisfying it is called a complete normed vector space or a *Banach space.* Thus, we have proved the following

Theorem 1.1. *Let X be a Banach space, and assume that K is an operator on X (i.e., maps X into itself) such that*

a) $K(v+w) = Kv + Kw$,

b) $K(-v) = -Kv$,

c) $\|Kv\| \leq M\|v\|$,
d) $\sum_0^\infty \|K^n v\| < \infty$

for all $v, w \in X$. *Then for each* $u \in X$ *there is a unique* $f \in X$ *such that*

$$f = u + Kf. \tag{1.22}$$

The uniqueness in Theorem 1.1 is trivial. In fact, suppose there were two solutions f_1 and f_2 of (1.22). Set $f = f_1 - f_2$. Then by a) and b) we have

$$f = Kf.$$

From this we get

$$f = K^2 f = K^3 f = \cdots = K^n f$$

for each n. Thus,

$$\|f\| = \|K^n f\| \to 0$$

as $n \to \infty$ by d). Since f does not depend on n, we have $\|f\| = 0$, and conclude that $f = 0$.

A special case of Theorem 1.1 is very important. If K satisfies a) and b) and

$$\|Kv\| \leq \theta\|v\|, \quad v \in X, \tag{1.23}$$

for some θ satisfying $0 < \theta < 1$, then c) and d) are both satisfied. For, we have

$$\|K^n v\| \leq \theta\|K^{n-1}v\| \leq \theta^2\|K^{n-2}v\| \leq \cdots \leq \theta^n\|v\|.$$

Thus,

$$\sum_0^\infty \|K^n v\| \leq \|v\| \sum_0^\infty \theta^n = \|v\|/(1-\theta).$$

As an example, let $k(x,t)$ be a continuous function in the square $a \leq x, t \leq b$. Then the equation

$$f(x) = u(x) + \int_a^b k(x,t) f(t)\, dt$$

is called a *Fredholm* integral equation of the second kind. If

$$\max_{a \leq x, t \leq b} |k(x,t)| < 1/(b-a),$$

then the operator

$$Kh = \int_a^b k(x,t) h(t)\, dt$$

satisfies (1.23), and Theorem 1.1 applies.

1.3. Examples of Banach spaces

We now consider some other Banach spaces.

Example 1. The most familiar is $\mathbb{R}^n$, Euclidean n-dimensional real space. It consists of sequences of n real numbers

$$f = (\alpha_1, \cdots, \alpha_n), \quad g = (\beta_1, \cdots, \beta_n),$$

where addition is defined by

$$f + g = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n),$$

and multiplication by a scalar is defined by

$$\gamma f = (\gamma\alpha_1, \cdots, \gamma\alpha_n).$$

Under these definitions, $\mathbb{R}^n$ is a vector space. If we set

$$\|f\| = (\alpha_1^2 + \cdots + \alpha_n^2)^{\frac{1}{2}}, \tag{1.24}$$

the only axioms of a Banach space which are not immediately verified are the triangle inequality (property (13)) and completeness (property (14)). We verify completeness first.

Let

$$f_j = (\alpha_1^{(j)}, \cdots, \alpha_n^{(j)})$$

be a Cauchy sequence in $\mathbb{R}^n$, i.e., assume

$$\|f_j - f_k\| \longrightarrow 0 \text{ as } j, k \longrightarrow \infty.$$

Thus, for any $\varepsilon > 0$, one can find a number N so large that

$$\|f_j - f_k\|^2 = (\alpha_1^{(j)} - \alpha_1^{(k)})^2 + \cdots + (\alpha_n^{(j)} - \alpha_n^{(k)})^2 < \varepsilon^2 \tag{1.25}$$

whenever j, $k > N$. In particular,

$$|\alpha_1^{(j)} - \alpha_1^{(k)}| < \varepsilon, \quad j,\, k > N.$$

This means that the sequence $\{\alpha_1^{(j)}\}$ is a Cauchy sequence of real numbers, which according to a well known theorem of advanced calculus has a limit. Thus, there is a real number α_1 such that

$$\alpha_1^{(j)} \longrightarrow \alpha_1 \text{ as } j \longrightarrow \infty.$$

The same reasoning shows that for each $l = 1, \cdots, n$,

$$\alpha_l^{(j)} \longrightarrow \alpha_l \text{ as } j \longrightarrow \infty.$$

Set $f = (\alpha_1, \cdots, \alpha_n)$. Then $f \in \mathbb{R}^n$. Now letting $k \to \infty$ in (1.25), we have

$$(\alpha_1^{(j)} - \alpha_1)^2 + \cdots + (\alpha_n^{(j)} - \alpha_n)^2 \leq \varepsilon^2 \; for \; j > N.$$

But this is precisely

$$\|f_j - f\|^2 \leq \varepsilon^2.$$

Thus,

$$\|f_j - f\| \longrightarrow 0 \text{ as } j \longrightarrow \infty.$$

Now for the triangle inequality. If it is to hold, we must have

$$\|f+g\|^2 \le (\|f\| + \|g\|)^2 = \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|.$$

In other words, we want to prove

$$\sum(\alpha_i^2 + 2\alpha_i\beta_i + \beta_i^2) \le \sum \alpha_i^2 + \sum \beta_i^2 + 2(\sum \alpha_i^2)^{\frac{1}{2}}(\sum \beta_i^2)^{\frac{1}{2}},$$

or

$$\sum \alpha_i\beta_i \le (\sum \alpha_i^2)^{\frac{1}{2}}(\sum \beta_i^2)^{\frac{1}{2}}. \tag{1.26}$$

Now, before things become more complicated, set

$$(f,g) = \sum_1^n \alpha_i\beta_i. \tag{1.27}$$

This expression has the following obvious properties:

i. $(\alpha f, g) = \alpha(f,g)$
ii. $(f+g,h) = (f,h) + (g,h)$
iii. $(f,g) = (g,f)$
iv. $(f,f) > 0$ unless $f = 0$,

and we want to prove

$$(f,g)^2 \le (f,f)(g,g). \tag{1.28}$$

Lemma 1.2. *Inequality (1.28) follows from the aforementioned properties.*

Proof. Let α be any scalar. Then

$$\begin{aligned}(\alpha f + g, \alpha f + g) &= \alpha^2(f,f) + 2\alpha(f,g) + (g,g)\\ &= (f,f)\left[\alpha^2 + 2\alpha\frac{(f,g)}{(f,f)} + \frac{(f,g)^2}{(f,f)^2}\right] + (g,g) - \frac{(f,g)^2}{(f,f)}\\ &= (f,f)\left[\alpha + \frac{(f,g)}{(f,f)}\right]^2 + (g,g) - \frac{(f,g)^2}{(f,f)},\end{aligned}$$

where we have completed the square with respect to α and tacitly assumed that $(f,f) \ne 0$. This assumption is justified by the fact that if $(f,f) = 0$, then (1.28) holds vacuously. We now note that the left-hand side of (1.29) is nonnegative by property iv listed above. If we now take $\alpha = -(f,g)/(f,f)$, this inequality becomes

$$0 \le (g,g) - \frac{(f,g)^2}{(f,f)},$$

which is exactly what we want. □

Returning to $\mathbb{R}^n$, we see that (1.26) holds and hence, the triangle inequality is valid. Thus, $\mathbb{R}^n$ is a Banach space.

An expression (f, g) that assigns a real number to each pair of elements of a vector space and satisfies the aforementioned properties is called a *scalar* (or *inner*) product. We have essentially proved

Lemma 1.3. *If a vector space X has a scalar product (f, g), then it is a normed vector space with norm $\|f\| = (f, f)^{\frac{1}{2}}$.*

Proof. Again, the only thing that is not immediate is the triangle inequality. This follows from (1.28) since

$$\begin{aligned}\|f+g\|^2 &= \|f\|^2 + \|g\|^2 + 2(f, g)\\ &\le \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\| = (\|f\| + \|g\|)^2.\end{aligned}$$

This gives the desired result. □

A vector space which has a scalar product and is complete with respect to the induced norm is called a *Hilbert space.* By Lemma 1.3, every Hilbert space is a Banach space, but we shall soon see that the converse is not true. Inequalities (1.26) and (1.28) are known as the *Cauchy-Schwarz* inequalities.

Now that we know that $\mathbb{R}^n$ is a Banach space, we can apply Theorem 1.1. In particular, if

$$Kf = (\frac{1}{2}\alpha_1, \frac{1}{3}\alpha_2, \cdots, \frac{1}{n+1}\alpha_n),$$

we know that K satisfies the hypotheses of that theorem and we can always solve

$$f = u + Kf$$

for any $u \in \mathbb{R}^n$. The same is true for the operator

$$Kf = \frac{1}{2}(\alpha_2, \cdots, \alpha_n, 0).$$

As another application of Theorem 1.1, consider the system of equations

$$x_j - \sum_{k=1}^{n} a_{jk} x_k = b_j, \quad j = 1, \cdots, n. \tag{1.29}$$

We are given the coefficients a_{jk} and the constants b_j, and we wish to solve for the x_j. One way of solving is to take $b = (b_1, \cdots, b_n)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and

$$Kx = (\sum_{k=1}^{n} a_{1k} x_k, \cdots, \sum_{k=1}^{n} a_{nk} x_k).$$

Then,

$$\begin{aligned}\|Kx\|^2 &= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} x_k \right)^2 \\ &\leq \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk}^2 \right) \left(\sum_{k=1}^{n} x_k^2 \right) \\ &\leq \left(\sum_{j,k=1}^{n} a_{jk}^2 \right) \|x\|^2.\end{aligned}$$

If we assume that

$$\sum_{j,k=1}^{n} a_{jk}^2 \leq \theta < 1,$$

then we can apply Theorem 1.1 to conclude that for every $b \in \mathbb{R}^n$ there is a unique $x \in \mathbb{R}^n$ such that $x = b + Kx$. This solves (1.29).

Example 2. Another example is given by the space l_∞ (the reason for the odd notation will be given later). It consists of infinite sequences of real numbers

$$f = (\alpha_1, \cdots, \alpha_n, \cdots) \tag{1.30}$$

for which

$$\sup_i |\alpha_i| < \infty.$$

(The sup of any set of real numbers is the least upper bound, i.e., the smallest number, which may be $+\infty$, that is an upper bound for the set. An important property of the real numbers is that every set of real numbers has a least upper bound.) If we define addition and multiplication by a scalar as in the case of $\mathbb{R}^n$, we come up with a vector space. But we want more. We want l_∞ to be a Banach space with norm

$$\|f\| = \sup_i |\alpha_i|.$$

As one will find in most examples, the only properties which are not immediately obvious are the triangle inequality and completeness. In this case the triangle inequality is not far from it, since

$$\sup_i |\alpha_i + \beta_i| \leq \sup_i (|\alpha_i| + |\beta_i|) \leq \sup_i |\alpha_i| + \sup_i |\beta_i|.$$

It remains to prove completeness. Suppose

$$f_j = (\alpha_1^{(j)}, \cdots, \alpha_i^{(j)}, \cdots)$$

is a Cauchy sequence in l_∞. Then for each $\varepsilon > 0$ there is an N so large that

$$\|f_j - f_k\| < \varepsilon \text{ when } j, k > N.$$

In particular for each i

$$|\alpha_i^{(j)} - \alpha_i^{(k)}| < \varepsilon \text{ for } j, k > N. \tag{1.31}$$

Thus, for each i the sequence $\{\alpha_i^{(j)}\}$ is a Cauchy sequence of real numbers, so that there is a number α_i for which

$$\alpha_i^{(j)} \longrightarrow \alpha_i \text{ as } j \longrightarrow \infty.$$

Set $f = (\alpha_1, \cdots, \alpha_n, \cdots)$. Is $f \in l_\infty$? Well, let us see. If we fix j and let $k \to \infty$ in (1.31), we have

$$|\alpha_i^{(j)} - \alpha_i| \leq \varepsilon \text{ for } j > N.$$

Thus,

$$|\alpha_i| \leq |\alpha_i - \alpha_i^{(j)}| + |\alpha_i^{(j)}| \leq \varepsilon + \|f_j\|$$

for $j > N$. This shows that $f \in l_\infty$. But this is not enough. We want

$$\|f_j - f\| \longrightarrow 0 \text{ as } j \longrightarrow \infty.$$

But by (1.31), we have

$$\|f_j - f\| = \sup_i |\alpha_i^{(j)} - \alpha_i| \leq \varepsilon \text{ for } j > N.$$

This is precisely what we want.

Again if we set

$$Kf = (\frac{1}{2}\alpha_1, \cdots, \frac{1}{i+1}\alpha_i, \cdots)$$

or

$$Kf = (\frac{1}{2}\alpha_2, \cdots, \frac{1}{2}\alpha_{i+1}, \cdots),$$

we can solve the equation $f = u + Kf$ by Theorem 1.1. We obtain the solutions as the limits of convergent series.

We can also consider the infinite system of equations

$$x_j - \sum_{k=1}^{\infty} a_{jk} x_k = b_j, \quad j = 1, \cdots. \tag{1.32}$$

As before, we are given the coefficients a_{jk} and the constants b_j, and we wish to solve for the x_j. One way of solving is to take $b = (b_1, \cdots)$, $x = (x_1, \cdots) \in l_\infty$ and

$$Kx = (\sum_{k=1}^{n} a_{1k} x_k, \cdots).$$

Then,

$$\begin{aligned}\|Kx\| &= \sup_j \left|\sum_{k=1}^{\infty} a_{jk}x_k\right| \\ &\le \sup_j \sum_{k=1}^{\infty} |a_{jk}| \cdot |x_k| \\ &\le \sup_j \sum_k |a_{jk}| \cdot \|x\|_\infty .\end{aligned}$$

If we assume that

$$\sup_j \sum_{k=1}^{\infty} |a_{jk}| \le \theta < 1,$$

then we can apply Theorem 1.1 to conclude that for every $b \in l_\infty$ there is a unique $x \in l_\infty$ such that $x = b + Kx$. This solves (1.32).

Example 3. A similar example is the space l_2. (Recall the remark concerning l_∞.) It consists of all sequences of the form (1.30) for which

$$\sum_1^\infty \alpha_i^2 < \infty.$$

Here we are not immediately sure that the sum of two elements of l_2 is in l_2. Looking ahead a moment, we intend to investigate whether l_2 is a Banach space. Now the candidate most likely to succeed as a norm is

$$\|f\| = \left(\sum_1^\infty \alpha_i^2\right)^{\frac{1}{2}}.$$

Thus, if we can verify the triangle inequality, we will also have shown that the sum of two elements of l_2 is in l_2. Now, since the triangle inequality holds in $\mathbb{R}^n$, we have

$$\begin{aligned}\sum_1^n (\alpha_i + \beta_i)^2 &\le \sum \alpha_i^2 + \sum \beta_i^2 + 2(\sum \alpha_i^2)^{\frac{1}{2}}(\sum \beta_i^2)^{\frac{1}{2}} \\ &\le \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\| = (\|f\| + \|g\|)^2\end{aligned}$$

for any n. Letting $n \to \infty$, we see that $f + g \in l_2$, and that the triangle inequality holds. Thus, we have a normed vector space. To check if l_2 is complete, let

$$f_j = (\alpha_1^{(j)}, \cdots, \alpha_i^{(j)}, \cdots)$$

be a Cauchy sequence in l_2. Thus, for each $\varepsilon > 0$, there is an N such that

$$\sum_{i=1}^{\infty} (\alpha_i^{(j)} - \alpha_i^{(k)})^2 < \varepsilon^2 \quad \text{for} \quad j, k > N. \tag{1.33}$$

In particular, (1.31) holds for each i, and hence, there is a number α_i which is the limit of $\alpha_i^{(j)}$ as $j \to \infty$. Now by (1.33), for each n

$$\sum_{i=1}^{n} (\alpha_i^{(j)} - \alpha_i^{(k)})^2 < \varepsilon^2 \text{ for } j, k > N. \tag{1.34}$$

Fix j in (1.34) and let $k \to \infty$. Then

$$\sum_{i=1}^{n} (\alpha_i^{(j)} - \alpha_i)^2 \leq \varepsilon^2 \text{ for } j > N.$$

Since this is true for each n, we have

$$\sum_{i=1}^{\infty} (\alpha_i^{(j)} - \alpha_i)^2 \leq \varepsilon^2 \text{ for } j > N. \tag{1.35}$$

Set

$$h_j = (\alpha_1^{(j)} - \alpha_1, \cdots, \alpha_i^{(j)} - \alpha_i, \cdots).$$

By (1.35), $h_j \in l_2$ and $\|h_j\| \leq \varepsilon$ when $j > N$. Hence,

$$f = f_j - h_j = (\alpha_1, \cdots, \alpha_i, \cdots)$$

is in l_2, and

$$\|f_j - f\| = \|h_j\| \leq \varepsilon \text{ for } j > N.$$

This means that

$$\|f_j - f\| \longrightarrow 0 \text{ as } j \longrightarrow \infty.$$

Thus, l_2 is a Banach space.

Since $\mathbb{R}^n$ is a Hilbert space, one might wonder whether the same is true of l_2. The scalar product for $\mathbb{R}^n$ is given by (1.27), so that the natural counterpart for l_2 should be

$$(f, g) = \sum_{1}^{\infty} \alpha_i \beta_i, \tag{1.36}$$

provided this series converges. By (1.26) we have

$$\sum_{1}^{n} |\alpha_i \beta_i| \leq (\sum_{1}^{n} \alpha_i^2)^{\frac{1}{2}} (\sum_{1}^{n} \beta_i^2)^{\frac{1}{2}} \leq \|f\| \, \|g\|,$$

showing that the series in (1.36) converges absolutely. Thus, (f, g) is defined for all f, $g \in l_2$, and satisfies i.– iv. (and hence (1.28)). Since $(f, f) = \|f\|^2$, we see that l_2 is a Hilbert space.

We can also solve (1.32) in l_2 for $b \in l_2$. In this case

$$\begin{aligned}\|Kx\|^2 &= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk}x_k\right)^2 \\ &\le \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk}^2\right)\left(\sum_{k=1}^{\infty} x_k^2\right) \\ &\le \left(\sum_{j,k=1}^{\infty} a_{jk}^2\right)\|x\|^2.\end{aligned}$$

If we assume that

$$\sum_{j,k=1}^{\infty} a_{jk}^2 \le \theta < 1,$$

then we can apply Theorem 1.1 to conclude that for every $b \in l_2$ there is a unique $x \in l_2$ such that $x = b + Kx$. This solves (1.32) for this case.

Example 4. Our last example is the set $B = B[a, b]$ of bounded real valued functions on an interval $[a, b]$. The norm is

$$\|\varphi\| = \sup_{a\le x\le b} |\varphi(x)|.$$

The verification that B is a normed vector space is immediate. To check that it is complete, assume that $\{\varphi_j\}$ is a sequence satisfying

$$\|\varphi_j - \varphi_k\| \longrightarrow 0 \text{ as } j, k \longrightarrow \infty.$$

Then for each $\varepsilon > 0$, there is an N satisfying

$$\sup |\varphi_j(x) - \varphi_k(x)| < \varepsilon \text{ for } j, k > N. \tag{1.37}$$

Thus, for each x in the interval $[a, b]$, the sequence $\{\varphi_j(x)\}$ has a limit c_x as $j \to \infty$. Define the function $\varphi(x)$ to have the value c_x at the point x. By (1.37), for each $x \in [a, b]$, we have

$$|\varphi_j(x) - \varphi_k(x)| < \varepsilon \text{ for } j, k > N.$$

Holding j fixed and letting $k \to \infty$, we get

$$|\varphi_j(x) - \varphi(x)| \le \varepsilon \text{ for } j > N.$$

Since this is true for each x, $\varphi_j - \varphi \in B$ for $j > N$ and

$$\|\varphi_j - \varphi\| \le \varepsilon \text{ for } j > N.$$

Thus, $\varphi \in B$, and it is the limit of the φ_j in B.

Let H be a Hilbert space. For any elements $f, g \in H$, we have

$$\|f + g\|^2 = \|f\|^2 + 2(f, g) + \|g\|^2$$

Figure 1.1

and

$$\|f-g\|^2 = \|f\|^2 - 2(f,g) + \|g\|^2,$$

giving

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2. \tag{1.38}$$

This is known as the *parallelogram law.* The name comes from the special case which states that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals. Thus, it follows that in any Hilbert space, (1.38) must hold for all elements f, g. This gives us a convenient way of checking whether a given Banach space is a Hilbert space as well. If one can exhibit two elements f, g of the Banach space which violate (1.38), then it clearly cannot be a Hilbert space.

Recall the spaces C and B from Section 1.2 and Example 4. Let us show that they are not Hilbert spaces. For simplicity take $a = 0,\ b = 3$. Define f to be 1 from 0 to 1, to vanish from 2 to 3 and to be linear from 1 to 2 (see Figure 1.1). Similarly, let g vanish from 0 to 1, equal 1 from 2 to 3 and be linear from 1 to 2. Both f and g are continuous in the closed interval [0,3] and hence are elements of B and C. But

$$\|f\| = \|g\| = \|f+g\| = \|f-g\| = 1,$$

violating (1.38). Thus, B and C are not Hilbert spaces. In Section 9.1 we shall show that every Banach space whose elements satisfy (1.38) is also a Hilbert space.

1.4. Fourier series

Let $f(x)$ be a function having a continuous derivative in the closed interval $[0, 2\pi]$ and such that $f(2\pi) = f(0)$. We can extend $f(x)$ to the whole of $\mathbb{R}$ by making it periodic with period 2π. It will be continuous in all of $\mathbb{R}$, but its derivative may have jumps at the endpoints of the periods. Then according to the theory of Fourier series

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx, \tag{1.39}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \tag{1.40}$$

and the convergence of (1.39) is uniform. Now our first reaction to all this is that formulas (1.39) and (1.40) are very complicated. So we recommend the following simplifications. Set

$$\varphi_0(x) = (2\pi)^{-\frac{1}{2}}, \quad \alpha_0 = (\pi/2)^{\frac{1}{2}} a_0,$$
$$\varphi_{2k}(x) = \pi^{-1/2}\cos kx, \quad \varphi_{2k+1}(x) = \pi^{-1/2}\sin kx,$$
$$\alpha_{2k} = \pi^{1/2} a_k, \quad \alpha_{2k+1} = \pi^{1/2} b_k.$$

With these definitions, (1.39) and (1.40) become

$$f(x) = \sum_0^\infty \alpha_n \varphi_n, \quad \alpha_n = \int_0^{2\pi} f(x)\varphi_n(x)\,dx. \tag{1.41}$$

An important property of the functions φ_n which was used in deriving (1.39) and (1.40) is

$$\int_0^{2\pi} \varphi_m(x)\varphi_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \tag{1.42}$$

Now consider the sequence $(\alpha_0, \alpha_1, \cdots)$ of coefficients in (1.41). A fact which may or may not be surprising is that it is an element of l_2. For,

$$\sum_0^n \alpha_j^2 = \sum_0^n \alpha_j \int_0^{2\pi} f(x)\varphi_j(x)\,dx = \int_0^{2\pi} f(x) f_n(x)\,dx,$$

where

$$f_n(x) = \sum_0^n \alpha_j \varphi_j(x). \tag{1.43}$$

Since the Fourier series converges uniformly to $f(x)$ and $f_n(x)$ is just a partial sum, we have

$$\sum_0^n \alpha_j^2 \longrightarrow \int_0^{2\pi} f(x)^2 dx \ \text{ as } \ n \longrightarrow \infty.$$

Thus, $(\alpha_0, \alpha_1, \cdots)$ is an element of l_2, and its norm is

$$\left(\int_0^{2\pi} f(x)^2 dx\right)^{1/2}.$$

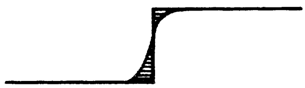

Figure 1.2

Hence for each $f \in C^1$ (the set of functions with period 2π having continuous first derivatives), there is a unique sequence $(\alpha_0, \alpha_1, \cdots)$ in l_2 such that

$$\sum_0^\infty \alpha_j^2 = \int_0^{2\pi} f(x)^2 dx, \tag{1.44}$$

and (1.41) holds. (In general, the symbol C^1 refers to all continuously differentiable functions. At this point we use it to refer only to periodic functions having period 2π.)

One might ask if we can go back. If we are given a sequence in l_2, does there exist a function $f \in C^1$ such that (1.41) and (1.44) hold? The answer is a resounding no. The reason is simple. Since l_2 is complete, every Cauchy sequence in l_2 has a limit in l_2. But does every sequence $\{f_j\}$ of functions in C^1 such that

$$\int_0^{2\pi} [f_j(x) - f_k(x)]^2 \, dx \longrightarrow 0 \text{ as } j, k \longrightarrow \infty \tag{1.45}$$

have a limit in C^1? We can see quite easily that this is not the case. In fact one can easily find a discontinuous function $g(x)$ which can be approximated by a smooth function $h(x)$ in such a way that

$$\int_0^{2\pi} [g(x) - h(x)]^2 dx$$

is as small as we like (see Figure 1.2). Another way of looking at it is that

$$\|f\| = \left(\int_0^{2\pi} f(x)^2 dx \right)^{1/2} \tag{1.46}$$

is a norm on the vector space C^1, but C^1 (or even C) is not complete with respect to this norm. What does one do in the situation when one has a normed vector space V which is not complete? In general one can complete the space by inventing fictitious or "ideal" elements and adding them to the space. This may be done as follows. Consider any Cauchy sequence of elements of V. If it has a limit in V, fine. Otherwise, we plug the "hole" by inventing a limit for it. One has to check that the resulting enlarged space satisfies all of the stipulations of a Banach space.

This can be done as follows: Let $\tilde{V}$ denote the set of all Cauchy sequences in V. If

$$G = \{g_k\}, \quad H = \{h_k\}$$

are members of $\tilde{V}$, we define

$$G + H = \{g_k + h_k\}, \quad \alpha G = \{\alpha g_k\}, \quad \|G\| = \lim_{k\to\infty} \|g_k\|.$$

We consider a Cauchy sequence equal to 0 if it converges to 0. With these definitions, $\tilde{V}$ becomes a normed vector space. Moreover, it is complete. To see this, let $G_n = (\{g_{nk}\})$ be a Cauchy sequence in $\tilde{V}$. Then for each n, there is an N_n such that

$$\|g_{nk} - g_{nl}\| \leq \frac{1}{n}, \quad k, l \geq N_n.$$

Pick an element g_{nl} with $l \geq N_n$ and call it f_n. Then

$$\|f_n - g_{nk}\| \leq \frac{1}{n}, \quad k \geq N_n.$$

Let F_n be the element in $\tilde{V}$ given by the constant sequence $(f_n, f_n, \cdots)$. (It qualifies as a Cauchy sequence.) Then

$$\|G_n - F_n\| \leq \frac{1}{n}.$$

Finally, let $F = \{f_n\}$. Then

$$\|G_n - F\| \leq \|G_n - F_n\| + \|F_n - F\| \to 0 \quad \text{as} \quad n \to \infty.$$

We note that in order to be logically precise, we had to consider equivalence classes of Cauchy sequences. If this sounds artificial, just remember that it is precisely the way we obtain the real numbers from the rationals. In Section 8.6 we shall give an "easy" proof of the fact that we can always complete a normed vector space to make it into a Banach space (without considering equivalence classes of Cauchy sequences).

In our present case, however, it turns out that we do not have to invent ideal elements. In fact, for each sequence satisfying (1.45) there is a bona fide, function $f(x)$ such that

$$\int_0^{2\pi} [f_j(x) - f(x)]^2 \, dx \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{1.47}$$

It may be that this function is very discontinuous, but its square is integrable in a generalized sense. No claim concerning pointwise convergence of the sequence $\{f_j\}$ is intended, but just that (1.47) holds. To summarize: the completion of C^1 with respect to the norm (1.46) consists of those functions having squares integrable in a generalized sense. We denote this space by

$L^2 = L^2([0, 2\pi])$. (When no confusion will result, we do not indicate the region over which the integration is taken. At the present time we are integrating over the interval $[0, 2\pi]$. At other times we shall use different regions.)

Now suppose $(\alpha_0, \alpha_1, \cdots)$ is a sequence in l_2. Set

$$f_n(x) = \sum_0^n \alpha_j \varphi_j(x). \tag{1.48}$$

Then $f \in C^1$, and for $m < n$

$$\|f_n - f_m\|^2 = \|\sum_{m+1}^n \alpha_j \varphi_j\|^2 = \sum_{m+1}^n \alpha_j^2 \longrightarrow 0 \text{ as } m, n \longrightarrow \infty. \tag{1.49}$$

Thus, there is an $f \in L^2$ such that

$$\|f_n - f\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

By (1.42), we see for $m < n$,

$$\alpha_m = \int_0^{2\pi} f_n(x)\varphi_m(x)\,dx.$$

We claim that

$$\alpha_m = \int_0^{2\pi} f(x)\varphi_m(x)\,dx. \tag{1.50}$$

This follows from the fact that

$$(f, g) = \int_0^{2\pi} f(x)g(x)\,dx$$

is a scalar product corresponding to (1.46). Hence,

$$|(f_n, \varphi_m) - (f, \varphi_m)| = |(f_n - f, \varphi_m)| \le \|f_n - f\| \cdot \|\varphi_m\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

But α_m does not depend on n. Therefore, (1.50) holds and

$$\sum_0^n \alpha_j^2 = \sum_0^n \alpha_j(f, \varphi_j) = (f, f_n).$$

Letting $n \to \infty$, we get

$$\sum_0^\infty \alpha_j^2 = \|f\|^2. \tag{1.51}$$

Conversely, let f be any function in L^2. Then there is a sequence $\{f_n\}$ of elements in C^1 which converges to f in L^2. Now each f_n can be expanded in a Fourier series. Thus,

$$f_n = \sum_{j=0}^\infty \alpha_j^{(n)} \varphi_j. \tag{1.52}$$

Since it is a Cauchy sequence, we have

$$\|f_n - f_m\|^2 = \sum_{j=0}^{\infty}(\alpha_j^{(n)} - \alpha_j^{(m)})^2 \longrightarrow 0 \text{ as } m,n \longrightarrow \infty. \tag{1.53}$$

Since l_2 is complete, there is a sequence $(\alpha_0, \alpha_1, \cdots)$ in l_2 such that

$$\sum_{j=0}^{\infty}(\alpha_j^{(n)} - \alpha_j)^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{1.54}$$

Set

$$\tilde{f} = \sum_0^{\infty} \alpha_j \varphi_j. \tag{1.55}$$

By what we have just shown, $\tilde{f} \in L^2$. Moreover

$$\|f_n - \tilde{f}\|^2 = \sum_{j=0}^{\infty}(\alpha_j^{(n)} - \alpha_j)^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence, $\tilde{f} = f$. Thus, we have

Theorem 1.4. *There is a one-to-one correspondence between l_2 and L^2 such that if $(\alpha_0, \alpha_1, \cdots)$ corresponds to f, then*

$$\|\sum_0^{n} \alpha_j \varphi_j - f\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{1.56}$$

and

$$\|f\|^2 = \sum_0^{\infty} \alpha_j^2, \quad \alpha_j = (f, \varphi_j). \tag{1.57}$$

Note that this correspondence is linear and one-to-one in both directions. We shall have more to say about this in Chapter 3. As usual, we are not satisfied with merely proving statements about L^2. We want to know if similar statements hold in other Hilbert spaces. So we examine the assumptions we have made. One property of the sequence $\{\varphi_n\}$ is

$$(\varphi_m, \varphi_n) = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \tag{1.58}$$

Such a sequence is called *orthonormal.* So suppose we have an arbitrary Hilbert space H and an orthonormal sequence $\{\varphi_j\}$ of elements in H. If f is an arbitrary element of H, set $\alpha_n = (f, \varphi_n)$. Then

$$\|f - \sum_1^n \alpha_i \varphi_i\|^2 = \|f\|^2 - 2\sum_1^n \alpha_i (f, \varphi_i) + \sum_1^n \alpha_i^2 = \|f\|^2 - \sum_1^n \alpha_i^2. \tag{1.59}$$

Hence,

$$\sum_1^n \alpha_i^2 \le \|f\|^2.$$

Letting $n \to \infty$, we have

$$\sum_1^\infty \alpha_i^2 \le \|f\|^2. \tag{1.60}$$

Equation (1.59) is called *Bessel's identity,* while (1.60) is called *Bessel's inequality.* We also have

Theorem 1.5. *Let* $(\alpha_1, \alpha_2, \cdots)$ *be a sequence of real numbers, and let* $\{\varphi_n\}$ *be an orthonormal sequence in* H. *Then*

$$\sum_1^n \alpha_i \varphi_i$$

converges in H *as* $n \to \infty$ *if, and only if,*

$$\sum_1^\infty \alpha_i^2 < \infty.$$

Proof. For $m < n$,

$$\left\| \sum_m^n \alpha_i \varphi_i \right\|^2 = \sum_m^n \alpha_i^2.$$

□

An orthonormal sequence $\{\varphi_j\}$ in a Hilbert space H is called *complete* if sums of the form

$$S = \sum_1^n \alpha_i \varphi_i \tag{1.61}$$

are dense in H; i.e., if for each $f \in H$ and $\varepsilon > 0$, there is a sum S of this form such that $\|f - S\| < \varepsilon$. We have

Theorem 1.6. *If* $\{\varphi_n\}$ *is complete, then for each* $f \in H$

$$f = \sum_1^\infty (f, \varphi_i)\varphi_i$$

and

$$\|f\|^2 = \sum_1^\infty (f, \varphi_i)^2. \tag{1.62}$$

Proof. Let f be any element of H, and let

$$f_n = \sum \alpha_j^{(n)} \varphi_j$$

be a sequence of sums of the form (1.61) which converges to f in H. In particular, (1.53) holds. Thus, there is a sequence $(\alpha_0, \alpha_1, \cdots)$ in l_2 such that (1.54) holds. If we define $\tilde{f}$ by (1.55), we know that $\tilde{f} \in H$ (Theorem 1.5) and $f_n \to \tilde{f}$ as $n \to \infty$. Hence, $f = \tilde{f}$. This proves the first statement. To prove (1.62), we have by Bessel's identity (1.59),

$$\|f - \sum_1^n \alpha_i \varphi_i\|^2 = \|f\|^2 - \sum_1^n \alpha_i^2.$$

Letting $n \to \infty$, we obtain (1.62). □

Equation (1.62) is known as *Parseval's equality.*

Theorem 1.6 has a trivial converse. If (1.62) holds for all f in a Hilbert space H, then the orthonormal sequence $\{\varphi_n\}$ is complete. This follows immediately from (1.59). We also have

Theorem 1.7. *If $\{\varphi_n\}$ is complete, then*

$$(f, g) = \sum_1^\infty (f, \varphi_n)(g, \varphi_n).$$

Proof. Set $\alpha_j = (f, \varphi_j)$. Since

$$f = \sum_1^\infty \alpha_j \varphi_j,$$

we have

$$(f, g) = \lim_{n \to \infty} (\sum_1^n \alpha_j \varphi_j, g) = \lim_{n \to \infty} \sum_1^n \alpha_j (\varphi_j, g).$$

□

1.5. Problems

(1) Show that statement (15) of Section 1.2 is implied by statements (1)–(13) of that section, but not by statements (1)–(9).

(2) Let $\varphi_1, \cdots, \varphi_n$ be an orthonormal set in a Hilbert space H. Show that

$$\|f - \sum_1^n \alpha_k \varphi_k\| \geq \|f - \sum_1^n (f, \varphi_k)\varphi_k\|$$

for all $f \in H$ and all scalars α_k.

(3) Show that an orthonormal sequence $\{\varphi_k\}$ is complete if, and only if, 0 is the only element orthogonal to all of them.

(4) Let c denote the set of all elements $(\alpha_1, \cdots) \in l_\infty$ such that $\{\alpha_n\}$ is a convergent sequence, and let c_0 be the set of all such elements for which $\alpha_n \to 0$ as $n \to \infty$. Show that c and c_0 are Banach spaces.

(5) Show that the operator given at the end of Section 1.1 satisfies the hypotheses of Theorem 1.1.

(6) Carry out the details of completing a normed vector space by the method described in Section 1.4.

(7) Prove Theorem 1.4 with L^2 replaced by any Hilbert space with a complete orthonormal sequence.

(8) Show that the norm of an element is never negative.

(9) If x, y are elements of a Hilbert space and satisfy $\|x + y\| = \|x\| + \|y\|$, show that either $x = cy$ or $y = cx$ for some scalar $c \geq 0$.

(10) If $x_k \to x,\ y_k \to y$ in a Hilbert space, show that $(x_k, y_k) \to (x, y)$.

(11) If $x_1, \cdots, x_n$ are elements of a Hilbert space, show that

$$\det[(x_j, x_k)] \geq 0.$$

(12) Show that in a normed vector space

$$\sum_{j=1}^{\infty} v_j = \sum_{j=1}^{n} v_j + \sum_{j=n+1}^{\infty} v_j.$$

(13) Let H be a Hilbert space, and let $\{\varphi_\nu\}$ be any collection of orthonormal elements in H (not necessary denumerable). If $x \in H$, show that there is at most a denumerable number of the φ_ν such that $(x, \varphi_\nu) \neq 0$.

(14) Show that $B[a, b]$ is a normed vector space.

(15) For $f = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, show that

$$\|f\|_0 = \max_k |\alpha_k|$$

and

$$\|f\|_1 = \sum_{k=1}^{n} |\alpha_k|$$

are norms. Is $\mathbb{R}^n$ complete with respect to either of them?

(16) If $\{\varphi_n\}$ is an orthonormal sequence in a Hilbert space H, and

$$\|f\|^2 = \sum_1^\infty (f, \varphi_n)^2$$

holds for each $f \in H$, show that $\{\varphi_n\}$ is complete.

(17) Prove: If $\{\varphi_n\}$ is an orthonormal sequence in a Hilbert space H and $(\alpha_1, \cdots) \in l_2$, then there is an $f \in H$ such that

$$\alpha_k = (f, \varphi_k), \quad k = 1, \cdots, \quad \text{and} \quad \|f\|^2 = \sum_{k=1}^{\infty} \alpha_k^2.$$

(18) Calculate

$$\sum_1^n a_k^2 \sum_1^n b_k^2 - \left(\sum_1^n a_k b_k \right)^2$$

and use the answer to prove the Cauchy-Schwarz inequality (1.26).

(19) Show that $\|cf\|_\infty = |c| \cdot \|f\|_\infty$ for $f \in l_\infty$.

(20) Show that the shortest distance between two points in a Hilbert space is along a straight line. Is this true in all Banach spaces?

(21) Define

$$\|f\|_* = \inf\{t > 0 : \|f\| < t\}, \quad f \in \mathbb{R}^n.$$

Is this a norm on $\mathbb{R}^n$?

(22) Does there exist a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that the set

$$B = \{x \in \mathbb{R}^2 : \|x\| < 1\}$$

is a rectangle? an ellipse?

(23) Show that (1.29) can be solved in $\mathbb{R}^n$ for arbitrary b if

$$|a_{jk}| < \frac{1}{n}, \quad 1 \le j,\ k \le n.$$

(24) Show that (1.32) can be solved in l_∞ for arbitrary b when

$$|a_{jk}| \le \frac{1}{2k^2}, \quad j,\ k = 1, 2, \cdots .$$

(25) Show that (1.32) can be solved in l_2 for arbitrary b if

$$|a_{jk}| \le \frac{1}{2jk}, \quad j,\ k = 1, 2, \cdots .$$

Chapter 2

DUALITY

2.1. The Riesz representation theorem

Let H be a Hilbert space and let (x, y) denote its scalar product. If we fix y, then the expression (x, y) assigns to each $x \in H$ a number. An assignment F of a number to each element x of a vector space is called a *functional* and denoted by $F(x)$. The scalar product is not the first functional we have encountered. In any normed vector space, the norm is also a functional.

The functional $F(x) = (x, y)$ has some very interesting and surprising features. For instance it satisfies

$$F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2) \tag{2.1}$$

for α_1, α_2 scalars. A functional satisfying (2.1) is called *linear.* Another property is

$$|F(x)| \leq M \|x\|, \tag{2.2}$$

which follows immediately from Schwarz's inequality (cf.(1.28)). A functional satisfying (2.2) is called *bounded.* Thus for y fixed, $F(x) = (x, y)$ is a bounded linear functional in the Hilbert space H.

"What is so surprising about this?" you are probably thinking to yourself; "I am sure that we can find many other examples of bounded linear functionals." Well, you may be surprised to learn that there aren't any others. In fact we have the following

Theorem 2.1. *For every bounded linear functional F on a Hilbert space H there is a unique element $y \in H$ such that*

$$F(x) = (x, y) \text{ for all } x \in H. \tag{2.3}$$

Moreover,

$$\|y\| = \sup_{x \in H,\, x \neq 0} \frac{|F(x)|}{\|x\|}. \tag{2.4}$$

Theorem 2.1 is known as the *Riesz representation theorem.* In order to get an idea how to go about proving it, let us examine (2.3) a bit more closely. If F assigns to each element x the value zero, then we can take $y = 0$, and the theorem is trivial. Otherwise the y we are searching for cannot vanish. However, it must be "orthogonal" to every x for which $F(x) = 0$; i.e., we must have $(x, y) = 0$ for all such x. Let N denote the set of those x satisfying $F(x) = 0$. Suppose we can find a $y \neq 0$ which is orthogonal to each $x \in N$. Then I claim that the theorem is proved. For clearly y is not in N (otherwise we would have $\|y\|^2 = (y, y) = 0$), and hence $F(y) \neq 0$. Moreover, for each $x \in H$, we have

$$F(F(y)x - F(x)y) = F(y)F(x) - F(x)F(y) = 0,$$

showing that $F(y)x - F(x)y$ is in N. Hence

$$(F(y)x - F(x)y, y) = 0,$$

or

$$F(x) = (x, \frac{F(y)}{\|y\|^2} y).$$

This gives (2.3) if we use $F(y)y/\|y\|^2$ in place of y. (This is to be expected since we made no stipulation on y other than that it be orthogonal to N.) We also note that the uniqueness and (2.4) are trivial. For if y_1 were another element of H satisfying (2.3), we would have

$$(x, y - y_1) = 0 \ \text{ for all } \ x \in H.$$

In particular this holds for $x = y - y_1$, showing that $\|y - y_1\| = 0$. Thus, $y_1 = y$. Now by Schwarz's inequality,

$$|F(x)| = |(x, y)| \leq \|x\| \, \|y\|.$$

Hence,

$$\|y\| \geq \sup_{x \in H,\, x \neq 0} \frac{|F(x)|}{\|x\|}.$$

However, we can obtain equality by taking $x = y$. In fact, $\|y\| = |F(y)|/\|y\|$. This gives (2.4).

All that is now needed to complete the proof of Theorem 2.1 is for us to find an element $y \neq 0$ which is orthogonal to N (i.e., to every element of N). In order to do this we examine N a little more closely. What kind of set is it? First of all, we notice that if x_1 and x_2 are elements of N, so is $\alpha_1 x_1 + \alpha_2 x_2$ for any scalars α_1, α_2. For, by the linearity of F,

$$F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2) = 0.$$

A subset U of a vector space V is called a *subspace* of V if $\alpha_1 x_1 + \alpha_2 x_2$ is in U whenever x_1, x_2 are in U and α_1, α_2 are scalars. Thus N is a subspace of H. There is another property of N which comes from (2.2) and is not so obvious. This is the fact that it is a closed subspace. A subset U of a normed vector space X is called *closed* if for every sequence $\{x_n\}$ of elements in U having a limit in X, the limit is actually in U. In our particular case, if $\{x_n\}$ is a sequence of elements in N which approaches a limit x in H, then by (2.2)

$$|F(x)| = |F(x) - F(x_n)| = |F(x - x_n)| \leq M\|x - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since x does not depend on n, we have $F(x) = 0$. Thus, $x \in N$, showing that N is closed in H.

Thus, we have a closed subspace N of H which is not the whole of H. We are interested in obtaining an element $y \neq 0$ of H which is orthogonal to N. For the special case of two-dimensional Euclidean space, we recall from our plane geometry that this can be done by drawing a perpendicular. We also recall that the shortest distance from a point (element) to a line (subspace) is along the perpendicular. The same thing is true in Hilbert space. We have

Theorem 2.2. *Let N be a closed subspace of a Hilbert space H, and let x be an element of H which is not in N. Set*

$$d = \inf_{z \in N} \|x - z\|. \tag{2.5}$$

Then there is an element $z \in N$ such that $\|x - z\| = d$.

Proof. By the definition of d, there is a sequence $\{z_n\}$ of elements of N such that $\|x - z_n\| \to d$. We apply the parallelogram law (cf.(1.38)) to $x - z_n$ and $x - z_m$. Thus

$$\|(x - z_n) + (x - z_m)\|^2 + \|(x - z_n) - (x - z_m)\|^2 = 2\|x - z_n\|^2 + 2\|x - z_m\|^2,$$

or

$$4\|x - [(z_n + z_m)/2]\|^2 + \|z_m - z_n\|^2 = 2\|x - z_n\|^2 + 2\|x - z_m\|^2. \tag{2.6}$$

Since N is a subspace, $(z_n + z_m)/2$ is in N. Hence, the left-hand side of (2.6) is not less than

$$4d^2 + \|z_m - z_n\|^2.$$

This implies

$$\|z_m - z_n\|^2 \leq 2\|x - z_n\|^2 + 2\|x - z_m\|^2 - 4d^2 \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Thus, $\{z_n\}$ is a Cauchy sequence in H. Using the fact that a Hilbert space is complete, we let z be the limit of this sequence. But N is closed in H. Hence, $z \in N$, and $d = \lim \|x - z_n\| = \|x - z\|$. $\square$

Theorem 2.3. *Let N be a closed subspace of a Hilbert space H. Then for each $x \in H$, there are a $v \in N$ and a w orthogonal to N such that $x = v + w$. This decomposition is unique.*

Proof. If $x \in N$, put $v = x$, $w = 0$. If $x \notin N$, let $z \in N$ be such that $\|x - z\| = d$, where d is given by (2.5). We set $v = z$, $w = x - z$ and must show that w is orthogonal to N. Let $u \neq 0$ be any element of N and α any scalar. Then

$$\begin{aligned} d^2 \leq \|w + \alpha u\|^2 &= \|w\|^2 + 2\alpha(w,u) + \alpha^2\|u\|^2 \\ = \|u\|^2 &\left[\alpha^2 + 2\alpha\frac{(w,u)}{\|u\|^2} + \frac{(w,u)^2}{\|u\|^4}\right] + d^2 - \frac{(w,u)^2}{\|u\|^2} \\ &= \|u\|^2\left[\alpha + \frac{(w,u)}{\|u\|^2}\right]^2 + d^2 - \frac{(w,u)^2}{\|u\|^2}, \end{aligned}$$

where we completed the square with respect to α. Take $\alpha = -(w,u)/\|u\|^2$. Thus, $(w,u)^2 \leq 0$, which can only happen if w is orthogonal to u. Since u was any arbitrary element of N, the first statement is proved. If $x = v_1 + w_1$, where $v_1 \in N$ and w_1 is orthogonal to N, then $v - v_1 = w_1 - w$ is both in N and orthogonal to N. In particular, it is orthogonal to itself and thus must vanish. This completes the proof. $\square$

The proof of Theorem 2.1 (the Riesz Representation Theorem) is now complete.

Corollary 2.4. *If N is a closed subspace of a Hilbert space H but is not the whole of H, then there is an element $y \neq 0$ in H which is orthogonal to N.*

Proof. Let x be any element of H which is not in N. By Theorem 2.3, $x = v + w$, where $v \in N$ and w is orthogonal to N. Clearly, $w \neq 0$, for otherwise x would be in N. We can take w as the element y sought. $\square$

Theorem 2.3 is called the *projection theorem* because of its obvious geometrical interpretation.

If F is a bounded linear functional on a normed vector space X, the norm of F is defined by

$$\|F\| = \sup_{x \in X,\ x \neq 0} \frac{|F(x)|}{\|x\|}. \tag{2.7}$$

It is equal to the smallest number M satisfying

$$|F(x)| \leq M\|x\| \quad \text{for all} \quad x \in X. \tag{2.8}$$

In this terminology (2.4) becomes

$$\|y\| = \|F\|. \tag{2.9}$$

2.2. The Hahn-Banach theorem

Now that we have shown that all of the bounded linear functionals on a Hilbert space are just the scalar products, you might begin to wonder whether Banach spaces which are not Hilbert spaces have any non-zero bounded linear functionals at all. (In fact, one might be tempted to reason as follows: if the only nontrivial bounded linear functionals on a Banach space that has a scalar product are precisely the scalar products, then Banach spaces which do not have scalar products should have none.) How can we go about trying to find any that might exist?

As we always do in such cases, we consider the simplest case possible. Let X be a Banach space, and let $x_0 \neq 0$ be a fixed element of X. The set of all elements of the form αx_0 forms a subspace X_0 of X. Do there exist bounded linear functionals on X_0? One candidate is

$$F(\alpha x_0) = \alpha.$$

Clearly this is a linear functional. It is also bounded, since

$$|F(\alpha x_0)| = |\alpha| = \frac{\|\alpha x_0\|}{\|x_0\|}.$$

So there are bounded linear functionals on such subspaces. If we could only extend them to the whole of X we would have what we want. However, difficulties immediately present themselves. Besides the fact that it is not obvious how to extend a bounded linear functional to larger subspaces, will the norm of the functional be increased in doing so? And what happens if one needs an infinite number of steps to complete the procedure? The answers to these questions are given by the celebrated Hahn-Banach theorem (Theorem 2.5 below).

Let V be a vector space. A functional $p(x)$ on V is called *sublinear* if

$$p(x+y) \leq p(x) + p(y), \quad x, y \in V, \tag{2.10}$$

$$p(\alpha x) = \alpha p(x), \quad x \in V,\ \alpha > 0. \tag{2.11}$$

Note that the norm in a normed vector space is a sublinear functional.

Theorem 2.5. *(The Hahn-Banach Theorem) Let V be a vector space, and let $p(x)$ be a sublinear functional on V. Let M be a subspace of V, and let $f(x)$ be a linear functional on M satisfying*

$$f(x) \leq p(x), \quad x \in M. \tag{2.12}$$

Then there is a linear functional $F(x)$ on the whole of V such that

$$F(x) = f(x), \quad x \in M, \tag{2.13}$$

$$F(x) \leq p(x), \quad x \in V. \tag{2.14}$$

Before attempting to prove the Hahn-Banach theorem, we shall show how it applies to our case. We have

Theorem 2.6. *Let M be a subspace of a normed vector space X, and suppose that $f(x)$ is a bounded linear functional on M. Set*

$$\|f\| = \sup_{x \in M,\, x \neq 0} \frac{|f(x)|}{\|x\|}.$$

Then there is a bounded linear functional $F(x)$ on the whole of X such that

$$F(x) = f(x), \quad x \in M, \tag{2.15}$$

$$\|F\| = \|f\|. \tag{2.16}$$

Proof. Set

$$p(x) = \|f\| \cdot \|x\|, \quad x \in X.$$

Then $p(x)$ is a sublinear functional and

$$f(x) \leq p(x), \quad x \in M.$$

Then by the Hahn-Banach theorem there is a functional $F(x)$ defined on the whole of X such that (2.15) holds and

$$F(x) \leq p(x) = \|f\| \cdot \|x\|, \quad x \in X.$$

Since

$$-F(x) = F(-x) \leq \|f\| \cdot \| - x\|, \quad x \in X,$$

we have

$$|F(x)| \leq \|f\| \cdot \|x\|, \quad x \in X.$$

Thus,

$$\|F\| \leq \|f\|.$$

Since F is an extension of f, we must have

$$\|f\| \leq \|F\|.$$

Hence, (2.16) holds, and the proof is complete. □

Since we have shown that every normed vector space having nonzero elements has a subspace having a nonzero bounded linear functional, it follows that every normed vector space having nonzero elements has nonzero bounded linear functionals.

Now we tackle the Hahn-Banach theorem. We first note that it says nothing if $M = V$. So we assume that there is an element x_1 of V which is not in M. Let M_1 be the set of elements of V of the form

$$\alpha x_1 + x, \quad \alpha \in \mathbb{R},\ x \in M. \tag{2.17}$$

Then one checks easily that M_1 is a subspace of V and that the representation (2.17) is unique. To feel our way, let us consider the less ambitious task of extending f to M_1 so as to preserve (2.12). If such an extension F exists on M_1, it must satisfy

$$F(\alpha x_1 + x) = \alpha F(x_1) + F(x) = \alpha F(x_1) + f(x).$$

Therefore, F is completely determined by the choice of $F(x_1)$. Moreover, we must have

$$\alpha F(x_1) + f(x) \leq p(\alpha x_1 + x) \tag{2.18}$$

for all scalars α and $x \in M$. If $\alpha > 0$, this means

$$F(x_1) \leq \frac{1}{\alpha}[p(\alpha x_1 + x) - f(x)] = p(x_1 + \frac{x}{\alpha}) - f(\frac{x}{\alpha}) = p(x_1 + z) - f(z),$$

where $z = x/\alpha$. If $\alpha < 0$, we have

$$F(x_1) \geq \frac{1}{\alpha}[p(\alpha x_1 + x) - f(x)] = f(y) - p(-x_1 + y),$$

where $y = -x/\alpha$. Thus we need

$$f(y) - p(y - x_1) \leq F(x_1) \leq p(x_1 + z) - f(z) \quad \text{for all } \ y, z \in M. \tag{2.19}$$

Conversely, if we can pick $F(x_1)$ to satisfy (2.19), then it will satisfy (2.18), and F will satisfy (2.14) on M_1. For if $F(x_1)$ satisfies (2.19), then for $\alpha > 0$, we have

$$\alpha F(x_1) + f(x) = \alpha[F(x_1) + f(\frac{x}{\alpha})] \leq \alpha p(x_1 + \frac{x}{\alpha}) = p(\alpha x_1 + x),$$

while for $\alpha < 0$ we have

$$\alpha F(x_1) + f(x) = -\alpha[-F(x_1) + f(-\frac{x}{\alpha})] \leq -\alpha p(-\frac{x}{\alpha} - x_1) = p(\alpha x_1 + x).$$

So we have now reduced the problem to finding a value of $F(x_1)$ to satisfy (2.19). In order for such a value to exist, we must have

$$f(y) - p(y - x_1) \leq p(x_1 + z) - f(z) \tag{2.20}$$

for all $y, z \in M$. In other words we need

$$f(y + z) \leq p(x_1 + z) + p(y - x_1).$$

This is indeed true by (2.12) and property (2.10) of a sublinear functional. Hence, (2.20) holds. If we fix y and let z run through all elements of M, we have

$$f(y) - p(y - x_1) \leq \inf_{z \in M}[p(x_1 + z) - f(z)] \equiv C.$$

Since this is true for any $y \in M$, we have

$$c \equiv \sup_{y \in M}[f(y) - p(y - x_1)] \leq C.$$

We now merely pick $F(x_1)$ to satisfy

$$c \leq F(x_1) \leq C.$$

Note that the extension F is unique only when $c = C$.

Thus we have been able to extend f from M to M_1 in the desired way. If $M_1 = V$, we are finished. Otherwise there is an element x_2 of V not in M. Let M_2 be the space "spanned" by x_2 and M_1. By repeating the process we can extend f to M_2 in the desired way. If $M_2 \neq V$, we keep a stiff upper lip and continue. We get a sequence M_k of subspaces each containing the preceding and such that f can be extended from one to the next. If, finally, we reach a k such that $M_k = V$, we are finished. Even if

$$V = \bigcup_{k=1}^{\infty} M_k, \tag{2.21}$$

then we are through because each $x \in V$ is in some M_k, and we can define F by induction. But what if (2.21) does not hold? We can complete the proof easily when V is a Hilbert space and $p(x) = \gamma\|x\|$ for some positive constant γ. For then one can extend f to the closure $\overline{M}$ of M by continuity. By this we mean that if $\{x_n\}$ is a sequence of elements in M which converges to $x \in V$, then $\{f(x_n)\}$ is a Cauchy sequence of real numbers and hence has a limit. We then define $F(x) = \lim f(x_n)$. The limit is independent of the sequence chosen. One checks easily that $F(x)$ is a bounded linear functional on the set $\overline{M}$ and coincides with $f(x)$ on M. Since $\overline{M}$ is a Hilbert space, there is an element $y \in \overline{M}$ such that $F(x) = (x, y)$ for all $x \in \overline{M}$ (Theorem 2.1). Moreover $\|y\| = \|f\| \leq \gamma$. But $F(x)$ can be defined as (x, y) on the whole of V, and its norm will not be increased.

What do we do when the space V is not a Hilbert space and (2.21) does not hold? This is not a trivial situation. In this case we need a statement known as Zorn's lemma concerning maximal elements of chains in partially ordered sets. This lemma is equivalent to the axiom of choice, the validity of which we cannot discuss here. We shall present Zorn's lemma in Section 9.5. There we shall complete the proof of the Hahn-Banach theorem for those spaces that require Zorn's lemma in the proof.

2.3. Consequences of the Hahn-Banach theorem

The Hahn-Banach theorem is one of the most important theorems in functional analysis and has many far-reaching consequences. One of them is

Theorem 2.7. *Let X be a normed vector space and let $x_0 \neq 0$ be an element of X. Then there is a bounded linear functional $F(x)$ on X such that*

$$\|F\| = 1, \quad F(x_0) = \|x_0\|. \tag{2.22}$$

Corollary 2.8. *If x_1 is an element of X such that $f(x_1) = 0$ for every bounded linear functional f on X, then $x_1 = 0$.*

First we prove Theorem 2.7.

Proof. Let M be the set of all vectors of the form αx_0. Then M is a subspace of X. Define f on M by

$$f(\alpha x_0) = \alpha\|x_0\|.$$

Then f is linear, and

$$|f(\alpha x_0)| = |\alpha| \cdot \|x_0\| = \|\alpha x_0\|.$$

Thus, f is bounded on M, and $\|f\| = 1$. By the Hahn-Banach Theorem, there is a bounded linear functional F on X such that $\|F\| = 1$, and $F(\alpha x_0) = \alpha\|x_0\|$. This is exactly what we want. □

Corollary 2.8 is an immediate consequence of Theorem 2.7. If $x_1 \neq 0$, there would be a bounded linear functional F on X such that $F(x_1) = \|x_1\|$. Thus, $x_1 = 0$.

Another consequence of Theorem 2.6 is

Theorem 2.9. *Let M be a subspace of a normed vector space X, and suppose x_0 is an element of X satisfying*

$$d = d(x_0, M) = \inf_{x\in M} \|x_0 - x\| > 0. \tag{2.23}$$

Then there is a bounded linear functional F on X such that $\|F\| = 1$, $F(x_0) = d$, and $F(x) = 0$ for $x \in M$.

Proof. Let M_1 be the set of all elements $z \in X$ of the form

$$z = \alpha x_0 + x, \quad \alpha \in \mathbb{R},\ x \in M. \tag{2.24}$$

Define the functional f on M_1 by $f(z) = \alpha d$. Now the representation (2.24) is unique, for if $z = \alpha_1 x_0 + x_1$, we have $(\alpha - \alpha_1)x_0 = x_1 - x \in M$, which contradicts (2.23) unless $\alpha_1 = \alpha$ and $x_1 = x$. Thus, f is well defined and linear on M_1. It also vanishes on M. Is it bounded on M_1? Yes, since

$$|f(\alpha x_0 + x)| = |\alpha|d \leq |\alpha| \cdot \|x_0 + \frac{x}{\alpha}\| = \|\alpha x_0 + x\|.$$

Hence, f is a bounded linear functional on M_1 with $\|f\| \leq 1$. However, for any $\varepsilon > 0$ we can find an $x_1 \in M$ such that $\|x_0 - x_1\| < d + \varepsilon$. Then $f(x_0 - x_1) = d$, and hence,

$$\frac{|f(x_0 - x_1)|}{\|x_0 - x_1\|} > \frac{d}{d+\varepsilon} = 1 - \frac{\varepsilon}{d+\varepsilon},$$

which is as close to one as we like. Hence, $\|f\| = 1$. We now apply Theorem 2.6 to conclude that there is a bounded linear functional F on X such that $\|F\| = 1$ and $F = f$ on M_1. This completes the proof. $\square$

Theorem 2.9 is a weak substitute in general Banach spaces for the projection theorem (Theorem 2.4) in Hilbert space. Sometimes it can be used as a replacement.

For any normed vector space X, let X' denote the set of bounded linear functionals on X. If $f, g \in X'$, we say that $f = g$ if

$$f(x) = g(x) \text{ for all } x \in X.$$

The "zero" functional is the one assigning zero to all $x \in X$. We define $h = f + g$ by

$$h(x) = f(x) + g(x), \quad x \in X,$$

and $g = \alpha f$ by

$$g(x) = \alpha f(x), \quad x \in X.$$

Under these definitions, X' becomes a vector space. We have been employing the expression

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}, \quad f \in X'. \tag{2.25}$$

This is easily seen to be a norm. In fact

$$\sup \frac{|f(x) + g(x)|}{\|x\|} \leq \sup \frac{|f(x)|}{\|x\|} + \sup \frac{|g(x)|}{\|x\|}.$$

Thus X' is a normed vector space. It is therefore natural to ask when X' will be complete. A rather surprising answer is given by

Theorem 2.10. *X' is a Banach space whether or not X is.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in X'. Thus for any $\varepsilon > 0$, there is an N such that

$$\|f_n - f_m\| < \varepsilon \quad \text{for } m, n > N,$$

or, equivalently,

$$|f_n(x) - f_m(x)| < \varepsilon\|x\| \quad \text{for } m, n > N,\ x \in X,\ x \neq 0. \tag{2.26}$$

Thus for each $x \neq 0$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers, and hence has a limit c_x depending on x. Define

$$f(x) = c_x.$$

Clearly f is a functional on X. It is linear, since

$$\begin{aligned} f(\alpha_1 x_1 + \alpha_2 x_2) &= \lim f_n(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \lim\{\alpha_1 f_n(x_1) + \alpha_2 f_n(x_2)\} \\ &= \alpha_1 f(x_1) + \alpha_2 f(x_2). \end{aligned}$$

It is also bounded. For let n be fixed in (2.26), and let $m \to \infty$. Then we have

$$|f_n(x) - f(x)| \le \varepsilon \|x\|, \quad n > N, \ x \in X. \tag{2.27}$$

Hence,

$$|f(x)| \le \varepsilon\|x\| + |f_n(x)| \le (\varepsilon + \|f_n\|)\|x\|$$

for $n > N$, $x \in X$. Hence, $f \in X'$. But we are not finished. We must show that f_n approaches f in X'. For this we use (2.27). It gives

$$\|f_n - f\| \le \varepsilon \quad \text{for } \ n > N.$$

Since ε was arbitrary, the result follows. □

We now give an interesting counterpart of (2.25). From it we see that

$$|f(x)| \le \|f\| \cdot \|x\|,$$

and hence,

$$\|x\| \ge \sup_{f \in X',\, f \ne 0} \frac{|f(x)|}{\|f\|}.$$

But by Theorem 2.7, for each $x \in X$ there is an $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Hence,

$$\|x\| = \max_{f \in X',\, f \ne 0} \frac{|f(x)|}{\|f\|}. \tag{2.28}$$

2.4. Examples of dual spaces

The space X' is called the *dual* (or *conjugate*) space of X. We consider some examples. If H is a Hilbert space, we know that every $f \in H'$ can be represented in the form

$$f(x) = (x, y), \quad x \in H.$$

The correspondence $f \leftrightarrow y$ is one-to-one, and $\|f\| = \|y\|$. Hence, we may identify H' with H itself.

We next consider the space l_p, where p is a real number satisfying $1 \le p < \infty$ (note that we have already mentioned l_2 and l_∞). It is the set of all infinite sequences $x = (x_1, \cdots, x_j, \cdots)$ such that

$$\sum_1^\infty |x_j|^p < \infty.$$

Set

$$\|x\|_p = \left(\sum_1^\infty |x_j|^p\right)^{1/p}. \tag{2.29}$$

The first question that comes to mind is whether l_p is a vector space. In particular, is the sum of two elements in l_p also in l_p? In showing this we might as well show that (2.29) is a norm, i.e., that

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p, \quad x,\, y \in l_p. \tag{2.30}$$

This will not only show that l_p is a vector space, but that it is a normed vector space. Inequality (2.30) is known as *Minkowski's inequality.*

To prove (2.30) we first note that it is trivial for $p = 1$, so we assume $p > 1$. To proceed one would expect to raise both sides to the p-th power and evaluate, as we did in the case $p = 2$. However, experience has shown that this leads to quite a complicated state of affairs, especially when p is not an integer. Since this course is devoted to the proposition that only easy things should be done, we follow an age-old trick discovered by our forefathers which notes that it is much simpler to try to prove the equivalent statement

$$\|x+y\|_p^p \leq (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1}. \tag{2.31}$$

Next, we note that we might as well assume that the components x_j and y_j of x and y are all nonnegative. For (2.30) is supposed to hold for such cases anyway, and once (2.30) is proved for such cases, it follows immediately that it holds in general. This assumption saves us the need to write absolute value signs all the way through the argument. Now (2.31) is equivalent to

$$\sum_1^\infty x_i(x_i+y_i)^{p-1} + \sum_1^\infty y_i(x_i+y_i)^{p-1} \leq \|x\|_p\|x+y\|_p^{p-1} + \|y\|_p\|x+y\|_p^{p-1},$$

so it suffices to show that

$$\sum_1^\infty x_i(x_i+y_i)^{p-1} \leq \|x\|_p\|x+y\|_p^{p-1}. \tag{2.32}$$

Set $z_i = (x_i + y_i)^{p-1}$. Then

$$\|x+y\|_p^p = \sum_1^\infty z_i^{p/p-1}.$$

Set $q = p/(p-1)$. Then

$$\|x+y\|_p^{p-1} = \left(\sum_1^\infty z_i^q\right)^{1/q} = \|z\|_q,$$

where $z = (z_1, \cdots, z_j, \cdots)$. Thus we want to prove

$$\sum_1^\infty x_i z_i \le \|x\|_p \|z\|_q. \tag{2.33}$$

This is known as *Hölder's inequality.* It implies (2.32), which in turn implies (2.31), which in turn implies Minkowski's inequality (2.30). Hence, if we prove (2.33), we can conclude that l_p is a normed vector space.

To prove (2.33), we note that it suffices to prove

$$\sum_1^\infty x_i z_i \le 1 \quad \text{when} \quad \|x\|_p = \|z\|_q = 1. \tag{2.34}$$

In fact, once we have proved (2.34), we can prove (2.33) for arbitrary $x \in l_p$ and $z \in l_q$ by applying (2.34) to the vectors

$$x' = x/\|x\|_p, \quad z' = z/\|z\|_q,$$

which satisfy $\|x'\|_p = \|z'\|_q = 1$. Multiplying through by $\|x\|_p\|z\|_q$ gives (2.33). Now (2.34) is an immediate consequence of

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \tag{2.35}$$

for positive numbers a, b. In fact, we have

$$\sum_1^\infty x_i z_i \le \sum \frac{x_i^p}{p} + \sum \frac{z_i^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

To prove (2.35), we graph the function $y = x^{p-1}$ and note that the rectangle with sides a, b is contained in the sum of the areas bounded by the curve and the x and y axes (cf. Figure 2.1). Since the curve is also given by $x = y^{q-1}$, we have

$$ab \le \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy = \frac{a^p}{p} + \frac{b^q}{q}.$$

This completes the proof. $\square$ A natural question to ask now is whether l_p is complete. We leave this question aside for the moment and discuss the dual space l_p' of l_p. Again we assume $p > 1$. A hint is given by Hölder's inequality (2.33). If $z \in l_q$, and we set

$$f(x) = \sum_1^\infty x_i z_i, \tag{2.36}$$

then f is a linear functional on l_p, and by (2.33) it is also bounded. We now show that all bounded linear functionals on l_p are of the form (2.36).

Theorem 2.11. *$l_p' = l_q$, where $1/p + 1/q = 1$.*

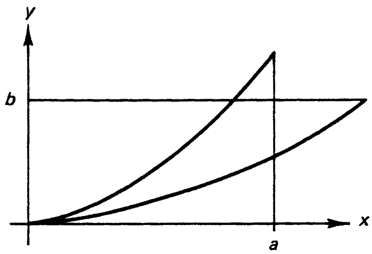

Figure 2.1

Proof. Suppose $x = (x_1, x_2, \cdots) \in l_p$, and $f \in l_p'$. Set $e_1 = (1, 0, \cdots)$, $e_2 = (0, 1, 0, \cdots)$, and in general e_j the vector having the j-th entry equal to one and all other entries equal to zero. Set

$$s_n = \sum_1^n x_j e_j.$$

Then $s_n \in l_p$, and

$$\|x - s_n\|_p^p = \sum_{n+1}^{\infty} |x_j|^p \longrightarrow 0.$$

Thus,

$$f(s_n) = f\left(\sum_1^n x_j e_j\right) = \sum_1^n x_j f(e_j),$$

and

$$|f(x) - f(s_n)| = |f(x - s_n)| \leq \|f\| \cdot \|x - s_n\|_p \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence,

$$f(x) = \sum_1^{\infty} x_j f(e_j).$$

Set $z_j = f(e_j)$ and $z = (z_1, z_2, \cdots)$. We must show that $z \in l_q$. Up until now, x has been completely arbitrary. Now we take a particular x. We set

$$x_i = \begin{cases} |z_i|^{q-2} z_i, & \text{when } z_i \neq 0; \\ 0, & \text{when } z_i = 0. \end{cases}$$

For this case

$$\|s_n\|_p^p = \sum_1^n |x_i|^p = \sum_1^n |z_i|^{p(q-1)} = \sum_1^n |z_i|^q,$$

since $p = q/(q-1)$. Moreover,

$$f(s_n) = \sum_1^n x_i z_i = \sum_1^n |z_i|^q,$$

and

$$|f(s_n)| \leq \|f\|(\sum_1^n |z_i|^q)^{1/p}.$$

Hence,

$$\sum_1^n |z_i|^q \leq \|f\|(\sum_1^n |z_i|^q)^{1/p},$$

or

$$(\sum_1^n |z_i|^q)^{1/q} \leq \|f\|.$$

Hence, $z \in l_q$, and $\|z\|_q \leq \|f\|$. But

$$|f(x)| = |\sum_1^n x_i z_i| \leq \|x\|_p \|z\|_q,$$

and thus,

$$\|f\| \leq \|z\|_q.$$

This shows that

$$\|f\| = \|z\|_q.$$

□

Thus, we have proved

Theorem 2.12. *If $f \in l'_p$, then there is a $z \in l_q$ such that*

$$f(x) = \sum_1^\infty x_i z_i, \quad x \in l_p,$$

and

$$\|f\| = \|z\|_q. \tag{2.37}$$

By the way, we now have the answer about completeness. For if we compute l'_q, we find that it is just l_p. Thus l_p is the dual space of a normed vector space and, hence, is complete by Theorem 2.10. It is for this reason that we waited until now to introduce the spaces l_p when $p \neq 2, \infty$. Remember that we have not proved completeness for the case $p = 1$. Since it is very easy, we leave it as an exercise.

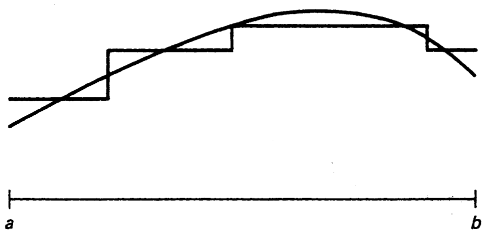

Figure 2.2

We wish to remark that the same ideas apply to the space L^p consisting of functions $x(t)$ on some interval $a \le t \le b$ such that $|x(t)|^p$ is integrable on this interval. The norm is given by

$$\|x\|_p = \left(\int_a^b |x(t)|^p dt \right)^{1/p}.$$

The same reasoning shows that L^p is a normed vector space and that Hölder's inequality

$$\int_a^b x(t)y(t)\,dt \le \|x\|_p \|y\|_q \tag{2.38}$$

holds. Moreover, for each $F \in (L^p)'$ there is a $y \in L^q$ such that

$$F(x) = \int_a^b x(t)y(t)\,dt, \quad x \in L^p, \tag{2.39}$$

and

$$\|F\| = \|y\|_q. \tag{2.40}$$

As another example, we consider the space $C = C[a,b]$ of continuous functions $x(t)$ in the interval $a \le t \le b$. Let f be a bounded linear functional on C and let $x(t)$ be a function in C. Now it is well known that we can approximate any continuous function as closely as desired by a step function, i.e., a function that is constant on a finite number of subintervals covering $[a,b]$ (cf. Figure 2.2). If f were also defined for step functions, we could compute it for a sequence approaching $x(t)$ and take the limit.

"Even so," you say, "what makes you believe that it would be easier to compute f for a step function than for a continuous function?" Well, just look. If f were defined on step functions, we would let

$$k_s(t) = \begin{cases} 1, & a \le t \le s \le b, \\ 0, & a < s < t \le b, \end{cases} \tag{2.41}$$

be the characteristic function for the interval $[a, s]$ for $s > a$, and define $g(s) = f(k_s)$. Then for each step function $y(t)$ there is a set of numbers

$$a = t_0 < t_1 < \cdots < t_n = b$$

(called a *partition* of $[a, b]$) such that

$$y(t) = \sum_1^n \alpha_i [k_{t_i}(t) - k_{t_{i-1}}(t)].$$

By linearity, we would have

$$f(y) = \sum_1^n \alpha_i [g(t_i) - g(t_{i-1})],$$

and f is determined once we know the function $g(s)$. But this is all wishful thinking, since f is not defined for step functions. "Well," you say, "can't we extend it to be defined for step functions?" That is an idea! After all, C is contained in the space B of bounded functions, which contains among other things the step functions. Thus by Theorem 2.6, there is a bounded linear functional F on B which coincides with f on C and such that $\|F\| = \|f\|$. In particular, F is defined on step functions so we can set

$$g(s) = F(k_s). \tag{2.42}$$

Now let $x(t)$ be any element of C. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x(t') - x(t)| < \varepsilon \quad \text{whenever} \quad |t' - t| < \delta$$

(in case you do not recognize it, this is just uniform continuity). Let

$$a = t_0 < t_1 < \cdots < t_n = b$$

be any partition of $[a, b]$ such that

$$\eta = \max_i |t_i - t_{i-1}| < \delta.$$

Then if t_i' is any point satisfying

$$t_{i-1} \le t_i' \le t_i,$$

we have

$$|x(t) - x(t_i')| < \varepsilon \quad \text{for} \quad t_{i-1} \le t \le t_i.$$

Let

$$y(t) = \sum_1^n x(t_i')[k_{t_i}(t) - k_{t_{i-1}}(t)].$$

Then

$$F(y) = \sum_1^n x(t_i')[g(t_i) - g(t_{i-1})],$$

and we know that

$$|F(x) - F(y)| \leq \|F\| \cdot \|x - y\| < \varepsilon \|F\|.$$

In other words,

$$\lim_{\eta \to 0} \sum_1^n x(t_i')[g(t_i) - g(t_{i-1})] \tag{2.43}$$

exists and equals $F(x)$. This limit, when it exists, is better known as the *Riemann-Stieltjes integral*

$$\int_a^b x(t)\, dg(t). \tag{2.44}$$

Thus, we have shown that the integral (2.44) exists and

$$F(x) = \int_a^b x(t)\, dg(t), \quad x \in C. \tag{2.45}$$

What kind of function is g? To answer this, let

$$y(t) = \sum_1^n \alpha_i [k_{t_i}(t) - k_{t_{i-1}}(t)]$$

be any step function on $[a, b]$. Then

$$F(y) = \sum_1^n \alpha_i [g(t_i) - g(t_{i-1})],$$

showing that

$$|\sum_1^n \alpha_i [g(t_i) - g(t_{i-1})]| \leq \|F\| \cdot \|y\| = \|f\| \max_i |\alpha_i|.$$

This is true for all choices of the α_i. Take $\alpha_i = 1$ if $g(t_i) \geq g(t_{i-1})$ and $\alpha_i = -1$, otherwise. This gives

$$\sum_1^n |g(t_i) - g(t_{i-1})| \leq \|f\|. \tag{2.46}$$

Since y was any step function on $[a, b]$, (2.46) is true for any partition $a = t_0 < t_1 < \cdots < t_n = b$. Functions having this property are said to be of *bounded variation.* The *total variation* of g is defined as

$$V(g) = \sup \sum_1^n |g(t_i) - g(t_{i-1})|,$$

where the supremum (least upper bound) is taken over all partitions of $[a, b]$. To summarize, we see that for each bounded linear functional f on $C[a, b]$, there is a function $g(t)$ of bounded variation on $[a, b]$ such that

$$f(x) = \int_a^b x(t)\, dg(t), \quad x \in C[a, b], \tag{2.47}$$

and

$$V(g) \le \|f\|.$$

Since

$$|\sum_1^n x(t_i')[g(t_i) - g(t_{i-1})]| \le \|x\| V(g)$$

for all partitions $a = t_0 < t_1 < \cdots < t_n = b$ and all choices of t_i', it follows that

$$\left| \int_a^b x(t)\, dg(t) \right| \le \|x\| V(g). \tag{2.48}$$

Hence,

$$V(g) = \|f\|. \tag{2.49}$$

It should be pointed out that the converse is also true. If $x \in C$ and g is of bounded variation, then the Riemann-Stieltjes integral (2.44) exists and clearly gives a linear functional on C. By (2.48), this functional is bounded and its norm is $\le V(g)$.

There is a question concerning the uniqueness of the function g. We know that each bounded linear functional f can be represented in the form (2.47), where g is of bounded variation. But suppose f can be represented in this way by means of two such functions g_1 and g_2. Set $g = g_1 - g_2$. Then clearly g is also of bounded variation and

$$\int_a^b x(t)\, dg(t) = 0, \quad x \in C. \tag{2.50}$$

What does this imply concerning g?

In order to answer this question we must recall two well-known properties of functions of bounded variation. The first is that the set of points of discontinuity of such a function is at most denumerable. The second is that at any point t of discontinuity of such a function, the right- and left-hand limits

$$g(t+) = \lim_{\delta \searrow 0} g(t + \delta),$$

$$g(t-) = \lim_{\delta \searrow 0} g(t - \delta)$$

both exist (i.e., the discontinuity is merely a jump). Now let c be any point of continuity of g, $a < c < b$. Let $\delta > 0$ be given, and let $z(t)$ be

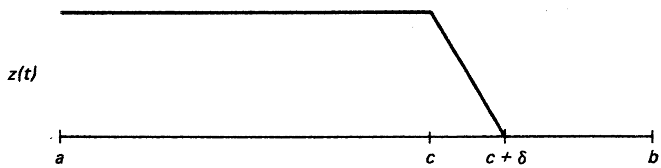

Figure 2.3

the continuous function which is identically one in $[a, c]$, identically zero in $[c+\delta, b]$ and linear in $[c, c+\delta]$ (cf. Figure 2.3). Then

$$\int_c^b z(t)\,dg(t) = g(c) - g(a) + \int_c^{c+\delta} z(t)\,dg(t).$$

By (2.48) this gives

$$|g(c) - g(a)| \le V_c^{c+\delta}(g),$$

the total variation of g in the interval $[c, c+\delta]$.

Now we make use of the fact (which we shall prove in a moment) that

$$V_c^{c+\delta}(g) \longrightarrow 0 \text{ as } \delta \longrightarrow 0, \tag{2.51}$$

whenever g is continuous at c. Hence, $g(c) = g(a)$ for all points c of continuity of g. Reasoning from the other end, we see that a necessary condition for (2.50) to hold is that

$$g(c) = g(a) = g(b) \tag{2.52}$$

for all points c of continuity of g. This condition is also sufficient. For if it is fulfilled, we have

$$\sum_1^n x(t_i')[g(t_i) - g(t_{i-1})] = 0$$

for all partitions which avoid the discontinuities of g. Since the integral exists, it is the limit of this expression no matter how the partitions are taken provided $\eta \to 0$. Since the discontinuities of g form at most a denumerable set, this can always be done while avoiding the discontinuities of g. Hence, (2.50) holds. Thus, we have proved

Lemma 2.13. *A necessary and sufficient condition for (2.50) to hold is that $g(c) = g(a) = g(b)$ at all points c of continuity of g.*

This shows us how to "normalize" the function g in (2.47) to make it unique. If we define

$$\omega(t) = \begin{cases} g(a), & t = a, \\ g(a) + g(t) - g(t+), & a < t < b, \\ g(a), & t = b, \end{cases}$$

then by Lemma 2.13,

$$\int_a^b x(t)\, d\omega(t) = 0, \quad x \in C.$$

If we now set

$$\hat{g}(t) = g(t) - \omega(t),$$

then $\hat{g}$ is of bounded variation and satisfies

$$\hat{g}(a) = 0, \tag{2.53}$$

$$\hat{g}(t) = \hat{g}(t+), \quad a \le t < b. \tag{2.54}$$

We call a function satisfying (2.53) and (2.54) *normalized.* By Lemma 2.13, there is at most one normalized function of bounded variation satisfying (2.47).

There is one more thing we must investigate. By (2.48) and (2.49),

$$V(g) = \|f\| \le V(\hat{g}).$$

We now show that $V(\hat{g}) \le V(g)$. To see this, let $a = t_0 < t_1 < \cdots < t_n = b$ be any partition of $[a, b]$, and let $\varepsilon > 0$ be given. Then there are points c_i such that $t_i < c_i < t_{i+1}$ and

$$|g(t_i+) - g(c_i)| < \varepsilon/2n.$$

Thus, if we take $c_0 = a$, $c_n = b$, we have

$$\sum_1^n |\hat{g}(t_i) - \hat{g}(t_{i-1})| \le \sum_1^{n-1} |g(t_i+) - g(c_i)| + \sum_1^n |g(c_i) - g(c_{i-1})|$$

$$+ \sum_2^n |g(c_{i-1}) - g(t_{i-1}+)| \le V(g) + \varepsilon.$$

Thus,

$$V(\hat{g}) \le V(g) + \varepsilon.$$

Since ε was arbitrary, we have the result.

To summarize, we have

Theorem 2.14. *For each bounded linear functional f on $C[a,b]$ there is a unique normalized function $\hat{g}$ of bounded variation such that*

$$f(x) = \int_a^b x(t)\, d\hat{g}(t), \quad x \in C[a,b], \tag{2.55}$$

and

$$V(\hat{g}) = \|f\|. \tag{2.56}$$

Conversely, every such normalized $\hat{g}$ gives a bounded linear functional on $C[a,b]$ satisfying (2.55) and (2.56).

Proof. In case you have forgotten, it remains to prove (2.51). Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ so small and a partition

$$c + \delta = \tau_0 < \tau_1 < \cdots < \tau_m = b$$

of the interval $[c+\delta, b]$ such that

$$V_c^b(g) - \varepsilon \leq |g(c+\delta) - g(c)| + \sum_1^m |g(\tau_k) - g(\tau_{k-1})|,$$

and

$$|g(c+\delta) - g(c)| < \varepsilon.$$

Now let

$$c = t_0 < t_1 < \cdots < t_n = c + \delta$$

be any partition of $[c, c+\delta]$. Then

$$\sum_1^n |g(t_i) - g(t_{i-1})| + \sum_1^m |g(\tau_k) - g(\tau_{k-1})| \leq V_c^b(g)$$

$$\leq 2\varepsilon + \sum_1^m |g(\tau_k) - g(\tau_{k-1})|.$$

Hence,

$$\sum_1^n |g(t_i) - g(t_{i-1})| \leq 2\varepsilon.$$

Since this is true for any partition of $[c, c+\delta]$, we have

$$V_c^{c+\delta}(g) \leq 2\varepsilon.$$

This completes the proof. □

Theorem 2.14 is due to F. Riesz.

2.5. Problems

(1) Prove the statement following (2.7).

(2) If F is a bounded linear functional on a normed vector space X, show that

$$\|F\| = \sup_{\|x\| \leq 1} |F(x)| = \sup_{\|x\|=1} |F(x)|.$$

(3) A functional $F(x)$ is called *additive* if $F(x+y) = F(x) + F(y)$. If F is additive, show that $F(\alpha x) = \alpha F(x)$ for all rational α.

(4) Show that an additive functional is continuous everywhere if it is continuous at one point.

(5) Prove that l_1 is complete.

(6) Prove the Hahn-Banach theorem for a Hilbert space following the procedure outlined at the end of Section 2.2.

(7) Let M be a subspace of a normed vector space X which is not dense. Show that there is a sequence $\{x_n\} \subset X$ such that $\|x_n\| = 1$ and $d(x_n, M) \to 1$.

(8) If $x \in l_p,\ y \in l_q,\ z \in l_r$, where $1/p + 1/q + 1/r = 1$, show that

$$\sum_1^\infty |x_i y_i z_i| \leq \|x\|_p \|y\|_q \|z\|_r.$$

(9) If f is a bounded linear functional on X, let N be the set of all $x \in X$ such that $f(x) = 0$. Show that there is an $x_0 \in X$ such that every element $x \in X$ can be expressed in the form $x = \alpha x_0 + x_1$, where $x_1 \in N$.

(10) Show that $c_0' = l_1$ (see Problem 4 of Chapter 1).

(11) Show that (2.11) implies $p(0) = 0$.

(12) Prove the Hahn-Banach theorem for the case when V satisfies (2.21) without using Zorn's lemma.

(13) Show that Theorem 2.1 implies Theorem 2.3.

(14) Let F, G be linear functionals on a vector space V, and assume that $F(v) = 0 \Rightarrow G(v) = 0, \; v \in V$. Show that there is a scalar C such that $G(v) = CF(v), \; v \in V$.

(15) Let F be a linear functional on a Hilbert space X with $D(F) = X$. Show that F is bounded if and only if $N(F)$ is closed.

(16) A subset U of a normed vector space is called *convex* if $\alpha x + (1-\alpha)y$ is in U for all $x, y \in U$ and $0 \leq \alpha \leq 1$. Show that Theorem 2.2 holds when N is a closed convex set.

(17) A set U is called *midpoint convex* if $(x+y)/2$ is in U for all $x, y \in U$. Show that a closed, midpoint convex subset of a normed vector space is convex, but if it is not closed, this need not be the case.

(18) For X a Banach space, show that a sequence $\{x_k\}$ converges in norm if and only if $x'(x_k)$ converges uniformly for each $x' \in X'$ satisfying $\|x'\| = 1$.

(19) If $\{x_k\}$ is a sequence of elements in a normed vector space X and $\{\alpha_k\}$ is a sequence of scalars, show that a necessary and sufficient condition for the existence of an $x' \in X'$ satisfying

$$x'(x_k) = \alpha_k \quad \text{and} \quad \|x'\| = M,$$

is that

$$|\sum_1^n \beta_k \alpha_k| \leq M \| \sum_1^n \beta_k x_k \|$$

holds for each n and scalars $\beta_1, \cdots, \beta_n$.

(20) If $p(x)$ is a sublinear functional on a vector space V and x_0 is any element of V, show that there is a linear functional F on V such that $F(x_0) = x_0$ and

$$-p(-x) \leq F(x) \leq p(x), \quad x \in V.$$

(21) Show that there is a functional $f \in l'_\infty$ satisfying

$$\liminf_{k\to\infty} x_k \le f(x) \le \limsup_{k\to\infty} x_k, \quad x = (x_1, x_2, \cdots) \in l_\infty.$$

(22) If X is a Hilbert space, show that the functional F given by Theorem 2.6 is unique.

(23) Show that l_p is not a Banach space when $0 < p < 1$. What goes wrong?

(24) If $f \in l_p$ for some $p < \infty$, show that

$$\|f\|_p \longrightarrow \|f\|_\infty \text{ as } p \longrightarrow \infty.$$

Chapter 3

LINEAR OPERATORS

3.1. Basic properties

Let X, Y be normed vector spaces. A mapping A which assigns to each element x of a set $D(A) \subset X$ a unique element $y \in Y$ is called an *operator* (or *transformation*). The set $D(A)$ on which A acts is called the *domain* of A. The operator A is called *linear* if

a) $D(A)$ is a subspace of X

and

b) $A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$ for all scalars α_1, α_2 and all elements x_1, $x_2 \in D(A)$.

Until further notice we shall only consider operators A with $D(A) = X$.

An operator A is called *bounded* if there is a constant M such that

$$\|Ax\| \le M\|x\|, \quad x \in X. \tag{3.1}$$

The norm of such an operator is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}. \tag{3.2}$$

Again, it is the smallest M which works in (3.1). An operator A is called *continuous* at a point $x_0 \in X$ if $x_n \to x$ in X implies $Ax_n \to Ax$ in Y. A bounded linear operator is continuous at each point. For if $x_n \to x$ in X, then

$$\|Ax_n - Ax\| \leq \|A\| \cdot \|x_n - x\| \longrightarrow 0.$$

We also have

Theorem 3.1. *If a linear operator A is continuous at one point $x_0 \in X$, then it is bounded, and hence continuous at every point.*

Proof. If A were not bounded, then for each n we could find an element $x_n \in X$ such that

$$\|Ax_n\| > n\|x_n\|.$$

Set

$$z_n = \frac{x_n}{n\|x_n\|} + x_0.$$

Then $z_n \to x_0$. Since A is continuous at x_0, we must have $Az_n \to Ax_0$. But

$$Az_n = \frac{Ax_n}{n\|x_n\|} + Ax_0.$$

Hence,

$$\frac{Ax_n}{n\|x_n\|} \longrightarrow 0.$$

But

$$\frac{\|Ax_n\|}{n\|x_n\|} > 1,$$

providing a contradiction. □

We let $B(X, Y)$ be the set of bounded linear operators from X to Y. Under the norm (3.2), one easily checks that $B(X, Y)$ is a normed vector space. As a generalization of Theorem 2.10, we have

Theorem 3.2. *If Y is a Banach space, so is $B(X, Y)$.*

Proof. Suppose $\{A_n\}$ is a Cauchy sequence of operators in $B(X, Y)$. Then for each $\varepsilon > 0$ there is an integer N such that

$$\|A_n - A_m\| < \varepsilon \text{ for } m, n > N.$$

Thus for each $x \neq 0$,

$$\|A_n x - A_m x\| < \varepsilon\|x\|, \quad m, n > N. \tag{3.3}$$

This shows that $\{A_n x\}$ is a Cauchy sequence in Y. Since Y is complete, there is a $y_x \in Y$ such that $A_n x \to y_x$ in Y. Define the operator A from X to Y by $Ax = y_x$. Then A is linear (see the proof of Theorem 2.10). Let $m \to \infty$ in (3.3). Then

$$\|A_n x - Ax\| \leq \varepsilon \|x\|, \quad n > N.$$

Hence,

$$\|Ax\| \leq \varepsilon \|x\| + \|A_n x\| \leq (\varepsilon + \|A_n\|)\|x\|, \quad n > N.$$

This shows that A is bounded. Moreover,

$$\|A_n - A\| \leq \varepsilon, \quad n > N.$$

Hence,

$$\|A_n - A\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This completes the proof. □

3.2. The adjoint operator

Suppose X, Y are normed vector spaces and $A \in B(X, Y)$. For each $y' \in Y'$, the expression $y'(Ax)$ assigns a scalar to each $x \in X$. Thus, it is a functional $F(x)$. Clearly F is linear. It is also bounded since

$$|F(x)| = |y'(Ax)| \leq \|y'\| \cdot \|Ax\| \leq \|y'\| \cdot \|A\| \cdot \|x\|.$$

Thus, there is an $x' \in X'$ such that

$$y'(Ax) = x'(x), \quad x \in X. \tag{3.4}$$

This functional x' is unique, for any other functional satisfying (3.4) would have to coincide with x' on each $x \in X$. Thus, to each $y' \in Y'$ we have assigned a unique $x' \in X'$. We designate this assignment by A' and note that it is a linear operator from Y' to X'. Thus, (3.4) can be written in the form

$$y'(Ax) = A'y'(x). \tag{3.5}$$

The operator A' is called the *adjoint* (or *conjugate*) of A, depending on the mood one is in. But no matter what the mood, the following is true.

Theorem 3.3. *$A' \in B(Y', X')$, and $\|A'\| = \|A\|$.*

Proof. We have by (3.5),

$$|A'y'(x)| = |y'(Ax)| \leq \|y'\| \cdot \|A\| \cdot \|x\|.$$

Hence,

$$\|A'y'\| = \sup_{x \neq 0} \frac{|A'y'(x)|}{\|x\|} \leq \|y'\| \cdot \|A\|.$$

This shows that $A' \in B(Y', X')$ and that $\|A'\| \le \|A\|$. To prove the reverse inequality we must show that

$$\|Ax\| \le \|A'\| \cdot \|x\|, \quad x \in X. \tag{3.6}$$

Again by (3.5), we have

$$|y'(Ax)| \le \|A'y'\| \cdot \|x\| \le \|A'\| \cdot \|y'\| \cdot \|x\|,$$

showing that

$$\sup_{y' \neq 0} \frac{|y'(Ax)|}{\|y'\|} \le \|A'\| \cdot \|x\|, \quad x \in X.$$

We now appeal to (2.28) to obtain (3.6). □

The adjoint has the following easily verified properties:

$$(A+B)' = A' + B'. \tag{3.7}$$

$$(\alpha A)' = \alpha A'. \tag{3.8}$$

$$(AB)' = B'A'. \tag{3.9}$$

Why should we consider adjoints? One reason is as follows. Many problems in mathematics and its applications can be put in the form: given normed vector spaces X, Y and an operator $A \in B(X, Y)$, one wishes to solve

$$Ax = y. \tag{3.10}$$

The set of all y for which one can solve (3.10) is called the *range* of A and is denoted by $R(A)$. The set of all x for which $Ax = 0$ is called the *null space* of A and is denoted by $N(A)$. Since A is linear, it is easily checked that $N(A)$ and $R(A)$ are subspaces of X and Y, respectively.

If $y \in R(A)$, there is an $x \in X$ satisfying (3.10). For any $y' \in Y'$ we have

$$y'(Ax) = y'(y).$$

Taking adjoints we get

$$A'y'(x) = y'(y).$$

If $y' \in N(A')$, this gives $y'(y) = 0$. Thus, a necessary condition that $y \in R(A)$ is that $y'(y) = 0$ for all $y' \in N(A')$. Obviously, it would be of great interest to know when this condition is also sufficient. We shall find the answer in the next section. In doing so we shall find it convenient to introduce some picturesque (if not grotesque) terminology.

3.3. Annihilators

Let S be a subset of a normed vector space X. A functional $x' \in X'$ is called an *annihilator* of S if

$$x'(x) = 0, \quad x \in S.$$

The set of all annihilators of S is denoted by S°. To be fair, this should not be a one-way proposition. So for any subset T of X', we call $x \in X$ an annihilator of T if

$$x'(x) = 0, \quad x' \in T.$$

We denote the set of such annihilators of T by ${}^\circ T$.

Now we can state our necessary condition of the last section in terms of annihilators. In fact, it merely states

$$R(A) \subset {}^\circ N(A'). \tag{3.11}$$

We are interested in determining when the two sets are the same. We first prove some statements about annihilators.

Lemma 3.4. *S° and ${}^\circ T$ are closed subspaces.*

Proof. We consider S°; the proof for ${}^\circ T$ is similar. Clearly, S° is a subspace, for if x_1, x_2 annihilate S, so does $\alpha_1 x_1 + \alpha_2 x_2$. Suppose $x'_n \in S^\circ$ and $x'_n \to x'$ in X'. Then

$$x'_n(x) \longrightarrow x'(x), \quad x \in X.$$

In particular, this holds for all $x \in S$, showing that $x' \in S^\circ$. $\square$

Lemma 3.5. *If M is a closed subspace of X, then ${}^\circ(M^\circ) = M$.*

Proof. Clearly, $x \in {}^\circ(M^\circ)$ if, and only if, $x'(x) = 0$ for all $x' \in M^\circ$. But this is satisfied by all $x \in M$. Hence, $M \subset {}^\circ(M^\circ)$. Now suppose x_1 is an element of X which is not in M. Since M is a closed subspace,

$$d(x_1, M) = \inf_{z \in M} \|x_1 - z\| > 0.$$

By Theorem 2.10, there is an $x'_1 \in X'$ such that $x'_1(x_1) = d(x_1, M)$, $\|x'_1\| = 1$, and $x'_1(x) = 0$ for all $x \in M$ (i.e., $x' \in M^\circ$). Since x_1 does not annihilate x'_1, it is not in ${}^\circ(M^\circ)$. Hence, ${}^\circ(M^\circ) \subset M$, and the proof is complete. $\square$

Let W be a subset of X. The subspace of X *spanned* (or *generated*) by W consists of the set of finite linear combinations of elements of W, i.e., sums of the form

$$\sum_1^n \alpha_j x_j,$$

where the α_j are scalars and the x_j are in W. The closure $\overline{W}$ of W consists of those points of X which are the limits of sequences of elements of W. One checks easily that $\overline{W}$ is a closed set in X. The closed subspace spanned by W is the closure of the subspace spanned by W. We have

Lemma 3.6. *If S is a subset of X, and M is the closed subspace spanned by S, then $M^\circ = S^\circ$ and $M = {}^\circ(S^\circ)$.*

Proof. The second statement follows from the first, since by Lemma 3.5, $M = {}^\circ(M^\circ) = {}^\circ(S^\circ)$. As for the first, since $S \subset M$, we have clearly $M^\circ \subset S^\circ$. Moreover, if $x_j \in S$ and $x' \in S^\circ$, then

$$x'(\sum_1^n \alpha_j x_j) = \sum_1^n \alpha_j x'(x_j) = 0,$$

showing that x' annihilates the subspace spanned by S. Moreover, if $\{z_n\}$ is a sequence of elements of this subspace, and $z_n \to z$ in X, then

$$x'(z_n) \longrightarrow x'(z).$$

Hence, x' annihilates M, and the proof is complete. $\square$

Now returning to our operator $A \in B(X,Y)$, we note that

$$R(A)^\circ = N(A'). \tag{3.12}$$

For, $y' \in R(A)^\circ$ if, and only if, $y'(Ax) = 0$ for all $x \in X$. This in turn is true if, and only if, $A'y'(x) = 0$ for all x, i.e., if $A'y' = 0$. Now applying Lemma 3.6, we have

$$\overline{R(A)} = {}^\circ[R(A)^\circ] = {}^\circ N(A'), \tag{3.13}$$

since $R(A)$ itself is a subspace. Thus we have

Theorem 3.7. *A necessary and sufficient condition that*

$$R(A) = {}^\circ N(A'), \tag{3.14}$$

is that $R(A)$ be closed in Y.

3.4. The inverse operator

Suppose we are interested in solving the equation

$$Ax = y, \tag{3.15}$$

where $A \in B(X,Y)$ and X, Y are normed vector spaces. If $R(A) = Y$, we know that we can solve (3.15) for each $y \in Y$. If $N(A)$ consists only of the vector 0, we know that the solution is unique. However, if the problem arises from applications, it may happen that y was determined by experimental means, which are subject to a certain amount of error. Of course, it is hoped that the error will be small. Thus, in solving (3.15) it is desirable to know that when the value of y is close to its “correct” value, the same will be true for the solution x. Mathematically, the question is: if y_1 is close to y_2 and

$Ax_i = y_i,\ i = 1, 2$, does it follow that x_1 is close to x_2? This question can be expressed very conveniently in terms of the inverse operator of A.

If $R(A) = Y$ and $N(A) = \{0\}$ (i.e., consists only of the vector 0), we can assign to each $y \in Y$ the unique solution of (3.15). This assignment is an operator from Y to X and is usually denoted by A^{-1} and called the *inverse operator* of A. It is linear because of the linearity of A. Our question of the last paragraph is equivalent to: when is A^{-1} continuous? By Theorem 3.1, this is equivalent to when is it bounded. A very important answer to this question is given by

Theorem 3.8. *If X, Y are Banach spaces, and $A \in B(X, Y)$ with $R(A) = Y$, $N(A) = \{0\}$, then $A^{-1} \in B(Y, X)$.*

This theorem is sometimes referred to as the *bounded inverse theorem.* Its proof is not difficult, but requires some preparation. The main tool is the *Baire category theorem* (Theorem 3.9 below).

Let X be a normed vector space and let W be a set of vectors in X. The set W is called *nowhere dense* in X if every sphere of the form

$$\|x - x_0\| < r,\ r > 0,$$

contains a vector not in the closure $\overline{W}$ of W (i.e., $\overline{W}$ contains no sphere). A set $W \subset X$ is of the *first category* in X if it is the denumerable union of nowhere dense sets, i.e., if

$$W = \bigcup_1^\infty W_k,$$

where each W_k is nowhere dense. Otherwise, W is said to be in the *second category.* The following is known as *Baire's category theorem.*

Theorem 3.9. *If X is complete, then it is of the second category.*

Proof. Suppose X were of the first category. Then

$$X = \bigcup_1^\infty W_k, \tag{3.16}$$

where each W_k is nowhere dense. Thus there is a point x_1 not in $\overline{W}_1$. Since x_1 is not a limit point of W_1, there is an r_1 satisfying $0 < r_1 < 1$ such that the closure of the sphere

$$S_1 = \{x : \|x - x_1\| < r_1\}$$

does not intersect W_1. This sphere contains a point x_2 not in $\overline{W}_2$, and hence contains a sphere of the form

$$S_2 = \{x : \|x - x_2\| < r_2\},\ 0 < r_2 < 1/2,$$

such that the closure of S_2 does not intersect W_2. Inductively, there is a sequence of spheres $S_k \subset S_{k-1}$ of the form

$$S_k = \{x : \|x - x_k\| < r_k\},\ 0 < r_k < 1/k,$$

such that the closure of S_k does not intersect W_k. Now for $j > k$, we have $x_j \in S_k$, and hence,

$$\|x_j - x_k\| < r_k < 1/k, \quad j > k. \tag{3.17}$$

This shows that $\{x_k\}$ forms a Cauchy sequence in X. Since X is complete, this sequence has a limit $x_0 \in X$. Letting $j \to \infty$ in (3.17), we get

$$\|x_0 - x_k\| \leq r_k.$$

Thus x_0 is in the closure of S_k for each k, showing that x_0 is not in any of the W_k, and hence not in

$$\bigcup_1^\infty W_k.$$

This contradicts (3.16). □

Let X, Y be normed vector spaces, and let A be a linear operator from X to Y. We now officially lift our restriction that $D(A) = X$. However, if $A \in B(X, Y)$, it is still to be assumed that $D(A) = X$.

The operator A is called *closed* if whenever $\{x_n\} \subset D(A)$ is a sequence satisfying

$$x_n \longrightarrow x \text{ in } X, \quad Ax_n \longrightarrow y \text{ in } Y, \tag{3.18}$$

then $x \in D(A)$ and $Ax = y$. Clearly, all operators in $B(X, Y)$ are closed. Another obvious statement is that if A is closed, then $N(A)$ is a closed subspace of X. A statement which is not so obvious is

Theorem 3.10. *If X, Y are Banach spaces, and A is a closed linear operator from X to Y, with $D(A) = X$, then*

a) There are positive constants M, r such that $\|Ax\| \leq M$ whenever $\|x\| < r$.

b) $A \in B(X, Y)$.

Theorem 3.10 is called the *closed graph theorem.* The geometrical significance of the terminology will be discussed later. We note here that the theorem immediately implies Theorem 3.8. For if A^{-1} exists, it is obviously a closed operator from Y to X, and by hypothesis $D(A^{-1}) = Y$. Hence, we can apply Theorem 3.10 to conclude that A^{-1} is bounded. In making these observations, we note that it was even unnecessary to assume in Theorem

3.8 that $A \in B(X, Y)$. This hypothesis was only used to show that A^{-1} is closed. But from the definition of a closed operator we see that A^{-1} is closed if, and only if, A is. So we might as well have assumed in the first place that A is merely closed. We can therefore replace Theorem 3.8 by the seemingly stronger

Theorem 3.11. *If X, Y are Banach spaces and A is a closed linear operator from X to Y with $R(A) = Y$, $N(A) = \{0\}$, then $A^{-1} \in B(Y, X)$.*

The reason we said "seemingly" is that Theorem 3.8 actually implies Theorem 3.11. To see this we use a little trick. Suppose A is a closed linear operator from X to Y. As we noted, $D(A)$ is a subspace of X. If we use the norm of X, then $D(A)$ is a normed vector space. The only trouble is that it is not complete (unless $A \in B(X, Y)$). So we use another norm, namely

$$\|x\|_A = \|x\| + \|Ax\|. \tag{3.19}$$

Now don't laugh. This is a norm on $D(A)$. But that is not all. If X and Y are Banach spaces and A is closed, then $D(A)$ is complete with respect to this norm. For if $\{x_n\}$ is a Cauchy sequence with respect to this norm, then $\{x_n\}$ is a Cauchy sequence in X and $\{Ax_n\}$ is a Cauchy sequence in Y. Hence, there are $x \in X$, $y \in Y$ such that (3.18) holds. Since A is closed, $x \in D(A)$ and $Ax = y$. Thus, $\|x_n - x\|_A \to 0$. Now we forget about X and consider A as an operator from the Banach space $D(A)$ with its new norm (3.19) to Y. Moreover, $A \in B(D(A), Y)$, since

$$\|Ax\| \leq \|x\|_A.$$

Thus we can apply Theorem 3.8 to conclude that $A^{-1} \in B(Y, D(A))$, i.e., that

$$\|A^{-1}y\|_A \leq \|y\|_Y, \quad y \in Y,$$

or

$$\|A^{-1}y\|_X + \|y\|_Y \leq C\|y\|_Y.$$

This gives Theorem 3.11.

A similar trick can be used to show that Theorem 3.8 also implies the closed graph theorem (Theorem 3.10). This is done by introducing the *cartesian product* $X \times Y$ of X and Y. This is defined as the set of all ordered pairs $\langle x, y\rangle$ of elements $x \in X$, $y \in Y$. They are added by means of the formula

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$

Under the norm

$$\|\langle x, y\rangle\| = \|x\|_X + \|y\|_Y,$$

$X \times Y$ becomes a normed vector space provided X and Y are normed vector spaces. Moreover, it is a Banach space when X and Y are Banach spaces.

If A is an operator from X to Y, the *graph* G_A of A is the subset of $X \times Y$ consisting of pairs of the form

$$\langle x, Ax \rangle.$$

It is clearly a subspace of $X \times Y$. Moreover, G_A is a closed subspace if, and only if, A is a closed operator (this is the reason for the terminology). Equipped with this knowledge, we now show how Theorem 3.10 is a consequence of Theorem 3.8. Let A be a closed operator from X to Y defined on the whole of X, where X and Y are Banach spaces. Let E be the linear operator from G_A to X defined by

$$E(\langle x, Ax \rangle) = x.$$

By what we have just seen, G_A is a Banach space. Moreover, $E \in B(G_A, X)$, and $R(E) = X$. Also $N(E) = \langle 0, 0 \rangle$. Hence, we can apply Theorem 3.8 to conclude that E^{-1} is bounded, i.e., that

$$\|x\| + \|Ax\| \le C\|x\|,$$

from which we conclude that $A \in B(X, Y)$.

We have shown that Theorems 3.8, 3.10, and 3.11 are equivalent. It therefore suffices to prove only one of them. We choose Theorem 3.10.

Proof of Theorem 3.10. We first note that (b) follows from (a). For if $x \neq 0$ is any element of X, set $z = rx/2\|x\|$. Then $\|z\| < r$, so that $\|Az\| \le M$. But $Az = rAx/2\|x\|$, showing that

$$\|Ax\| \le 2Mr^{-1}\|x\|.$$

Thus, (a) implies (b). In order to prove (a) we employ Theorem 3.9. Set

$$U_n = \{x \ : \ \|Ax\| < n\}.$$

Then

$$X = \bigcup_1^\infty U_n.$$

Since X is complete, it must be of the second category (Theorem 3.9), and hence, at least one of the U_n, say U_k, is not nowhere dense (note the double negative). This means that $\overline{U}_k$ contains a sphere of the form

$$V_t = \{x \ : \ \|x - x_0\| < t\}, \quad t > 0,$$

i.e., U_k is dense in V_t. We may take the center x_0 of V_t to be in U_k by shifting it slightly. From this it follows that the set of vectors of the form $z - x_0$, where $z \in U_k$, is dense in the sphere

$$S_t = \{x \ : \ \|x\| < t\}.$$

For, if $x \in S_t$, then $x + x_0 \in V_t$, and for any $\varepsilon > 0$ there is a $z \in U_k$ such that

$$\|x + x_0 - z\| < \varepsilon.$$

Now, all vectors of the form $z - x_0,\ z \in U_k$, are contained in U_{2k}, since

$$\|A(z - x_0)\| \leq \|Az\| + \|Ax_0\| < 2k.$$

Hence, U_{2k} is dense in S_t. Since $x \in U_m$ if, and only if, $x/m \in U_1$, U_1 is dense in

$$S_r = \{x \ : \ \|x\| < r = t/2k\},$$

and for each $\alpha > 0$, U_α is dense in $S_{\alpha r}$. Let δ be any number satisfying $0 < \delta < 1$. I claim that

$$S_r \subset U_{1/(1-\delta)}. \tag{3.20}$$

This means that $\|x\| < r$ implies $\|Ax\| < 1/(1 - \delta)$, which is exactly what we want to prove. To prove (3.20), let x be any point in S_r. Since U_1 is dense in S_r, there is an $x_1 \in U_1 \cap S_r$ such that

$$\|x_1 - x\| < \delta r,$$

i.e., $x_1 - x \in S_{\delta r}$. Now U_δ is dense in $S_{\delta r}$. Hence, there is an $x_2 \in U_\delta \cap S_{\delta r}$ such that

$$\|x_2 + x_1 - x\| < \delta^2 r,$$

i.e., $x_2 + x_1 - x \in S_{\delta^2 r}$. But U_{δ^2} is dense in $S_{\delta^2 r}$, and there is an $x_3 \in U_{\delta^2} \cap S_{\delta^2 r}$ such that

$$\|x_3 + x_2 + x_1 - x\| < \delta^3 r,$$

i.e., $x_3 + x_2 + x_1 - x \in S_{\delta^3 r}$. Continuing in this manner, there is an $x_{n+1} \in U_{\delta^n} \cap S_{\delta^n r}$ such that

$$\|\sum_1^{n+1} x_k - x\| < \delta^{n+1} r.$$

Since $x_{n+1} \in U_{\delta^n}$, we have

$$\|Ax_{n+1}\| < \delta^n,$$

so that

$$\|A \sum_j^k x_n\| \leq \sum_j^k \|Ax_n\| \leq \delta^{j-1} \frac{1 - \delta^{k-j+1}}{1 - \delta} \longrightarrow 0 \ \text{ as } \ j, k \longrightarrow \infty.$$

Thus, $A \sum_1^k x_n$ is a Cauchy sequence in Y. Since Y is complete, this converges to some $y \in Y$. But

$$\|\sum_1^k x_n - x\| < \delta^k r \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty.$$

Since A is closed, $Ax = y$, and

$$\|y\| \leq \sum_1^\infty \|Ax_n\| \leq \sum_1^\infty \delta^{n-1} = 1/(1-\delta).$$

This completes the proof. $\square$

3.5. Operators with closed ranges

Let X, Y be normed vector spaces and let A be an operator in $B(X, Y)$. Theorem 3.7 tells us that $R(A)$ consists precisely of the annihilators of $N(A')$, provided $R(A)$ is closed in Y. It would therefore be of interest to know when this is so. There is a simple answer when X, Y are complete.

Suppose that $N(A) = \{0\}$, i.e., that A is one-to-one (we shall remove this assumption later). Then the inverse A^{-1} of A exists and is defined on $R(A)$. If Y is complete and $R(A)$ is closed in Y, then $R(A)$ itself is a Banach space with the same norm. If X is complete as well, we can apply Theorem 3.8 with Y replaced by $R(A)$ to conclude that $A^{-1} \in B(R(A), X)$. Thus there is a constant C such that

$$\|A^{-1}y\| \leq C\|y\|, \quad y \in R(A). \tag{3.21}$$

Another way of writing (3.21) is

$$\|x\| \leq C\|Ax\|, \quad x \in X. \tag{3.22}$$

Thus, if X, Y are complete, then a necessary condition that $R(A)$ be closed in Y is that (3.22) holds. A moment's reflection shows that it is also sufficient. For if $y_n \to y$ in Y, where $y_n \in R(A)$, set $x_n = A^{-1}y_n$. Then, by (3.22),

$$\|x_n - x_m\| \leq C\|y_n - y_m\| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X, and therefore approaches a limit $x \in X$. Since A is bounded, $y_n = Ax_n \to Ax$, showing that $Ax = y$.

Thus, we have shown that when X, Y are complete and A is a one-to-one operator in $B(X, Y)$, then a necessary and sufficient condition that $R(A)$ be closed in Y is that (3.22) hold. As you know, we have a great weakness for trying to "generalize" statements in hopes that they may apply to other situations. At this point we give way to ourselves and ask whether the same conclusion can be drawn without assuming that $A \in B(X, Y)$. True, we used Theorem 3.8, which assumes $A \in B(X, Y)$. But we could have used Theorem 3.11, which comes to the same conclusion with only assuming that A is a closed operator from X to Y. So let us try to get away with assuming only this. Checking the rest of the proof, we find that the only other place we used the boundedness of A is in concluding that $Ax_n \to Ax$. But I claim this is true even when A is only closed. For, remember that $Ax_n \to y$ in Y

while $x_n \to x$ in X. By the definition of a closed operator, $x \in D(A)$ and $Ax = y$, showing that $y \in R(A)$. Thus, we have proved

Theorem 3.12. *Let X, Y be Banach spaces, and let A be a one-to-one closed linear operator from X to Y. Then a necessary and sufficient condition that $R(A)$ be closed in Y is that (3.22) hold.*

Of course this is all well and good if $N(A)$ is just the zero element. But what does one do in case it is not? Do not despair. There is a method, but it needs a bit of preparation.

We first examine $N(A)$ a bit more closely. As we have remarked several times, it is a subspace of X. But more than that, when A is a closed operator, $N(A)$ is a closed subspace of X. For if $x_n \in N(A)$ and $x_n \to x$ in X, then $0 = Ax_n \to 0$. Since A is closed, $x \in D(A)$ and $Ax = 0$, i.e., $x \in N(A)$.

Now let M be any closed subspace of a normed vector space X. Suppose we say that $x \in X$ is equivalent to $u \in X$ and write $x \sim u$ whenever $x - u \in M$. Then one checks that $x \sim x$, and $x \sim u$ is the same as $u \sim x$. Moreover, $x \sim u$, $u \sim v$ implies $x \sim v$. A relationship having these properties is called an *equivalence relation.*

Next, for each $x \in X$, let $[x]$ denote the set of all $u \in X$ such that $u \sim x$. Such a set is called a *coset.* Clearly $[x] = [u]$ if, and only if, $x \sim u$. Moreover, if x is not equivalent to u, then $[x]$ and $[u]$ have no elements in common. We define the "sum" of two cosets by

$$[x] + [u] = [x + u].$$

However, we have to ascertain whether this definition makes sense. For suppose $[x_1] = [x]$ and $[u_1] = [u]$. Then $x_1 \sim x$, $u_1 \sim u$, and hence, $x_1 + u_1 \sim x + u$, showing that $[x_1 + u_1] = [x + u]$. We also define multiplication of a coset by a scalar by $\alpha[x] = [\alpha x]$. Under these definitions, one can verify easily that the collection of cosets forms a vector space with zero element $[0] = M$. You may think that it is rather strange to form a vector space in which each element is a set. But there is nothing in the rules that prevents us from doing it.

We are going to do something even crazier. We are going to give this vector space a norm by setting

$$\|[x]\| = d(x, M) = \inf_{z \in M} \|x - z\|.$$

Of course, we must check to see if this satisfies all of the properties of a norm. For instance, if $\|[x]\| = 0$, then $d(x, M) = 0$. Here we need the fact that M is closed in X to conclude that $x \in M$ so that $[x] = [0]$. The triangle inequality is also easily verified. In fact, for x, $u \in X$, there are sequences $\{z_n\}$ and $\{w_n\}$ in M such that $\|x - z_n\| \to d(x, M)$, $\|u - w_n\| \to d(u, M)$.

Hence,

$$\begin{aligned} d(x+u, M) &= \inf_{z \in M} \|x+u-z\| \\ &\le \|x+u-z_n-w_n\| \\ &\le \|x-z_n\| + \|u-w_n\| \\ &\longrightarrow d(x,M)+d(u,M). \end{aligned}$$

It follows that the collection of cosets forms a normed vector space, which inappropriately is called a *quotient* (or *factor*) space and is denoted by X/M. A very important property of such spaces is given by

Theorem 3.13. *If M is a closed subspace of a Banach space X, then X/M is a Banach space.*

Before proving Theorem 3.13, we show how quotient spaces help us find a counterpart for Theorem 3.12 when A is not one-to-one. Assume that X, Y are Banach spaces and that A is a closed linear operator from X to Y. Define the operator $\hat{A}$ from $X/N(A)$ to Y as follows. $D(\hat{A})$ is to consist of those cosets $[x] \subset D(A)$. (Note that $[x] \in D(\hat{A})$ if, and only if, $x \in D(A)$. For, if $x \in D(A)$ and $u \in [x]$, then $x-u \in N(A)$, showing that $u \in D(A)$.) For $[x] \in D(\hat{A})$, we define $\hat{A}[x]$ to be Ax. This definition makes sense, since $Au = Ax$ whenever $u \in [x]$. Moreover, $\hat{A}$ is a closed operator from $X/N(A)$ to Y. For if $[x_n] \to [x]$ in $X/N(A)$ and $\hat{A}[x_n] \to y$ in Y, then there is a sequence $\{z_n\} \subset N(A)$ such that $x_n + z_n \to x$ in X and $A(x_n+z_n) \to y$ in Y. Since A is closed, $x \in D(A)$ and $Ax = y$. Hence, $[x] \in D(\hat{A})$ and $\hat{A}[x] = y$.

What is $N(\hat{A})$? If $A[x] = 0$, then $Ax = 0$, and hence, $x \in N(A)$. Thus, $[x] = [0]$. The accomplishment of all this madness is that $\hat{A}$ is a one-to-one, closed linear operator from $X/N(A)$ to Y. Moreover, $X/N(A)$ is a Banach space by Theorem 3.13. Thus, by Theorem (3.12), $R(\hat{A}) = R(A)$ is closed in Y if, and only if,

$$\|[x]\| \le C\|\hat{A}[x]\|, \quad [x] \in D(\hat{A}), \tag{3.23}$$

for some constant C. Rewriting (3.23) in our old terminology, we get

$$d(x, N(A)) \le C\|Ax\|, \quad x \in D(A). \tag{3.24}$$

Hence, we have

Theorem 3.14. *If X, Y are Banach spaces, and A is a closed linear operator from X to Y, then $R(A)$ is closed in Y if, and only if, there is a constant C such that (3.24) holds.*

It remains to prove Theorem 3.13. To that end, let $\{x_n\}$ be a Cauchy sequence in X/M. Then for each k, there is a number $N(k) \ge N(k-1)$ such that

$$\|[x_m] - [x_n]\| < 2^{-k}, \quad m,\, n \geq N(k).$$

Set $u_k = x_{N(k)}$. Then

$$\|[u_{k+1}] - [u_k]\| < 2^{-k}.$$

Thus, there is a $z_k \in M$ such that

$$\|u_{k+1} - u_k + z_k\| < 2^{-k}.$$

Set $v_k = u_{k+1} - u_k + z_k$, and

$$w_n = \sum_1^n v_k = u_{n+1} - u_1 + \sum_1^n z_k.$$

Then

$$\|v_k\| < 2^{-k},$$

and consequently for $n > m$,

$$\|w_n - w_m\| \leq \sum_{m+1}^n \|v_k\| < \sum_{m+1}^n 2^{-k} \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Hence, $\{w_n\}$ is a Cauchy sequence in X, and since X is complete, $w_n \to w$ in X. Thus,

$$\|[u_n] - [w + u_1]\| \leq \|u_n - w - u_1 + \sum_1^{n-1} z_k\| = \|w_{n-1} - w\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence, $[u_n]$ converges to a limit in $X/N(A)$, and the proof is complete.

"Wait a minute!" you cry. "It is explicitly stated in the definition of completeness that the whole Cauchy sequence is to converge to a limit. Here $\{[u_n]\}$ is merely a subsequence of the original Cauchy sequence $\{[x_n]\}$." Relax! There is nothing to worry about. A Cauchy sequence cannot go off in different directions. In fact, we have

Lemma 3.15. *If a subsequence of a Cauchy sequence converges, then the whole sequence converges.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a normed vector space X, and let $\varepsilon > 0$ be given. Then there is an N so large that

$$\|x_n - x_m\| < \varepsilon, \quad m, n > N.$$

Now if $\{x_n\}$ has a subsequence converging to x in X, then there is an $m > N$ such that

$$\|x_m - x\| < \varepsilon.$$

Thus,

$$\|x_n - x\| \leq \|x_n - x_m\| + \|x_m - x\| < 2\varepsilon$$

for all $n > N$. Now the proof is complete. $\square$

We now use Theorem 3.14 to prove a result related to Theorem 3.7.

Theorem 3.16. *Let X, Y be Banach spaces, and assume that $A \in B(X,Y)$. If $R(A)$ is closed in Y, then*

$$R(A') = N(A)^{\circ}, \tag{3.25}$$

and hence, $R(A')$ is closed in X' (Lemma 3.4).

Proof. If $x' \in R(A')$, then there is a $y' \in Y'$ such that $A'y' = x'$. For $x \in N(A)$,

$$x'(x) = A'y'(x) = y'(Ax) = 0.$$

Hence, $x' \in N(A)^{\circ}$. Conversely, assume that $x' \in N(A)^{\circ}$. Let y be any element of $R(A)$, and let x be any element of X such that $Ax = y$. Set

$$f(y) = x'(x).$$

This, believe it or not, is a functional on $R(A)$. For, if we had chosen another $x_1 \in X$ such that $Ax_1 = y$, then $x_1 - x \in N(A)$, and hence,

$$x'(x_1) = x'(x),$$

showing that f depends only on y and not on the particular x chosen. Clearly f is a linear functional. But it is also bounded. For, if z is any element in $N(A)$,

$$f(y) = x'(x - z),$$

and hence,

$$|f(y)| \le \|x'\| \cdot \|x - z\|.$$

Thus,

$$|f(y)| \le \|x'\|\, d(x, N(A)) \le C\|x'\| \cdot \|y\|$$

by Theorem 3.14. By the Hahn-Banach Theorem (Theorem 2.6), f can be extended to the whole of Y. Thus there is a $y' \in Y'$ such that

$$f(y) = y'(y), \quad y \in Y.$$

In particular, this holds for $y \in R(A)$. Hence,

$$x'(x) = y'(Ax), \quad x \in X.$$

By the definition of A', this becomes

$$x'(x) = A'y'(x), \quad x \in X.$$

Since this is true for all $x \in X$, we have

$$x' = A'y'.$$

Hence, $x' \in R(A')$, and the proof is complete. $\square$

3.6. The uniform boundedness principle

As an application of the closed graph theorem, we shall prove an important result know either as the *Uniform Boundedness Principle* or the *Banach-Steinhaus Theorem.*

Theorem 3.17. *Let X be a Banach space, and let Y be a normed vector space. Let W be any subset of $B(X,Y)$ such that for each $x \in X$,*

$$\sup_{A\in W} \|Ax\| < \infty.$$

Then there is a finite constant M such that $\|A\| \le M$ for all $A \in W$.

Proof. For each positive integer n, let S_n denote the set of all $x \in X$ such that $\|Ax\| \le n$ for all $A \in W$. Clearly, S_n is closed. For if $\{x_k\}$ is a sequence of elements in S_n, and $x_k \to x$, then for each $A \in W$ we have $\|Ax\| = \lim \|Ax_k\| \le n$. Thus, $x \in S_n$. Now, by hypothesis, each $x \in X$ is in some S_n. Therefore,

$$X = \bigcup_1^\infty S_n.$$

Since X is of the second category (Theorem 3.9), at least one of the S_n, say S_N, contains a sphere (note that each S_n is closed). Thus, there is an $x_0 \in X$ and an $r > 0$ such that $x \in S_N$ when $\|x - x_0\| < r$. Let x be an element of X, and set $z = x_0 + rx/2\|x\|$. Then $\|z - x_0\| < r$. Thus, $\|Az\| \le N$ for all $A \in W$. Since $x_0 \in S_N$, this implies

$$\|Ax\| \le \frac{4N\|x\|}{r}, \quad A \in W,\ x \in X,$$

which is precisely what we want to show. The proof is complete. $\square$

3.7. The open mapping theorem

Another useful consequence of the previous theorems can be stated as follows.

Theorem 3.18. *Let A be a closed operator from a Banach space X to a Banach space Y such that $R(A) = Y$. If Q is any open subset of $D(A)$, then the image $A(Q)$ of Q is open in Y.*

Proof. Let $D = D(A)/N(A)$, and define the operator $\hat{A}$ from D to Y by $\hat{A}[x] = Ax$. Then $\hat{A}$ is a one-to-one operator from the Banach space D onto the Banach space Y. Consequently, it has a bounded inverse:

$$\|\hat{A}^{-1}y\| \le C_0\|y\|, \quad y \in Y.$$

Let Q be an open set in $D(A)$, and let $y_0 \in A(Q)$. Then there is an $x_0 \in Q$ such that $Ax_0 = y_0$. Let $\varepsilon > 0$ be such that $\|x - x_0\|_A < \varepsilon$ implies that $x \in Q$. Take $\delta = \varepsilon / C_0$. Then $\|y - y_0\| < \delta$ implies

$$\|\hat{A}^{-1}(y - y_0)\|_A \leq C_0\|y - y_0\| < \varepsilon.$$

Consequently, there is a $[z] \in D$ such that $\|[z]\|_A < \varepsilon$ and $\hat{A}[z] = y - y_0$. This implies that there is a $z \in D(A)$ such that $\|z\|_A < \varepsilon$ and $Az = y - y_0$. Let $x = z + x_0$. Then $x \in D(A)$ and $\|x - x_0\|_A = \|z\|_A < \varepsilon$. Thus, $x \in Q$ and $Ax = y$. This means that $\|y - y_0\| < \delta$ implies that $y \in A(Q)$. Hence, $A(Q)$ is open. $\square$

Theorem 3.18 is known as the *Open Mapping Theorem.* A simple consequence is

Corollary 3.19. *Let A be a closed operator from a Banach space X to a Banach space Y such that $R(A) = Y$. If Q is any open subset of X, then the image $A(Q \cap D(A))$ of $Q \cap D(A)$ is open in Y.*

This follows from the fact that $Q \cap D(A)$ is open in $D(A)$.

3.8. Problems

(1) Prove that a normed vector space X is complete if and only if every series in X satisfying

$$\sum_1^\infty \|x_j\| < \infty$$

converges to a limit in X.

(2) Let A be a linear operator from a normed vector space X onto a normed vector space Y. Show that A^{-1} exists and is bounded if and only if there is a number $M > 0$ such that

$$\|x\| \leq M\|Ax\|, \quad x \in D(A).$$

(3) If X, Y are Hilbert spaces, is $B(X, Y)$ a Hilbert space?

(4) Prove: (a) $N(A) = {}^\circ R(A')$; (b) $R(A') \subset N(A)^\circ$.

(5) Let X, Y be two normed vector spaces, and let $x \neq 0$ be any element of X and y any element of Y. Show that there is an operator $A \in B(X, Y)$ such that $Ax = y$.

(6) Let X, Y be normed vector spaces having the property that there is a one-to-one operator $A \in B(X,Y)$ such that $A^{-1} \in B(Y,X)$. Show that X is complete if and only if Y is.

(7) If a normed vector space X has a subspace M such that M and X/M are complete, show that X is a Banach space.

(8) Let X, Y be Banach spaces, and assume that $A \in B(X,Y)$, $B \in B(Y,X)$ are such that $BA = I$. If $T \in B(X,Y)$ is such that $\|T - A\| \cdot \|B\| < 1$, show that there is an $S \in B(Y,X)$ such that $ST = I$.

(9) Let M be a closed subspace of a normed vector space X. Show that

$$M' = \frac{X'}{M^\circ}, \qquad \left(\frac{X}{M}\right)' = M^\circ.$$

Why did we assume M closed?

(10) If X, Y are normed vector spaces and $B(X,Y)$ is complete, show that Y is complete.

(11) Show that a subset G of $X \times Y$ is the graph of an operator from X to Y if and only if G is a subspace such that $\langle 0, y\rangle \in G$ implies $y = 0$.

(12) If X, Y are normed vector spaces, are the norms of $(X \times Y)'$ and $X' \times Y'$ equivalent?

(13) Prove that if $\{T_n\}$ is a sequence in $B(X,Y)$ such that $\lim T_n x$ exists for each $x \in X$, then there is a $T \in B(X,Y)$ such that $T_n x \to Tx$ for all $x \in X$.

(14) Let X, Y be normed vector spaces. Show that an operator A from X to Y is bounded if and only if

$$\sup_{\substack{x \in X, \|x\|=1 \\ y' \in Y', \|y'\|=1}} |y'(Ax)| < \infty.$$

(15) If W is a subspace of X', show that $\overline{W} \subset ({}^\circ W)^\circ$.

(16) If X, Y are normed vector spaces and A is a linear operator from X to Y which is bounded on any nonempty open set, show that A is bounded.

(17) Let X, Y be Banach spaces, and let A be a linear operator from X to Y such that $N(A)$ and $R(A)$ are closed and (3.24) holds. Show that A is a closed operator.

(18) For X, Y normed vector spaces and $A \in B(X, Y)$ satisfying (3.24), show that

$$\inf_{x \in X} \frac{\|Ax\|}{d(x, N(A))} = \inf_{y' \in Y'} \frac{\|A'y'\|}{d(y', N(A'))}$$

and that $R(A')$ is closed.

(19) Assume that X, Y are Banach spaces and that A is a closed linear operator from X to Y which maps bounded closed sets onto closed sets. Show that $R(A)$ is closed.

(20) If X, Y are normed vector spaces and $\{A_k\}$ is a sequence of operators in $B(X, Y)$ such that $A_k x \to Ax$ for each $x \in X$, show that $A \in B(X, Y)$ and $\|A\| \leq \liminf \|A_k\|$.

(21) Let X, Y be Banach spaces, and assume that A is a closed operator from X to Y having an inverse $A^{-1} \in B(Y, X)$. Let B be an operator from X to Y such that $D(A) \subset D(B)$ and $\|BA^{-1}\| < 1$. Show that $A + B$ has an inverse $(A + B)^{-1}$ in $B(Y, X)$.

(22) Under the above hypotheses, show that

$$\|(A + B)^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\| \cdot \|BA^{-1}\|}{1 - \|BA^{-1}\|}.$$

(23) If X is a Hilbert space, $A \in B(X)$ and $\|A\| \leq 1$, show that $Ax = x$ if and only if $A'x = x$.

(24) Let X, Y be Banach spaces, and let $F(x, y)$ be a linear functional on $X \times Y$ which is continuous in x for each fixed y and continuous in y for each fixed x. Show that $F \in (X \times Y)'$.

(25) Suppose A is a linear operator on a Hilbert space X such that $D(A) = X$ and

$$(x, Ay) = (Ax, y), \quad x, y \in X.$$

Show that $A \in B(X)$.

(26) Let $\{D_k\}$ be a sequence of dense open subsets of a Banach space X. Show that

$$D = \bigcap_1^\infty D_k$$

is dense in X.

(27) Let X, Y be Banach spaces, and let A be an operator in $B(X, Y)$ with $R(A)$ dense in Y. If $R(A)$ is of the second category in Y, show that $A(Q)$ is open in $R(A)$ for each open set $Q \subset X$.

(28) Let S be a subset of a Banach space X such that

$$\sup_{x \in S} |x'(x)| < \infty, \quad x' \in X'.$$

Show that

$$\sup_{x \in S} \|x\| < \infty.$$

THE RIESZ THEORY FOR COMPACT OPERATORS

4.1. A type of integral equation

In Chapter I, we found a method for solving an integral equation of the form

$$x(t) = y(t) + \int_a^t k(t,s)x(s)\,ds. \tag{4.1}$$

We now want to examine another integral equation of a slightly different form, namely

$$x(t) = y(t) + v(t)\int_a^b w(s)x(s)\,ds, \tag{4.2}$$

which also comes up frequently in applications. Of course, now we are more sophisticated and can recognize that (4.2) is of the form

$$x - Kx = y, \tag{4.3}$$

where K is an operator on a normed vector space X defined by

$$Kx = x_1'(x)x_1, \tag{4.4}$$

where x_1 is a given element of X, and x_1' is a given element of X'. Clearly, K is a linear operator defined everywhere on X. It is also bounded since

$$\|Kx\| = \|x_1'(x)\| \cdot \|x_1\| \leq \|x_1'\| \cdot \|x_1\| \cdot \|x\|.$$

Now clearly in order to solve (4.3), it suffices to find Kx, i.e., $x_1'(x)$. Since $x = y + Kx$,

$$x_1'(x) = x_1'(y) + x_1'(x)x_1'(x_1),$$

or

$$[1 - x_1'(x_1)]x_1'(x) = x_1'(y). \tag{4.5}$$

Now if $x_1'(x_1) \neq 1$, we can solve for $x_1'(x)$ and substitute into (4.3) to obtain

$$x = y + \frac{x_1'(y)}{1 - x_1'(x_1)} x_1. \tag{4.6}$$

By substituting (4.6) into (4.3), we see that it is indeed a solution. Hence a solution of (4.2) is

$$x(t) = y(t) + \frac{\int_a^b w(s)y(s)\,ds}{1 - \int_a^b w(s)v(s)\,ds} v(t)$$

provided

$$\int_a^b w(s)v(s)\,ds \neq 1.$$

(I'll bet you never would have dreamt of solving (4.2) in this way without first changing it to (4.3).)

Concerning uniqueness, we see from (4.5) that if $y = 0$, then $x_1'(x)$ must vanish, and hence, so must x.

Now, suppose $x_1'(x_1) = 1$. By (4.5), we see that we must have

$$x_1'(y) = 0 \tag{4.7}$$

in order that (4.3) have a solution. So let us assume that (4.7) holds. On the other hand, any solution of (4.3) must be of the form

$$x = y + \alpha x_1, \tag{4.8}$$

where α is a scalar. For such x we have

$$Kx = x_1'(y)x_1 + \alpha x_1'(x_1)x_1 = \alpha x_1,$$

showing that any element of the form (4.8) is a solution. How about uniqueness? If $x = Kx = x_1'(x)x_1$, then x is of the form $x = \alpha x_1$. Consequently,

$$Kx = \alpha x_1'(x_1)x_1 = \alpha x_1,$$

showing that there is no uniqueness.

Set $A = I - K$, where I is the identity operator, i.e., $Ix = x$ for all $x \in X$. Then (4.3) is the equation

$$Ax = y. \tag{4.9}$$

In terms of A we have shown that if $x_1'(x_1) \neq 1$, there is a unique solution of (4.9) for each $y \in X$. If $x_1'(x_1) = 1$, one can solve (4.9) only for those

y satisfying (4.7), i.e., those that annihilate x_1'. In this case there is no uniqueness, and $N(A)$ consists of all vectors of the form αx_1.

Let us take a look at A'. Clearly, I' is the identity operator on X'. We shall denote it by I as well. By definition,

$$x'(Kx) = K'x'(x),$$

so that

$$x_1'(x)x'(x_1) = K'x'(x).$$

Hence,

$$K'x' = x'(x_1)x_1'. \tag{4.10}$$

If $x' \in N(A')$, then $x' = K'x' = \beta x_1'$. Thus,

$$K'x' = \beta x_1'(x_1)x_1',$$

showing that

$$\beta(1 - x_1'(x_1))x_1' = 0.$$

If $x_1'(x_1) \neq 1$, we must have $\beta = 0$, i.e., $N(A') = \{0\}$. If $x_1'(x_1) = 1$, then $N(A')$ consists of all functionals of the form $\beta x_1'$.

We can put our results in the following form

Theorem 4.1. *Let X be a normed vector space, and let $A = I - K$, where K is of the form (4.4). If $N(A) = \{0\}$, then $R(A) = X$. Otherwise $R(A)$ is closed in X, and $N(A)$ is finite dimensional, having the same dimension as $N(A')$.*

Yes, I shall explain everything. When $x_1'(x_1) = 1$, then $R(A)$ consists of the annihilators of x_1', and hence, is closed (Lemma 3.4). Moreover, $N(A)$ is the subspace spanned by x_1, while $N(A')$ is the subspace spanned by x_1'. As we shall see in a moment, both of these spaces are of dimension one.

To introduce the concept of dimension, let V be a vector space. The elements $v_1, \cdots, v_n$ are called *linearly independent* if the only scalars $\alpha_1, \cdots, \alpha_n$ for which

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \tag{4.11}$$

are $\alpha_1 = \cdots = \alpha_n = 0$. Otherwise, they are called *linearly dependent.* V is said to be of *dimension* $n > 0$ if

(a) There are n linearly independent vectors in V

and

(b) Every set of $n+1$ elements of V are linearly dependent.

If there are no independent vectors, V consists of just the zero element and is said to be of dimension zero. If V is not of dimension n for any finite n, we say that it is infinite dimensional.

Now suppose $\dim V = n$ (i.e., V is of dimension n), and let $v_1, \cdots, v_n$ be n linearly independent elements. Then every $v \in V$ can be expressed uniquely in the form

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n. \tag{4.12}$$

To see this, note that the set $v, v_1, \cdots, v_n$ of $n+1$ vectors must be linearly dependent. Thus, there are scalars $\beta, \beta_1, \cdots, \beta_n$, not all zero, such that

$$\beta v + \beta_1 v_1 + \cdots + \beta_n v_n = 0.$$

Now β cannot vanish, for otherwise the $v_1, \cdots, v_n$ would be dependent. Dividing by β, we get an expression of the form (4.12). This expression is unique. For if

$$v = \alpha_1' v_1 + \cdots + \alpha_n' v_n,$$

then

$$(\alpha_1 - \alpha_1')v_1 + \cdots + (\alpha_n - \alpha_n')v_n = 0,$$

showing that $\alpha_i' = \alpha_i$ for each i. If $\dim V = n$, we call any set of n linearly independent vectors in V a *basis* for V.

We are now going to attempt to extend Theorem 4.1 to a more general class of operators. In doing so we shall make use of a few elementary properties of finite dimensional normed vector spaces.

Let X be a normed vector space, and suppose that it has two norms $\|\cdot\|_1$, $\|\cdot\|_2$. We call them *equivalent* and write $\|\cdot\|_1 \sim \|\cdot\|_2$ if there is a positive number a such that

$$a^{-1}\|x\|_1 \le \|x\|_2 \le a\|x\|_1, \quad x \in X. \tag{4.13}$$

Clearly, this is an equivalence relation (See Section 3.5), and a sequence $\{x_n\}$ converges in one norm if and only if it converges in the other. We shall prove

Theorem 4.2. *If X is finite dimensional, all norms are equivalent.*

Before proving Theorem 4.2 we state some important consequences.

Corollary 4.3. *A finite dimensional normed vector space is always complete.*

Proof. Suppose $\dim X = n$, and let $x_1, \cdots, x_n$ be a *basis* for X (i.e., a set of n linearly independent elements). Then each $x \in X$ can be written in the form

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

Set

$$\|x\|_1 = (\sum_1^n |\alpha_i|^2)^{1/2}.$$

This is a norm on X, and by Theorem 4.2, it is equivalent to the given norm of X. Thus if

$$x^{(k)} = \alpha_1^{(k)} x_1 + \cdots + \alpha_n^{(k)} x_n \tag{4.14}$$

is a Cauchy sequence in X, then $(\alpha_1^{(k)}, \cdots, \alpha_n^{(k)})$ is a Cauchy sequence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete, there is an element $(\alpha_1, \cdots, \alpha_n)$ which is the limit of $(\alpha_1^{(k)}, \cdots, \alpha_n^{(k)})$ in $\mathbb{R}^n$. Set

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n. \tag{4.15}$$

Then $x^{(k)} \to x$ in X. Hence X is complete. □

Corollary 4.4. *If M is a finite dimensional subspace of a normed vector space, then M is closed.*

Proof. If $\{x^{(k)}\}$ is a sequence of elements of M such that $x^{(k)} \to x$ in X, it is a Cauchy sequence in M. Since M is complete with respect to any norm, $x^{(k)}$ has a limit in M which must coincide with x. □

A subset W of a normed vector space is called *bounded* if there is a number b such that $\|x\| \le b$ for all $x \in W$. It is called *compact* if each sequence $\{x^{(k)}\}$ of elements of W has a subsequence which converges to an element of W.

Corollary 4.5. *If X is a finite dimensional normed vector space, then every bounded closed set T in X is compact.*

Proof. Let $x_1, \cdots, x_n$ be a basis for X. Then every element $x \in X$ can be expressed in the form (4.15). Set

$$\|x\|_0 = \sum_1^n |\alpha_i|. \tag{4.16}$$

This is a norm on X, and consequently it is equivalent to all others. Let $\{x^{(k)}\}$ be a sequence of elements of T. They can be written in the form

(4.14). Since T is bounded, there is a constant b such that $\|x^{(k)}\| \leq b$. Hence, there is a constant c such that

$$\|x^{(k)}\|_0 = \sum_{i=1}^{n} |\alpha_i^{(k)}| \leq c.$$

By the Bolzano-Weierstrass theorem, $\{\alpha_1^{(k)}\}$ has a subsequence which converges to a number α_1. Let us discard the $x^{(k)}$ not in this subsequence. For the remaining indices k, $\{\alpha_2^{(k)}\}$ has a convergent subsequence. Again, discard the rest of the sequence. Continuing in this manner, we have a subsequence such that $\alpha_i^{(k)} \to \alpha_i$ for each i. Set $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Since (4.16) is equivalent to the norm of X, this subsequence converges to x in X. Since T is closed, $x \in T$. $\square$

We now give the proof of Theorem 4.2.

Proof. Let $x_1, \cdots, x_n$ be a basis for X. Then every $x \in X$ can be written in the form (4.15). We shall show that any norm $\|\cdot\|$ on X is equivalent to the norm given by (4.16). In one direction we have

$$\|x\| = \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \leq \|\alpha_1 x_1\| + \cdots + \|\alpha_n x_n\| \leq \max_j \|x_j\| \cdot \|x\|_0.$$

Conversely, there is a constant C such that

$$\|x\|_0 \leq C\|x\|,$$

i.e.,

$$|\alpha_1| + \cdots + |\alpha_n| \leq C\|\alpha_1 x_1 + \cdots + \alpha_n x_n\|. \tag{4.17}$$

In proving (4.17), it clearly suffices to assume

$$|\alpha_1| + \cdots + |\alpha_n| = 1. \tag{4.18}$$

For if $\sum |\alpha_i|$ is not equal to one, we can divide each α_i by this value, apply (4.17) and then multiply out. Now, if (4.17) were not true for α_i satisfying (4.18), there would be a sequence of the form (4.14) satisfying

$$\|x^{(k)}\|_0 = 1$$

and such that $\|x^{(k)}\| \to 0$. As we just showed, there is a subsequence of $\{x^{(k)}\}$ for which $\alpha_i^{(k)} \to \alpha_i$ for each i. Let $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Then

$$\|x\|_0 = \sum_1^n |\alpha_i| = 1,$$

and

$$\|x^{(k)} - x\| = \|\sum_i (\alpha_i^{(k)} - \alpha_i) x_i\| \leq \sum_i |\alpha_i^{(k)} - \alpha_i| \cdot \|x_i\|$$

$$\leq \max_i \|x_i\| \sum_i |\alpha_i^{(k)} - \alpha_i| \to 0.$$

Thus $x = 0$, contradicting the fact that $\|x\|_0 = 1$. This completes the proof. $\square$

Corollary 4.5 has a converse.

Theorem 4.6. *If X is a normed vector space and the surface of its unit sphere (i.e., the set $\|x\| = 1$) is compact, then X is finite dimensional.*

In proving Theorem 4.6 we shall make use of a simple lemma due to F. Riesz.

Lemma 4.7. *Let M be a closed subspace of a normed vector space X. If M is not the whole of X, then for each number θ satisfying $0 < \theta < 1$ there is an element $x_\theta \in X$ such that*

$$\|x_\theta\| = 1 \text{ and } d(x_\theta, M) \geq \theta. \tag{4.19}$$

Note the difference between Lemma 4.7 and Theorem 2.2. The reason we cannot take $\theta = 1$ in Lemma 4.7 is that in general, for x not in M, there may not be a $z \in M$ satisfying

$$\|x - z\| = d(x, M). \tag{4.20}$$

This may be due to the fact that X is not complete. But even if X is a Banach space, (4.20) may fail because of the geometry of X. We leave it as an exercise to show that if M is finite dimensional, we may take $\theta = 1$ in Lemma 4.7.

We can now give the proof of Theorem 4.6.

Proof. Let x_1 be any element of X satisfying $\|x_1\| = 1$, and let M_1 be the subspace spanned by x_1. If $M_1 = X$, then X is finite dimensional and we are through. Otherwise, by Lemma 4.7, there is an $x_2 \in X$ such that $\|x_2\| = 1$ and $d(x_2, M_1) > \frac{1}{2}$. (Note that since M_1 is finite dimensional, it is closed. By the remark at the end of the preceding paragraph, we could have taken x_2 such that $d(x_2, M_1) = 1$, but we do not need this fact here.) Let M_2 be the subspace of X spanned by x_1 and x_2. Then M_2 is a closed subspace, and if $M_2 \neq X$, there is an $x_3 \in X$ such that $\|x_3\| = 1,\ d(x_3, M_2) > \frac{1}{2}$.

We continue in this manner. Inductively, if the subspace M_n spanned by $x_1, \cdots, x_n$ is not the whole of X, there is an x_{n+1} such that

$$\|x_{n+1}\| = 1, \text{ and } d(x_{n+1}, M_n) > \frac{1}{2}.$$

We cannot continue indefinitely, for then there would be an infinite sequence $\{x_n\}$ of elements such that $\|x_n\| = 1$ while

$$\|x_n - x_m\| > \frac{1}{2}, \quad n \neq m.$$

This sequence clearly has no convergent subsequence showing that the surface of the unit sphere in X is not compact. Thus there is a k such that $M_k = X$, and the proof is complete. □

It remains to give the proof of Lemma 4.7.

Proof. Since $M \neq X$, there is an $x_1 \in X \setminus M$ (i.e., in X but not in M). Since M is closed, $d = d(x_1, M) > 0$. For any $\varepsilon > 0$ there is an $x_0 \in M$ such that

$$\|x_1 - x_0\| < d + \varepsilon.$$

Take $\varepsilon = d(1 - \theta)/\theta$. Then $d + \varepsilon = d/\theta$. Set

$$x_\theta = \frac{x_1 - x_0}{\|x_1 - x_0\|}.$$

Then $\|x_\theta\| = 1$, and for any $x \in M$ we have

$$\|x - x_\theta\| = \frac{\|(\|x_1 - x_0\|x + x_0) - x_1\|}{\|x_1 - x_0\|} \geq \frac{d}{\|x_1 - x_0\|} > \theta,$$

since $\|x_1 - x_0\|x + x_0$ is in M. This completes the proof. □

Another simple but useful statement about finite-dimensional spaces is

Lemma 4.8. *If V is an n-dimensional vector space, then every subspace of V is of dimension $\leq n$.*

Proof. Let W be a subspace of V. If W consists only of the element 0, then $\dim W = 0$. Otherwise, there is an element $w_1 \neq 0$ in W. If there does not exist a $w \in W$ such that w_1 and w are linearly independent, then $\dim W = 1$. Otherwise, let w_2 be a vector in W such that w_1 and w_2 are linearly independent. Continue in this way until we have obtained linearly independent elements $w_1, \cdots, w_m$ in W such that $w, w_1, \cdots, w_m$ are dependent for all $w \in W$. This must happen for some $m \leq n$, for otherwise the dimension of V would be more than n. This means that $\dim W = m \leq n$, and the proof is complete. □

4.2. Operators of finite rank

Encouraged by our success in solving (4.3) when K is given by (4.4), we attempt to extend these results to more general operators. As we always do when we are unsure of ourselves, we consider an operator only slightly more difficult. The next logical step would be to take K of the form

$$Kx = \sum_{j=1}^{n} x_j'(x)x_j, \quad x_j \in X, \ x_j' \in X'. \tag{4.21}$$

Note that K is bounded, since

$$\|Kx\| \leq (\sum_1^n \|x_j'\| \cdot \|x_j\|)\|x\|.$$

Moreover, $R(K)$ is clearly seen to be finite dimensional. Conversely, any operator $K \in B(X) = B(X,X)$ such that $R(K)$ is finite dimensional must be of the form (4.21). For, let $x_1, \cdots, x_n$ be a basis for $R(K)$. Then for each $x \in X$,

$$Kx = \sum_1^n \alpha_j(x)x_j,$$

where the coefficients $\alpha_j(x)$ are clearly seen to be linear functionals. They are also bounded, since by Theorem 4.2, there is a constant C such that

$$\sum_1^n |\alpha_j| \leq C\|\sum_1^n \alpha_j x_j\|.$$

Hence,

$$\sum_1^n |\alpha_j(x)| \leq C\|Kx\| \leq C\|K\| \cdot \|x\|.$$

Thus there are functionals $x_j' \in X'$ such that $\alpha_j(x) = x_j'(x)$, and K is of the form (4.21).

An operator of the form (4.21) is said to be of *finite rank*. As a counterpart of Theorem 4.1, we have

Theorem 4.9. *Let X be a normed vector space, and let K be an operator of finite rank on X. Set $A = I - K$. Then $R(A)$ is closed in X, and the dimensions of $N(A)$ and $N(A')$ are finite and equal.*

Proof. Clearly, we may assume that the x_j and the x_j' are linearly independent in (4.21). If x is a solution of $Ax = y$, then

$$x - \sum_1^n x_k'(x)x_k = y. \tag{4.22}$$

In order to determine x, it suffices to find $x_1'(x), \cdots, x_n'(x)$. Operating on (4.22) with x_j', we have

$$x_j'(x) - \sum_{k=1}^{n} x_k'(x) x_j'(x_k) = x_j'(y), \quad 1 \le j \le n,$$

or

$$\sum_{k=1}^{n} \left[\delta_{jk} - x_j'(x_k)\right] x_k'(x) = x_j'(y), \quad 1 \le j \le n, \tag{4.23}$$

where δ_{jk} is the *Kronecker delta*

$$\delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

Now, (4.23) is a system of n linear equations in n unknowns. From the theory of such equations we know that if the determinant

$$\Delta = |\delta_{jk} - x_j'(x_k)|$$

does not vanish, then one can always solve (4.23) for the $x_k'(x)$, and the solution is unique. (If you are unfamiliar with this theory, do not despair. We shall prove more general results shortly.) On the other hand, if $\Delta = 0$, there is no uniqueness, and (4.23) can be solved only for those y which satisfy

$$\sum_{1}^{n} \alpha_j x_j'(y) = 0 \tag{4.24}$$

whenever

$$\sum_{j=1}^{n} \left[\delta_{jk} - x_j'(x_k)\right] \alpha_j = 0, \quad 1 \le k \le n. \tag{4.25}$$

Moreover, the number of linearly independent solutions of (4.25) is the same as the number of linearly independent solutions of

$$\sum_{k=1}^{n} \left[\delta_{jk} - x_j'(x_k)\right] \beta_k = 0, \quad 1 \le j \le n. \tag{4.26}$$

The only fact needed to complete the proof is that $x' \in N(A')$ if and only if

$$x' = \sum_{1}^{n} \alpha_j x_j', \tag{4.27}$$

where the α_j satisfy (4.25). For, by definition,

$$K'x'(x) = x'(Kx) = \sum_{1}^{n} x_k'(x) x'(x_k)$$

for any $x \in X$ and $x' \in X'$. Hence,

$$K'x' = \sum_{1}^{n} x'(x_k)x'_k. \tag{4.28}$$

This shows that any solution of $A'x' = x' - K'x' = 0$ must be of the form (4.27). Moreover, if x' is of this form, then

$$K'x' = \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{n} x'_j(x_k)x_k,$$

showing that (4.27) is in $N(A')$ if, and only if, the α_j satisfy (4.25). Hence, (4.23) can be solved for those y which annihilate $N(A')$, and $\dim N(A) = \dim N(A') \leq n$. This completes the proof. $\square$

Now that we can handle operators of finite rank, where can we go from here? It is natural to think about operators which are "close" to operators of finite rank such that

$$\|K_n - K\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.29}$$

Then for n sufficiently large,

$$\|K - K_n\| < 1. \tag{4.30}$$

If $A = I - K$, we have

$$A = I - (K - K_n) - K_n.$$

Set

$$B_n = I - (K - K_n).$$

If X is a Banach space, then, by Theorem 1.1, B_n has an inverse B_n^{-1} defined on the whole of X. One can verify directly or use Theorem 3.8 to show that B_n^{-1} is a bounded operator. Hence,

$$A = B_n(I - B_n^{-1}K_n).$$

We leave as an exercise the trivial fact that the product of a bounded operator and an operator of finite rank (in either order) is an operator of finite rank. Since the equation $Ax = y$ is equivalent to

$$(I - B_n^{-1}K_n)x = B_n^{-1}y, \tag{4.31}$$

Theorem 4.9 applies. We obtain

Theorem 4.10. *Let X be a Banach space, and assume that $K \in B(X)$ is the limit in norm of a sequence of operators of finite rank. If $A = I - K$, then $R(A)$ is closed in X, and* $\dim N(A) = \dim N(A') < \infty$.

Proof. From (4.31) we see that $N(A) = N(I - B_n^{-1}K_n)$. Since $A' = (I - B_n^{-1}K_n)'B_n'$, it follows that $\dim N(A') = \dim N[(I - B_n^{-1}K_n)']$. Hence, $\dim N(A) = \dim N(A') < \infty$. If $Ax_k \to y \in X$, then $(I - B_n^{-1}K_n)x_k \to B_n^{-1}y$, showing that there is an $x \in X$ such that $(I - B_n^{-1}K_n)x = B_n^{-1}y$, or equivalently, $Ax = y$. Hence, $R(A)$ is closed and the proof is complete. $\square$

What kind of operators are the limits in norm of operators of finite rank? We shall describe such operators in the next section.

4.3. Compact operators

Let X, Y be normed vector spaces. A linear operator K from X to Y is called *compact* (or *completely continuous*) if $D(K) = X$ and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \leq C$, the sequence $\{Kx_n\}$ has a subsequence which converges in Y. The set of all compact operators from X to Y is denoted by $K(X, Y)$.

A compact operator is bounded. Otherwise, there would be a sequence $\{x_n\}$ such that $\|x_n\| \leq C$, while $\|Kx_n\| \to \infty$. Then $\{Kx_n\}$ could not have a convergent subsequence. The sum of two compact operators is compact, and the same is true of the product of a scalar and a compact operator. Hence, $K(X, Y)$ is a subspace of $B(X, Y)$. An operator of finite rank is clearly compact. For it takes a bounded sequence into a bounded sequence in a finite-dimensional Banach space. Hence, the image of the sequence has a convergent subsequence.

If $A \in B(X, Y)$ and $K \in K(Y, Z)$, then $KA \in K(X, Z)$. Similarly, if $L \in K(X, Y)$ and $B \in B(Y, Z)$, then $BL \in K(X, Z)$.

Suppose $K \in B(X, Y)$, and there is a sequence $\{F_n\}$ of operators of finite rank such that

$$\|K - F_n\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \tag{4.32}$$

We claim that if Y is a Banach space, then K is compact.

Proof. Suppose $\{x_k\}$ is a bounded sequence in X, say $\|x_k\| \leq C$. Since F_1 is a compact operator, there is a subsequence $\{x_{k_1}\}$ for which $\{F_1 x_{k_1}\}$ converges. Since F_2 is compact, there is a subsequence $\{x_{k_2}\}$ of $\{x_{k_1}\}$ for which $\{F_2 x_{k_2}\}$ converges. Continuing, we see that there is a subsequence $\{x_{k_n}\}$ of $\{x_{k_{n-1}}\}$ for which $\{F_n x_{k_n}\}$ converges. Set $z_k = x_{k_k}$. Thus, $\{F_n z_k\}$ converges as $k \to \infty$ for each F_n. Now

$$\begin{aligned} \|Kz_j - Kz_k\| &\leq \|Kz_j - F_n z_j\| + \|F_n z_j - F_n z_k\| + \|F_n z_k - Kz_k\| \\ &\leq 2C\|K - F_n\| + \|F_n z_j - F_n z_k\|. \end{aligned}$$

Now let $\varepsilon > 0$ be given. We take n so large that

$$2C\|K - F_n\| < \varepsilon/2.$$

Since $\{F_n z_k\}$ converges, we can take j, k so large that

$$\|F_n z_j - F_n z_k\| < \varepsilon/2.$$

Hence, $\{Kz_k\}$ is a Cauchy sequence. Since Y is a Banach space, the proof is complete. □

Note that in this proof we only made use of the fact that the F_n are compact. Hence, we have proved

Theorem 4.11. *Let X be a normed vector space and Y a Banach space. If L is in $B(X, Y)$ and there is a sequence $\{K_n\} \subset K(X,Y)$ such that*

$$\|L - K_n\| \longrightarrow 0 \text{ as } n \longrightarrow 0,$$

then L is in $K(X,Y)$.

To illustrate Theorem 4.11, consider the operator K on l_p given by

$$K(x_1, x_2, \cdots, x_k, \cdots) = (x_1, x_2/2, \cdots, x_k/k, \cdots).$$

We shall see that K is a compact operator. To that end, set

$$F_n(x_1, x_2, \cdots) = (x_1, x_2/2, \cdots, x_n/n, 0, \cdots), \quad n = 1, 2, \cdots.$$

For each n the operator F_n is of finite rank. The verification is simple; we shall leave it as an exercise. Moreover, for $1 \leq p < \infty$,

$$\|(K - F_{n-1})x\|_p^p = \sum_n^\infty \frac{|x_k|^p}{k^p} \leq \frac{\|x\|_p^p}{n^p},$$

showing that

$$\|K - F_{n-1}\|_p \leq 1/n.$$

For the case $p = \infty$, we have

$$\|(K - F_{n-1})x\|_\infty = \sup_{k \geq n} \frac{|x_k|}{k} \leq \frac{\|x\|_\infty}{n}.$$

Thus, (4.32) holds for $1 \leq p \leq \infty$. By Theorem 4.11, we can now conclude that K is a compact operator. Compact integral operators are considered in Section 11.4.

We see from Theorem 4.11 that $K(X,Y)$ is a closed subspace of $B(X,Y)$. Now that we know that, in a Banach space, the limit in norm of operators of finite rank is compact, we wonder whether the converse is true. If K is compact, can we find a sequence of operators of finite rank that converges to K in norm? As we shall see later, the answer is yes, if we are in a Hilbert space. (See Section 12.3.) It is also affirmative for many well-known Banach spaces. For many years it was believed to be true for a general Banach space.

However, examples of Banach spaces were found for which the assertion is false.

Thus, if X is a Hilbert space and $K \in K(X) = K(X, X)$, then Theorem 4.10 applies. If X is a Banach space, the hypotheses of that theorem may not be fulfilled for some $K \in K(X)$. However, we are going to show that, nevertheless, the conclusion is true. We shall prove

Theorem 4.12. *Let X be a Banach space and let K be an operator in $K(X)$. Set $A = I - K$. Then, $R(A)$ is closed in X and* $\dim N(A) = \dim N(A')$ *is finite. In particular, either $R(A) = X$ and $N(A) = \{0\}$, or $R(A) \neq X$ and $N(A) \neq \{0\}$.*

The last statement of Theorem 4.12 is known as the *Fredholm alternative.* To show that $R(A)$ is closed we make use of the trivial

Lemma 4.13. *Let X, Y be normed vector spaces, and let A be a linear operator from X to Y. Then for each x in $D(A)$ and $\varepsilon > 0$ there is an element $x_0 \in D(A)$ such that*

$$Ax_0 = Ax, \quad d(x_0, N(A)) = d(x, N(A)),$$

$$d(x, N(A)) \leq \|x_0\| \leq d(x, N(A)) + \varepsilon.$$

Proof. There is an $x_1 \in N(A)$ such that $\|x - x_1\| < d(x, N(A)) + \varepsilon$. Set $x_0 = x - x_1$. □

Returning to $A = I - K$, we show that $R(A)$ is closed by proving that for some C,

$$d(x, N(A)) \leq C\|Ax\|, \quad x \in X, \tag{4.33}$$

and then applying Theorem 3.14. If (4.33) did not hold, there would be a sequence $\{x_n\} \in X$ such that

$$d(x_n, N(A)) = 1, \quad \|Ax_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

By Lemma 4.13, there is a sequence $\{z_n\}$ such that

$$d(z_n, N(A)) = 1, \quad 1 \leq \|z_n\| \leq 2, \quad \|Az_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since K is compact, there is a subsequence $\{u_n\}$ of $\{z_n\}$ such that $Ku_n \to w \in X$. Now $u_n = Ku_n + Au_n \to w$, and, hence, $Aw = 0$. But this is impossible, since $\|u_n - w\| \geq d(u_n, N(A)) = 1$ for each n.

The proof that $\dim N(A)$ is finite is simple. We employ Theorem 4.6. $N(A)$ is a normed vector space. If we can show that the surface of the unit sphere in $N(A)$ is compact, it will follow from Theorem 4.6 that $N(A)$ is finite-dimensional. So suppose $\{x_n\}$ is a sequence in $N(A)$ such that $\|x_n\| = 1$. Then $\{Kx_n\}$ has a convergent subsequence. Since $x_n = Kx_n$, the same is true for $\{x_n\}$. Hence, the surface of the unit sphere in $N(A)$ is compact.

How about $N(A')$? Well, $A' = I - K'$. If it turns out that K' is also compact, then we have $\dim N(A') < \infty$ as well. The compactness of K' will be shown in the next section. We assume it here. To show that $\dim N(A) = \dim N(A')$, we first consider the case $\dim N(A') = 0$, i.e., $N(A') = \{0\}$. Since $R(A)$ is closed, we see, by Theorem 3.7, that $R(A) = {}^\circ N(A') = X$. Now suppose there is an $x_1 \neq 0$ in X such that $Ax_1 = 0$. Since $R(A) = X$, there is an $x_2 \in X$ such that $Ax_2 = x_1$ and an x_3 such that $Ax_3 = x_2$. Continuing, we can always find an x_n such that $Ax_n = x_{n-1}$.

Now A^n is a bounded operator with $\|A^n\| \leq \|A\|^n$ (see Section 1.2). This implies that $N(A^n)$ is a closed subspace of X (cf. Section 3.5). In addition, we have

$$N(A) \subset N(A^2) \subset \cdots \subset N(A^n) \subset \cdots ,$$

and what is more, these spaces are actually increasing, because

$$A^n x_n = A^{n-1}(Ax_n) = A^{n-1}x_{n-1} = \cdots = Ax_1 = 0,$$

and

$$A^{n-1}x_n = A^{n-2}(Ax_n) = A^{n-2}x_{n-1} = \cdots = Ax_2 = x_1 \neq 0.$$

Hence, we can apply Lemma 4.7 to find a $z_n \in N(A^n)$ such that

$$\|z_n\| = 1, \quad d(z_n, N(A^{n-1})) > 1/2.$$

Since K is compact, $\{Kz_n\}$ should have a convergent subsequence. But for $n > m$,

$$\|Kz_n - Kz_m\| = \|z_n - Az_n - z_m + Az_m\| = \|z_n - (z_m - Az_m + Az_n)\| > 1/2,$$

since $A^{n-1}(z_m - Az_m + Az_n) = 0$. This shows that $\{Kz_n\}$ cannot have a convergent subsequence. There must be something wrong. What caused the contradiction? Obviously, it was the assumption that $x_1 \neq 0$. Hence, $\dim N(A) = 0$.

We have just shown that $\dim N(A') = 0$ implies $\dim N(A) = 0$. Conversely, assume that $\dim N(A) = 0$. Then $R(A') = N(A)^\circ = X'$ by Theorem 3.16. Since $A' = I - K'$ and we know that K' is compact, the argument just given implies that $\dim N(A') = 0$. Hence, we have shown that $\dim N(A) = 0$ if, and only if, $\dim N(A') = 0$.

Next, suppose

$$\dim N(A) = n > 0, \quad \dim N(A') = m > 0.$$

Let $x_1, \cdots, x_n;\ x'_1, \cdots, x'_m$ span $N(A)$ and $N(A')$, respectively. I claim there are a functional x'_0 and an element x_0 such that

$$x'_0(x_j) = 0, \quad 1 \leq j < n, \quad x'_0(x_n) \neq 0. \tag{4.34}$$

$$x'_j(x_0) = 0, \quad 1 \le j < m, \quad x'_m(x_0) \ne 0. \tag{4.35}$$

Assume this for the moment and put

$$K_0 x = x'_0(x) x_0, \quad x \in X. \tag{4.36}$$

Then

$$K'_0 x' = x'(x_0) x'_0, \quad x' \in X'.$$

Set $A_1 = A - K_0$. We shall show that

$$\dim N(A_1) = n - 1, \quad \dim N(A'_1) = m - 1. \tag{4.37}$$

But $A_1 = I - (K + K_0)$, and $K + K_0$ is a compact operator. Thus, we have an operator A_1 of the same form as A with the dimensions of its null space and that of its adjoint exactly one less than those of A, A', respectively. If m and n are both greater than one, we can repeat the process and reduce $\dim N(A_1)$ and $\dim N(A'_1)$ each by one. Continuing in this way we eventually reach an operator $\hat{A} = I - \hat{K}$, where $\hat{K}$ is compact and either $\dim N(\hat{A}) = 0$ or $\dim N(\hat{A}') = 0$. It then follows from what we have proved that they must both be zero. Hence, $m = n$, and the proof is complete.

To prove (4.37), suppose $x \in N(A_1)$. Then $Ax = K_0 x = x'_0(x) x_0$. Now x_0 is not in $R(A) = {}^\circ N(A')$, since it does not annihilate x'_m. Hence, we must have $x'_0(x) = 0$, and consequently $Ax = 0$. Since $x \in N(A)$,

$$x = \sum_1^n \alpha_j x_j. \tag{4.38}$$

Therefore, by (4.34),

$$x'_0(x) = \sum_1^n \alpha_j x'_0(x_j) = \alpha_n x'_0(x_n) = 0,$$

showing that $\alpha_n = 0$. Hence, x is of the form

$$x = \sum_1^{n-1} \alpha_j x_j. \tag{4.39}$$

Conversely, every element of the form (4.39) is in $N(A_1)$. This follows from the fact that it is in $N(A)$ and satisfies $x'_0(x) = 0$. This shows that $\dim N(A_1) = n - 1$.

Suppose $x' \in N(A'_1)$, i.e., $A'x' = K'_0 x' = x'(x_0) x'_0$. But x'_0 is not in $R(A') = N(A)^\circ$, since it does not annihilate x_n. Hence, $x'(x_0) = 0$, and consequently, $A'x' = 0$. Thus,

$$x' = \sum_1^m \beta_j x'_j$$

and

$$x'(x_0) = \sum_1^m \beta_j x'_j(x_0) = \beta_m x'_m(x_0) = 0,$$

showing that $\beta_m = 0$. Hence, x' is of the form

$$x' = \sum_1^{m-1} \beta_j x'_j. \tag{4.40}$$

Conversely, every functional of the form (4.40) is in $N(A'_1)$, since it is in $N(A')$ and $N(K'_0)$. This proves (4.37).

It remains to show that we can find an x'_0 satisfying (4.34) and an x_0 satisfying (4.35). The functional x'_0 can be found easily. To see this, let M be the subspace spanned by $x_1, \cdots, x_{n-1}$ [i.e., the set of all x of the form (4.39)]. The element x_n is not in this finite dimensional (and, hence, closed) subspace. This implies that $d(x_n, M) > 0$. Then, by Theorem 2.9, there is an $x'_0 \in X'$ such that $\|x'_0\| = 1$, $x'_0(x_n) = d(x_n, M)$, and $x'_0(x) = 0$ for $x \in M$. This more than suffices for our needs.

We find x_0 by induction. In fact, we are going to prove

Lemma 4.14. *If $x'_1, \cdots, x'_m$ are linearly independent vectors in X', then there are vectors $x_1, \cdots, x_m$ in X such that*

$$x'_j(x_k) = \delta_{jk} = \begin{cases} 1 & j = k, \\ 0, & j \neq k, \end{cases} \qquad 1 \leq j, k \leq m. \tag{4.41}$$

Moreover, if $x_1, \cdots, x_m$ are linearly independent vectors in X, then there are vectors $x'_1, \cdots, x'_m$ in X' such that (4.41) holds.

Proof. Assume that the lemma is true for $m = l - 1 \geq 1$. Let $x'_1, \cdots, x'_l$ be linearly independent vectors in X'. Then, for any $x \in X$,

$$x'_j(x - \sum_1^{l-1} x'_k(x)x_k) = x'_j(x) - x'_j(x) = 0, \quad 1 \leq j < l. \tag{4.42}$$

Now if x'_l vanished on ${}^\circ[x'_1, \cdots, x'_{l-1}]$ (the set of annihilators of $x'_1, \cdots, x'_{l-1}$), then it would have to vanish on

$$x - \sum_1^{l-1} x'_k(x)x_k$$

for any $x \in X$ by (4.42). This would mean that

$$x'_l(x) = \sum_1^{l-1} x'_k(x)x'_l(x_k), \quad x \in X,$$

or

$$x'_l = \sum_{1}^{l-1} x'_l(x_k)x'_k.$$

But this is impossible, since $x'_1, \cdots, x'_l$ are linearly independent. Hence, x'_l does not vanish on the whole of $^\circ[x'_1, \cdots, x'_{l-1}]$. There must be at least one element x_l in this set which does not annihilate x'_l. Thus, we can choose x_l so that

$$x'_j(x_l) = 0, \quad 1 \le j < l, \; x'_l(x_l) = 1.$$

Now, for each $k \neq l$ we can rearrange the functionals $x'_1, \cdots, x'_l$ so that x'_k replaces x'_l as the last one. Since the x'_j with $j \neq k$ form a set of $l-1$ linearly independent functionals, we can again use our induction hypothesis and the argument given above to conclude that there is an $x_k \in X$ such that

$$x'_j(x_k) = 0, \quad j \neq k, \; x'_k(x_k) = 1.$$

This implies that (4.41) holds for $m = l$. Since it is trivial for $m = 1$, the proof is complete for the first part. For the second part, we use the argument given above. We let M_j denote the subspace of X spanned by $x_1, \cdots, x_m$ with x_j missing. The vector x_j is not in M_j, and consequently, $d(x_j, M_j) > 0$. Then by Theorem 2.9, there is a $x'_j \in M_j^\circ$ such that $x'_j(x_j) = 1$. This proves the second part of the lemma. $\square$

We still must prove the fact that $K' \in K(X')$ whenever $K \in K(X)$. This proof will be given in the next section.

In Section 4.2, when we considered operators of finite rank, we employed the theory of n linear equations in n unknowns. We also remarked that if you are unfamiliar with this theory, we shall prove it later from more general considerations. Our more general result is Theorem 4.12.

Suppose we have a system of the form

$$\sum_{j=1}^{n} a_{ij}x_j = y_i, \quad 1 \le i \le n. \tag{4.43}$$

We can consider this as an equation of the form $Ax = y$ in $\mathbb{R}^n$, where $x = (x_1, \cdots, x_n),\; y = (y_1, \cdots, y_n)$. The operator A is clearly linear and bounded (with $\|A\| \le \max_{i,j} |a_{ij}|$). Moreover, if we set $K = I - A$, then K is an operator of finite rank (since $R(K) \subset \mathbb{R}^n$), and, hence, compact. To find A', we have, by definition,

$$(A'u, x) = (u, Ax), \quad u, \; x \in \mathbb{R}^n.$$

(Here, we are using the fact that $\mathbb{R}^n$ is a Hilbert space.) Thus, if $A'u = v$, we see that

$$\sum_1^n v_j x_j = \sum_{i=1}^n u_i \sum_{j=1}^n a_{ij} x_j = \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} u_i.$$

Since this is true for all $x \in \mathbb{R}^n$, we have

$$\sum_{i=1}^n a_{ij} u_i = v_j, \quad 1 \le j \le n. \tag{4.44}$$

Now the range of A is evidently closed, being finite-dimensional. Hence, y is a solution of (4.43) if, and only if, it annihilates (or is orthogonal to) $N(A')$. This is the same as saying

$$\sum_1^n u_i y_i = 0 \tag{4.45}$$

for all vectors u satisfying

$$\sum_{i=1}^n a_{ij} u_i = 0, \quad 1 \le j \le n. \tag{4.46}$$

Moreover, by Theorem 4.12, $\dim N(A) = \dim N(A')$. This means that the number of linearly independent solutions of

$$\sum_{j=1}^n a_{ij} x_j = 0, \quad 1 \le i \le n, \tag{4.47}$$

is the same as the number of linearly independent solutions of (4.46). In particular, if the only solution of (4.47) is $x = 0$, then (4.43) can be solved for all y. We should remark that we have used much more than was necessary (we used a mountain to crush a flea).

4.4. The adjoint of a compact operator

In this section, we shall prove

Theorem 4.15. *Let X, Y be normed vector spaces, and assume that K is in $K(X, Y)$. Then K' is in $K(Y', X')$.*

In order to prove Theorem 4.15, we shall develop a few concepts. Let X be a normed vector space. A set $V \subset X$ is called *relatively compact* if every sequence of elements of V has a convergent subsequence. The limit of this subsequence, however, need not be in V. Let $\varepsilon > 0$ be given. A set of points $W \subset X$ is called an ε-net for a set $U \subset X$ if for every $x \in U$ there is a $z \in W$ such that $\|x - z\| < \varepsilon$. A subset $U \subset X$ is called *totally bounded* if for every $\varepsilon > 0$ there is a finite set of points $W \subset X$ which is an ε-net for

U. We leave it as an exercise to show that if U is totally bounded, then for each $\varepsilon > 0$ it has a finite ε-net $W \subset U$.

If A is a map from X to Y, then the *image* $A(U)$ of a set $U \subset X$ is the set of those $y \in Y$ for which there is an $x \in U$ satisfying $Ax = y$. We have

Lemma 4.16. *Let X, Y be normed vector spaces. A linear operator K from X to Y is compact if, and only if, the image K(U) of a bounded set $U \subset X$ is relatively compact in Y.*

Proof. Suppose $K \in K(X, Y)$. Since U is bounded, there is a constant C such that $\|x\| \leq C$ for all $x \in U$. Let $\{Kx_n\}$ be a sequence in $K(U)$. Then $\|x_n\| \leq C$. Since K is compact, $\{Kx_n\}$ has a convergent subsequence. This means that $K(U)$ is relatively compact. Conversely, assume that $K(U)$ is relatively compact for each bounded set U. Let $\{x_n\}$ be a bounded sequence in X. Then $\{Kx_n\}$ has a convergent subsequence. Hence, $K \in K(X, Y)$. □

Theorem 4.17. *If a set $U \subset X$ is relatively compact, then it is totally bounded. If X is complete and U is totally bounded, then U is relatively compact.*

Before proving Theorem 4.17 we shall show how it can be used to prove Theorem 4.15.

Proof. Since X' is complete (Theorem 2.10), it suffices by Theorem 4.17 to show that for any bounded set $W \subset Y'$, the image $K'(W)$ is totally bounded in X'. Let $\varepsilon > 0$ be given, and let S_1 denote the closed unit sphere $\|x\| \leq 1$ in X. Since K is a compact operator, $K(S_1)$ is totally bounded by Lemma 4.16 and Theorem 4.17. Thus, there are elements $x_1, \cdots, x_n$ of S_1 such that each $x \in S_1$ satisfies

$$\|Kx - Kx_i\| < \varepsilon \tag{4.48}$$

for some i. Let B be the mapping from Y' to $\mathbb{R}^n$ defined by

$$By' = (y'(Kx_1), \cdots, y'(Kx_n)).$$

Since $R(B)$ is finite dimensional, B is an operator of finite rank. Let W be a bounded set in Y', say $\|y'\| \leq C_0$ for $y' \in W$. Then $B(W)$ is totally bounded. Thus, there are functionals $y'_1, \cdots, y'_m$ such that every $y' \in W$ satisfies

$$\|By' - By'_j\| < \varepsilon$$

for some j. Hence,

$$|y'(Kx_i) - y'_j(Kx_i)| < \varepsilon \tag{4.49}$$

for each i. Now let y' be any element of W. Then there is a j for which (4.49) holds for each i. Let x be any element of S_1. Then there is an i such that (4.48) holds. Hence,

$$\begin{aligned}|y'(Kx) - y'_j(Kx)| &\leq |y'(Kx) - y'(Kx_i)| \\ &\quad + |y'(Kx_i) - y'_j(Kx_i)| + |y'_j(Kx_i) - y'_j(Kx)| \\ &\leq \|y'\| \cdot \|Kx - Kx_i\| + \varepsilon + \|y'_j\| \cdot \|Kx_i - Kx\| \\ &\leq 2C_0\varepsilon + \varepsilon.\end{aligned}$$

Since this is true for any $x \in S_1$,we have

$$\|K'y' - K'y'_j\| < (2C_0 + 1)\varepsilon.$$

This means that $K'(W)$ is totally bounded, and the proof is complete. $\square$

It remains to give the proof of Theorem 4.17.

Proof. Assume that U is relatively compact, and let $\varepsilon > 0$ be given. We shall show that U has an ε-net consisting of a finite number of points of U. (We actually do not need to show that the points belong to U.) Let x_1 be any point of U. If U is contained in the set $\|x - x_1\| < \varepsilon$, we are through. Otherwise, let $x_2 \in U$ be such that $\|x_2 - x_1\| \geq \varepsilon$. If every $x \in U$ satisfies

$$\|x - x_1\| < \varepsilon \quad \text{or} \quad \|x - x_2\| < \varepsilon, \tag{4.50}$$

we are also through. Otherwise, let x_3 be a point of U not satisfying (4.50). Inductively, if $x_1, \cdots, x_n$ are chosen and no point of U satisfies

$$\|x - x_1\| \geq \varepsilon, \cdots, \|x - x_n\| \geq \varepsilon,$$

then the first statement is proved. Otherwise, let x_{n+1} be such a point. We eventually stop after a finite number of steps. Otherwise, $\{x_j\}$ would be a sequence of elements of U satisfying

$$\|x_j - x_k\| \geq \varepsilon \ \text{ for } \ j \neq k.$$

This sequence would have no convergent subsequence, contradicting the fact that U is relatively compact. Hence, U is totally bounded.

Conversely, assume now that U is totally bounded and that X is complete. Let $\{x_n\}$ be any sequence of points of U. Since U is totally bounded, it is covered by (i.e., contained in the union of) a finite number of spheres of radius 1. At least one of those spheres contains an infinite number of elements of $\{x_n\}$ (counting repetitions). Let S_1 be any one of the spheres containing an infinite number of elements of $\{x_n\}$. Throw out all elements of $\{x_n\}$ not in S_1, and denote the resulting subsequence by $\{x_{n1}\}$. There is a finite number of spheres of radius $1/2$ which cover U. At least one of these contains an infinite number of points of $\{x_{n1}\}$. Choose one, and denote it by S_2. Throw out all points of $\{x_{n1}\}$ not in S_2, and denote the resulting sequence by $\{x_{n2}\}$. Continue inductively, and set $z_n = x_{nn}$. Then for each $\varepsilon > 0$, there is an N so large that all z_k are in a sphere of radius less than ε for $k > N$. In other words, $\{z_n\}$ is a Cauchy sequence. We now use the completeness of X to conclude that it has a limit in X. This completes the proof of Theorem 4.17. $\square$

4.5. Problems

(1) Show that the set I^ω consisting of all elements $f = (\alpha_1, \cdots)$ in l_∞ such that $|\alpha_n| \leq 1/n$ is compact in l_∞. The set I^ω is called the *Hilbert cube.*

(2) Let M be a totally bounded subset of a normed vector space X. Show that for each $\varepsilon > 0$, M has a finite ε-net $N \subset M$ (not just merely $N \subset X$).

(3) Prove that if M is finite dimensional, then we can take $\theta = 1$ in Lemma 4.7.

(4) Show that if X is infinite dimensional and K is a one-to-one operator in $K(X)$, then $K - I$ cannot be in $K(X)$.

(5) Let X be a vector space which can be made into a Banach space by introducing either of two different norms. Suppose that convergence with respect to the first norm always implies convergence with respect to the second norm. Show that the norms are equivalent.

(6) Suppose $A \in B(X, Y)$, $K \in K(X, Y)$, where X, Y are Banach spaces. If $R(A) \subset R(K)$, show that $A \in K(X, Y)$.

(7) Suppose X, Y are Banach spaces and $K \in K(X, Y)$. If $R(K) = Y$, show that Y is finite dimensional.

(8) Show that if X is an infinite dimensional normed vector space, then there is a sequence $\{x_n\}$ such that $\|x_n\| = 1$, $\|x_n - x_m\| \geq 1$ for $m \neq n$.

(9) A normed vector space X is called strictly convex if $\|x+y\| < \|x\| + \|y\|$ whenever $x, y \in X$ are linearly independent. Show that every finite-dimensional vector space can be made into a strictly convex normed vector space.

(10) Let V be a vector space having n linearly independent elements $v_1, \cdots, v_n$ such that every element $v \in V$ can be expressed in the form (4.12). Show that $\dim V = n$.

(11) Let V, W be subspaces of a Hilbert space. If $\dim V < \infty$ and $\dim V < \dim W$, show that there is a $u \in W$ such that $\|u\| = 1$ and $(u, v) = 0$ for all $v \in V$.

(12) For X, Y Banach spaces, let A be an operator in $B(X, Y)$ such that $R(A)$ is closed and infinite-dimensional. Show that A is not compact.

(13) Suppose X is a Banach space consisting of finite linear combinations of a denumerable set of elements. Show that $\dim X < \infty$.

(14) Show that every linear functional on a finite dimensional normed vector space is bounded.

(15) If M is a subspace of a Banach space X, then we define

$$\operatorname{codim} M = \dim\ X/M.$$

If M is closed, show that $\operatorname{codim} M = \dim\ M^\circ$ and $\operatorname{codim} M^\circ = \dim\ M$.

(16) If M, N are subspaces of a Banach space X and $\operatorname{codim} N < \dim M$, show that $M \cap N \neq \{0\}$. (If $\dim M = \infty$, it is assumed that $\operatorname{codim} N < \infty$.)

(17) If $A \in B(X, Y)$, $R(A)$ is dense in Y and D is dense in X, show that A maps D onto a dense subset of Y.

(18) Show that every infinite dimensional normed vector space has an unbounded linear functional.

(19) If X, Y are Banach spaces and A is a closed linear operator from X to Y such that $R(A)$ is not closed in Y, show that for each $\varepsilon > 0$ there is an infinite dimensional closed subspace $M \subset D(A)$ such that the restriction of A to M is compact with norm $< \varepsilon$.

(20) Let $\{x'_k\}$ be a sequence of functionals in X', where X is a finite dimensional normed vector space, and assume that $x'_k(x) \to 0$ as $k \to \infty$ for each $x \in X$. Show that $\|x'_k\| \to 0$.

(21) If X is a Banach space and $A \in B(X)$, let

$$|A|_K = \inf_{K \in K(X)} \|A + K\|.$$

Show that $|A|_K < 1$ implies that $R(I - A)$ is closed in X and $\dim N(I - A) = \dim N(I - A') < \infty$.

(22) Show that the product of a bounded operator and an operator of finite rank (in either order) is an operator of finite rank.

FREDHOLM OPERATORS

5.1. Orientation

If X is a Banach space and $K \in K(X)$, we saw in the last chapter that $A = I - K$ has closed range and that both $N(A)$ and $N(A')$ are finite dimensional. Operators having these properties form a very interesting class and arise very frequently in applications. In this chapter, we shall investigate some of their properties.

Let X, Y be Banach spaces. An operator $A \in B(X, Y)$ is said to be a *Fredholm operator* from X to Y if

(1) $\alpha(A) = \dim N(A)$ is finite,

(2) $R(A)$ is closed in Y,

(3) $\beta(A) = \dim N(A')$ is finite.

The set of Fredholm operators from X to Y is denoted by $\Phi(X, Y)$ in deference to the Russians. If $X = Y$ and $K \in K(X)$, then, clearly, $I - K$ is a Fredholm operator. The *index* of a Fredholm operator is defined as

$$i(A) = \alpha(A) - \beta(A). \tag{5.1}$$

For $K \in K(X)$, we have shown that $i(I - K) = 0$ (Theorem 4.12).

For convenience, we shall assume throughout this chapter that X, Y, Z are Banach spaces (unless we state otherwise). If $A \in \Phi(X,Y)$ with $N(A) = \{0\}$ and $R(A) = Y$, then A has an inverse A^{-1} in $B(Y,X)$ (Theorem 3.11). Is there any corresponding statement that can be made for an arbitrary $A \in \Phi(X,Y)$? If $\alpha(A) \neq 0$, A cannot have an inverse. There is, however, a subspace X_0 of X such that the restriction of A to X_0 has an inverse (i.e., is one-to-one). This can be effected easily by means of

Lemma 5.1. *Let N be a finite dimensional subspace of a normed vector space X. Then there is a closed subspace X_0 of X such that*

(a) $X_0 \cap N = \{0\}$

(b) For each $x \in X$ there is an $x_0 \in X_0$ and an $x_1 \in N$ such that $x = x_0 + x_1$. This decomposition is unique.

If V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$, then the set of all sums $v_1 + v_2$, $v_i \in V_i$, is called the *direct sum* of V_1 and V_2 and is denoted by $V_1 \oplus V_2$. Thus, the conclusion of Lemma 5.1 can be stated as $X = X_0 \oplus N$. The proof of Lemma 5.1 will be given at the end of this section.

Now, by Lemma 5.1, there is a closed subspace X_0 such that

$$X = X_0 \oplus N(A). \tag{5.2}$$

Clearly, A is one-to-one on X_0. Let $\hat{A}$ be the restriction of A to X_0. Then $\hat{A} \in B(X_0, R(A))$. Moreover, $R(\hat{A}) = R(A)$. To see this, note that if $y \in R(A)$, then there is an $x \in X$ such that $Ax = y$. But $x = x_0 + x_1$, where $x_0 \in X_0$, $x_1 \in N(A)$. Hence, $Ax_0 = Ax - Ax_1 = y$, showing that $y \in R(\hat{A})$. Thus $\hat{A}$ has an inverse $\hat{A}^{-1}$ in $B(R(A), X_0)$ (here, we make use of the fact that $R(A)$ is closed and, hence, a Banach space).

We have found a sort of "inverse" for A. The only trouble with it is that $\hat{A}^{-1}$ is defined only on $R(A)$ and not on the whole of Y. This fact will make it extremely awkward in using the inverse. It would be more useful if there would be an operator $A_0 \in B(Y,X)$ that equals $\hat{A}^{-1}$ on $R(A)$. Such an operator can be found if there is a finite dimensional subspace Y_0 such that

$$Y = R(A) \oplus Y_0. \tag{5.3}$$

Then we can define A_0 to equal $\hat{A}^{-1}$ on $R(A)$ and to vanish (or do something else) on Y_0. That $A_0 \in B(Y,X)$ would follow from

Lemma 5.2. *Let X_1 be a closed subspace of a normed vector space X, and let M be a finite dimensional subspace such that $M \cap X_1 = \{0\}$. Then*

$X_2 = X_1 \oplus M$ is a closed subspace of X. Moreover, the operator P defined by

$$Px = \begin{cases} x, & x \in M, \\ 0, & x \in X_1 \end{cases} \tag{5.4}$$

is in $B(X_2)$.

Applying Lemma 5.2 to our situation, we see that there is an operator $P \in B(Y)$ such that

$$Py = \begin{cases} y, & y \in Y_0, \\ 0, & y \in R(A). \end{cases}$$

Thus, the operator $I - P$ is in $B(Y, R(A))$. Moreover, $A_0 = \hat{A}^{-1}(I-P)$, and, since $\hat{A}^{-1} \in B(R(A), X)$, we have $A_0 \in B(Y, X)$. That we can decompose Y into the form (5.3) follows from

Lemma 5.3. *Let X be a normed vector space, and let R be a closed subspace such that R° is of finite dimension n. Then there is an n-dimensional subspace M of X such that $X = R \oplus M$.*

Note that in our case $R(A)^\circ = N(A')$ [see (3.12)], which is finite dimensional. Hence, (5.3) is, indeed, possible. The proofs of Lemmas 5.2 and 5.3 will be given at the end of this section. We now summarize our progress so far.

Theorem 5.4. *If $A \in \Phi(X, Y)$, there is a closed subspace X_0 of X such that (5.2) holds and a subspace Y_0 of Y of dimension $\beta(A)$ such that (5.3) holds. Moreover, there is an operator $A_0 \in B(Y, X)$ such that*

(a) $N(A_0) = Y_0$,

(b) $R(A_0) = X_0$,

(c) $A_0 A = I$ on X_0,

(d) $AA_0 = I$ on $R(A)$.

In addition,

(e) $A_0 A = I - F_1$ on X,

(f) $AA_0 = I - F_2$ on Y,

where $F_1 \in B(X)$ *with* $R(F_1) = N(A)$ *and* $F_2 \in B(Y)$ *with* $R(F_2) = Y_0$. *Consequently, the operators* F_1 *and* F_2 *are of finite rank.*

To prove statement (e), we note that the operator $F_1 = I - A_0A$ is equal to I on $N(A)$ and vanishes on X_0. Hence, it is in $B(X)$ by Lemma 5.2. Similar reasoning gives (f).

We conclude this section by giving the proofs of Lemmas 5.1–5.3. First, we give the proof of Lemma 5.1.

Proof. Let $x_1, \cdots, x_n$ be a basis for N (see Section 4.1). By Lemma 4.14, we can find functionals $x_1', \cdots, x_n'$ in X' such that

$$x_j'(x_k) = \delta_{jk}, \quad 1 \leq j, k \leq n. \tag{5.5}$$

Let X_0 be the set of those $x \in X$ such that $x_j'(x) = 0$ for each j. Clearly, X_0 is a subspace of X. It is also closed. For if $z_n \in X_0$ and $z_n \to z$ in X, then $0 = x_j'(z_n) \to x_j'(z)$ for each j. Hence, $z \in X_0$. Moreover, $X_0 \cap N = \{0\}$. For if $x \in N$,

$$x = \sum_1^n \alpha_k x_k,$$

and hence, $x_j'(x) = \alpha_j$ for each j. If x is to be in X_0 as well, we must have each $\alpha_j = 0$. Finally, we must show that every $x \in X$ can be written in the form $x = x_0 + z$, where $x_0 \in X_0$ and $z \in N$. Let x be any element in X. Set

$$z = \sum_1^n x_k'(x) x_k.$$

Clearly, $z \in N$. Moreover, we have $x_j'(x - z) = x_j'(x) - x_j'(x) = 0$ for each j. Hence, $x - z \in X_0$. This gives the desired decomposition. The uniqueness is obvious. □

Next, we give the proof of Lemma 5.2.

Proof. We first prove

$$\|Px\| \leq C\|x\|, \quad x \in X_2. \tag{5.6}$$

If (5.6) did not hold, there would be a sequence $\{x_n\}$ of elements of X_2 such that $\|Px_n\| = 1$, while $x_n \to 0$ in X. Since M is finite dimensional, $\{Px_n\}$ has a subsequence converging in M (Corollary 4.5). For convenience, we assume that the whole sequence converges. Thus $Px_n \to z$ in M. Then $(I-P)x_n \to -z$ in X_1. Since both M and X_1 are closed, we have $z \in M \cap X_1$.

Hence, by hypothesis, we must have $z = 0$. But $\|z\| = \lim \|Px_n\| = 1$. This provides the contradiction that proves (5.6).

To prove that X_2 is closed, we let $\{x_n\}$ be a sequence of elements of X_2 converging to an element $x \in X$. By (5.6), $\{Px_n\}$ is a Cauchy sequence in M and, hence, converges to an element $z \in M$. Consequently, $\{(I - P)x_n\}$ converges to an element $w \in X_1$. Hence, $x_n = Px_n + (I - P)x_n$ converges to $z + w \in X_2$. This shows that $x \in X_2$, and the proof is complete. $\square$

Finally, we give the proof of Lemma 5.3.

Proof. Let $x'_1, \cdots, x'_n$ be a basis for R°. Since $R = {}^\circ(R^\circ)$ (Lemma 3.6), we see that $x \in R$ if, and only if, $x'_j(x) = 0$ for each j. Now, by Lemma 4.14, there are elements $x_1, \cdots, x_n$ of X such that (5.5) holds. The x_k are clearly linearly independent. For, if

$$\sum_1^n \alpha_k x_k = 0,$$

then for each j,

$$x'_j(\sum_1 \alpha_k x_k) = \alpha_j = 0.$$

Let M be the n-dimensional subspace of X spanned by the x_k. Then $R \cap M = \{0\}$. For, if $x \in M$, then x is of the form

$$x = \sum_1^n \alpha_k x_k,$$

and hence, $x'_j(x) = \alpha_j$ for each j. If x is also in R, then we must have $\alpha_j = 0$ for each j. Now, let x be any element in X. Set

$$w = \sum_1^n x'_k(x) x_k.$$

Then $w \in M$ and $x'_j(x - w) = 0$ for each j. Hence, $x - w \in R$ and we have the desired decomposition. This completes the proof. $\square$

5.2. Further properties

From the definition, it may not be easy to recognize a Fredholm operator when one sees one. A useful tool in this connection is the following converse of Theorem 5.4.

Theorem 5.5. *Let A be an operator in $B(X, Y)$ and assume that there are operators A_1, $A_2 \in B(Y, X)$, $K_1 \in K(X)$, $K_2 \in K(Y)$ such that*

$$A_1 A = I - K_1 \;\text{ on }\; X \tag{5.7}$$

and

$$AA_2 = I - K_2 \text{ on } Y. \tag{5.8}$$

Then $A \in \Phi(X, Y)$.

Proof. Since $N(A) \subset N(A_1A)$, we have $\alpha(A) \leq \alpha(I - K_1) < \infty$ (Theorem 4.12). Likewise, $R(A) \supset R(AA_2) = R(I - K_2)$. Hence, $N(A') \subset N(I - K_2')$ and $\beta(A) \leq \alpha(I - K_2') < \infty$ (here, we have made use of the fact that the adjoint of a compact operator is compact; see Theorem 4.15). It remains to prove that $R(A)$ is closed. This follows easily from

Lemma 5.6. *Let X be a normed vector space, and suppose that $X = N \oplus X_0$, where X_0 is a closed subspace and N is finite dimensional. If X_1 is a subspace of X containing X_0, then X_1 is closed.*

Applying Lemma 5.6 to our situation, we know that there is a finite-dimensional subspace Y_1 of Y such that $Y = R(I - K_2) \oplus Y_1$ (Lemma 5.3). Since $R(A) \supset R(I - K_2)$, we see by Lemma 5.6 that $R(A)$ is closed in Y. □

Now we give the proof of Lemma 5.6.

Proof. Set $M = N \cap X_1$. Then $X_1 = X_0 \oplus M$. For if $x \in X_1,\ x = x_0 + z$, where $x_0 \in X_0$ and $z \in N$. Since $x_0 \in X_1$, the same is true for z. Hence, $z \in M$. We now apply Lemma 5.2. □

We now come to a very important property of Fredholm operators.

Theorem 5.7. *If $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$, then $BA \in \Phi(X,Z)$ and*

$$i(BA) = i(B) + i(A). \tag{5.9}$$

Proof. By Theorem 5.4, there are operators

$$A_0 \in B(Y, X),\ B_0 \in B(Z, Y),\ F_1 \in K(X),\ F_2,\, F_3 \in K(Y),\ F_4 \in K(Z)$$

such that

$$A_0A = I - F_1 \text{ on } X, \quad AA_0 = I - F_2 \text{ on } Y \tag{5.10}$$

and

$$B_0B = I - F_3 \text{ on } Y, \quad BB_0 = I - F_4 \text{ on } Z \tag{5.11}$$

(actually, the F_i are of finite rank, but we do not need to know this here). Hence,

$$A_0B_0BA = A_0(I - F_3)A = I - F_1 - A_0F_3A = I - F_5 \text{ on } X,$$

and

$$BAA_0B_0 = B(I - F_2)B_0 = I - F_4 - BF_2B_0 = I - F_6 \text{ on } Z,$$

where $F_5 \in K(X)$ and $F_6 \in K(Z)$. Hence, by Theorem 5.5, we see immediately that $BA \in \Phi(X, Z)$.

To prove (5.9), let $Y_1 = R(A) \cap N(B)$. We find subspaces Y_2, Y_3, Y_4 such that

$$R(A) = Y_1 \oplus Y_2$$
$$N(B) = Y_1 \oplus Y_3,$$
$$Y = R(A) \oplus Y_3 \oplus Y_4,$$

(see Figure 5.1). This can be done by Lemmas 5.1 and 5.3. Note that Y_1, Y_3, Y_4 are finite dimensional and that Y_2 is closed. Let $d_i = \dim Y_i$, $i =$ 1, 3, 4.

Now,

$$N(BA) = N(A) \oplus X_1,$$
$$R(B) = R(BA) \oplus Z_4,$$

where $X_1 \subset X_0$ is such that $A(X_1) = Y_1$ and $Z_4 = B(Y_4)$. We claim that

$$\dim X_1 = d_1, \quad \dim Z_4 = d_4. \tag{5.12}$$

Assuming this for the moment, we have

$$\alpha(BA) = \alpha(A) + d_1,$$
$$\beta(BA) = \beta(B) + d_4,$$
$$\alpha(B) = d_1 + d_3,$$
$$\beta(A) = d_3 + d_4.$$

These relationships immediately give (5.9). □

To prove (5.12), we employ

Lemma 5.8. *Let V, W be finite-dimensional vector spaces, and let L be a linear operator from V to W. Then* $\dim R(L) \leq \dim D(L)$. *If L is one-to-one, then* $\dim R(L) = \dim D(L)$.

In our case, A is a one-to-one linear map of X_1 onto Y_1. Hence, $\dim X_1 = \dim Y_1$. Similarly, B is one-to-one on Y_4, and its range on Y_4 is (by definition) Z_4. Hence, $\dim Z_4 = \dim Y_4$. Thus, we complete the proof of Theorem 5.7 by giving the simple proof of Lemma 5.8.

Proof. The second statement follows from the first, since L^{-1} exists and

$$D(L^{-1}) = R(L), \quad R(L^{-1}) = D(L).$$

To prove the first statement, assume that $\dim D(L) < n$, and let $w_1, \cdots, w_n$ be any n vectors in $R(L)$. Let $v_1, \cdots, v_n$ be vectors in $D(L)$ such that

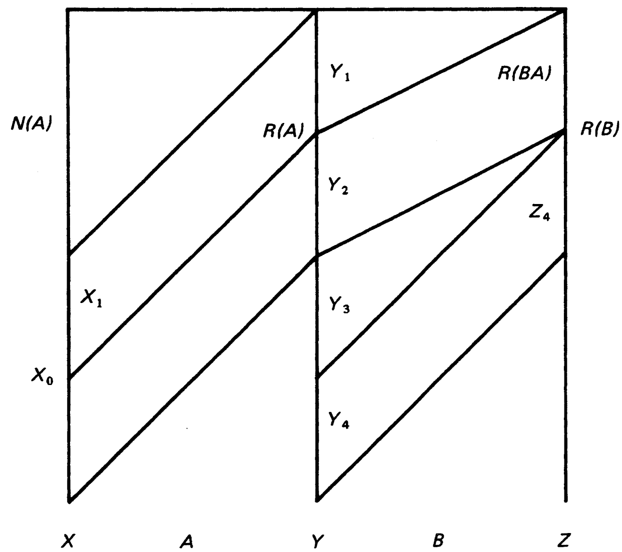

Figure 5.1

$Lv_i = w_i,\ 1 \leq i \leq n$. Since $\dim D(L) < n$, the v_i are linearly dependent. Hence, there are scalars $\alpha_1, \cdots, \alpha_n$, not all zero, such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

Consequently,

$$L(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \cdots + \alpha_n w_n = 0.$$

This shows that the w_i are linearly dependent. Since the w_i were any n vectors in $R(L)$, we see that $\dim R(L) < n$. Since this is true for any n, we must have $\dim R(L) \leq \dim D(L)$. The proof is complete. □

We end this section with a simple consequence of Theorems 5.5 and 5.7.

Lemma 5.9. *Suppose $A \in \Phi(X,Y)$, and let A_0 be any operator satisfying (e) and (f) of Theorem 5.4. Then $A_0 \in \Phi(Y,X)$ and $i(A_0) = -i(A)$.*

Proof. By hypothesis, (5.10) holds with $F_1 \in K(X)$ and $F_2 \in K(Y)$. Thus, by Theorem 5.5 (with X and Y interchanged), we see that $A_0 \in \Phi(Y,X)$.

Moreover, by Theorem 5.7,

$$i(A_0) + i(A) = i(I - F_1) = 0.$$

Hence, $i(A_0) = -i(A)$, and the proof is complete. □

5.3. Perturbation theory

If $K \in K(X)$, then $A = I - K$ is a Fredholm operator. If we add any compact operator to A, it remains a Fredholm operator. Is this true for any Fredholm operator? An affirmative answer is given by

Theorem 5.10. *If $A \in \Phi(X,Y)$ and $K \in K(X,Y)$, then $A + K \in \Phi(X,Y)$ and*

$$i(A + K) = i(A). \tag{5.13}$$

Proof. By Theorem 5.4, there are $A_0 \in B(Y,X)$, $F_1 \in K(X)$, $F_2 \in K(Y)$ such that

$$A_0 A = I - F_1 \text{ on } X, \quad AA_0 = I - F_2 \text{ on } Y. \tag{5.14}$$

Hence,

$$A_0(A + K) = I - F_1 + A_0 K = I - K_1 \text{ on } X,$$

and

$$(A + K)A_0 = I - F_2 + KA_0 = I - K_2 \text{ on } Y,$$

where $K_1 \in K(X)$, $K_2 \in K(Y)$. Hence, $A + K \in \Phi(X,Y)$ (Theorem 5.5). Moreover, by Theorem 5.7,

$$i[A_0(A + K)] = i(A_0) + i(A + K) = i(I - K_1) = 0.$$

But $i(A_0) = -i(A)$ (Lemma 5.9). Hence, (5.13) holds, and the proof is complete. □

Another result along these lines is given by

Theorem 5.11. *Assume that $A \in \Phi(X,Y)$. Then there is an $\eta > 0$ such that for any $T \in B(X,Y)$ satisfying $\|T\| < \eta$, one has $A + T \in \Phi(X,Y)$,*

$$i(A + T) = i(A), \tag{5.15}$$

and

$$\alpha(A + T) \le \alpha(A). \tag{5.16}$$

Proof. By (5.14), we have

$$A_0(A + T) = I - F_1 + A_0 T \text{ on } X$$

and

$$(A + T)A_0 = I - F_2 + TA_0 \text{ on } Y.$$

We take $\eta = \|A_0\|^{-1}$. Then $\|A_0T\| \leq \|A_0\| \cdot \|T\| < 1$, and similarly, $\|TA_0\| < 1$. Thus, the operators $I+A_0T$ and $I+TA_0$ have bounded inverses (Theorem 1.1). Consequently,

$$(I + A_0T)^{-1}A_0(A + T) = I - (I + A_0T)^{-1}F_1 \quad \text{on} \quad X \tag{5.17}$$

$$(A + T)A_0(I + TA_0)^{-1} = I - F_2(I + TA_0)^{-1} \quad \text{on} \quad Y. \tag{5.18}$$

We can now appeal to Theorem 5.5 to conclude that $A + T \in \Phi(X, Y)$. Moreover, by Theorem 5.7 applied to (5.17), we have

$$i[(I + A_0T)^{-1}] + i(A_0) + i(A + T) = 0.$$

Since $i[(I + A_0T)^{-1}] = 0$, we see that (5.15) holds. It remains to prove (5.16). To do this, we refer back to Theorem 5.4(c). From it, we see that

$$A_0(A + T) = I + A_0T \quad \text{on} \quad X_0,$$

and hence, this operator is one-to-one on X_0. Thus, $N(A + T) \cap X_0 = \{0\}$. Since $X = N(A) \oplus X_0$, we see that $\dim N(A+T) \leq \dim N(A)$. This follows from the next lemma (Lemma 5.12). □

Lemma 5.12. *Let X be a vector space, and assume that $X = N \oplus X_0$, where N is finite dimensional. If M is a subspace of X such that $M \cap X_0 = \{0\}$, then* $\dim M \leq \dim N$.

Proof. Suppose $\dim N < n$, and let $x_1, \cdots, x_n$ be any n vectors in M. By hypothesis,

$$x_k = x_{k0} + x_{k1}, \quad x_{k0} \in X_0, \ x_{k1} \in N, \ 1 \leq k \leq n.$$

Since $\dim N < n$, there are scalars $\alpha_1, \cdots, \alpha_n$ not all zero such that

$$\sum_1^n \alpha_k x_{k1} = 0.$$

Hence,

$$\sum_1^n \alpha_k x_k = \sum_1^n \alpha_k x_{k0} \in X_0.$$

Since the x_k are in M, this can happen only if

$$\sum_1^n \alpha_k x_k = 0,$$

showing that the x_k are linearly dependent. Thus $\dim < n$. Since this is true for any n, the proof is complete. □

We now note a converse of Theorem 5.7.

Theorem 5.13. *Assume that $A \in B(X, Y)$ and $B \in B(Y, Z)$ are such that $BA \in \Phi(X, Z)$. Then $A \in \Phi(X, Y)$ if, and only if, $B \in \Phi(Y, Z)$.*

Proof. First assume that $A \in \Phi(X, Y)$, and let A_0 be an operator satisfying Theorem 5.4. Thus,

$$BAA_0 = B - BF_2 \text{ on } Y,$$

where $F_2 \in K(Y)$. Now $A_0 \in \Phi(Y, X)$ by Lemma 5.9, while $BA \in \Phi(X, Z)$ by hypothesis. Hence, $BAA_0 \in \Phi(Y, Z)$ (Theorem 5.7). Since $BF_2 \in K(Y, Z)$, it follows that $B \in \Phi(Y, Z)$ (Theorem 5.10).

Next, assume that $B \in \Phi(Y, Z)$, and let $B_0 \in B(Z, Y)$ satisfy (5.11). Then,

$$B_0BA = A - F_3A \quad \text{on } X.$$

Now $BA \in \Phi(X, Z)$ by hypothesis, while $B_0 \in \Phi(Z, Y)$. Hence, $B_0BA \in \Phi(X, Y)$, and the same must be true of A. This completes the proof. $\square$

The following seemingly stronger statement is a simple consequence of Theorem 5.13.

Theorem 5.14. *Assume that $A \in B(X, Y)$ and B is in $B(Y,Z)$ are such that $BA \in \Phi(X, Z)$. If $\alpha(B) < \infty$, then $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$.*

Proof. Since $R(B) \supset R(BA)$, we see by Lemma 5.6 that $R(B)$ is closed. Moreover, $\beta(B) \leq \beta(BA)$, and, hence, $B \in \Phi(Y, Z)$. We now apply Theorem 5.13. $\square$

We shall see in the next section that we can substitute $\beta(A) < \infty$ in place of $\alpha(B) < \infty$ in the hypothesis of Theorem 5.14.

Before we continue, let us give some examples. Consider the space l_p, $1 \leq p \leq \infty$. If $x = (x_1, x_2, \cdots)$ is any element in l_p, define

$$L_1x = (x_2, x_3, \cdots).$$

We can check that $N(L_1)$ consists of those elements of the form

$$(x_1, 0, \cdots),$$

and hence, is of dimension 1. Let X_0 be the subspace consisting of elements of the form

$$(0, x_2, x_3, \cdots).$$

It is easily seen that X_0 is a closed subspace of l_p, and that

$$l_p = N(L_1) \oplus X_0.$$

Moreover, $R(L_1) = l_p$, so that $i(L_1) = 1$.

Another example is given by

$$L_2x = (0, x_1, x_2, \cdots).$$

In this case, we have $N(L_2) = \{0\},\ R(L_2) = X_0$. Thus, $i(L_2) = -1$. Finally, consider the operator

$$L_3 x = (0, x_3, x_2, x_5, x_4, x_7, x_6, \cdots).$$

Here, one has $N(L_3) = N(L_1),\ R(L_3) = X_0,\ i(L_3) = 0$.

Note that $L_1 L_2 = I$, while $L_2 L_1$ is I on X_0 and vanishes on $N(L_1)$. In Section 4.3 we showed that the operator K given by

$$Kx = (x_1, x_2/2, x_3/3, \cdots)$$

was compact on l_p. It follows therefore, that $L_i + K$ is a Fredholm operator, $i = 1, 2, 3$. Examine these operators.

5.4. The adjoint operator

If $A \in B(X, Y)$, then we know that $A' \in B(Y', X')$(Theorem 3.3). If $A \in \Phi(X, Y)$, what can be said about A'? This is easily handled by means of Theorems 5.4 and 5.5. In fact, by taking adjoints in (e) and (f) of Theorem 5.4, we have

$$A' A_0' = I - F_1' \text{ on } X', \quad A_0' A' = I - F_2' \text{ on } Y'.$$

Since $A_0' \in B(X', Y')$ and the F_i' are also of finite rank (see Section 4.2), we see, by Theorem 5.5, that $A' \in \Phi(Y', X')$. What about $i(A')$? Is it related in any way to $i(A)$? To answer this question, we must examine the operator A' a little more closely. Since $\alpha(A') = \dim N(A')$, we have $\alpha(A') = \beta(A)$. To determine $\beta(A')$, we set $A'' = (A')'$. Let X'' be the set of bounded linear functionals on X'. We know that X'' is a Banach space (Theorem 2.10). Moreover, A'' is the operator in $B(X'', Y'')$ defined by

$$A'' x''(y') = x''(A' y'), \quad x'' \in X'',\ y' \in Y'. \tag{5.19}$$

Now, the set of elements in X'' that annihilate $R(A')$ is precisely $N(A'')$ [(3.12) applied to A']. Hence, by Lemma 5.3, there is a subspace $W \subset X'$ of dimension $\alpha(A'')$ such that

$$X' = R(A') \oplus W. \tag{5.20}$$

On the other hand, if $\alpha(A) = n$ and $x_1, \cdots, x_n$ form a basis for $N(A)$, then there are functionals $x_1', \cdots, x_n'$ such that (5.5) holds (Lemma 4.14). Let Z be the n-dimensional subspace spanned by the x_j'. Then $Z \cap R(A') = \{0\}$. For if $z' \in Z$, then

$$z' = \sum_1^n \alpha_j x_j',$$

and hence,

$$z'(x_k) = \alpha_k, \quad 1 \le k \le n.$$

If $z' \in R(A')$ as well, then $z'(x_k) = 0$ for each k, since $R(A') = N(A)^\circ$ (Theorem 3.16). Hence, $z' = 0$. Now for any $x' \in X'$, set

$$u' = x' - \sum_1^n x'(x_j)x_j'.$$

Then $u'(x_k) = 0$ for each k by (5.5). Hence, $u' \in R(A')$. Since $x' - u' \in Z$, we have

$$X' = R(A') \oplus Z, \tag{5.21}$$

and $\dim Z = \alpha(A)$. We now apply Lemma 5.12 to conclude that $\dim Z = \dim W$. But $\dim Z = \alpha(A)$, and $\dim W = \alpha(A'') = \beta(A')$. Hence, we have proved

Theorem 5.15. *If $A \in \Phi(X,Y)$, then $A' \in \Phi(Y',X')$ and*

$$i(A') = -i(A). \tag{5.22}$$

We have just seen that

$$\alpha(A'') = \alpha(A) \tag{5.23}$$

whenever $A \in \Phi(X,Y)$. It should be noted that, in general,

$$\alpha(A'') \geq \alpha(A) \tag{5.24}$$

for arbitrary $A \in B(X,Y)$. This can be seen as follows. Let x_0 be any element of X, and set

$$F(x') = x'(x_0), \quad x' \in X'.$$

Then,

$$|F(x')| \leq \|x_0\| \cdot \|x'\|.$$

This means that $F(x')$ is a bounded linear functional on X'. Hence, there is an $x_0'' \in X''$ such that

$$x_0''(x') = x'(x_0), \quad x' \in X'. \tag{5.25}$$

The element x_0'' is unique. For if x_1'' also satisfies (5.25), then

$$x_0''(x') - x_1''(x') = 0, \quad x' \in X',$$

showing that $x_0'' = x_1''$. We set $x_0'' = Jx_0$. Clearly, J is a linear mapping of X into X'' defined on the whole of X. Moreover, it is one-to-one. For if $Jx_0 = 0$, we see by (5.25) that $x'(x_0) = 0$ for all $x' \in X'$. Hence, $x_0 = 0$. In our new notation, (5.25) becomes

$$Jx(x') = x'(x), \quad x \in X,\ x' \in X'. \tag{5.26}$$

We also note that J is a bounded operator. In fact, we have

$$\|Jx\| = \sup \frac{|Jx(x')|}{\|x'\|} = \sup \frac{|x'(x)|}{\|x'\|} = \|x\|, \tag{5.27}$$

where the least upper bound is taken over all nonvanishing $x' \in X'$, and we have used (2.28).

In particular, we note that if $x \in N(A)$, then $Jx \in N(A'')$. This follows from

$$A''Jx(y') = Jx(A'y') = A'y'(x) = y'(Ax), \quad y' \in Y'. \tag{5.28}$$

Since J is one-to-one, we have by Lemma 5.8 that

$$\alpha(A) = \dim N(A) = \dim J[N(A)] \leq \dim N(A'') = \alpha(A'').$$

This gives (5.24).

We now can make good our promise made at the end of the last section. We shall prove

Theorem 5.16. *Assume that A in $B(X,Y)$ and B in $B(Y,Z)$ are such that $BA \in \Phi(X,Z)$. If $\beta(A) < \infty$, then $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$.*

Proof. Taking adjoints, we have $A'B' \in \Phi(Z',X')$ by Theorem 5.15. Moreover, $\alpha(A') = \beta(A) < \infty$. We can now apply Theorem 5.14 to conclude that $A' \in \Phi(Y',X')$ and $B' \in \Phi(Z',Y')$. In particular, $\alpha(B'') < \infty$. But by (5.24), we know that

$$\alpha(B) \leq \alpha(B'') < \infty.$$

Hence, the hypotheses of Theorem 5.14 are satisfied, and the conclusion follows. □

5.5. A special case

Suppose $K \in K(X)$ and $A = I - K$. Then, for each positive integer n, the operator A^n is in $\Phi(X) \equiv \Phi(X,X)$ and, hence, $\alpha(A^n)$ is finite. Since $N(A^n) \subset N(A^{n+1})$, we have $\alpha(A^n) \leq \alpha(A^{n+1})$, and we see that $\alpha(A^n)$ approaches either a finite limit or ∞. Which is the case? The answer is given by

Theorem 5.17. *If K is in $K(X)$ and $A = I - K$, then there is an integer $n \geq 1$ such that $N(A^n) = N(A^k)$ for all $k \geq n$.*

Proof. First we note that the theorem is true if there is an integer k such that $N(A^k) = N(A^{k+1})$. For if $j > k$ and $x \in N(A^{j+1})$, then $A^{j-k}x \in N(A^{k+1}) = N(A^k)$, showing that $x \in N(A^j)$. Hence, $N(A^j) = N(A^{j+1})$ for each $j > k$. This means that if the theorem were not true, then $N(A^k)$ would be a proper subspace of $N(A^{k+1})$ for each $k > 1$. This would mean, by Lemma 4.7, that, for each k, there would be a $z_k \in N(A^k)$ such that

$$\|z_k\| = 1, \quad d(z_k, N(A^{k-1})) > 1/2.$$

Since K is a compact operator in X, $\{Kz_k\}$ would have a convergent subsequence. But for $j > k$,

$$\|Kz_j - Kz_k\| = \|z_j - Az_j - z_k + Az_k\|$$
$$= \|z_j - (z_k - Az_k + Az_j)\| > 1/2,$$

since $A^{j-1}(z_k - Az_k + Az_j) = 0$. This shows that $\{Kz_k\}$ cannot have a convergent subsequence. Whence, a contradiction. This completes the proof. □

Note the similarity to the proof of Theorem 4.12 of the previous chapter.

For $A \in \Phi(X)$, set

$$r(A) = \lim_{n\to\infty} \alpha(A^n), \quad r'(A) = \lim_{n\to\infty} \beta(A^n).$$

If $A = I - K$, we have just shown that $r(A) < \infty$. Since $i(A) = 0$, we have $i(A^n) = ni(A) = 0$, and hence, $\beta(A^n) = \alpha(A^n)$. This shows that $r'(A) < \infty$ as well.

Are there other operators $A \in \Phi(X)$ with both $r(A)$ and $r'(A)$ finite? Let us examine such an operator a bit more closely. If $r(A) < \infty$, then there must be an integer $n \geq 1$ for which the conclusion of Theorem 5.17 must hold. To see this, note that $\alpha(A^k)$ is a nondecreasing sequence of integers bounded from above. Similarly, if $r'(A) < \infty$, there must be an integer $m > 1$ such that $N(A'^k) = N(A'^m)$ for $k \geq m$. If both $r(A) < \infty$ and $r'(A) < \infty$, let $j = \max[m, n]$. Then $\alpha(A^k) = \alpha(A^j)$, $\beta(A^k) = \beta(A^j)$ for $k > j$. Hence, $i(A^k) = \alpha(A^k) - \beta(A^k) = \alpha(A^j) - \beta(A^j) = i(A^j)$ for $k > j$. But $i(A^k) = ki(A)$. Hence, $(k - j)i(A) = 0$ for all $k > j$, showing that we must have $i(A) = 0$. But if this is the case, we must also have $m = n$.

To recapitulate, if $A \in \Phi(X)$, $r(A) < \infty$ and $r'(A) < \infty$, then $i(A) = 0$, and there is an $n \geq 1$ such that

$$\alpha(A^k) = \alpha(A^n), \;\; \beta(A^k) = \beta(A^n), \quad k \geq n. \tag{5.29}$$

Moreover, we claim that

$$N(A^n) \cap R(A^n) = \{0\}. \tag{5.30}$$

This follows from the fact that if x is in this intersection, we have, on one hand, that $A^n x = 0$ and, on the other, that there is an $x_1 \in X$ such that $x = A^n x_1$. Thus, $A^{2n} x_1 = 0$, or $x_1 \in N(A^{2n}) = N(A^n)$. Hence, $x = A^n x_1 = 0$. Next, let $x_1, \cdots, x_s$ be a basis for $N(A^n)$. Then there are bounded linear functionals $x_1', \cdots, x_s'$, which annihilate $R(A^n)$ and satisfy

$$x_j'(x_k) = \delta_{jk}, \quad 1 \leq j, k \leq s. \tag{5.31}$$

To see this, we note that for each j, the subspace spanned by $R(A^n)$ and the x_k with $k \neq j$ is closed (see Lemma 5.2). Hence, by Theorem 2.9, there is an $x'_j \in X'$ which satisfies $x'_j(x_j) = 1$ and annihilates $R(A^n)$ and all the x_k such that $k \neq j$.

Next, set

$$V x = \sum_{1}^{n} x'_k(x) x_k. \tag{5.32}$$

Since V is of finite rank (and, hence, compact), we see that $A^n + V \in \Phi(X)$ and $i(A^n + V) = 0$ (Theorem 5.10). We note that $\alpha(A^n + V) = 0$. If $(A^n + V)x = 0$, then $Vx \in R(A^n) \cap N(A^n)$. Thus, $Vx = 0$. But then $A^n x = 0$, which means that $x \in N(A^n)$. On the other hand, $Vx = 0$ implies $x'_k(x) = 0$ for each k, and this can only happen if $x = 0$. The upshot of all this is that $A^n + V$ has a bounded inverse E. Moreover, since $R(V) \subset N(A^n)$ and $R(A^n) \subset N(V)$, we have $A^n V = V A^n = 0$. Hence,

$$(A^n + V)V = V(A^n + V) = V^2,$$

showing that

$$V = EV^2 = V^2 E,$$

or

$$VE = EV^2 E = EV,$$

whence,

$$EA^n = A^n E = I - EV.$$

Since E is bounded, $EV \in K(X)$. We have proved the "necessary" part of

Theorem 5.18. *A necessary and sufficient condition that $A \in \Phi(X)$ with $r(A) < \infty$ and $r'(A) < \infty$ is that there exist an integer $n \geq 1$ and operators E in $B(X)$ and K in $K(X)$ such that*

$$EA^n = A^n E = I - K. \tag{5.33}$$

Proof. To prove the sufficiency of (5.33), set $W = A^n$. Then by Theorem 5.5, $W \in \Phi(X)$. Now by Theorem 5.17, there is an integer m such that

$$N[(I - K)^j] = N[(I - K)^m], \ R[(I - K)^j] = R[(I - K)^m], \quad j \geq m.$$

Thus,

$$N(W^j) \subset N(E^j W^j) = N[(EW)^j] = N[(I - K)^j] = N[(I - K)^m],$$

since E and W commute by (5.33). Hence, $\alpha(W^j)$ is bounded from above, and $r(A) = r(W) < \infty$. Similarly,

$$R(W^j) \supset R(W^j E^j) = R[(WE)^j] = R[(I - K)^j] = R[(I - K)^m],$$

showing that

$$N(W'^j) \subset N[(I - K')^m],$$

and hence, $\beta(W^j)$ is bounded from above. This gives $r'(A) = r'(W) < \infty$. The proof is complete. $\square$

"Not so fast!" you exclaim. "We have shown that $W \in \Phi(X)$, but we have not shown that $A \in \Phi(X)$." Right you are! This we do by proving

Lemma 5.19. *Let $A_1, \cdots, A_n$ be operators in B(X) which commute, and suppose that their product $A = A_1 \cdots A_n$ is in $\Phi(X)$. Then each A_k is in $\Phi(X)$.*

Proof. Clearly, $N(A_k) \subset N(A)$ and $R(A_k) \supset R(A)$. Thus, $\alpha(A_k)$ and $\beta(A_k)$ are both finite. Moreover, by (5.3), $X = R(A) \oplus Y_0$, where Y_0 is a finite-dimensional subspace of X. Since $R(A_k) \supset R(A)$, we see by Lemma 5.6 that $R(A_k)$ is closed. $\square$

In connection with Theorem 5.18, we wish to make the observation that if $A \in \Phi(X)$, $i(A) \geq 0$ and $r(A) < \infty$, then the conclusions of Theorem 5.18 hold. In fact, since

$$i(A^k) = ki(A) \geq 0,$$

we always have $\alpha(A^k) \geq \beta(A^k)$, showing that $r'(A) < \infty$, and Theorem 5.18 applies directly. A similar statement holds if we assume $i(A) \leq 0$, $r'(A) < \infty$.

5.6. Semi-Fredholm operators

One might wonder whether something can be said when some of the stipulations made for Fredholm operators are relaxed. The answer is yes, and we give some examples in this section. Let $\Phi_+(X, Y)$ denote the set of all $A \in B(X, Y)$ such that $R(A)$ is closed in Y and $\alpha(A) < \infty$. Such operators are called *semi-Fredholm*. If $A \in \Phi_+(X, Y)$, then (5.2) still holds. Let P be the (projection) operator defined by

$$P = \begin{cases} I & \text{on } N(A), \\ 0 & \text{on } X_0. \end{cases} \tag{5.34}$$

By Lemma 5.2, $P \in B(X)$. We also have

Lemma 5.20. *Assume that A is in B(X,Y) and that $\alpha(A) < \infty$. Let P be defined by (5.34). Then R(A) is closed in Y if and only if*

$$\|(I - P)x\| \leq \|Ax\|, \quad x \in X. \tag{5.35}$$

Proof. If $R(A)$ is closed, then the restriction of A to X_0 is one-to-one and has closed range. Hence, by Theorem 3.12,

$$\|x\| \leq C\|Ax\|, \quad x \in X_0. \tag{5.36}$$

But for any $x \in X$, $(I-P)x \in X_0$ and $A(I-P)x = Ax$. This proves (5.35). Conversely, assume (5.35) holds. If $y \in Y$ and $Ax_n \to y$, then $\{Ax_n\}$ is a Cauchy sequence in Y. By (5.35), $\{(I - P)x_n\}$ is a Cauchy sequence in X.

Since X is complete, there is an $x \in X$ such that $(I - P)x_n \to x$. Thus, $A(I - P)x_n \to Ax$. But $A(I - P)x_n = Ax_n \to y$. Hence, $Ax = y$, and $y \in R(A)$. Accordingly, $R(A)$ is closed in Y, and the proof is complete. $\square$

Suppose $A \in \Phi_+(X, Y)$. Then by (5.35),

$$\|x\| \le C\|Ax\| + \|Px\|.$$

Set

$$|x| = \|Px\|. \tag{5.37}$$

Then we have

$$|\alpha x| = |\alpha| \cdot |x|, \tag{5.38}$$

$$|x + y| \le |x| + |y|, \tag{5.39}$$

but we do not have $|x| = 0$ only if $x = 0$. Thus, $|x|$ is not a norm. A functional satisfying (5.38) and (5.39) is called a *seminorm.* The seminorm given by (5.37) also has the following property. If $\{x_n\}$ is a sequence of elements of X such that $\|x_n\| \le C$, then it has a subsquence which is a Cauchy sequence in the seminorm $|\cdot|$(this follows from the compactness of P). A seminorm having this property is said to be *compact relative to* (or *completely continuous with respect to*) the norm of X. Thus, we have shown that if $A \in \Phi_+(X, Y)$, then there is a seminorm compact relative to the norm of X such that

$$\|x\| \le C\|Ax\| + |x|, \quad x \in X. \tag{5.40}$$

The converse is also true. In fact, we have

Theorem 5.21. *Suppose A is in $B(X,Y)$. Then $A \in \Phi_+(X, Y)$ if and only if there is a seminorm $|\cdot|$ compact relative to the norm of X such that (5.40) holds.*

Proof. We have proved the "only if" part. To prove the "if" part, assume that (5.40) holds. Then $\alpha(A) < \infty$. To see this, let $\{x_n\}$ be a sequence of elements in $N(A)$ such that $\|x_n\| = 1$. Then there is a subsequence (which we assume is the whole sequence) such that

$$|x_n - x_m| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Hence, by (5.40),

$$\|x_n - x_m\| \le |x_n - x_m| \longrightarrow 0.$$

Thus, $\{x_n\}$ converges. But this shows that $N(A)$ is finite dimensional (Theorem 4.6). Next, let P be defined by (5.34). Then we claim that (5.35) holds. For if it did not, there would be a sequence $\{x_n\}$ such that

$$\|(I - P)x_n\| = 1, \quad Ax_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Set $z_n = (I - P)x_n$. Then $z_n \in X_0$, and

$$\|z_n\| = 1, \quad Az_n \longrightarrow 0.$$

Since the sequence $\{z_n\}$ is bounded, it has a subsequence (assumed to be the whole sequence) such that

$$|z_n - z_m| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty.$$

Hence, by (5.40),

$$\|z_n - z_m\| \le C\|A(z_n - z_m)\| + |z_n - z_m| \longrightarrow 0.$$

Since X_0 is closed, there is a $z \in X_0$ such that $z_n \to z$. Hence, $Az_n \to Az$. But $Az_n \to 0$. This shows that $z \in N(A)$. On the other hand, $z \in X_0$. The only way these statements can be reconciled is if $z = 0$. But this gives further trouble, since $\|z\| = \lim \|z_n\| = 1$. Our only way out is to conclude that the assumption that (5.35) does not hold cannot be true. We now make use of Lemma 5.20. □

Now we can prove the counterparts of Theorems 5.10 and 5.11.

Theorem 5.22. *If $A \in \Phi_+(X,Y)$ and K is in $K(X,Y)$, then $A + K \in \Phi_+(X,Y)$ and (5.13) holds.*

Proof. By Theorem 5.21,

$$\|x\| \le C\|(A + K)x\| + |x| + C\|Kx\|, \quad x \in X.$$

Set

$$|x|_0 = |x| + C\|Kx\|.$$

Then, $|x|_0$ is a seminorm which is compact relative to the norm of X. Thus, $A + K \in \Phi_+(X,Y)$ by Theorem 5.21. If $A + K \in \Phi(X,Y)$, then $A = (A + K) - K \in \Phi(X,Y)$ by Theorem 5.10, and (5.13) holds. If $A + K \notin \Phi(X,Y)$, i.e., if $i(A + K) = -\infty$, then the same is true of A by Theorem 5.10. In this case, both sides of (5.13) are equal to $-\infty$. □

We also have

Theorem 5.23. *Suppose $A \in \Phi_+(X,Y)$. Then there is a number $\eta > 0$ such that for each T in $B(X,Y)$ satisfying $\|T\| < \eta$, we have $A + T \in \Phi_+(X,Y)$,*

$$\alpha(A + T) \le \alpha(A), \quad \beta(A + T) \le \beta(A), \tag{5.41}$$

and (5.15) holds.

Proof. By Lemma 5.20,

$$\|x\| \le C\|(A + T)x\| + C\|Tx\| + \|Px\|, \quad x \in X.$$

Take $\eta = 1/2C$. Then

$$\|x\| \leq C\|(A+T)x\| + \|Px\| + \frac{1}{2}\|x\|,$$

or

$$\|x\| \leq 2C\|(A+T)x\| + 2\|Px\|.$$

This immediately shows that $A+T \in \Phi_+(X,Y)$(Theorem 5.21). Moreover, since $Px = 0$ for $x \in X_0$, we have

$$\|x\| \leq 2C\|(A+T)x\|, \quad x \in X_0,$$

which shows that $N(A+T) \cap X_0 = \{0\}$. Since $X = X_0 \oplus N(A)$, we see by Lemma 5.12 that the first inequality in (5.41) holds. If $\beta(A) = \infty$, then the second inequality holds trivially. Otherwise, $A \in \Phi(X,Y)$, and there is an $\eta > 0$ such that $A+T \in \Phi(X,Y)$ and (5.15) holds (Theorem 5.11). Thus,

$$\beta(A+T) = \alpha(A+T) - \alpha(A) + \beta(A) \leq \beta(A).$$

It remains to prove (5.15) for the general case. Assume first that $N(A) = \{0\}$. Then there is a constant C_0 such that

$$\|x\| \leq C_0\|Ax\|, \quad x \in X$$

(Theorem 3.12). Take $\eta < 1/3C_0$. Then

$$\|x\| \leq C_0\|Ax + Tx\| + C_0\|T\| \cdot \|x\|,$$

and consequently,

$$\|x\| \leq \frac{3}{2}C_0\|(A+T)x\|, \quad x \in X.$$

This implies that $R(A+T)$ is closed and that (5.41) holds (in particular, $\alpha(A+T) = 0$). Reversing the procedure, we have

$$\|x\| \leq \frac{3}{2}C_0(\|(A+T-T)x\| + \|T\| \cdot \|x\|), \quad x \in X,$$

which gives

$$\|x\| \leq 3C_0\|Ax\|, \quad x \in X.$$

Of course, we knew that this inequality held before (even without the factor 3), but we wanted to see if the value of η that we picked allowed us to work back from $A+T$ to A. As we see, it does, and we can conclude that

$$\alpha(A) \leq \alpha(A+T), \quad \beta(A) \leq \beta(A+T),$$

from which we conclude in conjunction with (5.41) that

$$\alpha(A+T) = \alpha(A) = 0, \quad \beta(A+T) = \beta(A).$$

Thus, in this case (5.15) holds. In the general case, we can find a closed subspace X_0 of X such that

$$X = N(A) \oplus X_0$$

(cf. Lemma 5.1). We let A_0 be the restriction of A to X_0. Then

$$A_0 \in \Phi_+(X_0, Y),\ N(A_0) = \{0\},\ \alpha(A_0) = 0,$$

and

$$R(A_0) = R(A),\ \beta(A_0) = \beta(A),\ i(A_0) = i(A) - \alpha(A).$$

If T_0 is the restriction of T to X_0, then we see that

$$\alpha(A_0 + T_0) = 0,\ i(A_0 + T_0) = i(A_0).$$

Since

$$A(z + x) = Ax,\ (A + T)(z + x) = Tz + (A + T)x, \quad z \in N(A),\ x \in X_0,$$

the following lemma will allow us to conclude that

$$\alpha(A + T) \le \alpha(A_0 + T_0) + \alpha(A) = \alpha(A),$$

$$i(A) = i(A_0) + \alpha(A),$$

$$i(A + T) = i(A_0 + T_0) + \alpha(A) = i(A).$$

This will complete the proof. □

Lemma 5.24. *Assume* $A \in \Phi_+(X, Y),\ \tilde{X} = N \oplus X,\ n = \dim N < \infty,\ \tilde{A} \in B(\tilde{X}, Y)$ *satisfies*

$$\tilde{A}(z + x) = Cz + Ax, \quad z \in N,\ x \in X,$$

where C is in B(N,Y). Then $\tilde{A} \in \Phi_+(\tilde{X}, Y)$, *and*

$$\alpha(\tilde{A}) \le \alpha(A) + n, \quad i(\tilde{A}) = i(A) + n.$$

Proof. We may assume that $n = 1$, i.e., $N = \{z_0\}$. There are three cases.

Case 1. $Cz_0 = 0$. In this case we have $N(\tilde{A}) = N(A) \oplus \{z_0\},\ \alpha(\tilde{A}) = \alpha(A) + 1,\ R(\tilde{A}) = R(A),\ \beta(\tilde{A}) = \beta(A),\ i(\tilde{A}) = i(A) + 1$.

Case 2. $Cz_0 \ne 0,\ Cz_0 \notin R(A)$. For this case we have $N(\tilde{A}) = N(A)$, $\alpha(\tilde{A}) = \alpha(A),\ R(\tilde{A}) = R(A) \oplus \{Cz_0\},\ \beta(\tilde{A}) = \beta(A) - 1,\ i(\tilde{A}) = i(A) + 1$.

Case 3. $Cz_0 \ne 0,\ Cz_0 \in R(A)$. In this case there is an element $x_0 \in X$ such that $Ax_0 = Cz_0$. Thus,

$$\tilde{A}(\lambda z_0 + x) = \lambda C z_0 + Ax.$$

Consequently, $N(\tilde{A}) = N(A) \oplus \{z_0 - x_0\},\ \alpha(\tilde{A}) = \alpha(A) + 1,\ R(\tilde{A}) = R(A),\ \beta(\tilde{A}) = \beta(A),\ i(\tilde{A}) = i(A) + 1$. We see that in all cases the conclusions of the lemma hold. □

We end this section by making the following trivial but sometimes useful observations.

Proposition 5.25. *$A \in \Phi(X,Y)$ if and only if $A \in \Phi_+(X,Y)$ and $A' \in \Phi_+(Y',X')$.*

Proof. By Theorem 5.15, $A \in \Phi(X,Y)$ implies $A' \in \Phi(Y',X')$, so that the "only if" part is immediate. Moreover, if $A \in \Phi_+(X,Y)$, then $R(A)$ is closed in Y and $\alpha(A) < \infty$. If, in addition, $A' \in \Phi_+(Y',X')$, then $\beta(A) = \alpha(A') < \infty$. Thus, $A \in \Phi(X,Y)$. $\square$

Theorem 5.26. *If $A \in \Phi_+(X,Y)$ and $B \in \Phi_+(Y,Z)$, then $BA \in \Phi_+(X,Z)$, and (5.9) holds.*

Proof. By Theorem 5.21,

$$\|x\| \le C_1\|Ax\| + |x|_1, \quad x \in X,$$

$$\|y\| \le C_2\|By\| + |y|_2, \quad y \in Y,$$

where the seminorms $|\cdot|_1$ and $|\cdot|_2$ are compact relative to the norms of X and Y, respectively. Thus

$$\|Ax\| \le C_2\|BAx\| + |Ax|_2, \quad x \in X,$$

and hence,

$$\|x\| \le C_1C_2\|BAx\| + C_1|Ax|_2 + |x|_1, \quad x \in X.$$

It is easily checked that the seminorm

$$|x|_3 = C_1|Ax|_2 + |x|_1$$

is compact relative to the norm of X. Thus, $BA \in \Phi_+(X,Z)$ by Theorem 5.21. Finally, we note that if $i(BA)$ is finite, i.e., if $BA \in \Phi(X,Z)$, then we must have $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$ by Theorem 5.14 with (5.9) holding by Theorem 5.7. On the other hand, if $i(BA) = -\infty$, then either $A \notin \Phi(X,Y)$ or $B \notin \Phi(Y,Z)$ (or both). This means that either $i(A) = -\infty$ or $i(B) = -\infty$(or both). In this case, both sides of (5.9) are equal to $-\infty$. $\square$

We note that the proof of (5.23) depends only on the fact that $\alpha(A) < \infty$. Thus, we can extend Theorem 5.15 to Φ_+ operators. We have

Theorem 5.27. *If $A \in \Phi_+(X,Y)$, then*

$$i(A') = \alpha(A') - \alpha(A'') = -i(A).$$

In particular, $i(A') = \infty$ if $A \notin \Phi(X,Y)$.

We define $\Phi_-(X,Y)$ to be the set of those $A \in B(X,Y)$ such that $A' \in \Phi_+(Y',X')$. We have

Theorem 5.28. *If $A \in \Phi_-(X,Y)$ and K is in $K(X,Y)$, then $A+K \in \Phi_-(X,Y)$ and (5.13) holds.*

Proof. Now, $A' \in \Phi_+(Y',X')$ by hypothesis, and $K' \in K(Y',X')$ by Theorem 4.15. Thus, $A'+K' \in \Phi_+(Y',X')$ and

$$i(A'+K') = i(A')$$

(Theorem 5.22). Hence, $A+K \in \Phi_-(X,Y)$ by definition, and (5.13) holds by Theorem 5.15. □

Theorem 5.29. *If $A \in \Phi_-(X,Y)$, then there is an $\eta > 0$ such that $\|T\| < \eta$ for T in $B(X,Y)$ implies that $A+T \in \Phi_-(X,Y)$, and (5.15), (5.41) hold.*

Proof. We know that $T' \in B(Y',X')$ and that $\|T'\| < \eta$ (Theorem 3.3). Hence, $A'+T' \in \Phi_+(Y',X')$ for η sufficiently small, and

$$i(A'+T') = i(A'), \quad \alpha(A'+T') \le \alpha(A'), \quad \beta(A'+T') \le \beta(A')$$

(Theorem 5.23). This translates into (5.15), (5.41), and the conclusion follows. □

As a counterpart of Theorem 5.26, we have

Theorem 5.30. *If $A \in \Phi_-(X,Y)$ and $B \in \Phi_-(Y,Z)$, then $BA \in \Phi_-(X,Z)$, and (5.9) holds.*

Proof. By definition, $A' \in \Phi_+(Y',X')$ and $B' \in \Phi_+(Z',Y')$. Thus, $(BA)' = A'B' \in \Phi_+(Z',X')$ and

$$i[(BA)'] = i(A') + i(B')$$

by Theorem 5.26. Hence, $BA \in \Phi_-(X,Z)$, and (5.9) holds. □

5.7. Products of operators

We also have the following

Theorem 5.31. *If A is in $B(X,Y)$, B is in $B(Y,Z)$ and $BA \in \Phi_-(X,Z)$, then $B \in \Phi_-(Y,Z)$.*

Proof. Since $\dim R(BA)^\circ < \infty$, there is a subspace Z_0 such that $\dim Z_0 < \infty$ and $Z = R(BA) \oplus Z_0$ (Lemma 5.3). Since $R(B) \supset R(BA)$, we know that $R(B)$ is closed (Lemma 5.6) and $R(B)^\circ \subset R(BA)^\circ$. Thus, $\dim R(B)^\circ < \infty$. Consequently, $B \in \Phi_-(Y,Z)$. □

In contrast, we have

Theorem 5.32. *If A is in $B(X,Y)$, B is in $B(Y,Z)$ and $BA \in \Phi_+(X,Z)$, then $A \in \Phi_+(X,Y)$.*

Proof. We merely note that $A'B' \in \Phi_-(Z',X')$. Theorem 5.31 now implies that $A' \in \Phi_-(Y',X')$, which means that $A \in \Phi_+(X,Y)$. □

Two closed subspaces X_1, X_2 of a Banach space X are called *complementary* when $X_1 \cap X_2 = \{0\}$ and $X = X_1 \oplus X_2$. Either subspace is called a *complement* of the other. We say that a subspace $X_1 \subset X$ is *complemented* if it has a complement. Some Banach spaces contain subspaces which are not complemented.

We note

Lemma 5.33. *If X_1 is a closed complemented subspace of a Banach space X, then there is a bounded projection P on X with $R(P) = X_1$.*

Proof. Let X_2 be a closed complement of X_1 in X. Then each $x \in X$ is of the form

$$x = x_1 + x_2, \quad x_k \in X_k.$$

Define $Px = x_1$. Then P is a projection, and it is easily verified that it is a closed operator. Hence, it is bounded by the closed graph theorem (Theorem 3.10). □

Theorem 5.34. *If $A \in \Phi_+(X,Y)$ and $R(A)$ is complemented in Y, then there is an $A_0 \in B(Y,X)$ such that $A_0A \in \Phi(X)$.*

Proof. Let P be a bounded projection from Y to $R(A)$. There is a closed subspace $X_0 \in X$ such that

$$X = X_0 \oplus N(A).$$

Then A has a bounded inverse $\hat{A}$ from $R(A)$ to X_0. Let $A_0 = \hat{A}P$. Then $A_0 \in B(Y,X)$, and

$$A_0A = \begin{cases} I & \text{on } X_0, \\ 0 & \text{on } N(A). \end{cases}$$

Thus, $A_0A \in \Phi(X)$. □

Theorem 5.35. *If $A \in \Phi_-(X,Y)$ with $N(A)$ complemented, then there is an $A_0 \in B(Y,X)$ such that $AA_0 \in \Phi(Y)$.*

Proof. There is a finite dimensional subspace $Y_0 \in Y$ such that

$$Y = R(A) \oplus Y_0.$$

Let P be the bounded projection onto $R(A)$ which vanishes on Y_0, and define A_0 as above. Then

$$AA_0 = \begin{cases} I & \text{on } R(A), \\ 0 & \text{on } Y_0. \end{cases}$$

Thus, $AA_0 \in \Phi(Y)$. □

We also have

Theorem 5.36. *If A is in $B(X,Y)$, B is in $B(Y,Z)$ and $BA \in \Phi(X,Z)$, then $A \in \Phi_+(X,Y)$, and $B \in \Phi_-(Y,Z)$. Moreover, $R(A)$ and $N(B)$ are complemented.*

Proof. The first statement follows from Theorems 5.31 and 5.32. Consequently, there is a closed subspace $X_0 \subset X$ such that

$$X = X_0 \oplus N(A).$$

Let

$$Y_1 = R(A) \cap N(B), \quad X_1 = A^{-1}(Y_1) \cap X_0.$$

Since $X_1 \subset N(BA)$, $\dim X_1 \leq \alpha(BA) < \infty$. Since A is one-to-one from X_1 onto Y_1, we see that $\dim Y_1 = \dim X_1 < \infty$. Hence, there are subspaces $Y_2 \subset R(A)$, $Y_3 \subset N(B)$ such that

$$R(A) = Y_1 \oplus Y_2, \quad N(B) = Y_1 \oplus Y_3.$$

We also know that

$$Z = R(BA) \oplus Z_0,$$

where $\dim Z_0 < \infty$. Let $Z_4 = R(B) \cap Z_0$. Then

$$R(B) = R(BA) \oplus Z_4,$$

and there is a subspace $Z_5 \subset Z_0$ such that $Z_0 = Z_4 \oplus Z_5$. Consequently

$$Z = R(BA) \oplus Z_4 \oplus Z_5, \quad R(B) = R(BA) \oplus Z_4.$$

Let $z_1, \cdots, z_n$ be a basis for Z_4. Then there are $y_1, \cdots, y_n \in Y$ such that

$$By_j = z_j, \quad 1 \leq j \leq n.$$

Let Y_4 be the subspace of Y spanned by $y_1, \cdots, y_n$. Then $\dim Y_4 \leq \dim Z_4 < \infty$. We note that

$$Y_2 \cap N(B) = \{0\}, \quad Y_4 \cap N(B) = \{0\}, \quad Y_2 \cap Y_4 = \{0\}.$$

Thus,

$$N(B) \cap [Y_2 \oplus Y_4] = \{0\}.$$

Since $\dim Y_4 < \infty$, the subspace $Y_2 \oplus Y_4$ is closed (Lemma 5.2). Let y be any element of Y. Then $By \in R(B) = R(BA) \oplus Z_4$. Hence, there are

$z_2 \in R(BA),\ z_4 \in Z_4$ such that $By = z_2 + z_4$. There are $y_2 \in Y_2,\ y_4 \in Y_4$ such that $By_2 = z_2,\ By_4 = z_4$. Then

$$B(y - y_2 - y_4) = By - z_2 - z_4 = 0.$$

Thus, $y - y_2 - y_4 \in N(B)$, and consequently

$$Y = Y_2 \oplus Y_4 \oplus N(B).$$

This shows that $N(B)$ is complemented. Moreover,

$$Y = Y_2 \oplus Y_4 \oplus Y_1 \oplus Y_3 = Y_3 \oplus Y_4 \oplus R(A),$$

showing that $R(A)$ is also complemented. □

As a consequence of the above, we have

Theorem 5.37. *An operator A in $B(X,Y)$ is in $\Phi_+(X,Y)$ with $R(A)$ complemented if and only if there is an operator $A_0 \in B(Y,X)$ such that $A_0A \in \Phi(X)$. It is in $\Phi_-(X,Y)$ with $N(A)$ complemented if and only if there is an operator $A_0 \in B(Y,X)$ such that $AA_0 \in \Phi(Y)$.*

5.8. Problems

(1) Let X be a normed vector space, and let Y be a Banach space. Show that if $A \in B(X,Y)$ and $A' \in K(Y',X')$, then $A \in K(X,Y)$.

(2) Show that $I - A$ is a Fredholm operator if there is a nonnegative integer k such that A^k is compact.

(3) Suppose M and N are closed subspaces of a Banach space X, and $X = M \oplus N$. Show that $X' = M^\circ \oplus N^\circ$.

(4) Let X be a normed vector space and let $F \neq 0$ be any element of X'. Show that there is a one-dimensional subspace M of X such that $X = N(F) \oplus M$.

(5) If A is a linear operator on a finite dimensional space X, show that $\alpha(A) = \beta(A)$.

(6) Prove the converse of Lemma 5.3. If R is closed and $X = R \oplus M$ with $\dim M = n$, then $\dim R^\circ = n$.

(7) Prove Lemma 5.2 for X a Hilbert space and M assumed only closed.

(8) Show that $A \in \Phi(X)$ if there is a positive integer k such that $A^k - I$ is compact.

(9) If X_1, X_2 are orthogonal subspaces of a Hilbert space X, show that $X_1 \oplus X_2$ is closed in X if and only if X_1 and X_2 are both closed.

(10) If X_1, X_2 are subspaces of a Banach space X with $\dim X_1 < \infty$ and X_2 closed, show that $X_1 + X_2$ is closed in X. We define $X_1 + X_2$ as the set of sums of the form $x_1 + x_2$, where $x_k \in X_k$.

(11) If D, N are subspaces of a Hilbert space X with $\dim N < \infty$ and D dense, show that $D \cap N^\circ$ is dense in N°.

(12) If M, N are closed subspaces of a Hilbert space X and $M \cap N^\circ = \{0\}$, show that $\dim M \leq \dim N$.

(13) Let W be a subset of a Banach space X such that

$$\sup_{x \in W} |x'(x)| < \infty$$

for each $x' \in X'$. Show that W is bounded.

(14) Let M, N be subspaces of a Banach space X, and assume that $X = M + N,\ M \cap N = \{0\}$. Define

$$P = \begin{cases} I, & \text{on } M, \\ 0, & \text{on } N. \end{cases}$$

Show that $P \in B(X)$ if and only if M, N are both closed.

(15) Let X, Y be normed vector spaces with $\dim X < \infty$, and let $\{A_k\}$ be a sequence of operators in $B(X, Y)$ such that $A_k x \to 0$ for each $x \in X$. Show that $\|A_k\| \to 0$.

(16) For X, Y Banach spaces and $A \in B(X, Y)$, let

$$|A|_c = \inf_{K \in K(X,Y)} \|A + K\|.$$

If $A \in \Phi(X, Y)$, show that there is an $\eta > 0$ such that $|T|_c < \eta$ for $T \in B(X, Y)$ implies that $A + T \in \Phi(X, Y)$ with $i(A + T) = i(A)$.

(17) For X, Y Banach spaces, $A \in \Phi(X, Y)$ and $B \in B(X, Y)$, show that there is an $\eta > 0$ such that $\alpha(A - \lambda B)$ is constant for $0 < |\lambda| < \eta$.

Chapter 6

SPECTRAL THEORY

6.1. The spectrum and resolvent sets

Let X be a Banach space, and suppose $K \in K(X)$. If λ is a nonzero scalar, then

$$\lambda I - K = \lambda(I - \lambda^{-1}K) \in \Phi(X). \tag{6.1}$$

For an arbitrary operator $A \in B(X)$, the set of all scalars λ for which $\lambda I - A \in \Phi(X)$ is called the Φ-set of A and is denoted by Φ_A. Thus, (6.1) gives

Theorem 6.1. *If X is a Banach space and K is in $K(X)$, then Φ_K contains all scalars $\lambda \neq 0$.*

Before proceeding further, let us take the ingenious step of, hence-forth, denoting the operator λI by λ. I am sure you will appreciate the great efficiencies effected by this bold move. Throughout this chapter, we shall assume that X is a Banach space.

We can also say something about $\alpha(K - \lambda)$ for $\lambda \in \Phi_K$.

Theorem 6.2. *Under the hypothesis of Theorem 6.1, $\alpha(K - \lambda) = 0$ except for, at most, a denumerable set S of values of λ. The set S depends on K and has 0 as its only possible limit point. Moreover, if $\lambda \neq 0$ and $\lambda \notin S$, then $\alpha(K - \lambda) = 0$, $R(K - \lambda) = X$ and $K - \lambda$ has an inverse in $B(X)$.*

Proof. In order to prove the first part we show that for any $\varepsilon > 0$, there is at most a finite number of values of λ for which $|\lambda| \geq \varepsilon$ and $\alpha(K - \lambda) \neq 0$. To see this, suppose $\{\lambda_n\}$ were an infinite sequence of distinct scalars such that $|\lambda_n| > \varepsilon$ and $\alpha(K-\lambda_n) \neq 0$. Let $x_n \neq 0$ be any element such that $(K-\lambda_n)x_n = 0$. We claim that, for any n, the elements $x_1, \cdots, x_n$ are linearly independent. Otherwise, there would be an m such that

$$x_{m+1} = \alpha_1 x_1 + \cdots + \alpha_m x_m, \tag{6.2}$$

while the $x_1, \cdots, x_m$ are linearly independent. By (6.2),

$$Kx_{m+1} = \alpha_1 Kx_1 + \cdots + \alpha_m Kx_m = \alpha_1\lambda_1 x_1 + \cdots + \alpha_m \lambda_m x_m.$$

On the other hand,

$$Kx_{m+1} = \lambda_{m+1}x_{m+1} = \lambda_{m+1}\alpha_1 x_1 + \cdots + \lambda_{m+1}\alpha_m x_m,$$

whence,

$$\alpha_1(\lambda_1 - \lambda_{m+1})x_1 + \cdots + \alpha_m(\lambda_m - \lambda_{m+1})x_m = 0. \tag{6.3}$$

Since $x_1, \cdots, x_m$ are independent and the λ_n are distinct, (6.3) implies $\alpha_1 = \cdots = \alpha_m = 0$, which is impossible because $x_{m+1} \neq 0$.

Next, let M_n be the subspace spanned by $x_1, \cdots, x_n$. Then M_n is a proper subset of M_{n+1} for each n. Hence, by Riesz's Lemma (Lemma 4.7), there is a $z_n \in M_n$ such that

$$\|z_n\| = 1 \quad \text{and} \quad d(z_n, M_{n-1}) > 1/2. \tag{6.4}$$

Clearly, K maps M_n into itself. If

$$x = \sum_1^n \alpha_j x_j,$$

then

$$Kx = \sum_1^n \alpha_j \lambda_j x_j.$$

Moreover,

$$(K-\lambda_n)x = \sum_1^{n-1} \alpha_j(\lambda_j - \lambda_n)x_j.$$

Hence, $K - \lambda_n$ maps M_n into M_{n-1}. Now if $n > m$, we have

$$Kz_n - Kz_m = (K-\lambda_n)z_n + \lambda_n z_n - Kz_m = \lambda_n[z_n - \lambda_n^{-1}(Kz_m - (K-\lambda_n)z_n)].$$

But $Kz_m \in M_m \subset M_{n-1}$, while $(K-\lambda_n)z_n \in M_{n-1}$. Hence,

$$\|Kz_n - Kz_m\| \geq |\lambda_n|/2 \geq \varepsilon/2,$$

showing that $\{Kz_n\}$ can have no convergent subsequence. This contradicts the fact that K is a compact operator. Thus, there cannot exist an infinite sequence $\{\lambda_n\}$ of distinct values such that $\alpha(K-\lambda_n) \neq 0$ and $|\lambda_n| > \varepsilon$. Now suppose $\lambda \neq 0$ and $\lambda \notin S$. Then $\alpha(K-\lambda) = 0$. From (6.1) and Theorem 4.12, we see that $R(K-\lambda) = X$. All we need now is the bounded inverse theorem (Theorem 3.8) to conclude that $K-\lambda$ has an inverse in $B(X)$. This completes the proof. □

For any operator $A \in B(X)$, a scalar λ for which $\alpha(A-\lambda) \neq 0$ is called an *eigenvalue* of A. Any element $x \neq 0$ of X such that $(A-\lambda)x = 0$ is called an *eigenvector* (or *eigenelement*). The points λ for which $(A-\lambda)$ has a bounded inverse in $B(X)$ comprise the *resolvent set* $\rho(A)$ of A. It is the set of those λ such that $\alpha(A-\lambda) = 0$ and $R(A-\lambda) = X$ (Theorem 3.8). The *spectrum* $\sigma(A)$ of A consists of all scalars not in $\rho(A)$. The set of eigenvalues of A is sometimes called the *point spectrum* of A and is denoted by $P\sigma(A)$. In terms of the above definitions, Theorem 6.2 states that

(1) The point spectrum of K consists of, at most, a denumerable set S having 0 as its only possible limit point.

(2) All other points $\lambda \neq 0$ belong to the resolvent set of K.

We are now going to examine the sets Φ_A, $\rho(A)$ and $\sigma(A)$ for arbitrary $A \in B(X)$.

Theorem 6.3. *Φ_A and $\rho(A)$ are open sets. Hence, $\sigma(A)$ is a closed set.*

Proof. If $\lambda_0 \in \Phi_A$, then by definition $A - \lambda_0 \in \Phi(X)$. By Theorem 5.11, there is a constant $\eta > 0$ such that $|\mu| < \eta$ implies that $A - \lambda_0 - \mu \in \Phi(X)$ and

$$i(A-\lambda_0-\mu) = i(A-\lambda_0), \quad \alpha(A-\lambda_0-\mu) \leq \alpha(A-\lambda_0).$$

Thus, if λ is any scalar such that $|\lambda-\lambda_0| < \eta$, then $A-\lambda \in \Phi(X)$ and

$$i(A-\lambda) = i(A-\lambda_0), \quad \alpha(A-\lambda) \leq \alpha(A-\lambda_0). \tag{6.5}$$

In particular, this shows that $\lambda \in \Phi_A$. Thus, Φ_A is an open set. To show the same for $\rho(A)$, we make the trivial observation that $\lambda \in \rho(A)$ if and only if

$$\lambda \in \Phi_A, \quad i(A-\lambda) = 0 \quad \text{and} \quad \alpha(A-\lambda) = 0.$$

Thus, if $\lambda_0 \in \rho(A)$ and $|\lambda - \lambda_0| < \eta$, then by (6.5), $i(A-\lambda) = \alpha(A-\lambda) = 0$. Hence $\lambda \in \rho(A)$, and the proof is complete. □

Does every operator $A \in B(X)$ have points in its resolvent set? Yes; in fact, we have

Theorem 6.4. *For A in $B(X)$, set*

$$r_\sigma(A) = \inf_n \|A^n\|^{1/n}. \tag{6.6}$$

Then $\rho(A)$ contains all scalars λ such that $|\lambda| > r_\sigma(A)$.

This theorem is an immediate consequence of the following two lemmas:

Lemma 6.5. *If* $|\lambda| > \|A\|$, *then* $\lambda \in \rho(A)$.

Lemma 6.6. *If* $\lambda \in \sigma(A)$, *then for each* n *we have* $\lambda^n \in \sigma(A^n)$.

Using these lemmas, we give the proof of Theorem 6.4.

Proof. If $\lambda \in \sigma(A)$, then for each n, we have $\lambda^n \in \sigma(A^n)$ (Lemma 6.6). Hence, we must have $|\lambda^n| \leq \|A^n\|$, for otherwise, we would have $\lambda^n \in \rho(A^n)$ (Lemma 6.5). Thus,

$$|\lambda| \leq \|A^n\|^{1/n}, \quad \lambda \in \sigma(A),\ n = 1, 2, \cdots. \tag{6.7}$$

Consequently, $|\lambda| \leq r_\sigma(A)$ for all $\lambda \in \sigma(A)$. Hence, all λ such that $|\lambda| > r_\sigma(A)$ must be in $\rho(A)$. □

Now we give the proof of Lemma 6.5.

Proof. We have

$$A - \lambda = -\lambda(I - \lambda^{-1}A). \tag{6.8}$$

Now if $|\lambda| > \|A\|$, then the norm of the operator $\lambda^{-1}A$ is less than one. Hence, $I - \lambda^{-1}A$ has an inverse in $B(X)$ (Theorem 1.1). The same is, therefore, true for $A - \lambda$. □

Now we prove Lemma 6.6.

Proof. Suppose $\lambda^n \in \rho(A^n)$. Now,

$$A^n - \lambda^n = (A - \lambda)B = B(A - \lambda), \tag{6.9}$$

where

$$B = A^{n-1} + \lambda A^{n-2} + \cdots + \lambda^{n-2}A + \lambda^{n-1}.$$

Thus, $\alpha(A - \lambda) = 0$ and $R(A - \lambda) = X$. This means that $\lambda \in \rho(A)$, contrary to the assumption. This completes the proof. □

Lemma 6.6 has an interesting generalization. Let $p(t)$ be a polynomial of the form

$$p(t) = \sum_0^n a_k t^k.$$

Then for any operator $A \in B(X)$, we define the operator

$$p(A) = \sum_0^n a_k A^k,$$

where we take $A^0 = I$. We have

Theorem 6.7. *If* $\lambda \in \sigma(A)$, *then* $p(\lambda) \in \sigma(p(A))$ *for any polynomial* $p(t)$.

Proof. Since λ is a root of $p(t) - p(\lambda)$, we have

$$p(t) - p(\lambda) = (t - \lambda)q(t),$$

where $q(t)$ is a polynomial with real coefficients. Hence,

$$p(A) - p(\lambda) = (A - \lambda)q(A) = q(A)(A - \lambda). \tag{6.10}$$

Now, if $p(\lambda)$ is in $\rho(p(A))$, then (6.10) shows that $\alpha(A - \lambda) = 0$ and $R(A - \lambda) = X$. This means that $\lambda \in \rho(A)$, and the theorem is proved. □

A symbolic way of writing Theorem 6.7 is

$$p(\sigma(A)) \subset \sigma(p(A)). \tag{6.11}$$

Note that, in general, there may be points in $\sigma(p(A))$ which may not be of the form $p(\lambda)$ for some $\lambda \in \sigma(A)$. As an example, consider the operator on $\mathbb{R}^2$ given by

$$A(\alpha_1, \alpha_2) = (-\alpha_2, \alpha_1).$$

A has no spectrum; $A - \lambda$ is invertible for all real λ. However, A^2 has -1 as an eigenvalue. What is the reason for this? It is simply that our scalars are real. Consequently, imaginary numbers cannot be considered as eigenvalues. We shall see later that in order to obtain a more complete theory, we shall have to consider Banach spaces with complex scalars.

Another question is whether every operator $A \in B(X)$ has points in its spectrum. An affirmative answer will be given in Section 9.1.

6.2. The spectral mapping theorem

Suppose we want to solve an equation of the form

$$p(A)x = y, \quad x, y \in X, \tag{6.12}$$

where $p(t)$ is a polynomial and $A \in B(X)$. If 0 is not in the spectrum of $p(A)$, then $p(A)$ has an inverse in $B(X)$ and, hence, (6.12) can be solved for all $y \in X$. So a natural question to ask is: What is the spectrum of $p(A)$? By Theorem 6.7 we see that it contains $p(\sigma(A))$, but by the remark at the end of the preceding section it can contain other points. What a pity! If it were true that

$$p(\sigma(A)) = \sigma(p(A)), \tag{6.13}$$

then we could say that (6.12) can be solved uniquely for all $y \in X$ if and only if $p(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$.

Do not despair. There are spaces where (6.13) holds. They are called *complex Banach spaces*. They are the same as the Banach spaces that we have been considering with the modification that their scalars consist of all complex numbers. By comparison, the spaces we have been considering are called *real Banach spaces.* The same distinctions are made for vector spaces and normed vector spaces. For a complex Banach space we have

Theorem 6.8. *If X is a complex Banach space, then $\mu \in \sigma(p(A))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(A)$, i.e., if (6.13) holds.*

Proof. We have proved it in one direction already (Theorem 6.7). To prove it in the other, let $\gamma_1, \cdots, \gamma_n$ be the (complex) roots of $p(t) - \mu$. For a complex Banach space they are all scalars. Thus,

$$p(A) - \mu = c(A - \gamma_1) \cdots (A - \gamma_n),\ c \neq 0.$$

Now suppose that all of the γ_j are in $\rho(A)$. Then each $A - \gamma_j$ has an inverse in $B(X)$. Hence, the same is true for $p(A) - \mu$. In other words, $\mu \in \rho(p(A))$. Thus, if $\mu \in \sigma(p(A))$, then at least one of the γ_j must be in $\sigma(A)$, say γ_k. Hence, $\mu = p(\gamma_k)$, where $\gamma_k \in \sigma(A)$. This completes the proof. $\square$

"Hold it a minute!" you exclaim. "You are using results for a complex Banach space that were proved only for real Banach spaces." Yes, you are right. However, if you go through all of the proofs given so far for real spaces, you will find that they all hold equally well for complex spaces with one important exception, and even in that case the proof can be fixed up to apply to complex spaces as well. For all other cases, all you have to do is to substitute "complex number" in place of "real number" whenever you see the word "scalar." What is the exception? I'll let you think a bit about it. In any event, if you do not wish to take my word for it, you are invited to go through the material so far and check for yourself. In short, all of the theorems proved so far for real spaces are true for complex spaces as well.

Theorem 6.8 is called the *spectral mapping theorem* for polynomials. As mentioned before, it has the useful consequence

Corollary 6.9. *If X is a complex Banach space, then equation (6.12) has a unique solution for every y in X if and only if $p(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$.*

6.3. Operational calculus

Other things can be done in a complex Banach space that cannot be done in a real Banach space. For instance, we can get a formula for $p(A)^{-1}$ when it exists. To obtain this formula, we first note

Theorem 6.10. *If X is a complex Banach space, then $(z-A)^{-1}$ is a complex analytic function of z for $z \in \rho(A)$.*

By this, we mean that in a neighborhood of each $z_0 \in \rho(A)$, the operator $(z - A)^{-1}$ can be expanded in a "Taylor series," which converges in norm to

$(z-A)^{-1}$, just like analytic functions of a complex variable. Yes, I promised that you did not need to know anything but advanced calculus in order to read this book. But after all, you know that every rule has its exceptions. Anyone who is unfamiliar with the theory of complex variables should skip this and the next section and go on to Section 6.5. The proof of Theorem 6.10 will be given at the end of this section.

Now, by Lemma 6.5, $\rho(A)$ contains the set $|z| > \|A\|$. We can expand $(z-A)^{-1}$ in powers of z^{-1} on this set. In fact, we have

Lemma 6.11. *If* $|z| > \limsup \|A^n\|^{1/n}$, *then*

$$(z-A)^{-1} = \sum_1^\infty z^{-n} A^{n-1}, \tag{6.14}$$

where the convergence is in the norm of $B(X)$.

Proof. By hypothesis, there is a number $\delta < 1$ such that

$$\|A^n\|^{1/n} \le \delta |z| \tag{6.15}$$

for n sufficiently large. Set $B = z^{-1}A$. Then, by (6.15), we have

$$\sum_0^\infty \|B^n\| < \infty. \tag{6.16}$$

Now,

$$I - B^n = (I-B)\sum_0^{n-1} B^k = (\sum_0^{n-1} B^k)(I-B), \tag{6.17}$$

where, as always, we set $B^0 = I$. By (6.16), we see that the operators

$$\sum_0^{n-1} B^k$$

converge in norm to an operator, which by (6.17) must be $(I-B)^{-1}$. Now

$$z - A = z(I-B),$$

and hence,

$$(z-A)^{-1} = z^{-1}(I-B)^{-1} = \sum_0^\infty z^{-k-1}A^k = \sum_1^\infty z^{-k}A^{k-1}.$$

This completes the proof. $\square$

Let C be any circle with center at the origin and radius greater than, say, $\|A\|$. Then, by Lemma 6.11,

$$\oint_C z^n(z-A)^{-1}dz = \sum_{k=1}^{\infty} A^{k-1}\oint_C z^{n-k}dz = 2\pi i A^n, \tag{6.18}$$

or

$$A^n = \frac{1}{2\pi i}\oint_C z^n(z-A)^{-1}dz, \tag{6.19}$$

where the line integral is taken in the right direction.

Note that the line integrals are defined in the same way as is done in the theory of functions of a complex variable. The existence of the integrals and their independence of path (so long as the integrands remain analytic) are proved in the same way. Since $(z-A)^{-1}$ is analytic on $\rho(A)$, we have

Theorem 6.12. *Let C be any closed curve containing $\sigma(A)$ in its interior. Then (6.19) holds.*

As a direct consequence of this, we have

Theorem 6.13. *$r_\sigma(A) = \max_{\lambda\in\sigma(A)}|\lambda|$ and $\|A^n\|^{1/n} \to r_\sigma(A)$ as $n\to\infty$.*

Proof. Set $m = \max_{\lambda\in\sigma(A)}|\lambda|$, and let $\varepsilon > 0$ be given. If C is a circle about the origin of radius $a = m+\varepsilon$, we have by Theorem 6.12,

$$\|A^n\| \leq \frac{1}{2\pi}a^n M(2\pi a) = Ma^{n+1},$$

where

$$M = \max_C \|(z-A)^{-1}\|.$$

This maximum exists because $(z-A)^{-1}$ is a continuous function on C. Thus,

$$\limsup \|A^n\|^{1/n} \leq a = m+\varepsilon.$$

Since this is true for any $\varepsilon > 0$, we have by (6.7),

$$r_\sigma(A) = \inf\|A^n\|^{1/n} \leq \limsup\|A^n\|^{1/n} \leq m \leq r_\sigma(A),$$

which gives the theorem. $\square$

We can now put Lemma 6.11 in the form

Theorem 6.14. *If $|z| > r_\sigma(A)$, then (6.14) holds with convergence in B(X).*

Now let b be any number greater than $r_\sigma(A)$, and let $f(z)$ be a complex valued function that is analytic in $|z| < b$. Thus,

$$f(z) = \sum_0^{\infty} a_k z^k, \quad |z| < b. \tag{6.20}$$

We can define $f(A)$ as follows: The operators

$$\sum_0^n a_k A^k$$

converge in norm, since

$$\sum_0^\infty |a_k| \cdot \|A^k\| < \infty.$$

This last statement follows from the fact that if c is any number satisfying $r_\sigma(A) < c < b$, then

$$\|A^k\|^{1/k} \leq c$$

for k sufficiently large, and the series

$$\sum_0^\infty |a_k| c^k$$

is convergent. We define $f(A)$ to be

$$\sum_0^\infty a_k A^k. \tag{6.21}$$

By Theorem 6.12, this gives

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \sum_0^\infty a_k \oint_C z^k (z-A)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_C \sum_0^\infty a_k z^k (z-A)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_C f(z)(z-A)^{-1} dz, \end{aligned} \tag{6.22}$$

where C is any circle about the origin with radius greater than $r_\sigma(A)$ and less than b.

We can now give the formula that we promised. Suppose $f(z)$ does not vanish for $|z| < b$. Set $g(z) = 1/f(z)$. Then $g(z)$ is analytic in $|z| < b$, and hence, $g(A)$ is defined. Moreover, we shall prove at the end of this section that

$$f(A)g(A) = \frac{1}{2\pi i} \oint_C f(z)g(z)(z-A)^{-1} dz = \frac{1}{2\pi i} \oint_C (z-A)^{-1} dz = I.$$

Since $f(A)$ and $g(A)$ clearly commute, we see that $f(A)^{-1}$ exists and equals $g(A)$. Hence,

$$f(A)^{-1} = \frac{1}{2\pi i} \oint_C \frac{1}{f(z)} (z-A)^{-1} dz. \tag{6.23}$$

In particular, if

$$g(z) = 1/f(z) = \sum_{0}^{\infty} c_k z^k, \quad |z| < b,$$

then

$$f(A)^{-1} = \sum_{0}^{\infty} c_k A^k. \tag{6.24}$$

Now, suppose $f(z)$ is analytic in an open set Ω containing $\sigma(A)$, but not analytic in a disk of radius greater than $r_\sigma(A)$. In this case, we cannot say that the series (6.21) converges in norm to an operator in $B(X)$. However, we can still define $f(A)$ in the following way: There exists an open set ω whose closure $\overline{\omega} \subset \Omega$ and whose boundary $\partial\omega$ consists of a finite number of simple closed curves that do not intersect, and such that $\sigma(A) \subset \omega$. (That such a set always exists is left as an exercise.) We now define $f(A)$ by

$$f(A) = \frac{1}{2\pi i} \oint_{\partial\omega} f(z)(z - A)^{-1} dz, \tag{6.25}$$

where the line integrals are to be taken in the proper directions. It is easily checked that $f(A) \in B(X)$ and is independent of the choice of the set ω. By (6.22), this definition agrees with the one given above for the case when Ω contains a disk of radius greater than $r_\sigma(A)$. Note that if Ω is not connected, $f(z)$ need not be the same function on different components of Ω.

Now suppose $f(z)$ does not vanish on $\sigma(A)$. Then we can choose ω so that $f(z)$ does not vanish on $\overline{\omega}$ (this is also an exercise). Thus, $g(z) = 1/f(z)$ is analytic on an open set containing $\overline{\omega}$ so that $g(A)$ is defined. Since $f(z)g(z) = 1$, one would expect that $f(A)g(A) = g(A)f(A) = I$, in which case, it would follow that $f(A)^{-1}$ exists and is equal to $g(A)$. At the end of this section, we shall prove

Lemma 6.15. *If f(z) and g(z) are analytic in an open set Ω containing $\sigma(A)$ and*

$$h(z) = f(z)g(z),$$

then h(A)=f(A)g(A).

Therefore, it follows that we have

Theorem 6.16. *If A is in B(X) and f(z) is a function analytic in an open set Ω containing $\sigma(A)$ such that $f(z) \neq 0$ on $\sigma(A)$, then $f(A)^{-1}$ exists and is given by*

$$f(A)^{-1} = \frac{1}{2\pi i} \oint_{\partial\omega} \frac{1}{f(z)} (z - A)^{-1} dz,$$

where ω is any open set such that *(a)* $\sigma(A) \subset \omega,\ \overline{\omega} \subset \Omega$,

(b) $\partial\omega$ *consists of a finite number of simple closed curves,*

(c) $f(z) \neq 0$ *on* $\overline{\omega}$.

Now that we have defined $f(A)$ for functions analytic in a neighborhood of $\sigma(A)$, we can show that the spectral mapping theorem holds for such functions as well (see Theorem 6.8). We have

Theorem 6.17. *If f(z) is analytic in a neighborhood of* $\sigma(A)$, *then*

$$\sigma(f(A)) = f(\sigma(A)), \tag{6.26}$$

i.e., $\mu \in \sigma(f(A))$ *if and only if* $\mu = f(\lambda)$ *for some* $\lambda \in \sigma(A)$.

Proof. If $f(\lambda) \neq \mu$ for all $\lambda \in \sigma(A)$, then the function $f(z) - \mu$ is analytic in a neighborhood of $\sigma(A)$ and does not vanish there. Hence, $f(A) - \mu$ has an inverse in $B(X)$, i.e., $\mu \in \rho(f(A))$. Conversely, if $\mu = f(\lambda)$ for some $\lambda \in \sigma(A)$, set

$$g(z) = \begin{cases} [f(z) - \mu]/(z - \lambda), & z \neq \lambda, \\ f'(\lambda), & z = \lambda. \end{cases}$$

Then $g(z)$ is analytic in a neighborhood of $\sigma(A)$ and $g(z)(z-\lambda) = f(z) - \mu$. Hence, $g(A)(A - \lambda) = (A - \lambda)g(A) = f(A) - \mu$. If μ were in $\rho(f(A))$, then we would have

$$h(A)(A - \lambda) = (A - \lambda)h(A) = I,$$

where

$$h(A) = g(A)[f(A) - \mu]^{-1}.$$

This would mean that $\lambda \in \rho(A)$, contrary to assumption. Thus $\mu \in \sigma(f(A))$, and the proof is complete. $\square$

It remains to prove Theorem 6.10 and Lemma 6.15. For both of these, we shall employ

Theorem 6.18. *If* λ, μ *are in* $\rho(A)$, *then*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{6.27}$$

Moreover, if $|\lambda - \mu| \cdot \|(\mu - A)^{-1}\| < 1$, *then*

$$(\lambda - A)^{-1} = \sum_1^\infty (\mu - \lambda)^{n-1}(\mu - A)^{-n}, \tag{6.28}$$

and the series converges in B(X).

Proof. Let x be an arbitrary element of X, and set $u = (\lambda - A)^{-1}x$. Thus, $(\lambda - A)u = x$ and $(\mu - A)u = x + (\mu - \lambda)u$. Hence,

$$u = (\mu - A)^{-1}x + (\mu - \lambda)(\mu - A)^{-1}u.$$

Substituting for u, we get (6.27). Note that it follows from (6.27) that $(\lambda - A)^{-1}$ and $(\mu - A)^{-1}$ commute. Substituting for $(\lambda - A)^{-1}$ in the right-hand side of (6.27), we get

$$(\lambda - A)^{-1} = (\mu - A)^{-1} + (\mu - \lambda)(\mu - A)^{-2} + (\mu - \lambda)^2(\lambda - A)^{-1}(\mu - A)^{-2}.$$

Continuing in this way, we get

$$(\lambda - A)^{-1} = \sum_1^k (\mu - \lambda)^{n-1}(\mu - A)^{-n} + (\mu - \lambda)^n(\lambda - A)^{-1}(\mu - A)^{-n}.$$

Since

$$\|(\mu - \lambda)^n(\lambda - A)^{-1}(\mu - A)^{-n}\|$$
$$\leq |\mu - \lambda|^n \cdot \|(\lambda - A)^{-1}\| \cdot \|(\mu - A)^{-1}\|^n \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

we see that (6.28) follows. □

The analyticity claimed in Theorem 6.10 is precisely the expression (6.28) of Theorem 6.18.

We can now give the proof of Lemma 6.15.

Proof. Suppose $h(z) = f(z)g(z)$, where both $f(z)$ and $g(z)$ are analytic in an open set $\Omega \supset \sigma(A)$. Let ω_1 and ω_2 be two open sets such that $\sigma(A) \subset \omega_1$, $\overline{\omega}_1 \subset \omega_2$, $\overline{\omega}_2 \subset \Omega$, and whose boundaries consist of a finite number of simple closed curves. Then

$$\begin{aligned}
f(A)g(A) &= \frac{1}{2\pi i}\oint_{\partial\omega_1} f(z)(z - A)^{-1}g(A)dz \\
&= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)(z - A)^{-1}\oint_{\partial\omega_2} g(\zeta)(\zeta - A)^{-1}\,d\zeta dz \\
&= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)\oint_{\partial\omega_2} g(\zeta)\frac{(z - A)^{-1} - (\zeta - A)^{-1}}{\zeta - z}\,d\zeta dz \\
&= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)(z - A)^{-1}\oint_{\partial\omega_2}\frac{g(\zeta)\,d\zeta}{\zeta - z}\,dz \\
&\quad - \frac{1}{4\pi}\oint_{\partial\omega_2} g(\zeta)(\zeta - A)^{-1}\oint_{\partial\omega_1}\frac{f(z)\,dz}{z - \zeta}\,d\zeta,
\end{aligned}$$

by (6.27). Since $\partial\omega_1$ is in the interior of ω_2,

$$\oint_{\partial\omega_2} \frac{g(\zeta)\,d\zeta}{\zeta - z} = 2\pi i g(z)$$

for z on $\partial\omega_1$, while

$$\oint_{\partial\omega_1} \frac{f(z)\,dz}{z - \zeta} = 0$$

for ζ on $\partial\omega_2$. Hence,

$$f(A)g(A) = \frac{1}{2\pi i} \oint_{\partial\omega_1} f(z)g(z)(z - A)^{-1} dz = h(A).$$

This completes the proof. □

6.4. Spectral projections

In the previous section, we saw that we can define $f(A)$ whenever $A \in B(X)$ and $f(z)$ is analytic in some open set Ω containing $\sigma(A)$. If $\sigma(A)$ is not connected, then Ω need not be connected, and $f(z)$ need not be the same on different components of Ω. This leads to some very interesting consequences.

For example, suppose σ_1 and σ_2 are two subsets of $\sigma(A)$ such that $\sigma(A) = \sigma_1 \cup \sigma_2$, and there are open sets $\Omega_1 \supset \sigma_1$ and $\Omega_2 \supset \sigma_2$ such that Ω_1 and Ω_2 do not intersect. Such sets are called *spectral sets* of A. We can then take $f(z)$ to be one function on Ω_1 and another on Ω_2, and $f(A)$ is perfectly well defined. In particular, we can take $f(z) = 1$ on Ω_1 and $f(z) = 0$ on Ω_2. We set $P = f(A)$. Thus, if ω is an open set containing σ_1 such that $\overline{\omega} \subset \Omega_1$ and such that $\partial\omega$ consists of a finite number of simple closed curves, then

$$P = \frac{1}{2\pi i} \oint_{\partial\omega} (z - A)^{-1} dz. \tag{6.29}$$

Clearly, $P^2 = P$ (Lemma 6.15). Any operator having this property is called a *projection*. For any projection P, $x \in R(P)$ if and only if $x = Px$. For if $x = Pz$, then $Px = P^2 z = Pz = x$. Thus, $N(P) \cap R(P) = \{0\}$. Moreover,

$$X = R(P) \oplus N(P), \tag{6.30}$$

for if $x \in X$, then $Px \in R(P)$, and $(I - P)x \in N(P)$, since $P(I - P)x = (P - P^2)x = 0$. In our case, P has the additional property of being in $B(X)$. Thus, $N(P)$ and $R(P)$ are both closed subspaces of X.

Now, A maps $N(P)$ and $R(P)$ into themselves. For if $Px = 0$, then $PAx = APx = 0$. Similarly, if $Px = x$, then $PAx = APx = Ax$. Let A_1 be the restriction of A to $R(P)$ and A_2 its restriction to $N(P)$. Thus, we can consider A split into the "sum" of A_1 and A_2. Moreover, if we consider A_1 as an operator on $R(P)$ and A_2 as an operator on $N(P)$, we have

Theorem 6.19. $\sigma(A_i) = \sigma_i, \quad i = 1, 2.$

Proof. Let μ be any point not in σ_1. Set $g(z) = f(z)/(\mu - z)$, where $f(z)$ is identically one on σ_1 and vanishes on σ_2. Then $g(z)$ is analytic in a neighborhood of $\sigma(A)$ and $fg = g$. Hence, $(\mu - A)g(A) = P$ and $Pg(A) = g(A)$. Thus, $g(A)$ maps $R(P)$ into itself, and its restriction to $R(P)$ is the inverse of $\mu - A_1$. Hence, $\mu \in \rho(A_1)$. Since $I - P$ is also a projection and $R(I - P) = N(P)$, $N(I - P) = R(P)$, we see, by the same reasoning, that if μ is not in σ_2, then it is in $\rho(A_2)$. Hence, $\sigma(A_i) \subset \sigma_i$ for $i = 1, 2$. Now, if $\mu \in \rho(A_1) \cap \rho(A_2)$, then it is in $\rho(A)$. In fact, we have

$$(\mu - A)^{-1} = (\mu - A_1)^{-1}P + (\mu - A_2)^{-1}(I - P).$$

Hence, the points of σ_1 must be in $\sigma(A_1)$ and those of σ_2 must be in $\sigma(A_2)$. This completes the proof. □

Next, suppose that σ_1 consists of just one isolated point λ_1. Then $\sigma(A_1)$ consists of precisely the point λ_1. Hence, $r_\sigma(A_1 - \lambda_1) = 0$, i.e.,

$$\|(A_1 - \lambda_1)^n\|^{1/n} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \tag{6.31}$$

(Theorem 6.13). In particular,

$$\|(A - \lambda_1)^n x\|^{1/n} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \tag{6.32}$$

for all $x \in R(P)$. Conversely, we note that every $x \in X$ which satisfies (6.32) is in $R(P)$. Suppose (6.32) holds. Then

$$\|(A - \lambda_1)^n (I - P)x\|^{1/n} \longrightarrow 0,$$

since $I - P$ commutes with $A - \lambda_1$ and is bounded. Set

$$z_n = (A - \lambda_1)^n (I - P)x.$$

Then

$$\|z_n\|^{1/n} \longrightarrow 0.$$

But $\lambda_1 \in \rho(A_2)$, and hence,

$$(I - P)x = (A_2 - \lambda_1)^{-n} z_n.$$

Thus,

$$\|(I - P)x\| \leq \|(A_2 - \lambda_1)^{-1}\|^n \cdot \|z_n\|,$$

or

$$\|(I - P)x\|^{1/n} \leq \|(A_2 - \lambda_1)^{-1}\| \cdot \|z_n\|^{1/n} \longrightarrow 0.$$

Hence, $(I - P)x = 0$, showing that $x \in R(P)$.

We call P the *spectral projection* associated with σ_1. As an application, we have

Theorem 6.20. *Suppose that A is in $B(X)$ and that λ_1 is an isolated point of $\sigma(A)$ (i.e., the set consisting of the point λ_1 is a spectral set of A). If $R(A-\lambda_1)$ is closed and $r(A-\lambda_1)<\infty$ (see Section 5.5), then $\lambda_1 \in \Phi_A$.*

Proof. We must show that $\beta(A-\lambda_1)<\infty$. Let P be the spectral projection associated with the point λ_1, and set

$$N_0 = \bigcup_1^\infty N[(A-\lambda_1)^n].$$

By hypothesis, N_0 is finite-dimensional. We must show that this implies that $N_0 = R(P)$. Once this is known, the theorem follows easily. For, by (6.30),

$$X = N_0 \oplus N(P), \tag{6.33}$$

and if A_2 denotes the restriction of A to $N(P)$, then $\lambda_1 \in \rho(A_2)$ (Theorem 6.19). In particular, this means that $R(A_2-\lambda_1)=N(P)$, and since $R(A-\lambda_1) \supset R(A_2-\lambda_1)$, we see that $N[(A-\lambda_1)'] \subset N(P)^\circ$. This latter set is finite-dimensional. For if $x'_1, \cdots, x'_n$ are linearly independent elements of $N(P)^\circ$, then there are elements $x_1, \cdots, x_n \in X$ such that

$$x'_j(x_k) = \delta_{jk}, \quad 1 \le j,\, k \le n$$

(Lemma 4.14). Clearly, the x_k are linearly independent, and if M is the n-dimensional subspace spanned by them, then $M \cap N(P) = \{0\}$ (see the proof of Lemma 5.3). Thus $n \le \dim N_0$ (Lemma 5.12), and consequently, $\beta(A-\lambda_1)<\infty$.

To prove that $N_0 = R(P)$, we first note that $N_0 \subset R(P)$, since $x \in R(P)$ if and only if it satisfies (6.32). Now, suppose N_0 is not all of $R(P)$. Set $X_0 = R(P)/N_0$ and define the operator B on X_0 by means of

$$B[x] = [(A-\lambda_1)x], \quad x \in R(P),$$

where $[x]$ is any coset in X_0 (see Section 3.5). Clearly, B is one-to-one on X_0, and

$$B^n[x] = [(A-\lambda_1)^n x], \quad n = 1, 2, \cdots .$$

If we can show that the range of B is closed in X_0, it will follow that there is a constant $c>0$ such that

$$\|[x]\| \le c\|B[x]\|, \quad x \in R(P)$$

(Theorem 3.12). Consequently, we shall have

$$\|[x]\| \le c^n \|[(A-\lambda_1)^n x]\| \le c^n \|(A-\lambda_1)^n x\|, \quad n = 1, 2, \cdots .$$

Now, if x is an element in $R(P)$ which is not in N_0, then $\|[x]\| \ne 0$. Hence,

$$\liminf \|(A-\lambda_1)^n x\|^{1/n} \ge 1/c > 0,$$

which contradicts (6.32). This shows that there is no such x.

Thus, it remains only to show that $R(B)$ is closed in X_0. To this end, we employ a simple lemma.

Lemma 6.21. *If X_1 and X_2 are subspaces of a normed vector space X, let $X_1 + X_2$ denote the set of all sums of the form $x_1 + x_2$, where $x_i \in X_i$, $i = 1, 2$. If X_1 is closed and X_2 is finite-dimensional, then $X_1 + X_2$ is a closed subspace of X.*

Returning to the proof of Theorem 6.20, let A_1 be the restriction of A to $R(P)$. Then $R(A_1 - \lambda_1) = R(A - \lambda_1) \cap R(P)$. For, if $y \in R(A - \lambda_1) \cap R(P)$, then $y = (A - \lambda_1)x$ for some $x \in X$, and $Py = y$. Thus, $(A - \lambda_1)Px = P(A - \lambda_1)x = Py = y$, showing that $y \in R(A_1 - \lambda_1)$. In particular, we see from this that $R(A_1 - \lambda_1)$ is closed in $R(P)$. Thus, by Lemma 6.21, the same is true of $R(A_1 - \lambda_1) + N_0$. Now all we need is the simple observation that $R(B) = [R(A_1 - \lambda_1) + N_0]/N_0$. This latter set is closed (i.e., a Banach space) by Theorem 3.13. Thus, the proof of Theorem 6.20 will be complete once we have given the simple proof of Lemma 6.21. □

Proof. Set $X_3 = X_1 \cap X_2$. Since X_3 is finite-dimensional, there is a closed subspace X_4 of X_1 such that $X_1 = X_3 \oplus X_4$ (Lemma 5.1). Clearly, $X_1 + X_2 = X_4 \oplus X_2$, and the latter is closed by Lemma 5.2. □

As a consequence of Theorem 6.20 we have

Corollary 6.22. *Under the hypotheses of Theorem 6.20, there are operators E in $B(X)$ and K in $K(X)$ and an integer $m > 0$ such that*

$$(A - \lambda_1)^m E = E(A - \lambda_1)^m = I - K. \tag{6.34}$$

Proof. By Theorem 6.20, Φ_A contains the point λ_1. Since λ_1 is an isolated point of $\sigma(A)$, $i(A - \lambda) = 0$ in a neighborhood of λ_1 (Theorem 5.11), and, hence, $r'(A - \lambda_1) < \infty$. We now apply Theorem 5.18. □

We note that a partial converse of Theorem 6.20 is contained in the more general theorem

Theorem 6.23. *Suppose A is in $B(X)$ and λ_1 is a point of $\sigma(A)$ such that $R(A - \lambda_1)$ is closed in X. Then any two of the following conditions imply the others.*

(a) $r(A - \lambda_1) < \infty$,

(b) $r'(A - \lambda_1) < \infty$,

(c) $\alpha(A - \lambda_1) = \beta(A - \lambda) < \infty$,

(d) λ_1 is an isolated point of $\sigma(A)$.

We shall prove most of the theorem here. The only case omitted will be the one in which (c) and (d) are given. It is more convenient to postpone this case until Chapter 9, where we consider Banach algebras.

Proof. We first note that Theorem 6.20 shows that (a) and (d) imply (b) and (c), because once it is known that $\lambda_1 \in \Phi_A$, then we know that $i(A - \lambda_1) = 0$, and consequently, $r'(A - \lambda_1) = 0$.

To show that (b) and (d) imply the rest, we note that $R[(A - \lambda_1)']$ is closed by Theorem 3.16. Moreover, λ_1 is an isolated point of $\sigma(A')$. This follows from

Theorem 6.24. *For A in $B(X)$, $\sigma(A') = \sigma(A)$.*

We shall prove Theorem 6.24 at the end of this section. To continue our argument, we now apply Theorem 6.20 to $A' - \lambda_1 = (A - \lambda_1)'$ to conclude that $\lambda_1 \in \Phi_{A'}$. In particular, this gives $\alpha(A - \lambda_1) \leq \beta(A' - \lambda_1) < \infty$ [see (5.24)]. Hence, $\lambda_1 \in \Phi_A$, and consequently $i(A - \lambda_1) = 0$ by (d). Thus, $r(A - \lambda_1) < \infty$ by (b).

To show that (a) and (b) imply the others, assume for convenience that $\lambda_1 = 0$. It was shown in Section 5.5 that (a) and (b) imply (c). Moreover, it was also shown there that there is an integer $n > 0$ such that

$$N(A^n) \cap R(A^n) = \{0\}, \tag{6.35}$$

and that

$$N(A^k) = N(A^n),\ R(A^k) = R(A^n), \quad k > n. \tag{6.36}$$

Clearly,

$$X = N(A^n) \oplus R(A^n). \tag{6.37}$$

If the right-hand side of (6.37) were not the whole of X, there would be a subspace Z of dimension one such that

$$\{Z \oplus N(A^n)\} \cap R(A^n) = \{0\},$$

from which it would follow that $\beta(A^n) > \alpha(A^n)$ (Lemmas 5.3 and 5.12). Hence, (6.35) holds. Now $\lambda \in \Phi_A$ in some neighborhood of the origin (Theorem 6.3), and $i(A - \lambda) = 0$ in this neighborhood. Consequently, to prove (d) it suffices to show that $\alpha(A - \lambda) = 0$ for $\lambda \neq 0$ in this neighborhood. To do this, we note that, by (6.36), A maps $N(A^n)$ and $R(A^n)$ into themselves. Hence, by (6.37), it suffices to prove

$$(A - \lambda)u = 0,\ u \in N(A^n) \implies u = 0, \tag{6.38}$$

$$(A - \lambda)v = 0,\ v \in R(A^n) \implies v = 0 \tag{6.39}$$

for $\lambda \neq 0$ in some neighborhood of the origin. To prove (6.38), we note that $u = \lambda^{-1}Au = \lambda^{-2}A^2u = \cdots = \lambda^{-n}A^nu = 0$. To prove (6.39), we let A_1 be the restriction of A to $R(A^n)$ and show that $0 \in \rho(A_1)$. From this, (6.39) follows via Theorem 6.3. Now, suppose $Aw = 0$ for some $w \in R(A^n)$. Since $w = A^n g$ for some $g \in X$, we have $A^{n+1}g = 0$. By (6.36), $A^n g = 0$, showing that $w = 0$. Next, let f be any element in $R(A^n)$. By (6.36), $f \in R(A^{n+1})$. Therefore, there exists an $h \in X$ such that $A^{n+1}h = f$. Thus, $y = A^n h \in R(A^n)$ and $Ay = f$. Hence, A is one-to-one and onto, on $R(A^n)$. This means that $0 \in \rho(A_1)$.

Clearly, (a) and (c) imply (b), so that this case is resolved as well. Similarly, (b) and (c) imply (a). Thus, it remains only to show that (c) and (d) imply the others. This will be done in Section 9.2. $\square$

We now give the proof of Theorem 6.24.

Proof. Suppose $\lambda \in \rho(A)$. Then there is an operator $B \in B(X)$ such that

$$(A - \lambda)B = B(A - \lambda) = I$$

(Theorem 3.8). Taking adjoints we get

$$B'(A' - \lambda) = (A' - \lambda)B' = I \text{ on } X',$$

showing that $\lambda \in \rho(A')$. Conversely, suppose $\lambda \in \rho(A')$. Then there is an operator $C \in B(X')$ such that

$$(A' - \lambda)C = C(A' - \lambda) = I \text{ on } X'.$$

Taking adjoints gives

$$C'(A'' - \lambda) = (A'' - \lambda)C' = I \text{ on } X''. \tag{6.40}$$

Let J be the operator from X to X'' defined in Section 5.4, satisfying

$$Jx(x') = x'(x), \quad x \in X,\ x' \in X'.$$

By (5.27), J is one-to-one, and J^{-1} is bounded from $R(J)$ to X. Moreover,

$$A''Jx(x') = Jx(A'x') = A'x'(x) = x'(Ax), \quad x \in X,\ x' \in X',$$

showing that

$$A''Jx = JAx, \quad x \in X. \tag{6.41}$$

Combining this with (6.40), we obtain

$$Jx = C'(A'' - \lambda)Jx = C'J(A - \lambda)x, \quad x \in X.$$

Consequently,

$$\|x\| = \|Jx\| = \|C'J(A - \lambda)x\| \leq \|C'J\| \cdot \|(A - \lambda)x\|.$$

This shows that $N(A - \lambda) = \{0\}$, and that $R(A - \lambda)$ is closed (Theorem 3.12). If we can show that $R(A - \lambda) = X$, it will follow that $\lambda \in \rho(A)$, and the proof will be complete. To this end, let x' be any element in $R(A-\lambda)^\circ$. Then

$$(A'-\lambda)x'(x) = x'[(A-\lambda)x] = 0$$

for all $x \in X$. Thus, $x' \in N(A'-\lambda) = \{0\}$. Since $R(A-\lambda)$ is closed, and the only functional that annihilates it is 0, it must be the whole of X. This completes the proof. $\square$

6.5. Complexification

What we have just done is valid for complex Banach spaces. Suppose, however, we are dealing with a real Banach space. What can be said then?

Let X be a real Banach space. Consider the set Z of all ordered pairs $\langle x, y\rangle$ of elements of X. We set

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$$

$$(\alpha + i\beta)\langle x, y\rangle = \langle(\alpha x - \beta y), (\beta x + \alpha y)\rangle, \quad \alpha, \beta \in \mathbb{R}.$$

With these definitions, one checks easily that Z is a complex vector space. The set of elements of Z of the form $\langle x, 0\rangle$ can be identified with X. We would like to introduce a norm on Z that would make Z into a Banach space and satisfy

$$\|\langle x, 0\rangle\| = \|x\|, \quad x \in X.$$

An obvious suggestion is

$$(\|x\|^2 + \|y\|^2)^{1/2}.$$

However, it is soon discovered that this is not a norm on Z (why?). We have to be more careful. One that works is given by

$$\|\langle x, y\rangle\| = \max_{\alpha^2+\beta^2=1} (\|\alpha x - \beta y\|^2 + \|\beta x + \alpha y\|^2)^{1/2}.$$

With this norm, Z becomes a complex Banach space having the desired properties.

Now let A be an operator in $B(X)$. We define an operator $\hat{A}$ in $B(Z)$ by

$$\hat{A}\langle x, y\rangle = \langle Ax, Ay\rangle.$$

Then

$$\begin{aligned}\|\hat{A}\langle x, y\rangle\| &= \max_{\alpha^2+\beta^2=1} (\|\alpha Ax - \beta Ay\|^2 + \|\beta Ax + \alpha Ay\|^2)^{1/2} \\ &= \max_{\alpha^2+\beta^2=1} (\|A(\alpha x - \beta y)\|^2 + \|A(\beta x + \alpha y)\|^2)^{1/2} \\ &\le \|A\| \cdot \|\langle x, y\rangle\|.\end{aligned}$$

Thus

$$\|\hat{A}\| \le \|A\|.$$

But,

$$\|\hat{A}\| \geq \sup_{x \neq 0} \frac{\|\langle Ax, 0\rangle\|}{\|\langle x, 0\rangle\|} = \|A\|.$$

Hence,

$$\|\hat{A}\| = \|A\|.$$

If λ is real, then

$$(\hat{A} - \lambda)\langle x, y\rangle = \langle (A-\lambda)x, (A-\lambda)y\rangle.$$

This shows that $\lambda \in \rho(\hat{A})$ if and only if $\lambda \in \rho(A)$. Similarly, if $p(t)$ is a polynomial with real coefficients, then

$$p(\hat{A})\langle x, y\rangle = \langle p(A)x, p(A)y\rangle,$$

showing that $p(\hat{A})$ has an inverse in $B(Z)$ if and only if $p(A)$ has an inverse in $B(X)$. Hence, we have

Theorem 6.25. *Equation (6.12) has a unique solution for each y in X if and only if $p(\lambda) \neq 0$ for all $\lambda \in \sigma(\hat{A})$.*

In the example given at the end of Section 6.1, the operator $\hat{A}$ has eigenvalues i and $-i$. Hence, -1 is in the spectrum of $\hat{A}^2$ and also in that of A^2. Thus, the equation

$$(A^2 + 1)x = y$$

cannot be solved uniquely for all y.

6.6. The complex Hahn-Banach theorem

In case you have not already figured it out, the only theorem that we proved in the preceding chapters that needs modification for complex vector spaces is the Hahn-Banach theorem. We now give a form that is true for a complex Banach space.

Theorem 6.26. *Let V be a complex vector space, and let p be a real valued functional on V such that*

(i) $p(u+v) \leq p(u) + p(v), \quad u, v \in V,$

(ii) $p(\alpha u) = |\alpha| p(u), \quad \alpha$ *complex*, $u \in V.$

Suppose that there is a linear subspace M of V and a linear (complex valued) functional f on M such that

$$\Re e\, f(u) \leq p(u), \quad u \in M. \tag{6.42}$$

Then there is a linear functional F on the whole of V such that

$$F(u) = f(u), \quad u \in M, \tag{6.43}$$

$$|F(u)| \le p(u), \quad u \in V. \tag{6.44}$$

Proof. Let us try to reduce the "complex" case to the "real" case. To be sure, we can consider V as a real vector space by allowing multiplication by real scalars only. If we do this, M becomes a subspace of a real vector space V. Next, we can define the real valued functional

$$f_1(u) = \Re e\, f(u), \quad u \in M.$$

Then by (6.42),

$$f_1(u) \le p(u), \quad u \in M.$$

We can now apply the "real" Hahn-Banach theorem (Theorem 2.5) to conclude that there is a real functional $F_1(u)$ on V such that

$$F_1(u) = f_1(u), \quad u \in M,$$

$$F_1(u) \le p(u), \quad u \in V.$$

Now, this is all very well and good, but where does it get us? We wanted to extend the whole of f, not just its real part. The trick that now saves us is that there is an intimate connection between the real and imaginary part of a linear functional on a complex vector space. In fact,

$$f_1(iu) = \Re e\, f(iu) = \Re e\, if(u) = -\Im m\, f(u).$$

Hence,

$$f(u) = f_1(u) - if_1(iu).$$

This suggests a candidate for $F(u)$. Set

$$F(u) = F_1(u) - iF_1(iu), \quad u \in V.$$

$F(u)$ is clearly linear if real scalars are used. To see that it is linear in the "complex" sense, we note that

$$F(iu) = F_1(iu) - iF_1(-u) = i[F_1(u) - iF_1(iu)] = iF(u).$$

Second, we note that $F(u) = f(u)$ for $u \in M$. To complete the proof, we must show that (6.44) holds. Observe that

$$p(u) \ge 0, \quad u \in V. \tag{6.45}$$

In fact, by (ii) we see that $p(0) = 0$, while by (i), we see that $p(0) \le p(u) + p(-u) = 2p(u)$. Hence, (6.44) holds whenever $F(u) = 0$. If $F(u) \neq 0$, we write it in polar form $F(u) = |F(u)|e^{i\theta}$. Then

$$|F(u)| = e^{-i\theta}F(u) = F(e^{-i\theta}u) = F_1(e^{-i\theta}u) \le p(e^{-i\theta}u) = p(u).$$

This completes the proof. $\square$

A functional satisfying (i) and (ii) of Theorem 6.1 is called a *seminorm.* As a corollary to Theorem 6.26 we have

Corollary 6.27. *Let M be a subspace of a complex normed vector space X. If f is a bounded linear functional on M, then there is a bounded linear functional F on X such that*

$$F(x) = f(x), \quad x \in M,$$

$$\|F\| = \|f\|.$$

This corollary follows from Theorem 6.26 as in the real case.

In all of our future work, if we do not specify real or complex vector spaces, normed vector spaces, etc., it will mean that our statements hold for both.

6.7. A geometric lemma

In Section 6.3, we made use of the following fact:

Lemma 6.28. *Let Ω be an open set in $\mathbb{R}^2$, and let K be a bounded closed set in Ω. Then there exists a bounded open set ω such that*

(1) $\omega \supset K$,

(2) $\overline{\omega} \subset \Omega$,

(3) $\partial\omega$ consists of a finite number of simple polygonal closed curves which do not intersect.

Proof. By considering the intersection of Ω with a sufficiently large disk, we may assume that Ω is bounded. Let δ be the distance from K to $\partial\Omega$. Since both of these sets are compact and do not intersect, we must have $\delta > 0$. Cover $\mathbb{R}^2$ with a honeycomb of regular closed hexagons each having edges of length less than $\delta/4$. Thus, the diameter of each hexagon is less than $\delta/2$. Let R be the collection of those hexagons contained in Ω, and let W be the union of all hexagons in R. Let ω be the interior of W. Then $\omega \supset K$. If $x \in K$, then its distance to $\partial\Omega$ is $\geq \delta$. Thus, the hexagon containing x and all adjacent hexagons are in Ω. This implies that $x \in \omega$. Next we note that $\overline{\omega} \subset \Omega$. For if $x \in \overline{\omega}$, then x is in some hexagon contained in Ω.

Thus, it remains only to show that $\partial\omega$ satisfies (3). Clearly, $\partial\omega$ consists of a finite number of sides of hexagons. Thus, we can prove (3) by showing

(a) that $\partial\omega$ never intersects itself and (b) that no point of $\partial\omega$ is an end point of $\partial\omega$. Now, every point $x \in \partial\omega$ is either on an edge of some hexagon or at a vertex. If it is on an edge, then the whole edge is in $\partial\omega$, in which case, all points on one side of the edge are in ω and all points on the other side are not in ω. If x is a vertex, then there are three hexagons meeting at x. Either one or two of these hexagons are in Ω. In either case, exactly two of the three edges meeting at x belong to $\partial\omega$. Thus (a) and (b) are true, and the proof is complete. $\square$

6.8. Problems

(1) Show that if X is infinite dimensional and $K \in K(X)$, then $0 \in \sigma(K)$.

(2) Let A and B be operators in $B(H)$ which commute (i.e., $AB = BA$). Show that

$$r_\sigma(AB) \le r_\sigma(A)r_\sigma(B), \quad r_\sigma(A+B) \le r_\sigma(A) + r_\sigma(B).$$

(3) Suppose $A \in B(X)$, and $f(z)$ is an analytic function on an open set Ω containing $|z| \le r_\sigma(A)$. If

$$|f(z)| \le M, \quad |z| \le r_\sigma(A),$$

show that

$$r_\sigma[f(A)] \le M.$$

(4) Let A be an operator in X and $p(t)$ a polynomial. Show that if Φ_A is not empty, then $p(A)$ is a closed operator.

(5) Suppose that λ_0 is an isolated point of $\sigma(A)$, $A \in B(X)$, and $f(z)$ is analytic in a neighborhood of λ_0. If $f(A) = 0$, show that $f(z)$ has a zero at λ_0.

(6) Let λ_0 be an isolated point of $\sigma(A)$, $A \in B(X)$, with $\lambda_0 \in \Phi_A$. Let x' be any functional in X'. Show that $f(z) = x'[(z - A)^{-1}]$ has a pole at $z = \lambda_0$.

(7) Show that a projection $P \in B(X)$ is compact if and only if it is of finite rank.

(8) If $A \in B(X)$, show that

$$z(z-A)^{-1} \longrightarrow I \text{ as } |z| \longrightarrow \infty.$$

(9) Let A be an operator in $B(X)$, and suppose $\sigma(A)$ is contained in the half-plane $\Re e\, z > \delta > 0$. Let Γ be a simple closed curve in $\Re e\, z \geq \delta$ containing $\sigma(A)$ in its interior. Consider the operator

$$T = \frac{1}{2\pi i} \oint_{\Gamma} z^{1/2}(z-A)^{-1} dz.$$

Show that T is well defined and that $T^2 = A$. What can you say about $\sigma(T)$?

(10) Suppose A, $B \in B(X)$ with $0 \in \rho(A)$ and $\|A-B\| < 1/\|A^{-1}\|$. Show that $0 \in \rho(B)$ and

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\| \cdot \|A-B\|}.$$

(11) Show that every orthonormal sequence in a separable Hilbert space can be made part of a complete orthonormal sequence.

(12) Let M be a closed subspace of a Banach space X. Show that there is a projection $P \in B(X)$ such that $R(P) = M$ if and only if there is a closed subspace $N \in X$ such that $X = M \oplus N$.

(13) Let P, Q be bounded projections on a Banach space X such that $\|P-Q\| < 1$. Show that there is an operator $A \in B(R(P), R(Q))$ such that $A^{-1} \in B(R(Q), R(P))$.

(14) If $p(x)$ is a seminorm, show that

$$|p(x_1) - p(x_2)| \leq p(x_1 - x_2).$$

(15) If $p(x)$ is a seminorm on a vector space V, let

$$Q_c = \{x \in V : p(x) \leq c\}, \quad c > 0.$$

Show that

(a) $0 \in Q_c$

and

(b) $|\alpha| \leq 1$ implies $\alpha x \in Q_c$ for $x \in Q_c$.

(16) Under the same hypothesis, show that for each $x \in V$, there is an $\alpha > 0$ such that $\alpha x \in Q_c$.

(17) Under the same hypothesis, show that

$$p(x) = \inf\{\alpha > 0,\ \alpha^{-1}x \in Q_1\}.$$

Chapter 7

UNBOUNDED OPERATORS

7.1. Unbounded Fredholm operators

In many applications, one runs into unbounded operators instead of bounded ones. This is particularly true in the case of differential equations. For instance, if we consider the operator d/dt on $C[0,1]$, it is a closed operator with domain consisting of continuously differentiable functions. It is clearly unbounded. In fact, the sequence $x_n(t) = t^n$ satisfies $\|x_n\| = 1$, $\|dx_n/dt\| = n \to \infty$ as $n \to \infty$. It would, therefore, be useful if some of the results that we have proved for bounded operators would also hold for unbounded ones. We shall see in this chapter that, indeed, many of them do. Unless otherwise specified, X, Y, Z, and W will denote Banach spaces in this chapter.

Let us begin by trying to enlarge the set $\Phi(X, Y)$ of Fredholm operators to include unbounded ones. If we examine the definition of Chapter 5, we notice one immediate obstacle; we have not as yet defined A' for an unbounded operator. This obstacle is easily overcome. We just follow the definition for bounded operators, and exercise a bit of care. We want

$$A'y'(x) = y'(Ax), \quad x \in D(A). \tag{7.1}$$

Thus, we say that $y' \in D(A')$ if there is an $x' \in X'$ such that

$$x'(x) = y'(Ax), \quad x \in D(A). \tag{7.2}$$

Then we define $A'y'$ to be x'. In order that this definition make sense, we need x' to be unique, i.e., that $x'(x) = 0$ for all $x \in D(A)$ should imply that $x' = 0$. This is true if and only if $D(A)$ is dense in X. To summarize, we can define A' for any linear operator from X to Y provided $D(A)$ is dense in X. We take $D(A')$ to be the set of those $y' \in Y'$ for which there is an $x' \in X'$ satisfying (7.2). This x' is unique, and we set $A'y' = x'$. Note that if

$$|y'(Ax)| \leq C\|x\|, \quad x \in D(A),$$

then a simple application of the Hahn-Banach theorem shows that $y' \in D(A')$.

Now that this is done, we can attempt to define unbounded Fredholm operators. If you recall, in Chapter 5, we used the closed graph theorem (or its equivalent, the bounded inverse theorem) on a few occasions. Thus, it seems reasonable to define Fredholm operators in the following way: Let X, Y be Banach spaces. Then the set $\Phi(X, Y)$ consists of linear operators from X to Y such that

(1) $D(A)$ is dense in X,

(2) A is closed,

(3) $\alpha(A) < \infty$,

(4) $R(A)$ is closed in Y,

(5) $\beta(A) < \infty$.

We now ask what theorems of Chapter 5 hold for this larger class of operators. Surprisingly enough, most of them do. To begin with, we have, as before,

$$X = N(A) \oplus X_0, \tag{7.3}$$

where X_0 is a closed subspace of X. Since $N(A) \subset D(A)$, this gives

$$D(A) = N(A) \oplus [X_0 \cap D(A)]. \tag{7.4}$$

Similarly, we see from (7.1), just as in the bounded case, that $N(A') = R(A)^{\circ}$. Hence,

$$Y = R(A) \oplus Y_0, \tag{7.5}$$

where Y_0 is a subspace of Y of dimension $\beta(A)$. As before, the restriction of A to $X_0 \cap D(A)$ has a closed inverse defined everywhere on $R(A)$ (which is a Banach space), and hence, the inverse is bounded. This gives

$$\|x\|_X \leq C\|Ax\|_Y, \quad x \in X_0 \cap D(A). \tag{7.6}$$

Thus, we have

Theorem 7.1. *If $A \in \Phi(X,Y)$, then there is an $A_0 \in B(Y,X)$ such that*

(a) $N(A_0) = Y_0$,

(b) $R(A_0) = X_0 \cap D(A)$,

(c) $A_0 A = I \text{ on } X_0 \cap D(A)$,

(d) $AA_0 = I \text{ on } R(A)$.

Moreover, there are operators $F_1 \in B(X)$, $F_2 \in B(Y)$ such that

(e) $A_0 A = I - F_1 \text{ on } D(A)$,

(f) $AA_0 = I - F_2 \text{ on } Y$,

(g) $R(F_1) = N(A), \quad N(F_1) = X_0$,

(h) $R(F_2) = Y_0, \quad N(F_2) = R(A)$.

The proof of Theorem 7.1 is the same as that of Theorem 5.4. We also have

Theorem 7.2. *Let A be a densely defined closed linear operator from X to Y. Suppose there are operators A_1, $A_2 \in B(Y,X)$, $K_1 \in K(X)$, $K_2 \in K(Y)$ such that*

$$A_1 A = I - K_1 \text{ on } D(A), \tag{7.7}$$

and

$$AA_2 = I - K_2 \text{ on } Y. \tag{7.8}$$

Then $A \in \Phi(X,Y)$.

The proof is identical to that of Theorem 5.5. Note that for any operators, A, B, we define $D(BA)$ to be the set of those $x \in D(A)$ such that $Ax \in D(B)$. Corresponding to Theorem 5.7 we have

Theorem 7.3. *If $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$, then $BA \in \Phi(X,Z)$ and*

$$i(BA) = i(A) + i(B). \tag{7.9}$$

Proof. We must show that

(a) $D(BA)$ is dense in X,

(b) BA is a closed operator,

(c) $R(BA)$ is closed in Z,

(d) $\alpha(BA) < \infty$, $\beta(BA) < \infty$, and (7.9) holds.

The only part that can be carried over from the bounded case is (d). To prove (a), we first note that $D(A) \cap X_0$ is dense in X_0, where X_0 is any closed subspace of X satisfying (7.3). To prove this, let P be the operator which equals I on $N(A)$ and vanishes on X_0. Then P is in $B(X)$ by Lemma 5.2. Since $D(A)$ is dense in X, if $x_0 \in X_0$, then there is a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x_0$. Thus, $(I-P)x_n \to (I-P)x_0 = x_0$. But $(I-P)x_n \in X_0 \cap D(A)$ by (7.4). Hence, $D(A) \cap X_0$ is dense in X_0.

Since $N(A) \subset N(BA)$, it suffices to show that each element $x \in X_0 \cap D(A)$ can be approximated as closely as desired by an element of $X_0 \cap D(BA)$. Now $R(A) \cap D(B)$ is dense in $R(A)$. This can be seen by showing that we can take Y_0 in (7.5) to be contained in $D(B)$. In fact, we have

Lemma 7.4. *Let R be a closed subspace of a normed vector space X such that R° is of dimension $n < \infty$. Let D be any dense subspace of X. Then there is an n-dimensional subspace W of D such that $X = R \oplus W$.*

The proof of Lemma 7.4 will be given at the end of this section. Returning to our proof, if $x \in X_0 \cap D(A)$, then for any $\varepsilon > 0$ we can find a $y \in R(A) \cap D(B)$ such that $\|y - Ax\| < \varepsilon$. There is an $x_1 \in X_0 \cap D(A)$ such that $Ax_1 = y$. Hence, $x_1 \in D(BA) \cap X_0$, and by (7.6), $\|x_1 - x\| < C\varepsilon$. This proves (a).

To prove (b), suppose $\{x_n\}$ is a sequence in $D(BA)$ such that

$$x_n \longrightarrow x \text{ in } X \text{ and } BAx_n \longrightarrow z \text{ in } Z.$$

Now

$$Y = N(B) \oplus Y_1, \tag{7.10}$$

where Y_1 is a closed subspace of Y. Let Q be the operator which equals I on $N(B)$ and vanishes on Y_1. Then $Q \in B(Y)$ (Lemma 5.2). Thus, $B(I-Q)Ax_n \to z$. Hence, by (7.6) applied to B, there is $y_1 \in Y_1$ such that $(I-Q)Ax_n \to y_1$. We shall show that

$$\|QAx_n\| \leq C.$$

Assuming this for the moment, we know from the finite dimensionality of $N(B)$ that there is a subsequence of $\{x_n\}$ (which we assume is the whole sequence) such that QAx_n converges in Y to some element $y_2 \in N(B)$ (Corollary 4.5). Thus, $Ax_n \to y_1+y_2$. Since A is a closed operator, $x \in D(A)$ and $Ax = y_1 + y_2$. Since B is closed, $y_1 + y_2 \in D(B)$, and $B(y_1 + y_2) = z$. Hence, $x \in D(BA)$ and $Ax = z$.

To show that $\{QAx_n\}$ is a bounded sequence, suppose that $\gamma_n = \|QAx_n\| \to \infty$. Set $u_n = \gamma_n^{-1}QAx_n$. Then $\|u_n\| = 1$. Since $N(B)$ is finite-dimensional, there is a subsequence of $\{u_n\}$ (you guessed it; we assume it is the whole sequence) that converges to some element $u \in N(B)$. Moreover,

$$A(\gamma_n^{-1}x_n) - u_n = \gamma_n^{-1}(I - Q)Ax_n \longrightarrow 0,$$

since $(I - Q)Ax_n \to y_1$. Hence, $A(\gamma_n^{-1}x_n) \to u$. Since $\gamma_n^{-1}x_n \to 0$ and A is closed, we must have $u = 0$. But this is impossible, since

$$\|u\| = \lim \|u_n\| = 1.$$

This completes the proof of (b).

It remains to prove (c). To this end, we note that

$$Y = N(B) \oplus Y_2 \oplus Y_4 \tag{7.11}$$

(see Section 5.2). Now, $R(BA)$ is just the range of B on $Y_2 \cap D(B)$. If $y_n \in Y_2 \cap D(B)$ and $By_n \to z$ in Z, then by (7.6) applied to B, we have

$$\|y_n - y_m\| \le C\|B(y_n - y_m)\| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Since Y_2 is closed, y_n converges to some $y \in Y_2$. Since B is a closed operator, $y \in D(B)$ and $By = z$. Hence, $z \in R(BA)$. This shows that $R(BA)$ is closed, and the proof of the theorem is complete. □

In order to prove the counterparts of Theorems 5.10 and 5.11, we need a bit of preparation.

Lemma 7.5. *Suppose that $A \in \Phi(X,Y)$ and P is in $B(W,X)$. If P is one-to-one, $R(P) \supset D(A)$ and $P^{-1}(D(A))$ is dense in W, then $AP \in \Phi(W,Y)$, $\alpha(AP) = \alpha(A)$, and R(AP)= R(A). In particular, i(AP)=i(A).*

Proof. Since $D(AP) = P^{-1}(D(A))$, it is dense in W by assumption. Moreover, AP is a closed operator. For if $w_n \to w$ in W and $APw_n \to y$ in Y, then $Pw_n \to Pw$ in X. Since A is a closed operator, this shows that $Pw \in D(A)$ and $APw = y$. In other words, $w \in D(AP)$ and $APw = y$. Thus, AP is closed. Since P is a one-to-one map of $N(AP)$ onto $N(A)$, we see that the dimensions of these two spaces are equal (Lemma 5.8). The rest of the proof is trivial. □

We shall say that a normed vector space W is (or can be) *continuously embedded* in another normed vector space X if there is a one-to-one operator $P \in B(W,X)$. We can then "identify" each element $w \in W$ with the element

$Pw \in X$. If $R(P)$ is dense in X we say that W is dense in X. We can put Lemma 7.5 in the form

Corollary 7.6. *Assume that $A \in \Phi(X, Y)$ and W is continuously embedded in X in such a way that $D(A)$ is dense in W. Then $A \in \Phi(W, Y)$ with $N(A)$ and $R(A)$ the same.*

Another useful variation is

Lemma 7.7. *Assume that W is continuously embedded and dense in X. If $A \in \Phi(W, Y)$, then $A \in \Phi(X, Y)$ with $N(A)$ and $R(A)$ unchanged.*

Proof. Let P be the operator embedding W into X. Let Q be the linear operator from X to W with $D(Q) = R(P)$ defined by $Qx = w$ when $x = Pw$. Since P is one-to-one, Q is well defined. Moreover, one checks easily that $Q \in \Phi(X, W)$. Hence, by Theorem 7.3, $AQ \in \Phi(X, Y)$. But in the new terminology, AQ is the same as A. This completes the proof. $\square$

We now give the proof of Lemma 7.4.

Proof. By Lemma 5.3, there is an n-dimensional subspace V of X such that $X = R \oplus V$. Let $x_1, \cdots, x_n$ be a basis for V. Then by Lemma 5.2,

$$\|x_0\| + \sum_1^n |\alpha_k| \leq C\|x_0 + \sum_1^n \alpha_k x_k\|, \quad x_0 \in R, \ \alpha_k \ \text{scalars}, \tag{7.12}$$

where we have made use of the fact that all norms are equivalent on subspaces that are finite-dimensional (Theorem 4.2). Now by the density of D, there is for each k an element $\hat{x}_k \in D$ such that

$$\|x_k - \hat{x}_k\| \leq 1/2C.$$

Thus,

$$\|x_0\| + \sum_1^n |\alpha_k| \leq C\|x_0 + \sum_1^n \alpha_k \hat{x}_k\| + \frac{1}{2}\sum_1^n |\alpha_k|,$$

or

$$2\|x_0\| + \sum_1^n |\alpha_k| \leq 2C\|x_0 + \sum_1^n \alpha_k \hat{x}_k\|, \quad x_0 \in R, \ \alpha_k \ \text{scalars}. \tag{7.13}$$

The $\hat{x}_k$ are linearly independent, for if

$$\sum_1^n \alpha_k \hat{x}_k = 0,$$

then each $\alpha_k = 0$ by (7.13). Let W be the n-dimensional subspace spanned by the $\hat{x}_k$. Then $W \cap R = \{0\}$. For if $x \in R$ and

$$x = \sum_1^n \alpha_k \hat{x}_k,$$

then, by (7.13),

$$\|x\| + \sum_1^n |\alpha_k| \leq 2C\|x - x\| = 0.$$

We claim that $X = R \oplus W$. To see this, note that if $\tilde{x}$ is an element of X not in $R \oplus W$, let W_1 be the subspace spanned by $\tilde{x}$ and the $\hat{x}_k$. Then $W_1 \cap R = \{0\}$, and by Lemma 5.12, $\dim W_1 \leq \dim V = n$, which provides a contradiction. This completes the proof. □

7.2. Further properties

We now prove the counterparts of Theorems 5.10 and 5.11. First we have

Theorem 7.8. *If $A \in \Phi(X, Y)$ and K is in $K(X, Y)$, then $A + K \in \Phi(X, Y)$ and*

$$i(A + K) = i(A). \tag{7.14}$$

Proof. The fact that $A + K \in \Phi(X, Y)$ follows from Theorems 7.1 and 7.2 as before. To prove (7.14), we need a trick. Since A is closed, one can make $D(A)$ into a Banach space W by equipping it with the *graph norm*

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

Moreover, W is continuously embedded in X and is dense in X. Hence, $A \in \Phi(W, Y)$ (Lemma 7.7). In addition, an operator A_0, given by Theorem 7.1, is in $\Phi(Y, W)$ [see (a) and (b)]. Moreover, by (7.9),

$$i(A_0) + i(A + K) = i(I - F_2 + KA_0) = 0,$$

and we see that (7.14) holds. This completes the proof. □

We also have

Theorem 7.9. *For $A \in \Phi(X, Y)$, there is an $\eta > 0$ such that for every T in $B(X, Y)$ satisfying $\|T\| < \eta$, one has $A + T \in \Phi(X, Y)$,*

$$i(A + T) = i(A), \tag{7.15}$$

and

$$\alpha(A + T) \leq \alpha(A). \tag{7.16}$$

The proof of Theorem 7.9 is almost identical to that of Theorem 5.11.

A linear operator B from X to Y is called *A-compact* if $D(B) \supset D(A)$ and from every sequence $\{x_n\} \subset D(A)$ such that

$$\|x_n\|_{D(A)} \leq C,$$

one can extract a convergent subsequence from $\{Bx_n\}$. In other words, $B \in K(W, Y)$, where W is the Banach space formed from $D(A)$ by equipping it with the graph norm. We show now that this is all that is really needed in Theorem 7.8. For any operators A, B, we always take

$$D(A + B) = D(A) \cap D(B).$$

We have

Theorem 7.10. *If $A \in \Phi(X, Y)$ and B is A-compact, then $A+B \in \Phi(X, Y)$ and $i(A + B) = i(A)$.*

Proof. By Corollary 7.6, $A \in \Phi(W, Y)$. Since $B \in K(W, Y)$, we see that $A + B \in \Phi(W, Y)$ (Theorem 7.8). We now apply Lemma 7.7 to conclude that $A + B \in \Phi(X, Y)$. □

Next we have

Theorem 7.11. *If $A \in \Phi(X, Y)$, then there is an $\eta > 0$ such that for every linear operator B from X to Y satisfying $D(B) \supset D(A)$ and*

$$\|Bx\| \leq \eta(\|x\| + \|Ax\|), \quad x \in D(A),$$

we have $A + B \in \Phi(X, Y)$,

$$i(A + B) = i(A), \tag{7.17}$$

and

$$\alpha(A + B) \leq \alpha(A). \tag{7.18}$$

Proof. We introduce W as before and apply the same proof as in Theorem 7.9. □

As the counterpart of Theorem 5.13, we have, in one direction,

Theorem 7.12. *If $A \in \Phi(X, Y)$ and B is a densely defined closed linear operator from Y to Z such that $BA \in \Phi(X, Z)$, then $B \in \Phi(Y, Z)$.*

Proof. Here we must exercise a bit of care. By Lemma 7.4, there is a finite-dimensional subspace $Y_0 \subset D(B)$ such that (7.5) holds. Let A_0 be an operator given by Theorem 7.1. Because $Y_0 \subset D(B)$, AA_0 maps $D(B)$ into itself. Hence, $D(BAA_0) = D(B)$. As before, let W be $D(A)$ done up as a Banach space under the graph norm. Then $A_0 \in \Phi(Y, W)$. We claim that $BA \in \Phi(W, Z)$. Assuming this for the moment, we see that $BAA_0 \in \Phi(Y, Z)$. Now, by Theorem 7.1(f),

$$BAA_0 = B - BF_2,$$

where $F_2 \in B(Y)$ and $R(F_2) = Y_0$. Since $Y_0 \subset D(B)$, B is defined everywhere on Y_0 and, hence, bounded there. This means that the operator $BF_2 \in K(Y, Z)$. Applying Theorem 7.8, we see that $BAA_0 + BF_2 \in \Phi(Y, Z)$. But this is precisely the operator B.

Therefore, it remains only to prove that $BA \in \Phi(W, Z)$. This follows from Corollary 7.6 if we can show that $D(BA)$ is dense in W. Now, if $x \in W$, then $F_1 x \in N(A) \subset D(BA)$. Moreover, for any $\varepsilon > 0$, there is a $y \in R(A) \cap D(B)$ such that $\|y - Ax\| < \varepsilon$. This follows from (7.5) and the fact that $Y_0 \subset D(B)$. Since $y \in R(A)$, there is an $\hat{x} \in W$ such that $A\hat{x} = y$. Clearly, $\hat{x} \in D(BA)$. Moreover, by (7.6),

$$\|(I - F_1)(\hat{x} - x)\| \le C\|A(\hat{x} - x)\| \le C\varepsilon.$$

Set

$$\tilde{x} = F_1 x + (I - F_1)\hat{x} \in D(BA).$$

Then $A\tilde{x} = A\hat{x} = y$. Hence,

$$\|\tilde{x} - x\| + \|A(\tilde{x} - x)\| = \|(I - F_1)(\hat{x} - x)\| + \|y - Ax\| \le (C + 1)\varepsilon.$$

This completes the proof. □

In the other direction, Theorem 5.13 does not have a strict counterpart. All that can be said is the following:

Theorem 7.13. *If $B \in \Phi(Y, Z)$ and A is a densely defined closed linear operator from X to Y such that $BA \in \Phi(X, Z)$, then the restriction of A to D(BA) is in $\Phi(X, V)$, where V is D(B) equipped with the graph norm. Thus, if B is in B(Y, Z), then $A \in \Phi(X, Y)$.*

Proof. By Theorem 7.1, there is an operator $B_0 \in B(Z, Y)$ such that

$$B_0 B = I - F_3 \quad \text{on} \quad D(B),$$

where $F_3 \in B(Y)$ and $R(F_3) = N(B)$. Thus,

$$B_0 BA = A - F_3 A \quad \text{on} \quad D(BA).$$

Now $BA \in \Phi(D(BA), Z)$ (Corollary 7.6) and $B_0 \in \Phi(Z, V)$ (Theorem 7.1). Hence, $B_0 BA \in \Phi(D(BA), V)$ (Theorem 7.3). Moreover, we know that $A \in B(D(A), Y)$ and $F_3 \in B(Y, N(B))$. Consequently, $F_3 A \in B(D(A), N(B))$, and it is compact on $D(BA)$. Thus, the restriction of A to $D(BA)$ is in $\Phi(D(BA), V)$. By Lemma 7.7, it is in $\Phi(X, V)$. □

To see that one cannot conclude that $A \in \Phi(X,Y)$, let $X = Y = Z = C = C[0,1]$, and let $B = d/dt$ with $D(B)$ consisting of continuously differentiable functions. Then $N(B)$ consists of the constant functions, and $R(B) = C$. Clearly, $D(B)$ is dense in C. Moreover, B is a closed operator. If $\{x_n\}$ is a sequence of functions in $D(B)$ such that

$$x_n(t) \longrightarrow x(t), \quad x_n'(t) \longrightarrow y(t)$$

uniformly in [0,1], then

$$\int_0^t x_n'(s)\,ds \longrightarrow \int_0^t y(s)\,ds$$

uniformly in [0,1]. But the left-hand side is just $x_n(t) - x_n(0)$. Hence,

$$x(t) - x(0) = \int_0^t y(s)\,ds,$$

showing that $x \in D(B)$ and $Bx = y$. Thus, B is in $\Phi(C)$. Next, define the operator A by

$$Ax = \int_0^t x(s)\,ds.$$

Then one checks easily that $BA = I$, and hence, is in $\Phi(C)$. But it is not true that $A \in \Phi(C)$.

An interesting variation of Theorem 5.14 is

Theorem 7.14. *Let A be a densely defined closed linear operator from X to Y. Suppose B is in $B(Y,Z)$ with $\alpha(B) < \infty$ and $BA \in \Phi(X,Z)$. Then $A \in \Phi(X,Y)$.*

Proof. We have $R(B) \supset R(BA)$, which is closed and such that $R(BA)^\circ$ is finite-dimensional. Hence, $R(B)$ is closed (Lemmas 5.3 and 5.6), and $R(B)^\circ \subset R(BA)^\circ$ is finite-dimensional. Thus, $B \in \Phi(Y,Z)$. We now apply Theorem 7.13, making use of the fact that B is bounded. $\square$

7.3. Operators with closed ranges

In Section 3.5, we proved that if $A \in B(X,Y)$ and $R(A)$ is closed in Y, then $R(A')$ is closed in X'. If we examine our proof there, we will observe that it applies equally well to closed operators. We record this fact as

Theorem 7.15. *Let A be a densely defined closed linear operator from X to Y. If $R(A)$ is closed in Y, then $R(A') = N(A)^\circ$, and hence is closed in X'.*

Conversely, if we know that $R(A')$ is closed in X', does it follow that $R(A)$ is closed in Y? An affirmative answer is given by

Theorem 7.16. *If A is a densely defined closed linear operator from X to Y and $R(A')$ is closed in X', then $R(A) = {}^\circ N(A')$, and hence, is closed in Y.*

The proof of Theorem 7.16 is a bit involved and requires a few steps. We begin by first showing that an adjoint operator is always closed. If $y'_n \in D(A')$ and $y'_n \to y'$ in Y' with $A'y'_n \to x'$ in X', then for $x \in D(A)$,

$$A'y'_n(x) = y'_n(Ax),$$

and hence,

$$x'(x) = y'(Ax), \quad x \in D(A).$$

This shows that $y' \in D(A')$ and $A'y' = x'$. Hence, A' is closed. Once this is known, we can apply Theorem 3.14 to conclude that there is a number $r > 0$ such that

$$rd(y', N(A')) \leq \|A'y'\|, \quad y' \in D(A). \tag{7.19}$$

Let S be the set of those $x \in D(A)$ such that $\|x\| \leq 1$. We shall show that

(i) If $y \in {}^\circ N(A')$ and $\|y\| < r$, then $y \in \overline{A(S)}$.

(ii) If $y \in \overline{A(S)}$ for all $y \in {}^\circ N(A')$ such that $\|y\| < r$, then $y \in A(S)$ for all such y.

Clearly, the theorem follows from (i) and (ii). For if $y \in {}^\circ N(A')$, then $\tilde{y} = ry/2\|y\|$ satisfies $\|\tilde{y}\| < r$, and hence, there is an $\tilde{x} \in S$ such that $A\tilde{x} = \tilde{y}$. If we set $x = 2\|y\|\tilde{x}/r$, then $Ax = y$. Thus, $y \in R(A)$.

The proof of (ii) is very similar to that of the closed graph theorem (Theorem 3.10). Suppose $y \in {}^\circ N(A')$ and $\|y\| < r$. Then there is an $\varepsilon > 0$ such that $\hat{y} = y/(1-\varepsilon)$ also satisfies $\|\hat{y}\| < r$. Let $S(\alpha)$ be the set of those $x \in D(A)$ such that $\|x\| \leq \alpha$. If we can show that $\hat{y} \in A(S[(1-\varepsilon)^{-1}])$, then it follows that $y \in A(S)$. Now if $y \in \overline{A(S)}$ for each $y \in {}^\circ N(A')$ such that $\|y\| < r$, then, clearly, $y \in \overline{A(S(\varepsilon^n))}$ for all $y \in {}^\circ N(A')$ such that $\|y\| < r\varepsilon^n$. In particular, there is an $x_0 \in S = S(1)$ such that $\|\hat{y} - Ax_0\| < r\varepsilon$. Hence, there is an $x_1 \in S(\varepsilon)$ such that $\|\hat{y} - Ax_0 - Ax_1\| < r\varepsilon^2$.

Continuing, we have a sequence $\{x_n\}$ of elements such that

$$x_n \in S(\varepsilon^n) \textit{ and } \|\hat{y} - \sum_0^n Ax_k\| < r\varepsilon^{n+1}. \tag{7.20}$$

In particular, we have

$$\sum_0^\infty \|x_n\| \leq \frac{1}{1-\varepsilon}, \tag{7.21}$$

showing that

$$z_n = \sum_0^n x_k$$

is a Cauchy sequence in X. Hence, $z_n \to z \in X$. Moreover, $Az_n \to \hat{y}$, and since A is a closed operator, we have $z \in D(A)$ and $Az = \hat{y}$. By (7.21), $\|z\| \leq 1/(1-\varepsilon)$. Hence, $z \in S[(1-\varepsilon)^{-1}]$, and (ii) is proved.

The proof of (i) is more involved. It depends on the fact that $A(S)$ is convex. A subset U of a normed vector space is called *convex* if $\alpha x + (1-\alpha)y$ is in U for each $x, y \in U$, $0 < \alpha < 1$. Clearly, the closure of a convex set is convex. We shall use the following consequence of the Hahn-Banach theorem. It is sometimes referred to as the "geometric form of the Hahn-Banach Theorem."

Theorem 7.17. *If U is a closed, convex subset of a normed vector space X and $x_0 \in X$ is not in U, then there is an $x' \in X'$ such that*

$$\Re e\, x'(x_0) \geq \Re e\, x'(x), \quad x \in U, \tag{7.22}$$

and $\Re e\, x'(x_0) \neq \Re e\, x'(x_1)$ for some $x_1 \in U$.

Before proving Theorem 7.17, let us show how it implies (i). We first note that $A(S)$ and, hence, $\overline{A(S)}$ are convex. Thus if $y \notin \overline{A(S)}$, then by Theorem 7.17, there is a nonvanishing functional $y' \in Y'$ such that

$$\Re e\, y'(y) \geq \Re e\, y'(Ax), \quad x \in S.$$

If $x \in S$, set $y'(Ax) = |y'(Ax)|e^{i\theta}$. Then $xe^{-i\theta} \in S$, and hence,

$$\Re e\, y'(y) \geq \Re e\, y'(e^{-i\theta}Ax) = |y'(Ax)|, \quad x \in S.$$

Thus, $y'(y) \neq 0$ and

$$|y'(Ax)| \leq |y'(y)| \cdot \|x\|, \quad x \in D(A).$$

This shows that $y' \in D(A')$ and

$$|A'y'(x)| \leq |y'(y)| \cdot \|x\|, \quad x \in D(A),$$

or

$$\|A'y'\| \leq |y'(y)|. \tag{7.23}$$

On the other hand, if $y \in {}^\circ N(A')$ and $y' \in Y'$, then

$$|y'(y)| = |(y' - y_0')(y)| \leq \|y\| \cdot \|y' - y_0'\|, \quad y_0' \in N(A').$$

Since this is true for all $y_0' \in N(A')$, we have, by (7.19),

$$|y'(y)| \leq \|y\| d(y', N(A')) \leq r^{-1}\|y\| \cdot \|A'y'\|. \tag{7.24}$$

Combining (7.23) and (7.24), we get $\|y\| \geq r$. Thus, we have shown that if $y \in {}^\circ N(A')$ and $y \notin \overline{A(S)}$, then $\|y\| \geq r$. This is equivalent to (i).

It remains to prove Theorem 7.17. To do this, we introduce a few concepts. Let U be a convex subset of a normed vector space X. Assume that 0 is an interior point of U, i.e., there is an $\varepsilon > 0$ such that all x satisfying $\|x\| < \varepsilon$ are in U. For each $x \in X$, set

$$p(x) = \inf_{\alpha > 0,\, \alpha x \in U} \alpha^{-1}.$$

This is called the *Minkowski functional* of U. Since $\alpha x \in U$ for α sufficiently small, $p(x)$ is always finite. Other properties are given by

Lemma 7.18. *$p(x)$ has the following properties:*

(a) $p(x + y) \leq p(x) + p(y), \quad x, y \in X;$

(b) $p(\alpha x) = \alpha p(x), \quad x \in X,\ \alpha > 0;$

(c) $p(x) < 1$ *implies that* x *is in* U*;*

(d) $p(x) \leq 1$ *for all* x *in* U*.*

Proof. (a) Suppose $\alpha x \in U$ and $\beta y \in U$, where $\alpha > 0$ and $\beta > 0$. Since U is convex,

$$\frac{\alpha^{-1}\alpha x + \beta^{-1}\beta y}{\alpha^{-1} + \beta^{-1}} = \frac{x + y}{\alpha^{-1} + \beta^{-1}}$$

is in U. Hence,

$$p(x + y) \leq \alpha^{-1} + \beta^{-1}.$$

Since this is true for all $\alpha > 0$ such that $\alpha x \in U$ and $\beta > 0$ such that $\beta y \in U$, (a) follows.

To prove (b) we first note that $p(0) = 0$. If $\alpha > 0$, then

$$p(\alpha x) = \inf_{\beta > 0,\, \alpha\beta x \in U} \beta^{-1} = \alpha \inf_{r > 0,\, r x \in U} r^{-1} = \alpha p(x).$$

Concerning (c), assume that $p(x) < 1$. Then there is an $\alpha > 1$ such that $\alpha x \in U$. Since U is convex and $0 \in U$, we see that $x \in U$. Moreover, if $x \in U$, $1x \in U$, and hence, $p(x) \leq 1$. This proves (d). $\square$

Now we can give the proof of Theorem 7.17.

Proof. Since U is closed and $x_0 \notin U$, there is an $\eta > 0$ such that $\|x - x_0\| < \eta$ implies that $x \notin U$. Let u_0 be a point of U, and let V be the set of all sums of the form

$$v = u + y - u_0,$$

where $u \in U$ and $\|y\| < \eta$. Clearly, V is a convex set. Moreover, it contains all y such that $\|y\| < \eta$. Hence, 0 is an interior point, showing that the Minkowski functional $p(x)$ for V is defined.

Assume for the moment that X is a real vector space, and set $w_0 = x_0 - u_0$. For all vectors of the form αw_0, define the linear functional

$$f(\alpha w_0) = \alpha p(w_0).$$

For $\alpha > 0$, we have $f(\alpha w_0) = p(\alpha w_0)$ by (b) of Lemma 7.18. For $\alpha < 0$, we have

$$f(\alpha w_0) = \alpha p(w_0) \leq 0 \leq p(\alpha w_0).$$

Hence,

$$f(\alpha w_0) \leq p(\alpha w_0), \quad \alpha \text{ real}.$$

Properties (a) and (b) of Lemma 7.18 show that $p(x)$ is a sublinear functional. We can now apply the Hahn-Banach theorem (Theorem 2.5) to conclude that there is a linear functional $F(x)$ on X such that

$$F(\alpha w_0) = \alpha p(w_0), \quad \alpha \text{ real},$$

and

$$F(x) \leq p(x), \quad x \in X.$$

The functional $F(x)$ is bounded. For if x is any element of X, then $y = \eta x/2\|x\|$ satisfies $\|y\| < \eta$, and hence, is in V. Thus, $p(y) \leq 1$ by Lemma 7.18. Hence, $F(y) \leq 1$, or $F(x) \leq 2\eta^{-1}\|x\|$.

Now, $w_0 \notin V$. Hence,

$$F(x_0) - F(u_0) = p(w_0) \geq 1. \tag{7.25}$$

On the other hand, if $u \in U$, then $u - u_0 \in V$, and, hence,

$$F(u) - F(u_0) \leq p(u - u_0) \leq 1.$$

Thus,

$$F(x_0) \geq F(u), \quad u \in U.$$

This proves (7.22) for the case when X is a real vector space. The complex case is easily resolved. In fact, we first treat X as a real space and find $F(x)$ as above. We then set

$$G(x) = F(x) - iF(ix)$$

and verify, as in the proof of Theorem 6.26, that $G(x)$ is a complex, bounded linear functional on X. Since, $\Re e\, G(x) = F(x)$, (7.22) is proved. The last statement follows from (7.25). $\square$

As a consequence of Theorem 7.16, we have

Theorem 7.19. *If A is in $B(X,Y)$, then $A \in \Phi(X,Y)$ if and only if $A' \in \Phi(Y',X')$.*

Proof. That $A \in \Phi(X,Y)$ implies that $A' \in \Phi(Y',X')$ is Theorem 5.15. If $A' \in \Phi(Y',X')$, then $\beta(A) = \alpha(A') < \infty$ and $\alpha(A) \le \alpha(A'') = \beta(A') < \infty$ by (5.24). We now use Theorem 7.16 to conclude that $R(A)$ is closed, and the proof is complete. $\square$

What can be said if A is not in $B(X,Y)$? We shall discuss this in the next section.

7.4. Total subsets

Suppose $A \in \Phi(X,Y)$. What can be said about A'? Of course, $\alpha(A') = \beta(A) < \infty$. If A'' exists, then the proof that $\beta(A') = \alpha(A)$ is the same in the unbounded case as in the bounded case (see Section 5.4). As we just noted in the last section, adjoint operators are always closed. Moreover, $R(A')$ is closed by Theorem 7.15. Thus, the only thing needed for A' to be in $\Phi(Y',X')$ is the density of $D(A')$ in Y'. For this to be true, the only element of $y \in Y$ which can annihilate $D(A')$ is the zero element of Y. A subset of Y' having this property is called *total.* We have the following.

Theorem 7.20. *Let A be a closed linear operator from X to Y with $D(A)$ dense in X. Then $D(A')$ is total in Y'.*

We shall give the simple proof of Theorem 7.20 at the end of the section.

We should note that there may be total subsets of Y' that are not dense in Y'. The reason for this is as follows: We know that a subset $W \subset Y'$ is dense in Y' if and only if the only element $y'' \in Y''$ which annihilates W is the zero element of Y''. Now you may recall (see Section 5.4) that we have shown that there is a mapping $J \in B(Y,Y'')$ such that

$$Jy(y') = y'(y), \quad y \in Y,\ y' \in Y'. \tag{7.26}$$

According to the terminology of Section 7.1, this gives a continuous embedding of Y into Y''. Now from (7.26) we see that $W \subset V'$ is total if and only if the only element of $R(J)$ which annihilates W is 0. If $R(J)$ is not the whole of Y'', it is conceivable that there is a $y'' \neq 0$ which is not in $R(J)$ which annihilates W. We shall show that this, indeed, can happen.

Of course, this situation cannot occur if $R(J) = Y''$. In this case, we say that Y is *reflexive*. Thus, we have

Lemma 7.21. *If Y is a reflexive Banach space, then a subset W of Y' is total if and only if it is dense in Y'.*

Combining Theorem 7.20 and Lemma 7.21, we have

Theorem 7.22. *If $A \in \Phi(X, Y)$ and Y is reflexive, then $A' \in \Phi(Y', X')$ and $i(A') = -i(A)$.*

The converse is much easier. In fact, we have

Theorem 7.23. *Let A be a closed linear operator from X to Y with D(A) dense in X. If $A' \in \Phi(Y', X')$, then $A \in \Phi(X, Y)$ with $i(A) = -i(A')$.*

Proof. Clearly, $\beta(A) = \alpha(A') < \infty$, and by (5.24), we have

$$\alpha(A) \leq \beta(A') < \infty.$$

We have just shown that $R(A)$ is closed (Theorem 7.16). Thus $A \in \Phi(X, Y)$. Now we know that $\alpha(A) = \beta(A')$ by (5.23). □

Now we give the proof of Theorem 7.20.

Proof. Let $y_0 \neq 0$ be any element of Y. Since A is a closed operator, its graph G_A is a closed subspace of $X \times Y$, and it does not contain the element $\langle 0, y_0 \rangle$ (see Section 3.4). Hence, by Theorem 2.9, there is a functional $z' \in (X \times Y)'$ which annihilates G_A and satisfies $z'\langle 0, y_0 \rangle \neq 0$. Set

$$x'(x) = z'\langle x, 0 \rangle, \quad x \in X,$$

and

$$y'(y) = z'\langle 0, y \rangle, \quad y \in Y.$$

Then, clearly, $x' \in X'$ and $y' \in Y'$. Since $z'(G_A) = 0$, we have, for any $x \in D(A)$,

$$z'\langle x, Ax \rangle = 0$$

or, equivalently,

$$y'(Ax) = -x'(x), \quad x \in D(A).$$

This shows that $y' \in D(A')$. Moreover, $y'(y_0) \neq 0$. Thus, y_0 does not annihilate $D(A')$. Since y_0 was any nonvanishing element of Y, it follows that $D(A')$ is total. This completes the proof. □

7.5. The essential spectrum

Let A be a linear operator on a normed vector space X. We say that $\lambda \in \rho(A)$ if $R(A-\lambda)$ is dense in X and there is a $T \in B(X)$ such that

$$T(A-\lambda) = I \text{ on } D(A), \quad (A-\lambda)T = I \text{ on } R(A-\lambda). \tag{7.27}$$

Otherwise, $\lambda \in \sigma(A)$. As before, $\rho(A)$ and $\sigma(A)$ are called the *resolvent set* and *spectrum* of A, respectively. To show the relationship of this definition to the one given in Section 6.1, we note the following.

Lemma 7.24. *If X is a Banach space and A is closed, then $\lambda \in \rho(A)$ if and only if*

$$\alpha(A-\lambda) = 0, \quad R(A-\lambda) = X. \tag{7.28}$$

Proof. The "if" part follows from the bounded inverse theorem (Theorem 3.11). To prove the "only if" part, we first note that

$$T(A-\lambda)x = x \text{ for } x \in D(A).$$

Hence, if $(A-\lambda)x = 0$, then $x = 0$. Secondly, if x is any element of X, then there is a sequence $\{x_n\} \subset R(A-\lambda)$ such that $x_n \to x$. Hence, $Tx_n \to Tx$. But $(A-\lambda)Tx_n = x_n \to x$. Since A is a closed operator, $Tx \in D(A)$, and $(A-\lambda)Tx = x$. This shows that $x \in R(A-\lambda)$. Hence, $R(A-\lambda) = X$, and the proof is complete. □

Throughout the remainder of this section, we shall assume that X is a Banach space, and that A is a densely defined, closed linear operator on X. We ask the following question: What points of $\sigma(A)$ can be removed from the spectrum by the addition to A of a compact operator? The answer to this question is closely related to the set Φ_A. As before, we define this to be the set of all scalars λ such that $A - \lambda \in \Phi(X)$. We have

Theorem 7.25. *The set Φ_A is open, and $i(A-\lambda)$ is constant on each of its components.*

Proof. That Φ_A is an open set follows as in the proof of Theorem 6.3 (we use Theorem 7.9 here). To show that the index is constant on each component, let λ_1, λ_2 be any two points in Φ_A which are connected by a smooth curve C whose points are all in Φ_A. Then for each $\lambda \in C$, there is an $\varepsilon > 0$ such that $\mu \in \Phi_A$ and $i(A-\mu) = i(A-\lambda)$ for all μ satisfying $|\mu - \lambda| < \varepsilon$. By the Heine-Borel theorem, there is a finite number of such sets which cover C. Since each of these sets overlaps with at least one other and $i(A-\mu)$ is constant on each one, we see that $i(A-\lambda_1) = i(A-\lambda_2)$. This completes the proof. □

We also have

Theorem 7.26. *$\Phi_{A+K} = \Phi_A$ for all K which are A-compact, and $i(A+K-\lambda) = i(A-\lambda)$ for all $\lambda \in \Phi_A$.*

Proof. The proof is an immediate consequence of Theorem 7.10. $\square$

Set

$$\sigma_e(A) = \bigcap_{K \in K(X)} \sigma(A+K).$$

We call $\sigma_e(A)$ the *essential spectrum* of A. It consists of those points of $\sigma(A)$ which cannot be removed from the spectrum by the addition to A of a compact operator. We now characterize $\sigma_e(A)$.

Theorem 7.27. *$\lambda \notin \sigma_e(A)$ if and only if $\lambda \in \Phi_A$ and $i(A-\lambda) = 0$.*

Proof. If $\lambda \notin \sigma_e(A)$, then there is a $K \in K(X)$ such that $\lambda \in \rho(A+K)$. In particular, $\lambda \in \Phi_{A+K}$ and $i(A+K-\lambda) = 0$ (Lemma 7.24). Adding the operator $-K$ to $A+K$, we see that $\lambda \in \Phi_A$ and $i(A-\lambda) = 0$ (Theorem 7.26). To prove the converse, suppose that $\lambda \in \Phi_A$ and that $i(A-\lambda) = 0$. Without loss of generality, we may assume $\lambda = 0$. Let $x_1, \cdots, x_n$ be a basis for $N(A)$ and $y_1', \cdots, y_n'$ be a basis for $R(A)^\circ$. Then by Lemma 4.14, there are $x_1', \cdots, x_n' \in X'$, $y_1, \cdots, y_n \in Y$ such that

$$x_j'(x_k) = \delta_{jk},\ y_j'(y_k) = \delta_{jk}, \quad 1 \leq j,\ k \leq n. \tag{7.29}$$

Set

$$Fx = \sum_1^n x_k'(x) y_k, \quad x \in X. \tag{7.30}$$

Then F is an operator of finite rank on X. We have picked it in such a way that

$$N(A) \cap N(F) = \{0\}, \quad R(A) \cap R(F) = \{0\}. \tag{7.31}$$

To show this, note that if $x \in N(A)$, then

$$x = \sum_1^n \alpha_k x_k,$$

and hence,

$$x_j'(x) = \alpha_j, \quad 1 \leq j \leq n.$$

On the other hand, if $x \in N(F)$, then

$$x_j'(x) = 0, \quad 1 \leq j \leq n.$$

This proves the first relation in (7.31). The second is similar. In fact, if $y \in R(F)$, then

$$y = \sum_1^n \alpha_k y_k,$$

and hence,

$$y_j'(y) = \alpha_j, \quad 1 \le j \le n.$$

But if $y \in R(A)$, then

$$y_j'(y) = 0, \quad 1 \le j \le n.$$

This gives the second relation in (7.31). Now $F \in K(X)$. Hence, $0 \in \Phi_{A+F}$ and $i(A+F) = 0$. If $x \in N(A+F)$, then Ax is in $R(A) \cap R(F)$ and hence, must vanish. This in turn shows that x is in $N(A) \cap N(F)$, and hence, $x = 0$. Thus, $\alpha(A+F) = 0$, showing that $R(A+F) = X$. Hence, $0 \in \rho(A+F)$, and the proof is complete. □

As a corollary to Theorems 7.26 and 7.27 we have

Theorem 7.28. *If B is A-compact, then*

$$\sigma_e(A+B) = \sigma_e(A).$$

7.6. Unbounded semi-Fredholm operators

Let $\Phi_+(X, Y)$ denote the set of all closed linear operators from X to Y, such that $D(A)$ is dense in X, $R(A)$ is closed in Y and $\alpha(A) < \infty$. These are semi-Fredholm operators that are not necessarily bounded. Many of the results of Section 5.6 hold for these operators as well. In particular, we have

Theorem 7.29. *Let A be a closed linear operator from X to Y with domain $D(A)$ dense in X. Then $A \in \Phi_+(X, Y)$ if and only if there is a seminorm $|\cdot|$ defined on $D(A)$, which is compact relative to the graph norm of A, such that*

$$\|x\| \le C\|Ax\| + |x|, \quad x \in D(A). \tag{7.32}$$

Proof. If $A \in \Phi_+(X, Y)$, then we can write

$$X = N(A) \oplus X_0, \tag{7.33}$$

where X_0 is a closed subspace of X (Lemma 5.1). Let P be the projection of X onto $N(A)$ along X_0, i.e., the operator defined by

$$P = \begin{cases} I & \text{on } N(A), \\ 0 & \text{on } X_0. \end{cases} \tag{7.34}$$

Then P is in $B(X)$ by Lemma 5.2. It now follows that

$$\|(I-P)x\| \le C\|Ax\|, \quad x \in D(A) \tag{7.35}$$

[see the proof of (5.35)]. This gives a stronger form of (7.32) inasmuch as $\|Px\|$ is defined on the whole of X and is compact relative to its norm.

Conversely, assume that (7.32) holds. Then $\alpha(A) < \infty$, as in the proof of Theorem 5.21. Also, there is a $P \in B(X)$ satisfying (7.34), and (7.35) again holds as in the proof of Theorem 5.21. This in turn implies that $A \in \Phi_+(X, Y)$ as in the proof of Lemma 5.20. $\square$

Before proving results similar to those of Section 7.2 for operators in $\Phi_+(X, Y)$, let us make some observations. Let A be a closed operator from X to Y, and let B be a linear operator from X to Y which is A-compact. Then, we claim that

(a) $A + B$ is closed

and

(b) B is $(A + B)$-compact.

Both of these statements follow from the inequality

(c) $\|Bx\| \le C_1(\|x\| + \|Ax\|) \le C_2(\|x\| + \|(A + B)x\|), \quad x \in D(A).$

In fact, if $x_n \to x$ in X and $(A + B)x_n \to y$ in Y, then (c) shows that $\{Bx_n\}$ and $\{Ax_n\}$ are Cauchy sequences in Y. Since Y is complete, Bx_n converges to some element $z \in Y$, and $Ax_n \to y - z$ in Y. Since A is a closed operator, we see that $x \in D(A)$ and $Ax = y - z$. Again, by (c), we have

$$\|B(x_n - x)\| \le C_1(\|x_n - x\| + \|A(x_n - x)\|),$$

showing that $Bx_n \to Bx$ in Y. Hence, $z = Bx$ and $(A + B)x = y$. This proves (a). To prove (b), let $\{x_n\}$ be a sequence in $D(A)$ such that

$$\|x_n\| + \|(A + B)x_n\| \le C_3.$$

By (c), we see that

$$\|x_n\| + \|Ax_n\| \le C_2C_3.$$

Since B is A-compact, $\{Bx_n\}$ has a convergent subsequence.

It thus remains to prove (c).

Proof. If the left-hand inequality were not true, there would be a sequence $\{x_n\}$ in $D(A)$ such that

$$\|Bx_n\| \longrightarrow \infty, \quad \|x_n\| + \|Ax_n\| \le C_4.$$

Clearly, this contradicts the A-compactness of B, since $\{Bx_n\}$ can have no convergent subsequence. If the right-hand inequality did not hold, then there would be a sequence $\{x_n\}$ in $D(A)$ such that

$$\|x_n\| + \|Ax_n\| = 1, \quad \|x_n\| + \|(A + B)x_n\| \longrightarrow 0.$$

Since B is A-compact, $\{Bx_n\}$ has a convergent subsequence (which we assume to be the whole sequence). Thus, $Bx \to y$ in Y. This means that $Ax_n \to -y$. Since A is a closed operator and $x_n \to 0$, we must have $y = 0$. But this would mean that $\|x_n\| + \|Ax_n\| \to 0$. This completes the proof. $\square$

Next, we have

Theorem 7.30. *If $A \in \Phi_+(X, Y)$ and B is A-compact, then $A + B \in \Phi_+(X, Y)$.*

Proof. By Theorem 7.29,

$$\|x\| \leq C\|(A + B)x\| + |x| + C\|Bx\|, \quad x \in D(A).$$

Set

$$|x|_0 = |x| + C\|Bx\|.$$

Then $|\cdot|_0$ is a seminorm defined on $D(A)$ which is compact relative to the graph norm of A. Hence, $A + B \in \Phi_+(X, Y)$ by Theorem 7.29. $\square$

Theorem 7.31. *If $A \in \Phi_+(X, Y)$, then there is an $\eta > 0$ such that $A+B \in \Phi_+(X, Y)$ and*

$$\alpha(A + B) \leq \alpha(A) \tag{7.36}$$

for each linear operator B from X to Y with $D(B) \supset D(A)$ satisfying

$$\|Bx\| \leq \eta(\|x\| + \|Ax\|), \quad x \in D(A). \tag{7.37}$$

Proof. By (7.35),

$$\|x\| + \|Ax\| \leq (C + 1)\|(A + B)x\| + \|Px\| + (C + 1)\|Bx\|$$

$$\leq (C + 1)\|(A + B)x\| + \|Px\| + (C + 1)\eta(\|x\| + \|Ax\|)$$

for $x \in D(A)$. Take $\eta = 1/2(C + 1)$. Then

$$\|x\| + \|Ax\| \leq 2(C + 1)\|(A + B)x\| + 2\|Px\|, \quad x \in D(A), \tag{7.38}$$

which shows that $A+B \in \Phi_+(X, Y)$ by Theorem 7.29. Moreover, by (7.38),

$$\|x\| \leq 2(C + 1)\|(A + B)x\|, \quad x \in X_0 \cap D(A), \tag{7.39}$$

which shows that $N(A + B) \cap X_0 = \{0\}$. Thus, (7.36) follows from Lemma 5.12. $\square$

Theorem 7.32. *If $A \in \Phi_+(X, Y)$, $B \in \Phi_+(Y, Z)$ and $D(BA)$ is dense in X, then $BA \in \Phi_+(X, Z)$.*

Proof. That BA is a closed operator follows as in the proof of Theorem 7.3 (in fact, there we used only the facts that $\alpha(B) < \infty$, $R(B)$ is closed, and that A and B are closed). To prove the rest, we note that

$$\|x\| \leq C_1\|Ax\| + |x|_1, \quad x \in D(A),$$

and

$$\|y\| \leq C_2\|By\| + |y|_2, \quad y \in D(B),$$

where $|\cdot|_1$ and $|\cdot|_2$ are seminorms that are defined on $D(A)$ and $D(B)$, respectively, and compact relative to the graph norms of A and B, respectively (Theorem 7.29). Thus,

$$\|Ax\| \leq C_2\|BAx\| + |Ax|_2, \quad x \in D(BA), \tag{7.40}$$

and hence,

$$\|x\| \leq C_1C_2\|BAx\| + C_1|Ax|_2 + |x|_1, \quad x \in D(BA). \tag{7.41}$$

We note that

$$\|Ax\| \leq C_3(\|x\| + \|BAx\|), \quad x \in D(BA). \tag{7.42}$$

Assuming this for the moment, we set

$$|x|_3 = C_1|Ax|_2 + |x|_1.$$

Then $|\cdot|_3$ is a seminorm defined on $D(BA)$ and is compact relative to the graph norm of BA. For if $\{x_n\}$ is a sequence in $D(BA)$ such that

$$\|x_n\| + \|BAx_n\| \leq C_4,$$

then $\|Ax_n\| \leq C_3C_4$ by (7.42), and hence there is a subsequence (assumed to be the whole sequence) such that

$$|x_n - x_m|_1 \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Of this sequence, there is a subsequence (need I say it?) such that

$$|Ax_n - Ax_m|_2 \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

This now gives the desired result by Theorem 7.29.

Thus, it remains to prove (7.42). Since BA is a closed operator, $D(BA)$ can be made into a Banach space U by equipping it with the graph norm. Now, the restriction of A to $D(BA)$ is a closed linear operator from U to Y which is defined on the whole of U. Hence, $A \in B(U, Y)$ by the closed graph theorem (Theorem 3.10). This is precisely the statement of (7.42). The proof is complete. $\square$

To see that we cannot conclude that $D(BA)$ is dense in X as we did in Theorem 7.3, we use the example given after Theorem 7.13. Let B be defined as it was there, and let $w(t)$ be any function in C that is not in $D(B)$ (take a function which does not have a continuous derivative). Let

N be the one-dimensional subspace of C generated by w, and let A be the operator from N to C defined by $Ax = x$, $x \in N$. Then $A \in \Phi_+(N, C)$ and $B \in \Phi(C)$. But $D(BA) = \{0\}$.

The following is useful.

Lemma 7.33. *$A \notin \Phi_+$ if and only if there is a bounded sequence $\{u_k\} \subset D(A)$ having no convergent subsequence such that $\{Au_k\}$ converges.*

Proof. Suppose $A \notin \Phi_+$. If $\alpha(A) = \infty$, then there is a bounded sequence in $N(A)$ having no convergent subsequence. If $\alpha(A) < \infty$, then $R(A)$ is not closed. Let P be a bounded projection onto $N(A)$. Then there is a sequence $\{w_k\} \subset D(A)$ such that

$$\frac{\|(I-P)w_k\|}{\|Aw_k\|} \to \infty.$$

Put

$$u_k = \frac{(I-P)w_k}{\|(I-P)w_k\|}.$$

Then $\|u_k\| = 1$ and $\|Au_k\| \to 0$. If $\{u_k\}$ had a convergent subsequence, the limit u would be in $N(A)$ while $Pu = 0$. Consequently, we would have $u = 0$. But this is impossible since $\|u_k\| = 1$. Hence, there is no convergent subsequence. On the other hand, if $A \in \Phi_+$ and such a sequence existed, the limit f of Au_k would be in $R(A)$. Thus, there would be a $u \in D(A)$ such that $Au = f$. If we take $v_k = u_k - u$, then $\{v_k\}$ is bounded, has no convergent subsequence, and $Av_k \to 0$. Set $g_k = (I-P)v_k$. Then $\{g_k\}$ has no convergent subsequence (otherwise, $\{v_k\}$ would have one, since P is a compact operator). Thus, there is a $c_0 > 0$ such that $\|g_k\| \geq c_0$ for k sufficiently large. Hence,

$$\frac{\|g_k\|}{\|Ag_k\|} \geq \frac{c_0}{\|Av_k\|} \to \infty.$$

This contradicts the fact that $R(A)$ is closed. $\square$

Corollary 7.34. *$A \notin \Phi_+$ if and only if there is a bounded sequence $\{u_k\} \subset D(A)$ having no convergent subsequence such that $Au_k \to 0$.*

Other statements of Section 5.6 hold for unbounded operators as well. The proofs are the same.

7.7. The adjoint of a product of operators

If A, B are bounded operators defined everywhere, it is easily checked that

$$(BA)' = A'B'. \tag{7.43}$$

However, if A, B are only densely defined, $D(AB)$ need not be dense, and consequently, $(BA)'$ need not exist. In fact, we gave an example in Section 7.6 in which $D(BA) = \{0\}$. If $D(BA)$ is dense, then it follows that $D(A'B') \subset D[(BA)']$ and

$$(BA)'z' = A'B'z', \quad z' \in D(A'B'). \tag{7.44}$$

We can prove this by noting that if $z' \in D(A'B')$, then

$$z'(BAx) = B'z'(Ax) = A'B'z'(x), \quad x \in D(BA). \tag{7.45}$$

Consequently, $z' \in D[(BA)']$ and (7.44) holds. If, in addition, B is bounded and defined everywhere, then (7.43) will hold. (We leave this as an exercise.)

Now we shall show that (7.43) can hold even when neither operator is bounded. We shall prove

Theorem 7.35. *Let X, Y, Z be Banach spaces, and assume that A is a densely defined, closed linear operator from X to Y such that $R(A)$ is closed in Y and $\beta(A) < \infty$ (i.e., $A \in \Phi_-(X, Y)$). Let B be a densely defined linear operator from Y to Z. Then $(BA)'$ exists and (7.43) holds.*

That $D(BA)$ is dense in X follows from

Lemma 7.36. *If A, B satisfy the hypotheses of Theorem 7.35 and x is any element in $D(A)$, then there is a sequence $\{x_k\} \subset D(BA)$ such that $Ax_k \to Ax$ in Y and $x_k \to x$ in X. Consequently, $D(BA)$ is dense in X.*

Proof. Since $D(B)$ is dense in Y, we see by Lemma 7.4 that

$$Y = R(A) \oplus Y_0, \tag{7.46}$$

where $Y_0 \subset D(B)$. Let x be any element in $D(A)$. Then there is a sequence $\{y_k\} \subset D(B)$ such that $y_k \to Ax$ in Y. By (7.46), we can write $y_k = y_{1k} + y_{2k}$, where $y_{1k} \in R(A)$ and $y_{2k} \in Y_0$. Let

$$Py = \begin{cases} y, & y \in R(A), \\ 0, & y \in Y_0. \end{cases}$$

In view of Lemma 5.2, $P \in B(Y)$. Since y_k, y_{2k} are both in $D(B)$, the same is true of y_{1k}. Thus, $y_{1k} \in R(A) \cap D(B)$. Consequently, there is an $x_k \in D(BA)$ such that $Ax_k = y_{1k}$. Hence, $Ax_k = y_{1k} = Py_k \to PAx = Ax$. Since $N(A)$ is closed in X, we note that $X/N(A)$ is a Banach space. Let $[x]$ denote the coset containing x. Since $R(A)$ is closed in Y, there is a constant C such that

$$\|[x]\| \le C\|Ax\|, \quad x \in D(A). \tag{7.47}$$

By what we have shown, for each $x \in D(A)$ there is a sequence $\{x_k\} \subset D(BA)$ such that $Ax_k \to Ax$ in Y. By (7.47), $[x_k] \to [x]$ in $X/N(A)$. This means that there is a sequence $\{x_{0k}\} \subset N(A)$ such that $x_k - x_{0k} \to x$ in X. Since $x_k - x_{0k} \in D(BA)$ and $A(x_k - x_{0k}) = Ax_k \to Ax$, the proof is complete. $\square$

We can now give the proof of Theorem 7.35.

Proof. We have already proved that $D(A'B') \subset D[(BA)']$ and that (7.44) holds. We must show that $D[(BA)'] \subset D(A'B')$. Suppose $z' \in D[(BA)']$, and let y be any element in $D(B) \cap R(A)$. Then there is an $x \in D(BA)$ such that $Ax = y$. If $x_0 \in N(A)$, we have

$$z'(By) = z'[BA(x - x_0)] = (BA)'z'(x - x_0).$$

Consequently,

$$|z'(By)| \leq \|(BA)'z'\| \inf_{x_0 \in N(A)} \|x - x_0\| = \|(BA)'z'\| \cdot \|[x]\| \leq C\|(BA)'z'\| \cdot \|y\|$$

by (7.47). On the other hand, if $y \in Y_0$, then

$$|z'(By)| \leq C_1\|z'\| \cdot \|y\|,$$

since B is bounded on Y_0. By (7.46), every $y \in D(B)$ is the sum of an element in $D(B) \cap R(A)$ and an element in Y_0. Thus,

$$|z'(By)| \leq C_2\|y\|, \quad y \in D(B).$$

This shows that $z' \in D(B')$. To see that $B'z' \in D(A')$, note that

$$B'z'(Ax) = z'(BAx) = (BA)'z'(x), \quad x \in D(BA).$$

If x is any element in $D(A)$, then there is a sequence $\{x_k\} \subset D(BA)$ such that $Ax_k \to Ax$ in Y and $x_k \to x$ in X. Thus,

$$B'z'(Ax_k) = (BA)'z'(x_k), \quad k = 1, 2, \cdots.$$

Taking the limit as $k \to \infty$, we see that

$$B'z'(Ax) = (BA)'z'(x), \quad x \in D(A).$$

Hence, $B'z' \in D(A')$, and (7.43) holds. This completes the proof. $\square$

7.8. Problems

(1) Let A be a linear operator from X to Y with $D(A)\psi$ dense in X. If B is in $B(Y, Z)$, show that $(BA)' = A'B'$.

(2) Let A be a closed linear operator on X such that $(A - \lambda)^{-1}$ is compact for some $\lambda \in \rho(A)$. Show that $\sigma_e(A)$ is empty.

(3) Show that $p(\sigma_e(A)) \subset \sigma_e(p(A))$ for any polynomial $p(t)$ and any linear operator A on X.

(4) Let A be a closed operator on X such that $0 \in \rho(A)$. Show that $\lambda \neq 0$ is in $\sigma(A)$ if and only if $1/\lambda$ is in $\sigma(A^{-1})$.

(5) Show that if $A \in \Phi_+(X, Y)$ and M is a closed subspace of X, then $A(M)$ is closed in Y.

(6) Show that if U is a closed convex set and $p(x)$ is its Minkowski functional, then $u \in U$ if and only if $p(u) \leq 1$.

(7) Let U be a convex set, and let u be an interior point of U and y a boundary point of U. If $0 < \theta < 1$, show that $(1-\theta)u + \theta y$ is an interior point of U.

(8) Examine the proof of Theorem 7.3 to determine the exact hypotheses that were used in proving each of the statements (a), (b), and (c).

(9) In Lemma 7.4, show that $D \cap R$ is dense in R.

(10) Show that $y'(y) \neq 0$ in the proof of (i) in Theorem 7.16.

(11) Let A be the operator on l_2 defined by

$$A(x_1, x_2, \cdots, x_n, \cdots) = (x_1, 2x_2, \cdots, nx_n, \cdots),$$

where $D(A)$ consists of those elements $(x_1, \cdots)$ such that

$$\sum_1^\infty |nx_n|^2 < \infty.$$

(a) Is A closed? (b) Does A' exist? (c) What are $\sigma(A)$, Φ_A, $\sigma_e(A)$?

(12) Let X, Y be Banach spaces, and let A be a linear operator from X to Y such that $D(A) = X$. Show that $A \in B(X, Y)$ if and only if $D(A')$ is total in Y'.

(13) Prove: Every operator in $B(Y', X')$ is the adjoint of an operator in $B(X, Y)$ if and only if Y is reflexive.

(14) If W is a subspace of X', and X is reflexive, show that $\overline{W} = ({}^\circ W)^\circ$.

(15) Let A be a one-to-one, linear operator from X to Y with $D(A)$ dense in X and $R(A)$ dense in Y. Show that $(A')^{-1}$ exists and equals $(A^{-1})'$.

(16) Under the same hypotheses, show that A^{-1} is bounded if and only if $(A')^{-1}$ is.

(17) Let X, Y be normed vector spaces, and let A be a linear operator from X to Y such that $D(A)$ is dense in X and $N(A)$ is closed. Assume that $\gamma(A) > 0$, where

$$\gamma(A) = \inf_{x \in D(A)} \frac{\|Ax\|}{d(x, N(A))}.$$

Show that $\gamma(A') = \gamma(A)$ and that $R(A')$ is closed.

(18) If $p(x)$ is a seminorm on a vector space V, and $x_0 \in V$, show that there is a functional F on V satisfying $F(x_0) = p(x_0)$ and

$$|F(x)| \leq p(x), \quad x \in V.$$

(19) For X a normed vector space and $c > 0$, show that for every $x_0 \in X$ satisfying $\|x_0\| > c$ there is an $x' \in X'$ such that $x'(x_0) > 1$ and

$$|x'(x)| \leq 1, \quad \|x\| \leq c.$$

(20) Suppose X is a Banach space, $x'_1, \cdots, x'_n \in X'$ and $(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$. Show that for each $\varepsilon > 0$ one can find an element $x_0 \in X$ satisfying $\|x_0\| \leq M + \varepsilon$ and

$$x'_k(x_0) = \beta_k, \quad 1 \leq k \leq n,$$

if and only if

$$|\sum_1^n \alpha_k \beta_k| \leq M \|\sum_1^n \alpha_k x'_k\|$$

holds for every choice of the α_k.

(21) Show that the closure of a convex set is convex.

Chapter 8

REFLEXIVE BANACH SPACES

8.1. Properties of reflexive spaces

We touched briefly on reflexive spaces in Section 7.4 when we discussed total subsets. Recall that we showed, in Section 5.4, that corresponding to each element x in a Banach space X there is an element Jx in X'' such that

$$Jx(x') = x'(x), \quad x' \in X', \tag{8.1}$$

and

$$\|Jx\| = \|x\|, \quad x \in X. \tag{8.2}$$

Then we called X *reflexive* if $R(J) = X''$, i.e., if for every $x'' \in X''$ there is an $x \in X$ such that $Jx = x''$. One advantage of reflexivity was already noticed, namely, the fact that total subsets are dense. This allowed us to conclude that if Y is reflexive and $A \in \Phi(X, Y)$, then $A' \in \Phi(Y', X')$ (Theorem 7.22). In this chapter, we shall see that there are other advantages as well.

Let us first mention some reflexive Banach spaces. Every Hilbert space is reflexive, since, by the Riesz representation theorem (Theorem 2.1), every $x' \in X'$ is of the form (x, z), where $z \in X$. Hence, X' can be made into a Hilbert space. Thus, elements of X'' are of the same form. If $p > 1$, we proved that l_p' can be identified with l_q, where $1/p+1/q = 1$ (Theorem 2.11). The same theorem shows that l_q' can be identified with l_p. Hence, $l_p'' = l_p$, showing that l_p is reflexive for $p > 1$. We shall show that l_1 is not reflexive.

We now discuss some properties of reflexive spaces. Assume that X is a Banach space unless otherwise specified.

Theorem 8.1. *X is reflexive if and only if X' is.*

Proof. Suppose X is reflexive, and let J_1 be the mapping of X' into X''' given by

$$J_1x'(x'') = x''(x'), \quad x'' \in X''. \tag{8.3}$$

Let x''' be any element of X'''. Then by (8.2), $x'''(Jx)$ is a bounded linear functional on X. Hence, there is an $x' \in X'$ such that

$$x'''(Jx) = x'(x), \quad x \in X.$$

But

$$x'(x) = Jx(x'), \quad x \in X.$$

Hence,

$$x'''(Jx) = Jx(x'), \quad x \in X. \tag{8.4}$$

Now, since X is reflexive, we know that for any $x'' \in X''$ there is an $x \in X$ such that $Jx = x''$. Substituting in (8.4), we have

$$x'''(x'') = x''(x'), \quad x'' \in X''.$$

If we now compare this with (8.3), we see that $J_1x' = x'''$. Since x''' was any element of X''', we see that X' is reflexive.

Now assume that X' is reflexive. We note that $R(J)$ is a closed subspace of X''. For if $Jx_n \to x''$ in X'', then $\{x_n\}$ is a Cauchy sequence in X by (8.2). Since X is complete, there is an $x \in X$ such that $x_n \to x$ in X. Hence, $Jx_n \to Jx$ in X'', showing that $x'' = Jx$. Now if $R(J)$ is not the whole of X'', let x'' be any element of X'' not in $R(J)$. Then, by Theorem 2.9, there is an $x''' \in R(J)^\circ$ such that $x'''(x'') \neq 0$. Since X' is reflexive, there is an $x' \in X'$ such that $J_1x' = x'''$. Hence,

$$Jx(x') = J_1x'(Jx) = x'''(Jx) = 0, \quad x \in X, \tag{8.5}$$

and

$$x''(x') = J_1x'(x'') = x'''(x'') \neq 0. \tag{8.6}$$

By (8.5), we see that

$$x'(x) = Jx(x') = 0, \quad x \in X,$$

showing that $x' = 0$. But this contradicts (8.6). Hence, $R(J) = X''$, and the proof is complete. $\square$

We also have

Theorem 8.2. *Every closed subspace of a reflexive Banach space is reflexive.*

Proof. Let Z be a closed subspace of a reflexive space X. Then Z is a Banach space. Let z'' be any element of Z''. For any $x' \in X'$, the restriction x'_r of x' to Z is an element of Z'. Thus, $z''(x'_r)$ is defined, and

$$|z''(x'_r)| \leq \|z''\| \cdot \|x'_r\| \leq \|z''\| \cdot \|x'\|.$$

Hence, there is an $x'' \in X''$ such that

$$x''(x') = z''(x'_r).$$

Since X is reflexive, there is an $x \in X$ such that $x'' = Jx$. Hence,

$$z''(x'_r) = x'(x), \quad x' \in X'. \tag{8.7}$$

If we can show that $x \in Z$, it will follow that

$$z''(x'_r) = x'_r(x), \quad x' \in X'.$$

Since, for every $z' \in Z'$, there is an $x' \in X'$ such that $x'_r = z'$ (the Hahn-Banach theorem), we have

$$z''(z') = z'(x), \quad z' \in Z',$$

showing that Z is reflexive. Thus, it remains to show that $x \in Z$. If it were not, there would be an $x' \in Z^\circ$ such that $x'(x) \neq 0$. But this contradicts (8.7), which says that every $x' \in Z^\circ$ annihilates x as well. This completes the proof. $\square$

8.2. Saturated subspaces

If M is a closed subspace of X, then for each $x \in X \setminus M$, there is an $x' \in M^\circ$ such that $x'(x) \neq 0$ (Theorem 2.9). Now suppose W is a closed subspace of X'. Does W have the property that for each $x' \in X' \setminus W$ there is an $x \in {}^\circ W$ such that $x'(x) \neq 0$? Subspaces of X' having this property are called *saturated*. If X is reflexive, then any closed subspace W of X' is saturated, because there is an $x'' \in W^\circ$ such that $x''(x') \neq 0$. We then take $x = J^{-1}x''$. In this section, we shall see that this property characterizes reflexive spaces. We now investigate some properties of saturated subspaces.

Theorem 8.3. *A subspace W of X' is saturated if and only if $W = M^\circ$ for some subset M of X.*

Proof. If W is saturated, set $M = {}^\circ W$. Then clearly, $W \subset M^\circ$. Now suppose $x' \notin W$. Then there is an $x \in {}^\circ W = M$ such that $x'(x) \neq 0$. Therefore, x' is not in M°. This shows that $W = M^\circ$. Conversely, assume that $W = M^\circ$ for some set $M \subset X$. If $x' \notin W$, then there is an $x \in M$ such that $x'(x) \neq 0$. But $M \subset {}^\circ(M^\circ) = {}^\circ W$. Hence, W is saturated. $\square$

Corollary 8.4. *W is saturated if and only if $W = ({}^\circ W)^\circ$.*

A subset W of X' is called *weak** (pronounced "weak star") closed if $x' \in W$ whenever it has the property that for each $x \in X$, there is a sequence $\{x'_n\}$ of members of W such that

$$x'_k(x) \longrightarrow x'(x). \tag{8.8}$$

A weak* closed set is closed, since $x'_k \to x'$ implies (8.8). As we shall see, it is possible for a closed subset of X' not to be weak* closed. In particular, we have

Theorem 8.5. *A subspace W of X' is saturated if and only if it is weak* closed.*

Proof. Suppose W is saturated, and let $x' \in X'$ be such that for each $x \in X$, there is a $\{x'_k\} \subset W$ satisfying (8.8). If $x' \notin W$, then there is an $x \in {}^\circ W$ such that $x'(x) \neq 0$. But then (8.8) cannot hold for any $\{x'_k\} \subset W$ because $x'(x) = \lim x'_k(x) = 0$.

Now assume that W is weak* closed. If $W = X'$, there is nothing to prove. Otherwise, let x'_0 be some element in $X' \setminus W$. Thus, there is an element $x_0 \in X$ such that no sequence $\{x'_k\} \subset W$ satisfies (8.8) for $x = x_0$. Hence,

$$\inf_{x' \in W} |x'(x_0) - x'_0(x_0)| > 0. \tag{8.9}$$

Since $0 \in W$, we see that $x'_0(x_0) \neq 0$. Moreover, we claim that (8.9) implies that $x'(x_0) = 0$ for all $x' \in W$, i.e., $x_0 \in {}^\circ W$. For if $x' \in W$ and $x'(x_0) \neq 0$, set $\alpha = x'_0(x_0)/x'(x_0)$ and $y' = \alpha x'$. Then $y' \in W$ and $y'(x_0) = \alpha x'(x_0) = x'_0(x_0)$, contradicting (8.9). This shows that W is saturated, and the proof is complete. $\square$

We also have

Theorem 8.6. *A finite-dimensional subspace of X' is always saturated.*

Proof. Suppose that $x'_1, \cdots, x'_n$ form a basis for $W \subset X'$ and assume that $x'_0 \notin W$. Then the functionals $x'_0, x'_1, \cdots, x'_n$ are linearly independent. By Lemma 4.14, there are elements

$$x_0, x_1, \cdots, x_n \in X$$

such that

$$x'_j(x_k) = \delta_{jk}, \quad 0 \leq j,\, k \leq n. \tag{8.10}$$

In particular, $x'_0(x_0) = 1$ while $x'_j(x_0) = 0$ for $1 \leq j \leq n$. Hence, $x_0 \in {}^\circ W$, showing that W is saturated. $\square$

Theorem 8.7. *A Banach space X is reflexive if and only if every closed subspace of X' is saturated.*

In proving Theorem 8.7, we shall make use of the simple

Lemma 8.8. *Let X be a normed vector space, and let x' be an element of X'. Let M be the set of those x in X such that $x'(x) = 0$ (i.e., $M = {}^\circ[x']$). Let y be any element not in M, and let N be the one-dimensional subspace of X spanned by y (i.e., $N= [y]$). Then $X = N \oplus M$.*

Proof. Clearly, $N \cap M = \{0\}$. Moreover, for any $x \in X$, set

$$z = x - \frac{x'(x)}{x'(y)} y. \tag{8.11}$$

Then $x'(z) = 0$, showing that $z \in M$. Since $x = z + \alpha y$, the proof is complete. □

We can now give the proof of Theorem 8.7.

Proof. We have already given the simple argument that all closed subspaces of X' are saturated if X is reflexive. Now assume that all closed subspaces of X' are saturated. Let $x_0'' \neq 0$ be any element of X'', and let W be the set of all $x' \in X'$ which annihilate x_0''. Clearly, W is a closed subspace of X'. By hypothesis it is saturated. Since $x_0'' \neq 0$, W is not the whole of X'. Let x_1' be any element in $X' \setminus W$. Then there is an element $x_1 \in {}^\circ W$ such that $x_1'(x_1) \neq 0$. Moreover, by Lemma 8.8, every $x' \in X'$ can be written in the form

$$x' = \alpha x_1' + w', \tag{8.12}$$

where $w' \in W$. Set

$$g(x') = x_1'(x_1)x_0''(x') - x_0''(x_1')x'(x_1), \quad x' \in X'.$$

Then $g \in X''$, and $g(x') = 0$ for $x' \in W$ and for $x' = x_1'$. Hence, $g(x') \equiv 0$ by (8.12). This shows that

$$x_0''(x') = x'(\beta x_1), \quad x' \in X',$$

where $\beta = x_0''(x_1')/x_1'(x_1)$. Hence, $x_0'' = J(\beta x_1)$. Since x_0'' was arbitrary, it follows that X is reflexive. □

We also have

Corollary 8.9. *A finite-dimensional Banach space X is reflexive.*

In proving Corollary 8.9, we shall make use of

Lemma 8.10. *If* $\dim X = n < \infty$*, then* $\dim X' = n$.

Proof. Let $x_1, \cdots, x_n$ be a basis for X. Then there are functionals $x'_1, \cdots,$ x'_n in X' such that

$$x'_j(x_k) = \delta_{jk}, \quad 1 \leq j, k \leq n. \tag{8.13}$$

If $x \in X$, then

$$x = \sum_1^n \alpha_k x_k,$$

and consequently,

$$x'_j(x) = \alpha_j.$$

Substituting back in (8.13), we get

$$x = \sum_1^n x'_k(x) x_k, \quad x \in X.$$

Now, let x' be any functional in X'. Then

$$x'(x) = \sum_1^n x'_k(x) x'(x_k), \quad x \in X.$$

Thus,

$$x' = \sum_1^n x'(x_k) x'_k.$$

This shows that the x'_j form a basis for X'. □

We can now give the proof of Corollary 8.9.

Proof. Since all subspaces of X' are finite-dimensional, they are saturated (Theorem 8.6). Therefore, X is reflexive by Theorem 8.7. □

8.3. Separable spaces

In connection with reflexivity, another useful property of a Banach space is separability. A normed vector space is called *separable* if it has a dense subset that is denumerable. In other words, X is separable if there is a sequence $\{x_k\}$ of elements of X such that for each $x \in X$ and each $\varepsilon > 0$, there is an x_k satisfying $\|x - x_k\| < \varepsilon$. We investigate some properties of separable spaces. First we have

Theorem 8.11. *If X' is separable, so is X.*

Proof. Let $\{x'_n\}$ be a dense set in X'. For each n, there is an $x_n \in X$ such that $\|x_n\| = 1$ and

$$|x'_n(x_n)| \geq \|x'_n\|/2$$

by (2.28). Let $M = \overline{[\{x_n\}]}$, the closure of the set of linear combinations of the x_n. If $M \neq X$, let x_0 be any element of X not in M. Then there is an $x'_0 \in M^\circ$ such that $\|x'_0\| = 1$ and $x'_0(x_0) \neq 0$ (Theorem 2.9). In particular,

$$x'_0(x_n) = 0, \quad n = 1, 2, \cdots .$$

Thus,

$$\begin{aligned} \|x'_n\|/2 &\leq |x'_n(x_n)| = |x'_n(x_n) - x'_0(x_n)| \\ &\leq \|x'_n - x'_0\| \cdot \|x_n\| = \|x'_n - x'_0\|. \end{aligned}$$

Hence,

$$1 = \|x'_0\| \leq \|x'_n - x'_0\| + \|x'_n\| \leq 3\|x'_n - x'_0\|,$$

showing that none of the x'_n can come closer than a distance $1/3$ from x'_0. This contradicts the fact that $\{x'_n\}$ is dense in X'. Thus we must have $M = X$. But M is separable. To see this, we note that all linear combinations of the x_n with rational coefficients form a denumerable set. This set is dense in M. Hence, M is separable, and the proof is complete. □

We shall see later that it is possible for X to be separable without X' being so. However, this cannot happen if X is reflexive.

Corollary 8.12. *If X is reflexive and separable, then so is X'.*

Proof. Let $\{x_k\}$ be a sequence which is dense in X, and let x'' be any element of X''. Then there is an $x \in X$ such that $Jx = x''$. Moreover, for any $\varepsilon > 0$, there is an x_k such that $\|x_k - x\| < \varepsilon$. Thus, $\|Jx_k - x''\| < \varepsilon$ by (8.2). This means that X'' is separable. We now apply Theorem 8.11 to conclude that X' is separable. □

A sequence $\{x'_n\}$ of elements in X' is said to be *weak* convergent* to an element $x' \in X'$ if

$$x'_n(x) \longrightarrow x'(x) \text{ as } n \longrightarrow \infty, \quad x \in X. \tag{8.14}$$

A weak* convergent sequence is always bounded. This follows directly from the Banach-Steinhaus theorem (Theorem 3.17). There is a partial converse for separable spaces.

Theorem 8.13. *If X is separable, then every bounded sequence in X' has a weak* convergent subsequence.*

Proof. Let $\{x'_n\}$ be a bounded sequence in X', and let $\{x_k\}$ be a sequence dense in X. Now $x'_n(x_1)$ is a bounded sequence of scalars, and hence, it contains a convergent subsequence. Thus, there is a subsequence $\{x'_{n_1}\}$ of $\{x'_n\}$ such that $x'_{n_1}(x_1)$ converges as $n_1 \to \infty$. Likewise, there is a subsequence $\{x'_{n_2}\}$ of $\{x'_{n_1}\}$ such that $x'_{n_2}(x_2)$ converges. Inductively, there is a subsequence $\{x'_{n_k}\}$ of $\{x'_{n_{k-1}}\}$ such that $x'_{n_k}(x_k)$ converges. Set $z_n = x'_{n_n}$.

Then $\{z'_n\}$ is a subsequence of $\{x'_n\}$, and $z'_n(x_k)$ converges for each x_k. Let $x \in X$ and $\varepsilon > 0$ be given. There is an x_k such that $\|x - x_k\| < \varepsilon/3C$, where C is such that

$$\|x'_n\| \le C, \quad n = 1, 2, \cdots . \tag{8.15}$$

Thus,

$$|z'_n(x) - z'_m(x)| \le |z'_n(x) - z'_n(x_k)| + |z'_n(x_k) - z'_m(x_k)|$$
$$+|z'_m(x_k) - z'_m(x)| < 2\varepsilon/3 + |z'_n(x_k) - z'_m(x_k)|.$$

Now take m and n so large that the last term is less than $\varepsilon/3$. This shows that $z_n(x)$ is a convergent sequence for each $x \in X$. Set

$$F(x) = \lim_{n \longrightarrow \infty} z'_n(x).$$

Clearly, F is a bounded linear functional on X. This proves the theorem. $\square$

We also have

Theorem 8.14. *Every subspace of a separable space is separable.*

Proof. Let M be a subspace of a separable space X, and let $\{x_k\}$ be a dense sequence in X. For each pair of integers j, k, we pick an element $x_{jk} \in M$, if there is one, such that

$$\|x_{jk} - x_k\| < 1/j.$$

If there is not any, we forget about it. The set $\{x_{jk}\}$ of elements of M is denumerable. We claim that it is dense in M. To see this, let $x \in M$ and $\varepsilon > 0$ be given. Take j so large that $2 < j\varepsilon$. Since $\{x_k\}$ is dense in X, there is a k such that

$$\|x - x_k\| \le 1/j.$$

Since $x \in M$, this shows that there is an x_{jk} for this choice of j and k. Hence,

$$\|x - x_{jk}\| \le \|x - x_k\| + \|x_k - x_{jk}\| < 2/j < \varepsilon.$$

This completes the proof. $\square$

8.4. Weak convergence

A sequence $\{x_k\}$ of elements of a Banach space X is said to converge *weakly* to an element $x \in X$ if

$$x'(x_k) \longrightarrow x'(x) \text{ as } k \longrightarrow \infty \tag{8.16}$$

for each $x' \in X'$. We shall investigate how this convergence compares with convergence in norm (sometimes called *strong* convergence for contrast). Clearly, a sequence converging in norm also converges weakly. We have

Lemma 8.15. *A weakly convergent sequence is necessarily bounded.*

Proof. Consider the sequence $\{Jx_k\}$ of elements of X''. For each x' we have

$$\sup_k |Jx_k(x')| < \infty.$$

Then by the Banach-Steinhaus theorem (Theorem 3.17), there is a constant C such that $\|Jx_k\| \leq C$. Thus, $\|x_k\| \leq C$, and the proof is complete. $\square$

We also have

Theorem 8.16. *If X is reflexive, then every bounded sequence has a weakly convergent subsequence.*

Proof. Suppose X is reflexive, and let $\{x_n\}$ be a bounded sequence in X. Set $M = \overline{[\{x_n\}]}$, the closure of the set of linear combinations of the x_n. As we observed before, M is separable (see the proof of Theorem 8.11). Since it is a closed subspace of a reflexive space, it is reflexive (Theorem 8.2). Thus, M' is separable (Corollary 8.12). Now $\{Jx_n\}$ is a bounded sequence in M''. Hence, by Theorem 8.13, it has a subsequence (also denoted by $\{Jx_n\}$) such that $\{Jx_n(x')\}$ converges for each $x' \in M'$. This is the same as saying that $x'(x_n)$ converges for each $x' \in M'$. Now let x' be any element of X'. Then the restriction x'_r of x' to M is in M'. Thus, $x'(x_n) = x'_r(x_n)$ converges. This means that $\{x_n\}$ converges weakly, and the proof is complete. $\square$

We now realize that weak convergence cannot be equivalent to strong convergence in a reflexive, infinite dimensional space. The reason is that if X is reflexive, every sequence satisfying $\|x_n\| = 1$ has a weakly convergent subsequence (Theorem 8.16). If this subsequence converged strongly, then it would follow that X is finite-dimensional (Theorem 4.6). On the other hand, we have

Theorem 8.17. *If X is finite-dimensional, then a sequence converges weakly if and only if it converges in norm.*

Proof. Let $\{x_k\}$ be a sequence that is weakly convergent to x. Since X is finite-dimensional, so is X' (Lemma 8.10). Let $x'_1, \cdots, x'_n$ be a basis for X'. Then every $x' \in X'$ can be put in the form

$$x' = \sum_1^n \alpha_i x'_i.$$

Since all norms are equivalent on X' (Theorem 4.2), we can take

$$\|x'\| = \sum_1^n |\alpha_j|.$$

Now for each $\varepsilon > 0$, there is an N such that

$$|x_j'(x_k - x)| < \varepsilon, \quad 1 \leq j \leq n,$$

for all $k > N$. Then for any $x' \in X'$,

$$|x'(x_k - x)| \leq \sum_1^n |\alpha_j| \cdot |x_j'(x_k - x)| < \varepsilon \|x'\|$$

for such k. In view of (2.28), this gives

$$\|x_k - x\| < \varepsilon, \quad k > N,$$

which means that $\{x_k\}$ converges to x in norm. $\square$

8.5. Examples

Let us now apply the concepts of the preceding sections to some of the spaces we have encountered.

(1) l_p is separable for $1 < p < \infty$. Let W be the set of all elements of l_p of the form

$$x = (x_1, \cdots, x_j, \cdots), \tag{8.17}$$

where all of the x_j are rational and all but a finite number of them vanish. As is well known, W is a denumerable set. To see that it is dense in l_p, let $\varepsilon > 0$ and $x \in l_p$ be given. Then take N so large that

$$\sum_N^\infty |x_k|^p < \varepsilon/2.$$

Now, for each $k < N$, there is a rational number $\tilde{x}_k$ such that

$$|x_k - \tilde{x}_k|^p < \varepsilon/2N, \quad 1 \leq k < N. \tag{8.18}$$

Set

$$\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_{N-1}, 0, \cdots).$$

Then $\tilde{x} \in W$ and

$$\|x - \tilde{x}\|_p^p = \sum_1^{N-1} |x_k - \tilde{x}_k|^p + \sum_N^\infty |x_k|^p < \varepsilon,$$

showing that W is dense in l_p.

(2) l_∞ is not separable. Let

$$x^{(n)} = (x_1^{(n)}, \cdots, x_j^{(n)}, \cdots), \quad n = 1, 2, \cdots, \tag{8.19}$$

be any sequence of elements in l_∞. Define x by (8.17), where

$$x_j = \begin{cases} x_j^{(j)} + 1, & |x_j^{(j)}| \le 1, \\ 0, & |x_j^{(j)}| > 1. \end{cases}$$

Then $x \in l_\infty$. Moreover,

$$\|x - x^{(n)}\| \ge |x_n - x_n^{(n)}| \ge 1$$

for each n. This shows that the sequence $\{x^{(n)}\}$ cannot be dense in l_∞.

(3) $C[a,b]$ is separable. Let W be the set of piecewise linear functions of the form

$$\tilde{x}(t) = s_k + \frac{s_{k+1} - s_k}{t_{k+1} - t_k}(t - t_k), \quad t_k \le t \le t_{k+1}, \tag{8.20}$$

where

$$a = t_0 < t_1 < \cdots < t_n = b$$

is a partition of $[a,b]$, and the s_k, t_k are rational (with the possible exceptions of t_0 and t_n). The set W is denumerable. Let $x(t)$ be any continuous function in $[a,b]$, and let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|x(t) - x(s)| < \varepsilon/4 \quad \text{for} \quad |t - s| < \delta. \tag{8.21}$$

Let

$$a = t_0 < t_1 < \cdots < t_n = b$$

be a partition with $t_1, \cdots, t_{n-1}$ rational and such that

$$\max(t_{k+1} - t_k) < \delta.$$

Let $s_0, \cdots, s_n$ be rational numbers such that

$$|x(t_k) - s_k| < \varepsilon/4, \quad 0 \le k \le n. \tag{8.22}$$

Define $\tilde{x} \in C[a,b]$ by (8.20). Then for $t_k \le t \le t_{k+1}$, we have

$$\tilde{x}(t) - x(t) = \frac{t_{k+1} - t}{t_{k+1} - t_k}(s_k - x(t)) + \frac{t - t_k}{t_{k+1} - t_k}(s_{k+1} - x(t)), \quad t_k \le t \le t_{k+1},$$

and hence,

$$\begin{gathered} |\tilde{x}(t) - x(t)| \le |s_k - x(t_k)| + |x(t_k) - x(t)| \\ +|s_{k+1} - x(t_{k+1})| + |x(t_{k+1}) - x(t)| < \varepsilon, \quad t_k \le t \le t_{k+1}. \end{gathered}$$

Thus, W is dense in $C[a,b]$.

(4) Let $NBV[a,b]$ denote the set of normalized functions of bounded variation in $[a,b]$ (see Section 2.4). Under the norm $V(g)$ (the total variation of g), $NBV[a,b]$ becomes a normed vector space. It is complete because it represents the dual of $C[a,b]$ (Theorems 2.10 and 2.14). We claim that it is

not separable. We can see this as follows: For each s satisfying $a < s < b$, let $x_s(t)$ be the function defined by

$$x_s(t) = \begin{cases} 0, & a \le t < s, \\ 1, & s \le t \le b. \end{cases}$$

Then x_s is a normalized function of bounded variation in $[a,b]$. If $a < r < s < b$, then one easily checks that $V(x_r - x_s) = 2$. Thus, the spheres $V(x - x_r) < 1$, $V(x - x_s) < 1$ have no points in common for $r \neq s$. The set of all such spheres is nondenumerable. Since every dense subset of $NBV[a,b]$ must have at least one point in each of them, there can be no denumerable dense subsets.

(5) $l_1' = l_\infty$. To see this, we follow the proof of Theorem 2.11 until the definition of the vector z. To see that $z \in l_\infty$, we note that

$$|z_j| = |f(e_j)| \le \|f\| \cdot \|e_j\| = \|f\|.$$

In addition, we have

$$|f(x)| = |\sum_1^\infty x_j z_j| \le \sup |z_k| \sum_1^\infty |x_j| = \|z\|_\infty \|x\|_1.$$

This shows that $\|f\| = \|z\|_\infty$.

(6) $C[a,b]$ and l_1 are not reflexive. We know that they are separable. If they were also reflexive, then their duals would also be separable (Corollary 8.12). But their duals are $NBV[a,b]$ and l_∞, respectively, neither of which is separable.

(7) In l_1, weak convergence is equivalent to strong convergence. If they were not equivalent, there would be a sequence $\{x^{(n)}\}$ of the form (8.19) in l_1 such that

$$\sum_{j=1}^\infty z_j x_j^{(n)} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{8.23}$$

for each $z = (z_1, z_2, \cdots) \in l_\infty$, while there is an $\varepsilon > 0$ such that

$$\sum_{j=1}^\infty |x_j^{(n)}| > \varepsilon, \quad n = 1, 2, \cdots. \tag{8.24}$$

Taking $z_j = \delta_{jk}$, $j = 1, 2, \cdots$, we have by (8.23),

$$x_k^{(n)} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \quad k = 1, 2, \cdots.$$

Set $m_0 = n_0 = 0$, and inductively define the sequences $\{m_k\}$, $\{n_k\}$ as follows. If m_{k-1} and n_{k-1} are given, let n_k be the smallest integer $n > n_{k-1}$ such that

$$\sum_{j=1}^{m_{k-1}} |x_j^{(n_k)}| < \varepsilon/5, \tag{8.25}$$

and let m_k be the smallest integer $m > m_{k-1}$ such that

$$\sum_{j=m_k}^{\infty} |x_j^{(n_k)}| < \varepsilon/5. \tag{8.26}$$

Now, let $z = (z_1, z_2, \cdots)$ be the vector in l_∞ defined by

$$z_j = \operatorname{sgn} x_j^{(n_k)}, \quad m_{k-1} < j \le m_k,\ k = 1, 2, \cdots,$$

where $\operatorname{sgn}\alpha$ is the *signum function* defined to be $\alpha/|\alpha|$ for $\alpha \ne 0$ and 0 for $\alpha = 0$. Thus by (8.25) and (8.26),

$$\left|\sum_{j=1}^{\infty}(z_j x_j^{(n_k)} - |x_j^{(n_k)}|)\right| \le 2\sum_{j=1}^{m_{k-1}} |x_j^{(n_k)}| + 2\sum_{j=m_k}^{\infty} |x_j^{(n_k)}| < 4\varepsilon/5.$$

By (8.24), this gives

$$|\sum_{j=1}^{\infty} z_j x_j^{(n_k)}| > \varepsilon/5, \quad k = 1, 2, \cdots.$$

This contradicts (8.23), and the proof is complete.

(8) The last paragraph provides another proof that l_1 is not reflexive. If it were, every bounded sequence would have a weakly convergent subsequence (Theorem 8.16). But weak and strong convergence are equivalent in l_1. Hence, this subsequence converges in norm. This would show that the surface of the unit sphere is compact, and, consequently, that l_1 is finite-dimensional (Theorem 4.6). Since we know otherwise, it follows that l_1 is not reflexive.

(9) If X is a Banach space that is not reflexive, X' has closed subspaces that are not saturated (Theorem 8.7) and, hence, not weak* closed (Theorem 8.5). It also has total subspaces which are not dense. This follows from

Theorem 8.18. *If X is a Banach space such that every total subspace of X' is dense in X', then X is reflexive.*

Proof. If X were not reflexive, there would be an $x_0'' \in X''$ which is not in $R(J)$. Let W be the set of those $x' \in X'$ which annihilate x_0''. Since $x_0'' \neq 0$, W is a subspace of X' that is not dense in X'. We claim that it is total in X'. If we can substantiate this claim, it would follow that W violates the hypothesis of the theorem, providing a contradiction.

To prove that W is total, we must show that for each $x \neq 0 \in X$, there is an $x' \in W$ such that $x'(x) \neq 0$. Let $x_0' \in X'$ be such that $x_0''(x_0') \neq 0$, and suppose we are given an $x \neq 0$ in X. If $x_0'(x) = 0$, let x' be any element of X' such that $x'(x) \neq 0$. By Lemma 8.8,

$$x' = \alpha x_0' + x_1', \tag{8.27}$$

where $x_1' \in W$. Thus $x_1'(x) = x'(x) \neq 0$, and we are through. Otherwise, there is a $\beta \neq 0$ such that

$$x_0'(\beta x) = x_0''(x_0') \tag{8.28}$$

(just take $\beta = x_0''(x_0')/x_0'(x)$). Since x_0'' is not in $R(J)$, there is an $x' \in X'$ such that

$$x'(\beta x) \neq x_0''(x') \tag{8.29}$$

(otherwise we would have $x_0'' = J(\beta x)$). Decomposing x' in the form (8.27), we see by (8.28) and (8.29) that

$$x_1'(\beta x) \neq x_0''(x_1').$$

But, $x_0''(x_1') = 0$, since $x_1' \in W$. Hence, $x_1'(\beta x) \neq 0$, which means that $x_1'(x) \neq 0$. The proof is complete. $\square$

8.6. Completing a normed vector space

In Section 1.4, we mentioned that one could always complete a normed vector space X, i.e., find a Banach space Y containing X such that

(a) the norm of Y coincides with the norm of X on X.

(b) X is dense in Y.

We now give a proof of this fact.

Proof. Let X be any normed vector space. Consider the mapping J of X into X'' defined by (8.1). By (8.2), $R(J)$ is a normed vector space. Now X'' is complete (Theorem 2.10), and hence, the closure Y of $R(J)$ in X'' is a Banach space. Hence, Y is a Banach space containing $R(J)$ and satisfies

(a) and (b) with respect to it. Finally, we note that we can identify X with $R(J)$ by means of (8.1) and (8.2). This completes the proof. $\square$

8.7. Problems

(1) If M is a closed subspace of a separable normed vector space X, show that X/M is also separable.

(2) Suppose $\{x_n\}$ is a bounded sequence in a normed vector space X satisfying

$$x'(x_n) \longrightarrow x'(x) \text{ as } n \longrightarrow \infty$$

for all x' in a set $M \subset X'$ such that M is dense in X'. Show that x_n converges weakly to x.

(3) Show that the sequence $x_j = (\alpha_{1j}, \cdots)$ converges weakly in l_p, $1 < p < \infty$, if and only if it is bounded and for each n, α_{nj} converges as $j \to \infty$.

(4) Show that in a Hilbert space H, if u_n converges weakly to u and $\|u_n\| \to \|u\|$, then u_n converges strongly to u in H.

(5) Show that the range of a compact operator is separable.

(6) Let X be a Banach space and K an operator in $K(X)$. Show that if $\{x_n\}$ is a sequence converging weakly to x, then Kx_n converges strongly to Kx.

(7) Show that if $\{u_n\}$ is a sequence in a Hilbert space H which converges weakly to an element $u \in H$, then there is a subsequence $\{v_k\}$ of $\{u_n\}$ such that $(v_1 + \cdots + v_k)/k$ converges strongly to u in H.

(8) Let X, Y be Banach spaces, and let A be a linear operator from X to Y such that $D(A) = X$ and $D(A')$ is total in Y'. Show that $A \in B(X, Y)$.

(9) Let X, Y be Banach spaces with X reflexive. Show that if there is an operator $A \in B(X, Y)$ such that $R(A) = Y$, then Y is also reflexive.

(10) A linear operator which takes bounded sequences into sequences having weakly convergent subsequences is called *weakly compact.* Prove

(a) weakly compact operators are bounded

and

(b) if X or Y is reflexive, then every operator in $B(X, Y)$ is weakly compact.

(11) Show that c and c_0 are not reflexive.

(12) Show that if a sequence $\{x_n(t)\}$ of functions in $C[0, 1]$ converges weakly to $x(t)$, then the sequence is bounded and $x_n(t) \to x(t)$ for each t, $0 \le t \le 1$.

(13) Prove that a convex closed set is weakly closed.

(14) If $x_k \to x$ weakly in a Hilbert space and $y_k \to y$ strongly, show that $(x_k, y_k) \to (x, y)$.

(15) Show that a normed vector space X is not separable if and only if there are a $\delta > 0$ and a nondenumerable subset W of X such that $\|x - y\| \ge \delta$ for all $x, y \in W$, $x \ne y$.

(16) Let X be a separable Banach space. Show that there exists a map $A \in B(l_1, X)$ which is onto and such that A' maps X' isometrically into l_∞.

(17) If X is a Banach space, show that x'_k converges strongly if and only if $x'_k(x)$ converges uniformly for each $x \in X$ such that $\|x\| = 1$.

(18) For X a Banach space, show that $x_k \to x$ strongly and $x'_k \to x'$ in the weak* sense implies $x'_k(x_k) \to x'(x)$.

(19) Show that one can come to the same conclusion as in Problem 18 if $x_k \to x$ weakly and $x'_k \to x'$ strongly.

(20) If X, Y are Banach spaces and $K \in B(X,Y)$ is such that $K' \in K(Y', X')$, show that $K \in K(X,Y)$.

(21) Let X, Y be Banach spaces, with Y reflexive and X separable. If A_k is a bounded sequence in $B(X,Y)$, show that it has a renamed subsequence such that $A_k x$ converges weakly for each $x \in X$.

(22) If X, Y are Banach spaces and $K \in B(X,Y)$, show that $K \in K(X,Y)$ if and only if $Kx_n \to Kx$ strongly in Y whenever $x_n \to x$ weakly in X.

(23) If $z(t)$ is given by

$$z(t) = \begin{cases} 0, & a \le t < r, \\ 1, & r \le t < s, \\ 0, & s \le t < b, \end{cases}$$

for $a < r < s < b$, show that the total variation of $z(t)$ in $[a,b]$ is 2.

Chapter 9

BANACH ALGEBRAS

9.1. Introduction

If X and Y are Banach spaces, then we know that $B(X, Y)$ is a Banach space (Theorem 3.2). Moreover, if $Y = X$, then elements of $B(X)$ can be "multiplied;" i.e., if A, B are in $B(X)$, then $AB \in B(X)$ and

$$\|AB\| \leq \|A\| \cdot \|B\|, \tag{9.1}$$

since

$$\|ABx\| \leq \|A\| \cdot \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\|.$$

Moreover, we have trivially

$$(\alpha A + \beta B)C = \alpha(AC) + \beta(BC), \tag{9.2}$$

and

$$C(\alpha A + \beta B) = \alpha(CA) + \beta(CB). \tag{9.3}$$

A Banach space having a "multiplication" satisfying (9.1)–(9.3) is called a *Banach algebra.* In this chapter, we shall study some of the properties of such algebras. Among other things, we shall see that we can obtain a simple proof of that part of Theorem 6.23 not yet done.

Another property of $B(X)$ is that it has an element I such that

$$AI = IA = A \tag{9.4}$$

for all $A \in B(X)$. Such an element is called a *unit element.* Clearly, it is unique. Unless otherwise specified, whenever we speak of a Banach algebra, we shall assume that it has a unit element. This is no great restriction, for we can always add a unit element to a Banach algebra. In fact, if B is a Banach algebra without a unit element, consider the set C of pairs $\langle a, \alpha\rangle$, where $a \in B$ and α is a scalar. Define

$$\langle a, \alpha\rangle + \langle b, \beta\rangle = \langle a+b, \alpha+\beta\rangle$$

$$\beta\langle a, \alpha\rangle = \langle \beta a, \beta\alpha\rangle$$

$$\langle a, \alpha\rangle\langle b, \beta\rangle = \langle ab+\alpha b+\beta a, \alpha\beta\rangle,$$

and

$$\|\langle a, \alpha\rangle\| = \|a\| + |\alpha|.$$

We leave it as a simple exercise to show that C is a Banach algebra with unit element $\langle 0, 1\rangle$.

An element a of a Banach algebra B is called *regular* if there is an element $a^{-1} \in B$ such that

$$a^{-1}a = aa^{-1} = e, \tag{9.5}$$

where e is the unit element of B. The element a^{-1} is unique and is called the *inverse* of a.

The *resolvent* $\rho(a)$ of an element $a \in B$ is the set of those scalars λ such that $a - \lambda e$ is regular. The *spectrum* $\sigma(a)$ of a is the set of all scalars λ not in $\rho(a)$.

Some of the theorems for $B(X)$ are true for an arbitrary Banach algebra, and the proofs carry over. We list some of these here. In them, we assume that B is a complex Banach algebra.

Theorem 9.1. *If*

$$\sum_0^\infty \|a^n\| < \infty, \tag{9.6}$$

then e–a is a regular element, and

$$(e-a)^{-1} = \sum_0^\infty a^n. \tag{9.7}$$

(Theorem 1.1.)

Corollary 9.2. *If $\|a\| < 1$, then $e - a$ is regular, and (9.7) holds.*

Theorem 9.3. *For any a in B,*

$$r_\sigma(a) = \lim_{n\longrightarrow\infty} \|a^n\|^{1/n} \tag{9.8}$$

exists, and

$$r_\sigma(a) = \max_{\lambda\in\sigma(a)} |\lambda|. \tag{9.9}$$

If $|z| > r_\sigma(a)$*, then* $ze{-}a$ *is regular, and*

$$(ze-a)^{-1} = \sum_1^\infty z^{-n}a^{n-1}. \tag{9.10}$$

(Theorems 6.13 and 6.14.)

Our proof that $\rho(A)$ is an open set for $A \in B(X)$ depended on properties of Fredholm operators (see the proof of Theorem 6.3). Let us give another proof that works in Banach algebras.

Proof. Let a be a regular element, and suppose $\varepsilon > 0$ is such that $\varepsilon\|a^{-1}\| < 1$. Let x be any element satisfying $\|x-a\| < \varepsilon$. Then $\|xa^{-1}-e\| < 1$, showing that xa^{-1} is regular (Theorem 9.1). Let z be its inverse. Then $zxa^{-1} = xa^{-1}z = e$. Set $y = a^{-1}z$. Then $xy = e$ and $yx = a^{-1}zxa^{-1}a = a^{-1}ea = e$. This shows that x has y as an inverse. $\square$

Theorem 9.4. *If* λ, μ *are in* $\rho(a)$*, then*

$$(\lambda e-a)^{-1} - (\mu e-a)^{-1} = (\mu-\lambda)(\lambda e-a)^{-1}(\mu e-a)^{-1}. \tag{9.11}$$

If $|\lambda-\mu|\cdot\|(\mu e-a)^{-1}\| < 1$*, then*

$$(\lambda e-a)^{-1} = \sum_1^\infty (\mu-\lambda)^{n-1}(\mu e-a)^{-n}. \tag{9.12}$$

(Theorem 6.18.)

Theorem 9.5. *(Spectral mapping theorem) If p(t) is a polynomial, then*

$$\sigma[p(a)] = p[\sigma(a)]. \tag{9.13}$$

More generally, if f(z) is analytic in a neighborhood Ω *of* $\sigma(a)$*, define*

$$f(a) = \frac{1}{2\pi i}\oint_{\partial\omega} f(z)(ze-a)^{-1}dz, \tag{9.14}$$

where ω *is an open set containing* $\sigma(a)$ *such that* $\overline{\omega} \subset \Omega$*, and* $\partial\omega$ *consists of a finite number of simple closed curves which do not intersect. Then*

$$\sigma[f(a)] = f[\sigma(a)]. \tag{9.15}$$

(Theorem 6.17.)

A Banach algebra B is called *trivial* if $e = 0$. In this case, B consists of just the element 0. We have

Theorem 9.6. *If B is nontrivial, then for each a in B, $\sigma(a)$ is not empty.*

Proof. Suppose $a \neq 0$ and $\rho(a)$ is the whole complex plane. Let $a' \neq 0$ be any element of B' (the dual space of B considered as a Banach space). Since the series in (9.12) converges in norm for $|\lambda - \mu| \cdot \|(\mu e - a)^{-1}\| < 1$, we have

$$a'[(\lambda e - a)^{-1}] = \sum_1^\infty (\mu - \lambda)^{n-1} a'[(\mu e - a)^{-n}].$$

This shows that $f(z) = a'[(ze - a)^{-1}]$ is an entire function of z. Now, by (9.10),

$$f(z) = \sum_1^\infty z^{-n} a'(a^{n-1}). \tag{9.16}$$

If $|z| > k\|a\|,\ k > 1$, then

$$\begin{aligned} \left|\sum_1^N z^{-n} a'(a^{n-1})\right| &\leq \sum_1^N \frac{\|a'\| \cdot \|a\|^{n-1}}{k^n \|a\|^n} \\ &\leq \frac{\|a'\|}{\|a\|} \sum_1^\infty k^{-n} = \frac{\|a'\|}{(k-1)\|a\|}. \end{aligned}$$

Thus,

$$|f(z)| \leq \frac{\|a'\|}{(k-1)\|a\|}, \quad |z| > k\|a\|. \tag{9.17}$$

This shows that $f(z) \to 0$ as $|z| \to \infty$. In particular, $f(z)$ is bounded in the whole complex plane. By Liouville's theorem, $f(z)$ is constant, and since $f(z) \to 0$ as $z \to \infty$, we must have $f(z) = 0$. Since this is true for any $a' \in B'$, we see that

$$(ze - a)^{-1} \equiv 0.$$

[See (2.28).] But this is impossible since it would imply

$$e = (ze - a)(ze - a)^{-1} = 0.$$

Thus, $\sigma(a)$ cannot be empty. Since $\sigma(0) = \{0\}$ when B is nontrivial, the proof is complete. $\square$

Thus, we have proved

Corollary 9.7. *If X is a complex Banach space, and A is in $B(X)$, then $\sigma(A)$ is not empty.*

9.2. An example

Let X be a complex Banach space, and consider the Banach algebra $B(X)$. We know that the subspace $K(X)$ of compact operators on X is closed in $B(X)$ (Theorem 4.11). Let C be the factor space $B(X)/K(X)$ (see Section 3.5). Let $[A]$ denote the coset of C containing A. If we define

$$[A][B] = [AB], \tag{9.18}$$

then it is easily checked that C is a complex Banach algebra with unit element $[I]$. In this framework, we have

Theorem 9.8. $\rho([A]) = \Phi_A$.

Proof. By considering $A+\lambda$ in place of A, it suffices to prove that $A \in \Phi(X)$ if and only if $[A]$ is a regular element of C. If $[A]$ is a regular element of C, then there is an $A_0 \in B(X)$ such that

$$[A_0][A] = [A][A_0] = [I], \tag{9.19}$$

or

$$[A_0A] = [AA_0] = [I]. \tag{9.20}$$

Thus,

$$A_0A = I - K_1, \quad AA_0 = I - K_2, \tag{9.21}$$

where $K_1, K_2 \in K(X)$. By Theorem 5.5, we see that $A \in \Phi(X)$. Conversely, if $A \in \Phi(X)$, then there are $A_0 \in B(X)$ and $K_i \in K(X)$ such that (9.21) holds (Theorem 5.4). This leads to (9.20) and (9.19), which shows that $[A]$ is a regular element. This completes the proof. $\square$

We can now prove

Theorem 9.9. *If there is an operator A in $B(X)$ such that Φ_A consists of the whole complex plane, then X is finite dimensional.*

Proof. If Φ_A is the whole complex plane, then so is $\rho([A])$ (Theorem 9.8). By Theorem 9.6, C must be a trivial Banach algebra, i.e., $[I] = [0]$. This means that the identity I is a compact operator. Thus if $\{x_n\}$ is a sequence satisfying $\|x_n\| = 1$, it has a convergent subsequence. This implies that X is finite-dimensional (Theorem 4.6). $\square$

We can now complete the proof of Theorem 6.23.

Proof. We wish to show that (c) and (d) of that theorem imply the rest. Taking $\lambda_1 = 0$, we assume that $R(A)$ is closed, $\alpha(A) = \beta(A) < \infty$ and that 0 is an isolated point of $\sigma(A)$. Let σ_1 be the spectral set consisting of the point 0 alone, and let the operators P, A_1, A_2 be defined as in Section 6.4. Then $\sigma(A_1) = \{0\}$ (Theorem 6.19). Now

$$N(A_1) = N(A). \tag{9.22}$$

For if $A_1x = 0,\ x \in R(P)$, then $x = Px$, and $Ax = APx = A_1x = 0$. Hence, $N(A_1) \subset N(A)$. On the other hand,

$$N(A^n) \subset R(P), \quad n = 1, 2, \cdots, \tag{9.23}$$

by (6.32). This gives (9.22). Second, we have

$$R(A_1) = R(A) \cap R(P). \tag{9.24}$$

To see this, note that since A maps $R(P)$ into itself, we have $R(A_1) \subset R(A) \cap R(P)$. Moreover, if $y \in R(P)$ and $y = Ax$, then $y = Py = PAx = APx = A_1Px$, showing that $y \in R(A_1)$. This proves (9.24). Now we know that

$$X = R(P) \oplus N(P). \tag{9.25}$$

Hence, every bounded linear functional on $R(P)$ can be extended to a bounded linear functional on X by letting it vanish on $N(P)$. Thus, if one has k linearly independent functionals in $R(P)'$, their extensions which vanish on $N(P)$ form k linearly independent functionals in X'. If the functionals are in $R(A_1)^\circ$, then their extensions are in $R(A)^\circ$, since

$$R(PA) = R(AP) = R(A_1) = R(A) \cap R(P).$$

From this it follows that $\beta(A_1) \leq \beta(A)$. From (9.22), we have $\alpha(A_1) = \alpha(A)$, and $R(A_1)$ is closed by (9.24). Thus, $A_1 \in \Phi[R(P)]$. Moreover, we noted before that $A_1 - \lambda$ has a bounded inverse in $R(P)$ for all $\lambda \neq 0$. Hence, Φ_{A_1} is the whole complex plane. Therefore, it follows that $R(P)$ is finite-dimensional (Theorem 9.9). Moreover, (9.23) shows that

$$r(A) = \lim_{n \longrightarrow \infty} \alpha(A^n) \leq \dim R(P) < \infty.$$

Since $i(A) = 0,\ r'(A) = r(A) < \infty$, and the proof is complete. □

9.3. Commutative algebras

A Banach algebra B is called *commutative* (or *abelian*) if

$$ab = ba, \quad a, b \in B. \tag{9.26}$$

For such algebras some more interesting observations can be made. Throughout the remainder of this chapter, we shall assume that B is a nontrivial complex commutative Banach algebra. A linear functional m on B is called *multiplicative* if $m \neq 0$ and

$$m(ab) = m(a)m(b), \quad a,\ b \in B. \tag{9.27}$$

There is an interesting connection between the set M of multiplicative linear functionals and the spectrum of an element of B. In fact, we have

Theorem 9.10. *A complex number λ is in $\sigma(a)$ if and only if there is a multiplicative linear functional m on B such that $m(a) = \lambda$.*

This theorem is easy to prove in one direction. In fact, if $\lambda \in \rho(A)$, then there is a $b \in B$ such that

$$b(a - \lambda e) = e. \tag{9.28}$$

Then for any $m \in M$,

$$m(b)(m(a) - \lambda m(e)) = m(e). \tag{9.29}$$

Note that

$$m(e) = 1. \tag{9.30}$$

This follows from the fact that there is an $x \in B$ such that $m(x) \neq 0$ and $m(x) = m(ex) = m(e)m(x)$. Thus, (9.30) holds. Applying this to (9.29) we get $m(b)(m(a) - \lambda) = 1$, showing that we cannot have $m(a) = \lambda$. This proves the "if" part of the theorem. To prove the "only if" part, we shall need a bit of preparation.

Let us examine multiplicative functionals a bit more closely. If $m \in M$, let N be the set of those $x \in B$ such that $m(x) = 0$. From the linearity of m we know that N is a subspace of B. Moreover, if $a \in B$ and $x \in N$, then $m(ax) = m(a)m(x) = 0$, showing that $ax \in N$. A subspace having this property is called an *ideal*. By (9.30), we see that N is not the whole of B. In addition, if $a \in B$, then the element $a_1 = a - m(a)e$ is in N, since $m(a_1) = m(a) - m(a) = 0$. Thus, we can write a in the form

$$a = a_1 + \lambda e, \tag{9.31}$$

where $a_1 \in N$ and $\lambda = m(a)$. Note that a_1 and λ are unique. If $a = b + \mu e$, where $b \in N$, then $(\lambda - \mu)e \in N$. If $\lambda \neq \mu$, then $e \in N$. Thus, for any $x \in B$, $x = xe \in N$, showing that $N = B$. Since we know that $N \neq B$, we must have $\lambda = \mu$, and hence, $b = a_1$. An ideal $N \neq B$ having this property is called *maximal.*

A very important property of ideals is

Theorem 9.11. *Every ideal H which is not the whole of B is contained in a maximal ideal.*

The proof of Theorem 9.11 is not trivial, and makes use of *Zorn's lemma*, which is equivalent to the *axiom of choice.* We save the details for Section 9.5. Meanwhile, we use Theorem 9.11 to prove

Theorem 9.12. *If $H \neq B$ is an ideal in B, then there is an m in M such that m vanishes on H.*

Proof. By Theorem 9.11, there is a maximal ideal N containing H. Thus, if $a \in B$, then $a = a_1 + \lambda e$, where $a_1 \in N$. Define $m(a)$ to be λ. Clearly, m is a linear functional on B. It is also multiplicative. This follows from the fact that if $b = b_1 + \mu e$, $b_1 \in N$, then

$$m(ab) = m[(a_1 + \lambda e)b_1 + \mu a_1 + \lambda_1\lambda_2 e] = \lambda_1\lambda_2 = m(a)m(b).$$

Moreover, m vanishes on H, since it vanishes on $N \supset H$. This completes the proof. □

We now can supply the remainder of the proof of Theorem 9.10.

Proof. Suppose $\lambda \in \sigma(a)$. Then $a - \lambda e$ does not have an inverse. Thus, $b(a - \lambda e) \neq e$ for all $b \in B$. The set of all elements of the form $b(a - \lambda e)$ is an ideal $H \neq B$. By Theorem 9.12, there is an $m \in M$ which vanishes on H. Thus, $m(a - \lambda e) = 0$, or $m(a) = \lambda$. □

As an application of Theorem 9.10, let $a_1, \cdots, a_n$ be any elements of B. The vector $(\lambda_1, \cdots, \lambda_n)$ is said to be in the *joint resolvent set* $\rho(a_1, \cdots, a_n)$ of the a_k if there are elements $b_1, \cdots, b_n$ such that

$$\sum_1^n b_k(a_k - \lambda_k e) = e. \tag{9.32}$$

The set of all scalar vectors not in $\rho(a_1, \cdots, a_n)$ is called the *joint spectrum* of the a_k and is denoted by $\sigma(a_1, \cdots, a_n)$. Let

$$P(t_1, \cdots, t_n) = \sum \alpha_{k_1, \cdots, k_n} t_1^{k_1} \cdots t_n^{k_n}$$

be a polynomial in n variables. Then we can form the element $P(a_1, \cdots, a_n)$ of B. The following is a generalization of the spectral mapping theorem (Theorem 9.5):

Theorem 9.13. *A scalar μ is in $\sigma[P(a_1, \cdots, a_n)]$ if and only if there is a vector $(\lambda_1, \cdots, \lambda_n)$ in $\sigma(a_1, \cdots, a_n)$ such that $\mu = P(\lambda_1, \cdots, \lambda_n)$. In symbols,*

$$\sigma[P(a_1, \cdots, a_n)] = P[\sigma(a_1, \cdots, a_n)]. \tag{9.33}$$

Proof. By Theorem 9.10, $\mu \in \sigma[P(a_1, \cdots, a_n)]$ if and only if there is an $m \in M$ such that

$$m[P(a_1, \cdots, a_n)] = \mu. \tag{9.34}$$

But

$$m[P(a_1, \cdots, a_n)] = P[m(a_1), \cdots, m(a_n)]. \tag{9.35}$$

On the other hand, $(\lambda_1, \cdots, \lambda_n) \in \sigma(a_1, \cdots, a_n)$ if and only if

$$\sum_1^n b_k(a_k - \lambda_k e) \neq e \tag{9.36}$$

for all $b_k \in B$. By Theorem 9.12, this is true if and only if there is an $m \in M$ such that

$$m(a_k) = \lambda_k, \quad 1 \leq k \leq m. \tag{9.37}$$

Combining (9.34), (9.35), and (9.37), we get (9.33). □

9.4. Properties of maximal ideals

In proving Theorem 9.11, we shall make use of a few properties of maximal ideals. First we have

Lemma 9.14. *Maximal ideals are closed.*

Proof. Let N be a maximal ideal in B. Then

$$B = N \oplus \{e\}. \tag{9.38}$$

Let $\{a_n\}$ be a sequence of elements in N which approach an element a in B. Now

$$a = a_1 + \lambda e, \tag{9.39}$$

where $a_1 \in N$. Suppose $\lambda \neq 0$. Since $a_n - a_1 \to \lambda e$ in B, the element $a_n - a_1$ will be regular for n sufficiently large (Corollary 9.2). This would mean that $N = B$, since any element $b \in B$ can be written in the form $b(a_n - a_1)^{-1}(a_n - a_1)$, which is contained in the ideal N. Thus, the assumption $\lambda \neq 0$ leads to a contradiction. Hence, $\lambda = 0$, showing that $a = a_1 \in N$. Thus, N is closed. □

Next, we have

Theorem 9.15. *If m is in M, then m is bounded and $\|m\| \leq 1$.*

Proof. Let N be the set of those $x \in B$ such that $m(x) = 0$. Then N is a maximal ideal (see Section 9.3). If the inequality

$$|m(a)| \leq \|a\|, \quad a \in B, \tag{9.40}$$

were not true, there would exist an $a \in B$ such that $|m(a)| = 1, \ \|a\| < 1$. In this case, we would have

$$|m(a^n)| = 1, \ \|a^n\| \leq \|a\|^n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{9.41}$$

Thus, there would be a subsequence $\{a_k\}$ of $\{a^n\}$ such that

$$m(a_k) \longrightarrow \lambda, \quad a_k \longrightarrow 0, \quad \text{as} \quad k \longrightarrow \infty. \tag{9.42}$$

But

$$a_k = b_k + m(a_k)e, \tag{9.43}$$

where $b_k \in N$. Thus, $b_k = a_k - m(a_k)e \to -\lambda e$ as $k \to \infty$. Since N is closed, this can happen only if $\lambda = 0$. But this is impossible, since $|\lambda| = \lim |m(a_k)| = 1$. This contradiction shows that (9.40) holds. □

We also have

Theorem 9.16. *An ideal N is maximal if and only if the only ideal L satisfying $B \neq L \supset N$ is $L = N$.*

Proof. Suppose N is maximal and that L is an ideal satisfying $B \neq L \supset N$. Let a be any element in L. By the definition of a maximal ideal, $a = a_1 + \lambda e$, where $a_1 \in N$. Since a and a_1 are both in L, so is λe. If $\lambda \neq 0$, it follows that $e \in L$, from which we could conclude that $L = B$, contrary to assumption. Hence, $\lambda = 0$, and $a = a_1 \in N$. This shows that $L \subset N$. Since it was given that $L \supset N$, we have $L = N$.

Conversely, assume that N has the property described in the theorem. Let a be any element not in N, and let L be the set of all elements of the form $xa + y$, where $x \in B,\ y \in N$. Clearly, L is an ideal containing N. If $L \neq B$, we would have $L = N$, and it would follow that $a = ea + 0$ is in N, contrary to assumption. Hence, $L = B$. In particular, there are $\tilde{a} \in B,\ b \in N$ such that

$$e = \tilde{a}a + b. \tag{9.44}$$

Consider the quotient space B/N. It is a Banach space (Theorem 3.13). If we define

$$[x][y] = [xy],$$

it becomes a Banach algebra with unit element $[e]$. It is not trivial, since $[e] = [0]$ would imply that $e \in N$. Now (9.44) says that if $[a] \neq [0]$, then $[a]^{-1}$ exists (in fact, $[a]^{-1} = [a^{-1}]$). On the other hand, we know by Theorem 9.6 that $\sigma([a])$ is not empty for any $a \in B$. The only way these two statements can be reconciled is that for each $a \in B$ there is a scalar λ such that

$$[a] - \lambda[e] = [0],$$

which means that there is an $a_1 \in N$ such that

$$a - \lambda e = a_1,$$

which is precisely what we wanted to prove. □

The property mentioned in Theorem 9.16 is the reason for calling such ideals maximal.

9.5. Partially ordered sets

In this section, we shall present some of the theory of sets which is used in Theorem 9.11 and the Hahn-Banach theorem (Theorem 2.5).

A set S is called *partially ordered* if for some pairs of elements x, $y \in S$ there is an ordering relation $x \prec y$ such that

(1) $x \prec x, \quad x \in S,$

(2) $x \prec y,\ y \prec x \implies x = y,$

(3) $x \prec y,\ y \prec z \implies x \prec z.$

The set S is called *totally ordered* if for each pair x, y of elements of S, one has either $x \prec y$ or $y \prec x$ (or both).

A subset T of a partially ordered set S is said to have the element $x_0 \in S$ as an *upper bound* if $x \prec x_0$ for all $x \in T$. An element x_0 is said to be *maximal* for S if $x_0 \prec x$ implies $x = x_0$.

The following is equivalent to the axiom of choice and is called *Zorn's lemma.*

Principle 9.17. If S is a partially ordered set such that each totally ordered subset has an upper bound in S, then S has a maximal element.

We now show how Zorn's lemma can be used to give the proof of Theorem 9.11.

Proof. Let $H \neq B$ be the given ideal in B. Let S be the collection of all ideals L satisfying $B \neq L \supset H$. Among the ideals in S, $L_1 \subset L_2$ is a partial ordering. If W is a totally ordered subset of S, let

$$V = \bigcup_{L \in W} L.$$

V is a subspace of B. To see this, note that if a_1, a_2 are in V, then $a_1 \in L_1$, $a_2 \in L_2$ for L_1, $L_2 \in W$. Since W is totally ordered, we have either $L_1 \subset L_2$ or $L_2 \subset L_1$. We may suppose $L_1 \subset L_2$. Then a_1 and a_2 are both in L_2 and thus, so is $\alpha_1 a_1 + \alpha_2 a_2$, which must, therefore, also be in V. Moreover, V is also an ideal. For if $a \in V$, then $a \in L$ for some $L \in W$. Thus, $xa \in L$ for all $x \in B$, showing that $xa \in V$ for all $x \in B$. Clearly, V is an upper bound for W. Hence, S has the property that every totally ordered subset has an upper bound. By Zorn's lemma (Principle 9.17), S has a maximal element, i.e., an ideal N such that if $L \in S$ and $L \supset N$, then $L = N$. By Theorem 9.16, N is a maximal ideal in our sense. Since every ideal in S contains H, the proof is complete. □

We now give the proof of the Hahn-Banach theorem (Theorem 2.5).

Proof. Consider the collection S of all linear functionals g defined on subspaces $D(g)$ of V such that

(1) $D(g) \supset M$,

(2) $g(x) = f(x), \quad x \in M$,

(3) $g(x) \leq p(x), \quad x \in D(g)$.

Introduce a partial ordering in S as follows: If $D(g_1) \subset D(g_2)$ and $g_1(x) = g_2(x)$ for $x \in D(g_1)$, then write $g_1 \prec g_2$. If we can show that every totally ordered subset of S has an upper bound, it will follow from Principle 9.17 that S has a maximal element F. We claim that F is the desired functional. In fact, we must have $D(F) = V$. Otherwise, we have shown in our proof of Theorem 2.5 that there would be an $h \in S$ such that $F \prec h$ and $F \neq h$ [for we can take a vector $x \notin D(F)$ and extend F to $D(F) \oplus \{x\}$]. This would violate the maximality of F. Hence, $D(F) = V$, and F satisfies the stipulations of the theorem.

Therefore, it remains to show that every totally ordered subset of S has an upper bound. Let W be a totally ordered subset of S. Define the functional h by

$$D(h) = \bigcup_{g \in W} D(g)$$

$$h(x) = g(x), \quad g \in W,\ x \in D(g).$$

This definition is not ambigious, for if g_1 and g_2 are any elements of W, then either $g_1 \prec g_2$ or $g_2 \prec g_1$. At any rate, if $x \in D(g_1) \cap D(g_2)$, then $g_1(x) = g_2(x)$. Clearly, $h \in S$. Hence, it is an upper bound for W, and the proof is complete. □

9.6. Riesz operators

For a Banach space X, we call an operator $E \in B(X)$ a *Riesz operator* if $E - \lambda \in \Phi(X)$ for all scalars $\lambda \neq 0$. We denote the set of Riesz operators on X by $R(X)$. First we note

Lemma 9.18. *E is in $R(X)$ if and only if $I + \lambda E \in \Phi(X)$ for all scalars λ.*

Proof. If $E \in R(X)$, the statement is true for $\lambda = 0$. Otherwise, $E + I/\lambda \in \Phi(X)$. Hence, $I + \lambda E \in \Phi(X)$. Conversely, if $\mu \neq 0$, then $\mu(I + E/\mu) \in \Phi(X)$, showing that $E + \mu \in \Phi(X)$. □

As before, we let $K(X)$ denote the set of compact operators on X, and we let C be the factor space $B(X)/K(X)$. For an operator $A \in B(X)$, we denote the coset of C containing A by $[A]$. By Theorem 9.8 we have

Lemma 9.19. *$A \in \Phi(X)$ if and only if $[A]$ is invertible in C.*

We also have

Lemma 9.20. *E is in $R(X)$ if and only if $\|[E]^n\|^{1/n} \to 0$ as $n \to \infty$.*

Proof. This follows from the fact that $E \in R(X)$ if and only if $\lambda \in \Phi_E$ for all scalars $\lambda \neq 0$. By Theorem 9.8 this is true if and only if $\lambda \in \rho([E])$ for all $\rho \neq 0$. We now apply Theorem 6.13 to complete the proof. □

For any two operators $A, B \in B(X)$, we shall write $A \smile B$ to mean that $AB - BA \in K(X)$. The reason for this notation is that $[A]\,[B] = [B]\,[A]$ in this case. Such operators will be said to "almost commute." Next we have

Lemma 9.21. *If E is in $R(X)$ and K is in $K(X)$, then $E + K$ is in $R(X)$.*

Proof. We have $[E + K - \lambda] = [E - \lambda]$. □

Lemma 9.22. *If E is in $R(X)$, B is in $B(X)$ and $B \smile E$, then EB and BE are in $R(X)$.*

Proof. We note that $\|[EB]^n\|^{1/n} = \|[B]^n[E]^n\|^{1/n} \leq \|[B]\| \cdot \|[E]^n\|^{1/n} \to 0$ as $n \to \infty$. □

We also have

Lemma 9.23. *If $A \in \Phi(X)$, then there is an $A_0 \in \Phi(X)$ such that*

$$[A_0 A] = [A A_0] = [I]. \tag{9.45}$$

Proof. This follows from Theorem 5.4. □

Lemma 9.24. *If $E \in R(X), A \in \Phi(X)$ and $A \smile E$, then $A_0 + E \in \Phi(X)$.*

Proof. We have $[A(E + A_0)] = [(E + A_0)A] = [EA + I]$. Since $EA \in R(X)$(Lemma 9.22), $EA+I \in \Phi(X)$, and $[EA+I]$ is invertible in C. Hence, the same is true of $[E + A_0]$, showing that $E + A_0 \in \Phi(X)$. □

Lemma 9.25. *If $A \in \Phi(X)$, $E \in R(X)$, and $A \smile E$, then $A_0 \smile E$.*

Proof. We have $[A_0E] = [A_0EAA_0] = [A_0AEA_0] = [EA_0]$. □

This leads to

Theorem 9.26. *If $A \in \Phi(X)$, $E \in R(X)$ and $A \smile E$, then $A+E \in \Phi(X)$.*

Proof. We have $A_0 \in \Phi(X)$ and $A_0 \smile E$ (Lemmas 9.23 and 9.25). Thus $A + E \in \Phi(X)$(Lemma 9.24). □

Next, we note

Lemma 9.27. *Suppose $A \in \Phi(X)$ and $E \in B(X)$. Then $\lambda E + A \in \Phi(X)$ for all λ if and only if $EA_0 \in R(X)$.*

Proof. If $\lambda E+A \in \Phi(X)$, then $[(\lambda E+A)A_0] = [A_0(\lambda E+A)] = [\lambda EA_0+I]$ is invertible in C. Hence, $EA_0 \in R(X)$. Conversely, if $EA_0 \in R(X)$, then $[\lambda EA_0 + I]$ is invertible for each λ. Hence, so is $\lambda E + A$. □

We also have

Lemma 9.28. *Suppose $A \in \Phi(X)$ and $E \in B(X)$. Then $EA \in R(X) \Leftrightarrow AE \in R(X)$.*

Proof. If $EA \in R(X)$, then $\lambda EA+I \in \Phi(X)$ for all λ. Hence, so is $\lambda E+A_0$, and consequently, so is $\lambda AE + I$. Therefore, $AE \in R(X)$. □

We are led to

Theorem 9.29. *The operator E in $B(X)$ is in $R(X)$ if and only if $A+E \in \Phi(X)$ for all $A \in \Phi(X)$ such that $A \smile E$.*

Proof. By Theorem 9.26 we need only show the "if" part. To do this we merely take $A = \lambda \neq 0$. □

Theorem 9.30. *If $E_1, E_2 \in R(X)$ and $E_1 \smile E_2$, then $E_1 + E_2 \in R(X)$.*

Proof. If $\lambda \neq 0$, then $\lambda + E_1 \in \Phi(X)$. By Theorem 9.26, so is $\lambda + E_1 + E_2$. Thus, $E_1 + E_2 \in R(X)$. □

9.7. Fredholm perturbations

Let $F(X)$ denote the set of those $E \in B(X)$ such that $AE \in R(X)$ for all $A \in \Phi(X)$. We now characterize this set. When there is no chance of confusion, we shall write Φ in place of $\Phi(X)$. We have

Lemma 9.31. *The operator E is in F(X) if and only if $I + AE \in \Phi$ for all $A \in \Phi$.*

Proof. Use Lemma 9.18. $\square$

An operator $E \in B(X)$ is called a *Fredholm perturbation* if $A + E \in \Phi$ for all $A \in \Phi$. We have

Theorem 9.32. *E is in F(X) if and only if $A+E \in \Phi$ for all $A \in \Phi$. Thus F(X) coincides with the set of Fredholm perturbations.*

Proof. If $E \in F(X)$ and $A \in \Phi$, then $A_0E \in R(X)$ (Lemma 9.18). Thus, $(I+A_0E) \in \Phi$ (Lemma 9.18). Consequently, $A(I+A_0E) \in \Phi$, showing that $[A+E]$ is invertible in C. Hence, $A+E \in \Phi$. Conversely, suppose $A+E \in \Phi$ for all $A \in \Phi$. Let A be a particular operator in Φ. Then $\lambda A_0+E \in \Phi$ for all $\lambda \neq 0$. Hence, the same is true of $A(\lambda A_0 + E)$. This shows that $[\lambda + AE]$ is invertible for each $\lambda \neq 0$. Thus, $AE \in R(X)$. Since this is true for all $A \in \Phi$, we have $E \in F(X)$. $\square$

Corollary 9.33. *If $E_1, E_2 \in F(X)$, then $E_1 + E_2 \in F(X)$.*

Lemma 9.34. *For each B in B(X), there are operators $A_1, A_2 \in \Phi$ such that $B = A_1 + A_2$.*

Proof. For λ sufficiently large, $A_1 = \lambda + B$ is invertible (Lemma 6.5). Take $A_2 = -\lambda I$. $\square$

Corollary 9.35. *If E is in F(X), then BE is in F(X) for all B in B(X).*

Proof. By Lemma 9.34, each $B \in B(X)$ can be written in the form $B = A_1 + A_2$, where $A_j \in \Phi$. If A is any operator in Φ, then $AA_jE \in R(X)$. Thus, $A_jE \in F(X)$. Consequently, $BE = A_1E + A_2E \in F(X)$(Corollary 9.33). $\square$

Corollary 9.36. *If $E \in F(X)$, then $EA \in R(X)$ for all $A \in \Phi$.*

Proof. Use Lemma 9.28. $\square$

Corollary 9.37. *If E is in F(X), then EB is in F(X) for all B in B(X).*

Proof. Use Corollary 9.35. $\square$

Corollary 9.38. *If $E_n \in F(X)$ and $E_n \to E$ in $B(X)$, then $E \in F(X)$.*

Proof. If $A \in \Phi$, we can take n so large that $A - (E_n - E) \in \Phi$ (Theorem 5.11). Hence, $A - (E_n - E) + E_n \in \Phi$ (Theorem 9.32). This shows that $E \in F(X)$. □

Theorem 9.39. *$F(X)$ is a closed two-sided ideal.*

Proof. Use Corollaries 9.35, 9.37 and 9.38. □

9.8. Semi-Fredholm perturbations

In this section we examine operators which, when added to semi-Fredholm operators, do not remove them from the set. Again, we let X be a Banach space, and we use the notation Φ_+ to denote $\Phi_+(X)$ when there will be no confusion.

First, we shall need to gather some additional information concerning semi-Fredholm operators.

Lemma 9.40. *If P is a projection in $B(X)$ such that $\dim R(P) < \infty$, then there is a constant C such that*

$$\|x\| \leq Cd(x, R(P)), \quad x \in N(P).$$

Proof. Otherwise, there would be a sequence $\{x_k\} \in N(P)$ such that

$$\|x_k\| = 1, \quad d(x_k, R(P)) \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Hence, there is a sequence$\{z_k\} \in R(P)$ such that $x_k - z_k \to 0$. Since the x_k are bounded, the same is true of the z_k. Consequently, there is a renamed subsequence such that $z_k \to z \in R(P)$. Thus, $x_k \to z$ and $Px_k \to Pz$. But $Px_k = 0 \ \ \forall k$, and $Pz = z$. Hence, $0 \to Pz = z$. This means that $x_k \to 0$. But $\|x_k\| = 1$, and this contradiction proves the lemma. □

Theorem 9.41. *$A \in B(X)$ is in Φ_+ if and only if there are a projection P in $B(X)$ with $\dim R(P) < \infty$ and a constant C such that*

$$d(x, R(P)) \leq C\|Ax\|, \quad x \in N(P). \tag{9.46}$$

.

Proof. Assume $A \in \Phi_+$. By Lemmas 5.1 and 5.2, there is a projection $P \in B(X)$ such that $R(P) = N(A)$. Since $R(A)$ is closed, there is a constant C such that

$$d(x, N(A)) \leq C\|Ax\|, \quad x \in X$$

(Theorem 3.14). Hence,

$$d(x, R(P)) = d([I - P]x, R(P)) \leq C\|A(I - P)x\| = C\|Ax\|, \quad x \in X.$$

Conversely, if (9.46) holds for some $P \in B(X)$, then $N(A) \subset R(P)$. Consequently, $\alpha(A) < \infty$. Moreover,

$$\|(I-P)x\| \le cd([I-P]x, R(P)) \le C\|A(I-P)x\|, \quad x \in X,$$

or

$$\|x\| \le C\|Ax\| + \|Px\| + C\|APx\| = C\|Ax\| + |x|, \quad x \in X,$$

where $|\cdot|$ is a seminorm compact relative to the norm of X. Hence, $A \in \Phi_+$ (Theorem 5.21). $\square$

Theorem 9.42. *If A is not in Φ_+, then there are sequences $\{x_k\} \subset X$, $\{x'_k\} \subset X'$ such that*

$$x'_j(x_k) = \delta_{jk}, \quad \|x'_k\| \cdot \|Ax_k\| < 2^{-k}. \tag{9.47}$$

Proof. For $k \ge 1$, assume that $x_1, \cdots, x_{k-1},\ x'_1, \cdots, x'_{k-1}$ have been found, and set

$$Px = \sum_{j=1}^{k-1} x'_j(x)x_j, \quad x \in X,$$

when $k > 1$, and $P = 0$, otherwise. Then P is a projection in $B(X)$ with $\dim R(P) < \infty$. By Theorem 9.41, there is a sequence $\{z_i\} \subset N(P)$ such that

$$\|Az_i\|/d(z_i, R(P)) \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Pick $x_k \in N(P)$ so that

$$\|Ax_k\|/d(x_k, R(P)) < 2^{-k}.$$

By Theorem 2.9, there is a $\hat{x}'_k \in R(P)^\circ$ such that

$$\|\hat{x}'_k\| = 1, \quad \hat{x}'_k(x_k) = d(x_k, R(P)).$$

Take $x'_k = \hat{x}'_k/d(x_k, R(P))$. Then $x'_k \in R(P)^\circ$ and

$$x'_k(x_k) = 1, \quad \|x'_k\| = 1/d(x_k, R(P)) < 1/2^k\|Ax_k\|.$$

We now apply induction to obtain the complete sequences. $\square$

We can now prove

Theorem 9.43. *A in $B(X)$ is in Φ_+ if and only if $\alpha(A-K) < \infty$ for all $K \in K(X)$.*

Proof. If $A \in \Phi_+$ and $K \in K(X)$, then $A - K \in \Phi_+$ by Theorem 5.22. In particular, $\alpha(A-K) < \infty$. Conversely, suppose $A \notin \Phi_+$. Then by Theorem 9.42 there are sequences $\{x_k\}$, $\{x'_k\}$ satisfying (9.47). Define

$$K_n x = \sum_1^n x'_k(x)Ax_k, \quad n = 1, 2, \cdots. \tag{9.48}$$

Then K_n is a finite rank operator, and for $m < n$, we have

$$\|(K_n - K_m)x\| \leq \sum_{m+1}^{n} \|x'_k\| \cdot \|Ax_k\| \cdot \|x\| \leq \|x\| \sum_{m+1}^{n} 2^{-k},$$

showing that

$$\|K_n - K_m\| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

Hence,

$$Kx = \sum_{1}^{\infty} x'_k(x) Ax_k$$

is a compact operator on X (Theorem 4.11). Now $Kx = Ax$ for x equal to any of the x_k and consequently for x equal to any linear combination of the x_k. Since the x_k are linearly independent (why?), it follows that $\alpha(A - K) = \infty$. This completes the proof. □

Theorem 9.44. *$A \in \Phi$ if and only if $\alpha(A - K) < \infty$ and $\beta(A - K) < \infty$ for all $K \in K(X)$.*

Proof. If $A \in \Phi$, then $A - K \in \Phi \quad \forall K \in K(X)$. Consequently, $\alpha(A - K) < \infty$, $\beta(A - K) < \infty \quad \forall K \in K(X)$ (Theorem 5.10). Conversely, if $\alpha(A - K) < \infty \quad \forall K \in K(X)$, then $A \in \Phi_+$ (Theorem 9.43). If $\beta(A) < \infty$ as well, then $A \in \Phi$. □

Let $F_+(X)$ denote the set of all $E \in B(X)$ such that $A + E \in \Phi_+ \quad \forall A \in \Phi_+$. It is called the set of semi-Fredholm perturbations. We have

Corollary 9.45. *If $E_1, E_2 \in F_+(X)$, then $E_1 + E_2 \in F_+(X)$.*

Proof. Just use the definition. □

We also have

Theorem 9.46. *$E \in F_+(X)$ if and only if $\alpha(A - E) < \infty$ for all $A \in \Phi_+$.*

Proof. If $E \in F_+(X)$ and $A \in \Phi_+$, then $A - E \in \Phi_+$ by definition. Hence, $\alpha(A - E) < \infty$. If $A \in \Phi_+$ and $A - E \notin \Phi_+$, then there is a $K \in K(X)$ such that $\alpha(A - E - K) = \infty$ (Theorem 9.43). Set $B = A - K$. Then $B \in \Phi_+$ while $\alpha(B - E) = \infty$. Thus, $E \notin F_+(X)$. This proves the theorem. □

Theorem 9.47. *E is in $F(X)$ if and only if $\alpha(A - E) < \infty$ for all $A \in \Phi$.*

Proof. If $E \in F(X)$ and $A \in \Phi$, then $A - E \in \Phi$ (Theorem 9.32). Thus, $\alpha(A - E) < \infty$. Conversely, suppose $\alpha(A - E) < \infty \ \forall\ A \in \Phi$. Let A be any particular operator in Φ. Then $(A - K)/\lambda \in \Phi$ for each $K \in K(X)$ and $\lambda \neq 0$. Hence, $\alpha(A - \lambda E - K) < \infty$ for all λ and for all $K \in K(X)$. By Theorem 9.43, $A - \lambda E \in \Phi_+$ for each $\lambda \in \mathbb{R}$. In particular, this holds for

$0 \le \lambda \le 1$. It now follows from Theorem 5.23 that $i(A - \lambda E)$ is constant for λ in this interval. Now, if $\beta(A - E) = \infty$, it would follow that $\beta(A) = \infty$. But this is contrary to assumption. Hence, $A - E \in \Phi$. Since this is true for any $A \in \Phi$, the result follows. □

Corollary 9.48. $F_+(X) \subset F(X)$.

Lemma 9.49. *If $E_k \in F_+(X)$ and $E_k \to E$, then $E \in F_+(X)$.*

Proof. Use the same reasoning as in the proof of Corollary 9.38. Use Theorem 5.23 in place of Theorem 5.11. □

We also have

Lemma 9.50. *If $E \in F_+(X)$, then AE and EA are in $F_+(X)$ for all $A \in \Phi$.*

Proof. If $A \in \Phi$ and $C \in \Phi_+$, then $E + A_0C \in \Phi_+$ (Theorem 5.26). Thus, $A(E + A_0C) \in \Phi_+$ together with $AE + C$. This means that $AE \in F_+(X)$. A similar argument works for EA. □

Lemma 9.51. *If $E \in F_+(X)$, then BE and EB are in $F_+(X)$ for all $B \in B(X)$.*

Proof. See the proof of Corollary 9.35. □

Theorem 9.52. *$F_+(X)$ is a closed two-sided ideal.*

Proof. See the proofs of Lemmas 9.49 and 9.51. □

Similar statements can be made concerning Φ_- operators. In particular, we have

Theorem 9.53. *If A is not in Φ_-, then there are sequences $\{x_k\} \subset X$, $\{x'_k\} \subset X'$ such that*

$$x'_j(x_k) = \delta_{jk}, \ \|x'_k\| = 1, \ \|x_k\| \le a_k, \ \ \|x_k\| \cdot \|A'x'_k\| < 1/2^k, \tag{9.49}$$

where the a_k are given by

$$a_1 = 2, \quad a_n = 2(1 + \sum_1^{n-1} a_k), \quad n = 2, 3, \cdots .$$

Proof. For $n > 0$, assume that $x_1, \cdots, x_{n-1}, \ x'_1, \cdots, x'_{n-1}$ have been found, and set

$$Px' = \sum_1^{n-1} x'(x_k)x'_k.$$

Now by Theorem 9.41 there is an $x'_n \in N(P)$ such that

$$\|A'x'_n\| \leq d(x'_n, R(P))/2^n a_n C_n,$$

where C_n is such that

$$d(x', R(P)) \leq \|x' - Px'\| \leq C_n\|x'\|.$$

Consequently,

$$\|A'x'_n\| \leq \|x'_n\|/2^n a_n.$$

We can take $\|x'_n\| = 1$. By (2.25), there is an $x \in X$ such that $x'_n(x) = 1, \|x\| \leq 2$. Set

$$x_n = x - \sum_1^{n-1} x'_k(x)x_k.$$

Then

$$\|x_n\| \leq \|x\|(1 + \sum_1^{n-1} \|x_k\|) \leq a_n$$

by hypothesis. Moreover,

$$x'_n(x_n) = 1, \quad x'_n(x_k) = 0, \quad 1 \leq k < n$$

by the way x'_n and x_n were chosen. Also,

$$x'_k(x_n) = x_k(x) - x_k(x) = 0, \quad 1 \leq k < n.$$

This completes the proof. $\square$

Theorem 9.53 implies

Theorem 9.54. *A in B(X) is in Φ_- if and only if $\beta(A - K) < \infty$ for all K in K(X).*

Proof. If $A \in \Phi_-$ and $K \in K(X)$, then $A - K \in \Phi_-$ by Theorem 5.28. In particular, $\beta(A - K) < \infty$. Conversely, if $A \notin \Phi_-(X)$, then there are sequences $\{x_k\}, \{x'_k\}$ satisfying (9.49). Define

$$K_n x = \sum_1^n A'x'_k(x)x_k, \quad n = 1, 2, \cdots.$$

Then K_n is a finite rank operator, and for $m < n$, we have

$$\|(K_n - K_m)x\| \leq \sum_{m+1}^n \|A'x'_k\| \cdot \|x_k\| \cdot \|x\| \leq \|x\| \sum_{m+1}^n 2^{-k},$$

showing that

$$\|K_n - K_m\| \to 0 \text{ as } m, n \to \infty.$$

Hence,

$$Kx = \sum_1^\infty A'x'_k(x)x_k$$

is a compact operator on X (Theorem 4.11). Thus,

$$K'x' = \sum_1^\infty x'(x_k)A'x_k'$$

is a compact operator on X'. In particular, $K'x' = A'x'$ for x' equal to any of the x_k' and any of their linear combinations. Since the x_k' are linearly independent, it follows that $\alpha(A' - K') = \infty$. This completes the proof. $\square$

Let $F_-(X)$ denote the set of all $E \in B(X)$ such that $A + E \in \Phi_- \ \forall A \in \Phi_-$. It is also a set of semi-Fredholm perturbations. As before we have

Corollary 9.55. *If $E_1, E_2 \in F_-(X)$, then $E_1 + E_2 \in F_-(X)$.*

We also have

Theorem 9.56. *$E \in F_-(X)$ if and only if $\beta(A - E) < \infty$ for all $A \in \Phi_-$.*

Proof. If $E \in F_-(X)$ and $A \in \Phi_-$, then $A - E \in \Phi_-$ by definition. Hence, $\beta(A - E) < \infty$. If $A \in \Phi_-$ and $A - E \notin \Phi_-$, then there is a $K \in K(X)$ such that $\beta(A - E - K) = \infty$ (Theorem 9.54). Set $B = A - K$. Then $B \in \Phi_-$ while $\beta(B - E) = \infty$. Thus, $E \notin F_-(X)$. This proves the theorem. $\square$

Theorem 9.57. *E is in F(X) if and only if $\beta(A - E) < \infty$ for all $A \in \Phi$.*

Proof. If $E \in F(X)$ and $A \in \Phi$, then $A - E \in \Phi$ (Theorem 9.32). Thus, $\beta(A - E) < \infty$. Conversely, suppose $\beta(A - E) < \infty \ \forall A \in \Phi$. Let A be any particular operator in Φ. Then $(A - K)/\lambda \in \Phi$ for each $K \in K(X)$ and $\lambda \neq 0$. Hence, $\beta(A - \lambda E - K) < \infty$ for all λ and for all $K \in K(X)$. By Theorem 9.54, $A - \lambda E \in \Phi_-$ for each $\lambda \in \mathbb{R}$. In particular, this holds for $0 \leq \lambda \leq 1$. It now follows from Theorem 5.23 that $i(A - \lambda E)$ is constant for λ in this interval. Now, if $\alpha(A - E) = \infty$, it would follow that $\alpha(A) = \infty$. But this is contrary to assumption. Hence, $A - E \in \Phi$. Since this is true for any $A \in \Phi$, the result follows. $\square$

Corollary 9.58. *$F_-(X) \subset F(X)$.*

Lemma 9.59. *If $E_k \in F_-(X)$ and $E_k \to E$, then $E \in F_-(X)$.*

Proof. Use the same reasoning as in the proof of Corollary 9.38. Use Theorem 5.23 in place of Theorem 5.11. $\square$

We also have

Lemma 9.60. *If $E \in F_-(X)$, then AE and EA are in $F_-(X)$ for all $A \in \Phi$.*

Proof. If $A \in \Phi$ and $C \in \Phi_-$, then $E+A_0C \in \Phi_-$. Thus, $A(E+A_0C) \in \Phi_-$ together with $AE + C$. This means that $AE \in F_-(X)$. A similar argument works for EA. □

Lemma 9.61. *If $E \in F_-(X)$, then BE and EB are in $F_-(X)$ for all $B \in B(X)$.*

Proof. See the proof of Corollary 9.35. □

Theorem 9.62. *$F_-(X)$ is a closed two-sided ideal.*

Proof. See the proofs of Lemmas 9.49 and 9.51. □

9.9. Remarks

We note that $R(X)$ is not an ideal, and we see from Theorem 9.39 that $F(X)$ is the largest ideal contained in $R(X)$. Moreover, operators in $R(X)$ are characterized by the fact that each of them behaves like a Fredholm perturbation with respect to Fredholm operators which almost commute with it (Theorem 9.29).

Theorem 9.43 says that an operator in Φ_+ cannot coincide with a compact operator on any infinite dimensional subspace, and that this property characterizes these operators. Theorem 9.46 says that an operator is in $F_+(X)$ if and only if it does not coincide with a Φ_+ operator on any infinite dimensional subspace. Theorem 9.47 makes a similar statement for the set $F(X)$. Theorem 9.54 states that $A \in \Phi_-$ if and only if A' does not coincide with the adjoint of any compact operator on an infinite dimensional subspace of X'.

9.10. Problems

(1) Prove

$$r_\sigma(a^k) = [r_\sigma(a)]^k, \quad r_\sigma(\alpha a) = |\alpha| r_\sigma(a).$$

(2) Let $\{a_n\}$ be a sequence of elements of a Banach algebra such that a_n^{-1} exists for each n and $a_n \to a$. Show that either
(i) a^{-1} exists
or
(ii) there is a sequence $\{b_n\}$ such that $\|b_n\| = 1$ and $ab_n \to 0$.

(3) If $(\lambda_1, \cdots, \lambda_n) \in \sigma(a_1, \cdots, a_n)$ and $P(z_1, \cdots, z_n)$ is a polynomial, show that
$$|P(\lambda_1, \cdots, \lambda_n)| \leq \|P(a_1, \cdots, a_n)\|.$$

(4) Let B be a Banach algebra with unit. Show that for each $a_0 \in B$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that $\|a - a_0\| < \delta$ implies that $\sigma(a)$ lies in the set of all λ satisfying $d(\lambda, \sigma(a_0)) < \varepsilon$.

(5) If B is a Banach algebra without a unit element, consider the set C of pairs $\langle a, \alpha \rangle$, where $a \in B$ and α is a scalar. Define
$$\langle a, \alpha \rangle + \langle b, \beta \rangle = \langle a + b, \alpha + \beta \rangle$$
$$\beta \langle a, \alpha \rangle = \langle \beta a, \beta \alpha \rangle$$
$$\langle a, \alpha \rangle \langle b, \beta \rangle = \langle ab + \alpha b + \beta a, \alpha \beta \rangle,$$
and
$$\| \langle a, \alpha \rangle \| = \|a\| + |\alpha|.$$
Show that C is a Banach algebra with unit element $\langle 0, 1 \rangle$.

(6) Prove Lemma 9.51 and Corollary 9.52.

(7) Prove Lemma 9.61 and Corollary 9.62.

Chapter 10

SEMIGROUPS

10.1. A differential equation

Suppose A and u_0 are given numbers, and we want to solve the equation

$$\frac{du}{dt} = Au, \quad t > 0, \tag{10.1}$$

with the initial condition

$$u(0) = u_0 \tag{10.2}$$

for a function $u(t)$ in $t \geq 0$. The answer is

$$u(t) = u_0 e^{tA}. \tag{10.3}$$

Now we want to put the problem in a more general framework. Let X be a Banach space. We can define functions of a real variable t with values in X. This means that to each value t in the "domain" of the function u, we assign an element $u(t) \in X$. We say that $u(t)$ is continuous at t_0 if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies that $\|u(t) - u(t_0)\| < \varepsilon$. We say that $u(t)$ is differentiable at t_0 if there is an element $v \in X$ such that for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies

$$\left\| \frac{u(t) - u(t_0)}{t - t_0} - v \right\| < \varepsilon.$$

In such a case, we denote v by $u'(t_0)$ or $du(t_0)/dt$. Of course, a function differentiable at t_0 is continuous there.

To complete the picture, let A be any operator in $B(X)$, and let u_0 be any element of X. Then the equations

$$u'(t) = Au, \quad t > 0, \tag{10.4}$$

and

$$u(0) = u_0 \tag{10.5}$$

make sense. So we have created a differential problem in a Banach space. Does it have a solution?

Our first instinct is to look at (10.3). However, it does not seem to make much sense within the present framework. First, A is an operator and, hence, must operate on something. Second, what interpretation can we give to e^{tA}? Let us tackle the second point first. When A is a number,

$$e^{tA} = \sum_0^\infty \frac{1}{k!} t^k A^k, \tag{10.6}$$

and the series converges for each finite A. If the convergence is in norm, then the series in (10.6) makes sense for an operator $A \in B(X)$. And, indeed, it does converge in norm. This follows immediately from the fact that the series

$$\sum_0^\infty \frac{1}{k!} |t|^k \|A\|^k$$

converges, and hence,

$$\left\| \sum_m^n \frac{1}{k!} t^k A^k \right\| \leq \sum_m^n \frac{1}{k!} |t|^k \|A\|^k \to 0 \text{ as } m, n \to \infty.$$

Thus we can define e^{tA} by means of (10.6). With this definition, e^{tA} is in $B(X)$, and

$$\|e^{tA}\| \leq e^{|t|\|A\|}, \tag{10.7}$$

since

$$\left\| \sum_0^n \frac{1}{k!} t^k A^k \right\| \leq \sum_0^\infty \frac{1}{k!} |t|^k \|A\|^k = e^{|t|\|A\|}.$$

Once we have defined e^{tA}, we can resolve the other difficulty by replacing (10.3) by

$$u(t) = e^{tA} u_0. \tag{10.8}$$

Now, the question is whether or not (10.8) is a solution of (10.4) and (10.5). In order to answer this question, we must try to differentiate (10.8). Thus

$$\frac{u(t+h) - u(t)}{h} = \frac{e^{(t+h)A} - e^{tA}}{h} u_0 = \frac{e^{hA} - I}{h} e^{tA} u_0. \tag{10.9}$$

"Wait a minute!" you exclaim. "The property

$$e^{B+C} = e^B e^C \tag{10.10}$$

is well known for numbers, but far from obvious for operators." True, but let us assume for the moment that (10.10) does apply to operators in $B(X)$ that *commute* (i.e., that satisfy $BC = CB$). We shall prove this at the end of the section.

Since we suspect that $u'(t) = Au(t)$, let us examine the expression

$$\frac{u(t+h)-u(t)}{h} - Au = \left[\frac{e^{hA}-I}{h} - A\right] u.$$

Now,

$$\frac{e^{hA}-I}{h} = \sum_1^\infty \frac{1}{k!} h^{k-1} A^k,$$

and hence,

$$\left\| \frac{e^{hA}-I}{h} - A \right\| \leq \sum_2^\infty \frac{1}{k!} |h|^{k-1} \|A\|^k = \frac{e^{|h|\|A\|}-1}{|h|} - \|A\| \to 0 \text{ as } h \to 0.$$

This shows that $u'(t)$ exists for all t and equals $Au(t)$. Clearly, (10.5) holds. Hence, (10.8) is indeed a solution of (10.4) and (10.5).

Now, let us give the proof of (10.10) for operators in $B(X)$ that commute.

Proof. Since B, C commute,

$$(B+C)^k = \sum_{m=0}^k \frac{k!}{m!(k-m)!} B^m C^{k-m}.$$

Thus,

$$\sum_0^N \frac{1}{k!}(B+C)^k = \sum_{k=0}^N \sum_{m=0}^k \frac{1}{m!(k-m)!} B^m C^{k-m} = \sum_{m+n \leq N} \frac{1}{m!n!} B^m C^n.$$

Hence,

$$\sum_{m=0}^N \frac{1}{m!} B^m \sum_{n=0}^N \frac{1}{n!} C^n - \sum_{k=0}^N \frac{1}{k!}(B+C)^k = \sum_{\substack{m,n \leq N \\ m+n > N}} \frac{1}{m!n!} B^m C^n. \tag{10.11}$$

Moreover, the norm of this expression is not greater than

$$\begin{aligned} \sum_{\substack{m,n \leq N \\ m+n > N}} \frac{1}{m!n!} \|B\|^m \|C\|^n &= \sum_{m=0}^N \frac{1}{m!} \|B\|^m \sum_{n=0}^N \frac{1}{n!} \|C\|^n \\ &- \sum_{k=0}^N \frac{1}{k!} (\|B\| + \|C\|)^k \\ &\to e^{\|B\|} e^{\|C\|} - e^{\|B\|+\|C\|} = 0 \end{aligned}$$

as $N \to \infty$. But the left hand side of (10.11) converges in norm to

$$e^B e^C - e^{B+C},$$

and, hence, (10.10) holds. This completes the proof. □

10.2. Uniqueness

Now that we have solved (10.4) and (10.5), we may question whether the solution is unique. The answer is yes, but we must reason carefully. Suppose there were two solutions. Then their difference would satisfy

$$u'(t) = Au, \quad t > 0, \tag{10.12}$$

$$u(0) = 0. \tag{10.13}$$

In particular, we would have

$$e^{-tA}(u' - Au) = 0. \tag{10.14}$$

Now, if A were a number, this would reduce to

$$(e^{-tA}u)' = 0, \quad t > 0. \tag{10.15}$$

However, in the present case, care must be exercised. Set $v(t) = e^{-tA}u$. Then

$$\begin{aligned} \frac{v(t+h) - v(t)}{h} &= \frac{e^{-(t+h)A}u(t+h) - e^{-tA}u(t)}{h} \\ &= e^{-(t+h)A}\left[\frac{u(t+h) - u(t)}{h}\right] + \left[\frac{e^{-hA} - I}{h}\right] v(t), \end{aligned}$$

and this converges to the left hand side of (10.14) as $h \to 0$, since

$$\frac{e^{-hA} - I}{h} \to -A$$

in norm as $h \to 0$.

Once we have (10.15), we expect that it implies, in view of (10.13), that

$$e^{-tA}u = 0, \quad t \geq 0. \tag{10.16}$$

To see this, let f be any element of X', and set

$$F(t) = f(e^{-tA}u). \tag{10.17}$$

One easily checks that $F(t)$ is differentiable in $t > 0$, continuous in $t \geq 0$ and satisfies

$$F'(t) = 0, \quad t > 0,$$

and

$$F(0) = 0.$$

Since F is a scalar function, this implies that F vanishes identically. Since this is true for any $f \in X'$, we see that (10.16) does indeed hold. To complete the proof, we merely note that

$$u(t) = e^{tA}(e^{-tA}u) = 0, \quad t \geq 0.$$

If $u(t)$ is a continuous function in $t_0 \leq t \leq t_1$ with values in X, we can define the Riemann integral

$$\int_{t_0}^{t_1} u(s)ds = \lim_{\eta \to 0} \sum_1^n u(s_i')(s_i - s_{i-1}),$$

where $t_0 = s_0 < s_1 < \cdots < s_n = t_1$, $\eta = \max |s_i - s_{i-1}|$, and s_i' is any number satisfying $s_{i-1} \leq s_i' \leq s_i$. When $u(t)$ is continuous, the existence of the integral is proved in the same way as in the scalar case. We also have the estimate

$$\left\| \int_{t_0}^{t_1} u(s)ds \right\| \leq \int_{t_0}^{t_1} \|u(s)\|ds. \tag{10.18}$$

The existence of the integral on the right follows from the continuity of $\|u(s)\|$, which in turn follows from

$$|\, \|u(s)\| - \|u(t)\| \,| \leq \|u(s) - u(t)\|. \tag{10.19}$$

Moreover, the function

$$U(t) = \int_{t_0}^{t} u(s)ds$$

is differentiable in $t_0 \leq t \leq t_1$, and $U'(t) = u(t)$. Again, the proof is the same as in the scalar case. In particular, if $u'(t)$ is continuous in $t_0 \leq t \leq t_1$, then

$$u(t) - u(t_0) = \int_{t_0}^{t} u'(s)ds, \quad t_0 \leq t \leq t_1. \tag{10.20}$$

This follows from the fact that both sides of (10.20) are equal at t_0, and their derivatives are equal for all t in the interval.

10.3. Unbounded operators

In Section 10.1, we solved (10.4) and (10.5) for the case $A \in B(X)$. Suppose, as is more usual in applications, A is not bounded, but rather a closed linear operator on X with domain $D(A)$ dense in X. Can we solve (10.4), (10.5)? First, we must examine (10.4) a bit more closely. Since $D(A)$ need not be the whole of X, we must require $u(t)$ to be in $D(A)$ for each $t > 0$ in order that it be a solution of (10.4). With this interpretation, (10.4) makes sense.

Many sets of sufficient conditions are known for (10.4), (10.5) to have a solution. We shall consider one set. We have

Theorem 10.1. *Let A be a closed linear operator with dense domain $D(A)$ on X having the interval $[b,\infty)$ in its resolvent set $\rho(A)$, where $b \geq 0$, and such that there is a constant a satisfying*

$$\|(\lambda - A)^{-1}\| \leq (a+\lambda)^{-1}, \quad \lambda \geq b. \tag{10.21}$$

Then there is a family $\{E_t\}$ of operators in $B(X)$, $t \geq 0$, with the following properties:

(a) $E_sE_t = E_{s+t}, \quad s \geq 0,\ t \geq 0,$

(b) $E_0 = I,$

(c) $\|E_t\| \leq e^{-at}, \quad t \geq 0,$

(d) *E_tx is continuous in $t \geq 0$ for each $x \in X$,*

(e) *E_tx is differentiable in $t \geq 0$ for each $x \in D(A)$, and*

$$\frac{dE_tx}{dt} = AE_tx, \tag{10.22}$$

(f) $E_t(\lambda - A)^{-1} = (\lambda - A)^{-1}E_t, \quad \lambda \geq b,\ t \geq 0.$

Before proving the theorem, we show how it gives the solution to our problem provided $u_0 \in D(A)$. In fact,

$$u(t) = E_tu_0, \quad t \geq 0, \tag{10.23}$$

is a solution of (10.4) and (10.5). To see this, note that E_t maps $D(A)$ into itself. The reason for this is that if $v \in D(A)$, then by (f),

$$E_tv = E_t(b-A)^{-1}(b-A)v = (b-A)^{-1}E_t(b-A)v,$$

which is clearly an element of $D(A)$. Thus, if $u_0 \in D(A)$, so is $u(t)$ for each $t \geq 0$. By (b), we see that (10.5) is satisfied, while (10.22) implies (10.4).

A one-parameter family $\{E_t\}$ of operators satisfying (a) and (b) is called a *semigroup*. The operator A is called its *infinitesimal generator*. In proving Theorem 10.1 we shall make use of

Lemma 10.2. *Let D be a dense set in X, and let $\{B_\lambda\}$ be a family of operators in $B(X)$ satisfying*

$$\|B_\lambda\| \leq M, \quad \lambda \geq K. \tag{10.24}$$

If $B_\lambda x$ converges as $\lambda \to \infty$ for each $x \in D$, then there is a B in $B(X)$ such that

$$\|B\| \leq M \tag{10.25}$$

and

$$B_\lambda x \to Bx \text{ as } \lambda \to \infty, \quad x \in X. \tag{10.26}$$

Proof. Let $\varepsilon > 0$ be given, and let x be any element of X. Then we can find an element $\tilde{x} \in D$ such that

$$\|x - \tilde{x}\| < \frac{\varepsilon}{3M}. \tag{10.27}$$

Thus,

$$\|B_\lambda x - B_\mu x\| \le \|B_\lambda(x-\tilde{x})\| + \|B_\lambda\tilde{x} - B_\mu\tilde{x}\| + \|B_\mu(\tilde{x}-x)\| \le \frac{2\varepsilon}{3} + \|B_\lambda\tilde{x} - B_\mu\tilde{x}\|.$$

We now take λ, μ so large that

$$\|B_\lambda\tilde{x} - B_\mu\tilde{x}\| < \frac{\varepsilon}{3}.$$

Thus $Bx = \lim B_\lambda x$ exists for each $x \in X$. Clearly, B is a linear operator. Moreover,

$$\|Bx\| = \lim \|B_\lambda x\| \le M\|x\|,$$

and the proof is complete. □

We are now ready for the proof of Theorem 10.1.

Proof. Assume first that $a > 0$. Set

$$A_\lambda = \lambda A(\lambda - A)^{-1}, \quad \lambda \ge b. \tag{10.28}$$

We are going to show that

(1) $A_\lambda \in B(X), \quad \lambda \ge b,$

(2) $\|e^{tA_\lambda}\| \le \exp\left(\dfrac{-at\lambda}{a+\lambda}\right), \quad t \ge 0,\ \lambda \ge b,$

(3) $A_\lambda x \to Ax$ as $\lambda \to \infty, \quad x \in D(A),$

(4) $e^{tA_\lambda}x \to E_t x$ as $\lambda \to \infty, \quad x \in X,\ t \ge 0.$

The last assertion states that for each $t \ge 0$ and each $x \in X$, $e^{tA_\lambda}x$ converges to a limit in X as $\lambda \to \infty$. We define this limit to be $E_t x$. Clearly, E_t is a linear operator. It is in $B(X)$ by (2) and Lemma 10.2. Moreover,

$$\|E_t x\| \le \|e^{tA_\lambda}x\| + \|(E_t - e^{tA_\lambda})x\| \le \exp\left(\frac{-at\lambda}{\lambda + a}\right)\|x\| + \|(E_t - e^{tA_\lambda})x\|.$$

Letting $\lambda \to \infty$, we get (c). We now show that (a) – (f) follow from (1) – (4). We have just seen that (2) implies (c). To obtain (a), note that

$$\|E_{s+t}x - E_s E_t x\| \le \|(E_{s+t}e^{(s+t)A_\lambda})x\|$$

$$+\|e^{sA_\lambda}(e^{tA_\lambda}-E_t)x\|+\|(e^{sA_\lambda}-E_s)E_tx\|.$$

By (2) and (4), the right hand side tends to 0 as $\lambda\to\infty$.

To prove (d), note that

$$e^{tA_\lambda}x-e^{t_0A_\lambda}x=\int_{t_0}^t(e^{sA_\lambda}x)'ds=\int_{t_0}^t e^{sA_\lambda}A_\lambda x ds$$

by (10.20). Hence,

$$\|e^{tA_\lambda}x-e^{t_0A_\lambda}x\|\le\int_{t_0}^t\|e^{sA_\lambda}A_\lambda x\|ds\le\int_{t_0}^t\|A_\lambda x\|ds=(t-t_0)\|A_\lambda x\|.$$

Assume that $x\in D(A)$, and let $\lambda\to\infty$. Thus, we have by (3) and (4)

$$\|E_tx-E_{t_0}x\|\le(t-t_0)\|Ax\|.$$

This shows that

$$E_tx\to E_{t_0}x \text{ as } t\to t_0,\quad x\in D(A). \tag{10.29}$$

By (c) and Lemma 10.2, this implies (d). Moreover, once we know that E_tx is continuous in t, we can take the limits in the integrals to obtain

$$E_tx-E_{t_0}x=\int_{t_0}^t E_sAxds,\quad x\in D(A). \tag{10.30}$$

In particular, (10.30) implies that E_tx is differentiable, and

$$\frac{dE_tx}{dt}=E_tAx,\quad x\in D(A). \tag{10.31}$$

However, it follows from (f) that

$$AE_tx=E_tAx,\quad x\in D(A). \tag{10.32}$$

In fact, we have

$$AE_tx=bE_tx-(b-A)E_t(b-A)^{-1}(b-A)x=bE_tx-E_t(b-A)x=E_tAx.$$

This gives (e). (Yes, I know that I have not proved (f) yet. I just wanted to see if you would notice. As you will see, its proof will not involve (e).) To prove (f), note that

$$\sum_1^N\frac{1}{k!}t^kA_\lambda^k(\lambda-A)^{-1}x=(\lambda-A)^{-1}\sum_1^N\frac{1}{k!}t^kA_\lambda^kx$$

for each N, and hence,

$$e^{tA_\lambda}(\lambda-A)^{-1}x=(\lambda-A)^{-1}e^{tA_\lambda}x. \tag{10.33}$$

Taking the limit as $\lambda\to\infty$, we get (f).

It only remains to prove (1) – (4). Statement (1) follows from the fact that

$$\lambda(\lambda - A)^{-1} = I + A(\lambda - A)^{-1}, \tag{10.34}$$

and hence,

$$A_\lambda = -\lambda + \lambda^2(\lambda - A)^{-1}. \tag{10.35}$$

Thus,

$$e^{tA_\lambda} = e^{-t\lambda} e^{t\lambda^2(\lambda - A)^{-1}}, \tag{10.36}$$

so that

$$\|e^{tA_\lambda}\| \leq e^{-t\lambda} \exp\left(\frac{t\lambda^2}{a+\lambda}\right) = \exp\left(\frac{-ta\lambda}{a+\lambda}\right) \leq 1, \tag{10.37}$$

which is (2). Now by (10.34),

$$\|A(\lambda - A)^{-1}\| \leq 1 + \frac{\lambda}{a+\lambda} \leq 2, \tag{10.38}$$

while

$$\|A(\lambda - A)^{-1}x\| \leq \frac{\|Ax\|}{(a+\lambda)} \to 0 \text{ as } \lambda \to \infty, \quad x \in D(A). \tag{10.39}$$

Therefore, it follows from Lemma 10.2 that

$$A(\lambda - A)^{-1}x \to 0 \text{ as } \lambda \to \infty, \quad x \in X. \tag{10.40}$$

In view of (10.34), we get

$$\lambda(\lambda - A)^{-1}x \to x \text{ as } \lambda \to \infty, \quad x \in X. \tag{10.41}$$

Consequently,

$$\lambda(\lambda - A)^{-1}Ax \to Ax \text{ as } \lambda \to \infty, \quad x \in D(A).$$

This gives (3). To obtain (4), let λ, μ be any two positive numbers, and set

$$V_s = \exp[stA_\lambda + (1-s)tA_\mu].$$

If $v(s) = V_s x$, then

$$v'(s) = t(A_\lambda - A_\mu)v(s).$$

Consequently,

$$v(1) - v(0) = t(A_\lambda - A_\mu)\int_0^1 v(s)ds$$

by (10.20). This means

$$(e^{tA_\lambda} - e^{tA_\mu})x = t\int_0^1 V_s(A_\lambda - A_\mu)x ds.$$

Now

$$V_s = \exp[-st\lambda - (1-s)t\mu]\exp[st\lambda^2(\lambda - A)^{-1} + (1-s)t\mu^2(\mu - A)^{-1}],$$

so that

$$\begin{aligned}\|V_s\| &\leq \exp[-st\lambda-(1-s)t\mu]\exp\left[\left(\frac{st\lambda^2}{a+\lambda}\right)+\left(\frac{(1-s)t\mu^2}{a+\mu}\right)\right]\\ &= \exp\left[\left(\frac{-sta\lambda}{a+\lambda}\right)-\left(\frac{(1-s)ta\mu}{a+\mu}\right)\right]\leq 1.\end{aligned}$$

Hence,

$$\|(e^{tA_\lambda}-e^{tA_\mu})x\|\leq t\int_0^1 ds\|(A_\lambda-A_\mu)x\|,\quad x\in X.$$

Now if $x\in D(A)$, then this implies that

$$(e^{tA_\lambda}-e^{tA_\mu})x\to 0 \text{ as } \lambda,\mu\to\infty$$

in view of (3). Thus $e^{tA_\lambda}x$ approaches a limit as $\lambda\to\infty$ for each $x\in D(A)$. Denote this limit by E_tx, and apply Lemma 10.2. This completes the proof when $a>0$.

Now suppose that $a\leq 0$. Set

$$B=A+a-1.$$

Then $\rho(B)$ contains an interval of the form $[b_1,\infty)$ for some $b_1\geq 0$, and

$$\|(\lambda-B)^{-1}\|\leq(1+\lambda)^{-1},\quad \lambda\geq b_1.$$

By what has already been proved, there is a family $\{E_t\}$ of operators in $B(X)$ satisfying (a), (b), (d), (f),

$$\|E_t\|\leq e^{-t},$$

and

$$\frac{dE_tx}{dt}=BE_tx,\quad x\in D(A),\ t\geq 0.$$

Set

$$F_t=e^{t-at}E_t,\quad t\geq 0.$$

Then

$$\|F_t\|\leq e^{-at},$$

and

$$\frac{dF_tx}{dt}=(1-a)F_tx+e^{t-at}BE_tx=(B+1-a)F_tx=AF_tx.$$

Thus, the family $\{F_t\}$ satisfies all of the requirements of the theorem. This completes the proof. □

10.4. The infinitesimal generator

As noted before, a one-parameter family $\{E_t\}, t \geq 0$, of operators in $B(X)$ is called a *semigroup* if

$$E_s E_t = E_{s+t}, \quad s \geq 0,\ t \geq 0, \quad E_0 = I. \tag{10.42}$$

It is called *strongly continuous* if

$$E_t x \quad \text{is continuous in} \quad t \geq 0 \quad \text{for each} \quad x \in X. \tag{10.43}$$

A linear operator A in X is said to be an *infinitesimal generator* of $\{E_t\}$ if it is closed, $D(A)$ is dense in X and consists of those $x \in X$ for which $(E_t x)'$ exists, $t \geq 0$, and

$$(E_t x)' = A E_t x, \quad x \in D(A). \tag{10.44}$$

Clearly, an infinitesimal generator is unique. Theorem 10.1 states that every closed linear operator A satisfying (10.21) is an infinitesimal generator of a one-parameter, strongly continuous semigroup $\{E_t\}$ satisfying (c) and (f).

We now consider the opposite situation. Suppose we are given a one-parameter, strongly continuous semigroup. Does it have an infinitesimal generator? The answer is given by

Theorem 10.3. *Every strongly continuous, one-parameter semigroup $\{E_t\}$ of operators in $B(X)$ has an infinitesimal generator.*

Proof. Let W be the set of all elements of X of the form

$$x_s = \frac{1}{s} \int_0^s E_t x dt, \quad x \in X,\ s > 0.$$

Then W is dense in X, since

$$x_s \to E_0 x = x \ \text{ as } \ s \to 0.$$

Set

$$A_h = \frac{E_h - I}{h}, \quad h > 0. \tag{10.45}$$

Then

$$\begin{aligned} A_h x_s &= \frac{(E_h - I) \int_0^s E_t x dt}{sh} \\ &= \frac{\int_h^{s+h} E_t x dt - \int_0^s E_t x dt}{sh} \\ &= \frac{\int_s^{s+h} E_t x dt - \int_0^h E_t x dt}{sh} \\ &= A_s x_h. \end{aligned} \tag{10.46}$$

Thus,

$$A_h x_s \to A_s x \text{ as } h \to 0. \tag{10.47}$$

This shows that the set D of those $x \in X$ for which $A_h x$ converges as $h \to 0$ contains W and hence, is dense in X. For $x \in D$, we define Ax to be the limit of $A_h x$ as $h \to 0$. Clearly, A is a linear operator, and $D(A) = D$ is dense in X. It remains to show that A is closed and that (10.44) holds. The latter is simple, since

$$\frac{E_{t+h}x - E_t x}{h} = E_t A_h x = A_h E_t x. \tag{10.48}$$

This shows that $E_t x \in D(A)$ for $x \in D(A)$ and, hence, $(E_t x)' = E_t Ax = AE_t x$. To prove that A is closed, we make use of the facts that for each $s > 0$ we have

$$M_s = \sup_{0 \le t \le s} \|E_t\| < \infty \tag{10.49}$$

and

$$A_s z = Az_s = (Az)_s, \quad z \in D(A). \tag{10.50}$$

Assuming these for the moment, we note that

$$\|x_s\| \le M_s \|x\|, \quad x \in X. \tag{10.51}$$

Now suppose $\{x^{(n)}\}$ is a sequence of elements in $D(A)$ such that

$$x^{(n)} \to x, \; Ax^{(n)} \to y \; in \; X \text{ as } n \to \infty. \tag{10.52}$$

By (10.51), this implies $(Ax^{(n)})_s \to y_s$ as $n \to \infty$. But by (10.50),

$$(Ax^{(n)})_s = A_s x^{(n)},$$

showing that

$$A_s x^{(n)} \to y_s \text{ as } n \to \infty. \tag{10.53}$$

But

$$A_s x^{(n)} \to A_s x \text{ as } n \to \infty.$$

Hence

$$A_s x = y_s, \tag{10.54}$$

from which it follows that $A_s x \to y$ as $s \to 0$. Thus $x \in D(A)$ and $Ax = y$ by the definition of A. This shows that A is a closed operator.

Therefore, it remains only to prove (10.49) and (10.50). If (10.49) were not true, there would be a sequence $t_k \to t_0, \; 0 \le t_k \le s$, such that $\|E_{t_k}\| \to \infty$. By (10.43),

$$E_{t_k} x \to E_{t_0} x \text{ as } k \to \infty, \quad x \in X. \tag{10.55}$$

In particular,

$$\sup_k \|E_{t_k} x\| < \infty, \quad x \in X. \tag{10.56}$$

But this contradicts the Banach-Steinhaus theorem (Theorem 3.17), which says that (10.56) implies that there is a constant $M < \infty$ such that

$$\|E_{t_k} u\| \leq M. \tag{10.57}$$

Thus (10.49) holds. To prove (10.50), note that by (10.46),

$$A_s z_h = A_h z_s = (A_h z)_s. \tag{10.58}$$

Letting $h \to 0$, we obtain (10.50) in view of (10.51). This completes the proof of Theorem 10.3. □

In Theorem 10.1 the assumptions on A seem rather special. Are they necessary? The next theorem shows that they are.

Theorem 10.4. *If the family $\{E_t\}$ satisfies (a) – (d), then its infinitesimal generator A satisfies (10.21).*

Proof. We first note that for any $s > 0$ and $\lambda > 0$, we have $\lambda \in \rho(A_s)$ and

$$\|(\lambda - A_s)^{-1}\| \leq \frac{s}{1 + \lambda s - e^{-as}} \leq \frac{1}{\lambda}, \tag{10.59}$$

where A_s is defined by (10.45). In fact,

$$\lambda - A_s = \frac{(\lambda s - E_s + I)}{s} = \frac{(\lambda s + 1)[I - (E_s/\lambda s + 1)]}{s}.$$

Since $\|E_s\| \leq 1$ by (c), this operator is invertible for $\lambda > 0$, by Theorem 1.1, and a simple calculation gives (10.59). In particular, we have

$$\|(\lambda - A_s)x\| \geq \frac{(1 + \lambda s - e^{-as})\|x\|}{s}, \quad x \in X. \tag{10.60}$$

If $x \in D(A)$, we can take the limit as $s \to 0$, obtaining by l'Hospital's rule

$$\|(\lambda - A)x\| \geq (\lambda + a)\|x\|, \quad x \in D(A). \tag{10.61}$$

This shows that $\lambda - A$ has an inverse for $\lambda > 0$ and that its range is closed. If we can show that its range is also dense in X, it will follow that it is the whole of X, and (10.21) will follow from (10.61). Thus it remains to show that $R(\lambda - A)$ is dense in X. To do this, we shall show that it contains $D(A)$. In other words, we want to show that we can solve

$$(\lambda - A)x = y \tag{10.62}$$

for $x \in D(A)$ provided $y \in D(A)$. Set

$$x^{(s)} = (\lambda - A_s)^{-1} y, \quad s > 0. \tag{10.63}$$

We claim that $x^{(s)} \in D(A)$. This follows from the fact that

$$(\lambda - A_s)^{-1} A_h y = A_h (\lambda - A_s)^{-1} y. \tag{10.64}$$

Since $A_h y$ converges as $h \to 0$, then the same is true of $A_h(\lambda - A_s)^{-1}y$. In view of (10.59), we see by (10.63) that $x^{(s)}$ converges in X as $s \to 0$. In fact, we have

$$\begin{aligned} \|[(\lambda - A_s)^{-1} - (\lambda - A_t)^{-1}]y\| &= \|(\lambda - A_s)^{-1}(\lambda - A_t)^{-1}(A_t - A_s)y\| \\ &\le \lambda^{-2}\|(A_t - A_s)y\| \to 0 \text{ as } s, t \to 0. \end{aligned}$$

Moreover,

$$\|(\lambda - A)x^{(s)} - y\| = \|(\lambda - A_s)^{-1}(A_s - A)y\| \le \frac{\|(A_s - A)y\|}{\lambda} \to 0 \quad \text{as } s \to 0.$$

Thus $(\lambda - A)x^{(s)} \to y$ as $x \to 0$. Let x be the limit of $x^{(s)}$. Then $Ax^{(s)} \to \lambda x - y$. Since A is a closed operator, $x \in D(A)$ and $Ax = \lambda x - y$. Thus, x is a solution of (10.62). This shows that $R(\lambda - A) \supset D(A)$, and the proof is complete. □

10.5. An approximation theorem

We shall show how to approximate a semigroup by a family of the form $\{e^{tB}\}$, where $B \in B(X)$. We have

Theorem 10.5. *Let $\{E_t\}$ be a strongly continuous, one-parameter semigroup, and let A_h be defined by (10.45). Then*

$$e^{tA_h}x \to E_t x \text{ as } h \to 0, \quad t \ge 0,\ x \in X. \tag{10.65}$$

Moreover, for each $x \in X$, the convergence is uniform for t in a bounded interval.

Proof. We shall verify that

(i): A_h commutes with E_t for each $t \ge 0$,

(ii): for each $s > 0$, there is an N_s such that

$$\|e^{tA_h}\| \le N_s, \quad 0 \le t \le s,\ 0 < h < 1, \tag{10.66}$$

(iii): $A_h x \to Ax$ as $h \to 0$, $x \in D(A)$, where A is the infinitesimal generator of $\{E_t\}$.

Assuming these for the moment, let us see how they imply (10.65). Let n be a positive integer, and set $\tau = t/n$. Then

$$e^{tA_h} - E_t = e^{n\tau A_h} - E_\tau^n = (e^{\tau A_h} - E_\tau)\left[\sum_{k=0}^{n-1} e^{k\tau A_h} E_{(n-k-1)\tau}\right]$$

Hence, if $t \le s$,

$$\begin{aligned} \|(e^{tA_h} - E_t)x\| &\le \|(e^{\tau A_h} - E_\tau)x\| \left[\sum_{k=0}^{n-1} N_s M_s\right] = nN_s M_s \|(e^{\tau A_h} - E_\tau)x\| \\ &\le (s/\tau) N_s M_s \|(e^{\tau A_h} - E_\tau)x\|, \end{aligned}$$

where M_s is given by (10.49) and N_s is given by (10.66). Now if $x \in D(A)$, we have

$$\frac{(e^{\tau A_h} - E_\tau)x}{\tau} = \frac{(e^{\tau A_h} - I)x}{\tau} + \frac{(I - E_\tau)x}{\tau} \to (A_h - A)x \quad \text{as} \quad \tau \to 0.$$

Hence,

$$(e^{tA_h} - E_t)x \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0, \quad x \in D(A). \tag{10.67}$$

Moreover, if $t \le s$ and x is fixed, this convergence is uniform. To see this, first take $(A_h - A)x$ as small as you like, and then let $\tau \to 0$. Since

$$\|e^{tA_h} - E_t\| \le N_s + M_s,$$

(10.67) follows from Lemma 10.2. We leave it as an exercise to show that the convergence is uniform in t on bounded sets for each $x \in X$.

It remains to prove (i) – (iii). Now, (i) follows from (a), and (iii) is just (10.22). To prove (ii), note that

$$\|E_t\| \le M^{t+1}, \quad t \ge 0, \tag{10.68}$$

where $M = M_1$ is M_s for $s = 1$. This can be proved by noting that if j is the largest integer less than or equal to t, then

$$\|E_t\| \le \|E_1^j\| \cdot \|E_{t-j}\| \le M^{j+1}.$$

Now,

$$\|e^{\alpha E_h}\| \le \sum_0^\infty \frac{1}{k!}|\alpha|^k \|E_{hk}\| \le \sum_0^\infty \frac{1}{k!}|\alpha|^k M^{hk+1} \le M e^{|\alpha| M^h}.$$

Hence, in view of (10.45),

$$\|e^{tA_h}\| \le M \exp\left[\frac{t(M^h - 1)}{h}\right].$$

Since $M \ge 1$, we have

$$\frac{(M^h - 1)}{h} \le \max_{0 \le h \le 1} \left|\frac{dM^h}{dh}\right| = M \log M,$$

and hence,

$$\|e^{tA_h}\| \le M^{s^{(M+1)}}, \quad 0 \le t \le s,\ 0 < h \le 1. \tag{10.69}$$

This gives (10.66), and the proof is complete. $\square$

As an application, let us prove the Weierstrass approximation theorem, which states that every function continuous in the interval $0 \le t \le 1$ can be uniformly approximated as closely as desired by a polynomial. By defining

$$x(t) = x(1), \quad t > 1,$$

we can consider every function in $C = C[0,1]$ to be contained in the Banach space of functions bounded and uniformly continuous in the interval $0 \le t < \infty$ with norm

$$\|x\| = \sup_{0 \le t < \infty} |x(t)|.$$

The operators

$$E_s x(t) = x(t+s)$$

form a one-parameter semigroup. Since

$$\|E_s x - E_t x\| \le \sup_{0 \le \tau < \infty} |x(\tau + s) - x(\tau + t)| \to 0 \quad \text{as } s \to t,$$

the semigroup is strongly continuous. By Theorem 10.5,

$$x(t+s) = \lim_{h \to 0} e^{sA_h} x = \lim_{h \to 0} \sum_{k=0}^{\infty} \frac{1}{k!} s^k A_h^k x(t).$$

Thus, for any $\varepsilon > 0$, there are an $h > 0$ and an N such that

$$\left| x(t+s) - \sum_{k=0}^{N} \frac{1}{k!} s^k A_h^k x(t) \right| < \varepsilon, \quad 0 \le s, \ t \le 1.$$

Setting $t = 0$, we obtain precisely the desired result.

10.6. Problems

(1) Let $\{E_t\}$ be a strongly continuous semigroup of operators. Suppose E_{t_1} is compact for some fixed $t_1 > 0$. Show that E_t is compact for all $t > t_1$.

(2) Show that

$$\frac{d^k}{dt^k} E_t x = E_t A^k x, \quad x \in D(A^k), \ t \ge 0,$$

where $\{E_t\}$ is a strongly continuous semigroup and A is its infinitesimal generator.

(3) Show that the set of functions bounded and uniformly continuous in the interval $0 \le t < \infty$ with norm

$$\|x\| = \sup_{0 \le t < \infty} |x(t)|$$

forms a Banach space.

(4) Show that (10.17) is differentiable in $t > 0$, continuous in $t \geq 0$ and satisfies

$$F'(t) = 0, \quad t > 0, \quad F(0) = 0.$$

(5) Show that the Riemann integral

$$\int_{t_0}^{t_1} u(s)\, ds$$

exists for $u(t)$ continuous in $[t_0, t_1]$ with values in a Banach space X.

(6) Prove the estimate (10.18).

(7) Under the same hypotheses, show that the function

$$U(t) = \int_{t_0}^{t} U(s)\, ds$$

is differentiable in the interval and satisfies $U'(t) = u(t)$.

(8) Prove (10.20) for $u(t)$ satisfying the same hypotheses.

(9) Show that if the convergence in (10.67) is uniform in t on bounded sets for each $x \in D(A)$, then it is uniform for each $x \in X$.

(10) Prove the following. Let $\{E_t\}$ be a strongly continuous, one-parameter semigroup of operators on X with infinitesimal generator A. Let $\{G_r\}$ be a family of bounded operators depending on a parameter r, $0 \leq r \leq r_0$ such that

$$G_r E_t = E_t G_r, \quad t \geq 0,\ 0 \leq r \leq r_0,$$

$$\sup_{0 \leq t \leq T} \|e^{tG_r}\| \leq M_T, \quad 0 \leq r \leq r_0,$$

and

$$G_r u \to Au \ \text{ as } \ r \to 0, \quad u \in D(A).$$

Then

$$e^{tG_r} u \to E_t u \ \text{ as } \ r \to 0, \quad x \in X,\ 0 \leq t \leq T.$$

Moreover, the convergence is uniform.

Chapter 11

HILBERT SPACE

11.1. When is a Banach space a Hilbert space?

Since a Hilbert space has a scalar product, one should expect that many more things are true in a Hilbert space than are true in a Banach space that is not a Hilbert space. This supposition is, indeed, correct, and we shall describe some of these properties in the next few chapters.

There will be many occasions when it will be preferable to work in a *complex* Hilbert space. This is a complex Banach space having a scalar product $(\cdot, \cdot)$ which is complex valued and satisfies

(i): $(\alpha u, v) = \alpha(u, v), \quad \alpha$ complex,

(ii): $(u + v, w) = (u, w) + (v, w)$,

(iii): $(v, u) = \overline{(u, v)}$,

(iv): $(u, u) = \|u\|^2$.

Before studying Hilbert spaces, we should learn how to recognize them. It may seem simple. We first need to see if it has a scalar product. However, this is not as simple as it may first appear. If we are given a Banach space and not given a scalar product for it, this does not necessarily mean that one does not exist. Perhaps we could find one if we searched hard enough, and of course, as soon as we find one, the Banach space becomes a Hilbert space.

The problem reduces to the following: Given a Banach space X, does there exist a scalar product on X which will convert it into a Hilbert space with the same norm? We saw way back in Section 1.3 that a necessary condition is that the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X \tag{11.1}$$

should hold. We shall see that this is also sufficient. We have

Theorem 11.1. *A Banach space X can be converted into a Hilbert space with the same norm if and only if (11.1) holds.*

Proof. The simple proof of the "only if" part was given in Section 1.3. To prove the "if" part, assume that (11.1) holds. We must find a scalar product for X. First, assume that X is a real Banach space. If a scalar product existed, it would satisfy

$$\|x+y\|^2 = \|x\|^2 + 2(x,y) + \|y\|^2. \tag{11.2}$$

This suggests that we define the scalar product to be

$$(x,y) = [\|x+y\|^2 - \|x\|^2 - \|y\|^2]/2. \tag{11.3}$$

This does not look like a scalar product, but things are not always what they seem. If we take this as the definition, we get immediately that

$$(x,y) = (y,x), \tag{11.4}$$

$$(x,x) = \|x\|^2, \tag{11.5}$$

$$(x_n, y) \to (x,y) \quad \text{if} \ \ x_n \to x \ \ \text{in} \ \ X, \tag{11.6}$$

$$(0,y) = 0. \tag{11.7}$$

Moreover,

$$2(x,y) + 2(z,y) = \|x+y\|^2 - \|x\|^2 + \|z+y\|^2 - \|z\|^2 - 2\|y\|^2.$$

By the parallelogram law,

$$2\|x+y\|^2 + 2\|z+y\|^2 = \|x+z+2y\|^2 + \|x-z\|^2$$

and

$$2\|x\|^2 + 2\|z\|^2 = \|x+z\|^2 + \|x-z\|^2.$$

Hence,

$$(x,y) + (z,y) = [\|x+z+2y\|^2 - \|x+z\|^2 - 4\|y\|^2]/4 \tag{11.8}$$

$$= \|\frac{1}{2}(x+z) + y\|^2 - \|\frac{1}{2}(x+z)\|^2 - \|y\|^2 = 2\left(\frac{1}{2}(x+z), y\right).$$

Now, if we take $z=0$ in (11.8), we see by (11.7) that

$$(x,y) = 2\left(\frac{x}{2}, y\right). \tag{11.9}$$

Applying this to the right hand side of (11.8), we obtain

$$(x,y) + (z,y) = (x+z,y). \tag{11.10}$$

It only remains to prove that

$$\alpha(x, y) = (\alpha x, y), \quad x, y \in X \tag{11.11}$$

for any $\alpha \in \mathbb{R}$. By repeated applications of (11.10), we see that

$$n(x, y) = (nx, y), \quad x, y \in X$$

for any positive integer n, and hence, if m, n are positive integers,

$$\begin{aligned}\left(\frac{n}{m}\right)(x, y) &= \left(\frac{n}{m}\right)\left(m\left(\frac{x}{m}\right), y\right) = \left(\frac{n}{m}\right) m\left(\frac{x}{m}, y\right) \\ &= n\left(\frac{x}{m}, y\right) = \left(\frac{nx}{m}, y\right).\end{aligned}$$

Thus, (11.11) holds for α a positive rational number. Moreover, by (11.10)

$$(x, y) + (-x, y) = (x - x, y) = 0.$$

Hence,

$$(-x, y) = -(x, y).$$

This shows that (11.11) holds for all rational numbers α. Now if α is any real number, there is a sequence $\{\alpha_n\}$ of rational numbers converging to α. Hence, by (11.6),

$$\alpha(x, y) = \lim \alpha_n(x, y) = \lim(\alpha_n x, y) = (\alpha x, y).$$

This completes the proof for a real Banach space. If X is a complex Banach space, then any prospective scalar product must satisfy

$$\|x + y\|^2 = \|x\|^2 + 2\Re e\,(x, y) + \|y\|^2.$$

Hence, we must have

$$2\Re e\,(x, y) = \|x + y\|^2 - \|x\|^2 - \|y\|^2. \tag{11.12}$$

The arguments given above show that $\Re e\,(x, y)$ has all of the properties of a real scalar product. Moreover, if (x, y) is a complex scalar product, then

$$\Im m\,(x, y) = \Re e\,(-i)(x, y) = -\Re e\,(ix, y).$$

Thus, it seems reasonable to define (x, y) by

$$(x, y) = \Re e\,(x, y) - i\,\Re e\,(ix, y), \tag{11.13}$$

where $\Re e\,(x, y)$ is given by (11.12). We then have

$$(ix, y) = \Re e\,(ix, y) - i\,\Re e\,(-x, y) = i[\Re e\,(x, y) - i\,\Re e\,(ix, y)] = i(x, y).$$

This shows that (11.11) holds with respect to (x, y) for all complex α. Now by (11.12),

$$\Re e\,(ix, iy) = \Re e\,(x, y).$$

Hence,

$$\begin{aligned}(y, x) &= \Re e\,(y, x) - i\,\Re e\,(iy, x) \\ &= \Re e\,(x, y) - i\,\Re e\,(-y, ix) \\ &= \Re e\,(x, y) + i\,\Re e\,(ix, y) \\ &= \overline{(x, y)}.\end{aligned}$$

Thus (x, x) is real and must, therefore, equal $\Re e(x, x)$, which equals $\|x\|^2$. The other properties of (x, y) follow from those of $\Re e\,(x, y)$, and the proof is complete. $\square$

11.2. Normal operators

Let $\{\varphi_n\}$ be an orthonormal sequence (finite or infinite) in a Hilbert space H. Let $\{\lambda_k\}$ be a sequence (of the same length) of scalars satisfying

$$|\lambda_k| \leq C.$$

Then for each element $f \in H$, the series

$$\sum \lambda_k(f, \varphi_k)\varphi_k$$

converges in H (Theorem 1.5). Define the operator A on H by

$$Af = \sum \lambda_k(f, \varphi_k)\varphi_k. \tag{11.14}$$

Clearly, A is a linear operator. It is also bounded, since

$$\|Af\|^2 = \sum |\lambda_k|^2|(f, \varphi_k)|^2 \leq C^2\|f\|^2, \tag{11.15}$$

by Bessel's inequality (cf.(1.60)). For convenience, let us assume that each $\lambda_k \neq 0$ (just remove those φ_k corresponding to the λ_k that vanish). In this case, $N(A)$ consists of precisely those $f \in H$ which are orthogonal to all of the φ_k. Clearly, such f are in $N(A)$. Conversely, if $f \in N(A)$, then

$$0 = (Af, \varphi_k) = \lambda_k(f, \varphi_k).$$

Hence, $(f, \varphi_k) = 0$ for each k. Moreover, each λ_k is an eigenvalue of A with φ_k the corresponding eigenvector. This follows immediately from (11.14). Since $\sigma(A)$ is closed, it also contains the limit points of the λ_k.

Next, we shall see that if $\lambda \neq 0$ is not a limit point of the λ_k, then $\lambda \in \rho(A)$. To show this, we solve

$$(\lambda - A)u = f \tag{11.16}$$

for any $f \in H$. Any solution of (11.16) satisfies

$$\lambda u = f + Au = f + \sum \lambda_k(u, \varphi_k)\varphi_k. \tag{11.17}$$

Hence,

$$\lambda(u, \varphi_k) = (f, \varphi_k) + \lambda_k(u, \varphi_k),$$

or

$$(u, \varphi_k) = \frac{(f, \varphi_k)}{\lambda - \lambda_k}. \tag{11.18}$$

Substituting back in (11.17), we obtain

$$\lambda u = f + \sum \frac{\lambda_k (f, \varphi_k) \varphi_k}{\lambda - \lambda_k}. \tag{11.19}$$

Since λ is not a limit point of the λ_k, there is a $\delta > 0$ such that

$$|\lambda - \lambda_k| \geq \delta, \quad k = 1, 2, \cdots .$$

Hence, the series in (11.19) converges for each $f \in H$. It is an easy exercise to verify that (11.19) is indeed a solution of (11.16). To see that $(\lambda - A)^{-1}$ is bounded, note that

$$|\lambda| \cdot \|u\| \leq \|f\| + C\|f\|/\delta \tag{11.20}$$

(cf. (11.15)). Thus, we have proved

Lemma 11.2. *If the operator A is given by (11.14), then $\sigma(A)$ consists of the points λ_k, their limit points and possibly* 0. *$N(A)$ consists of those u which are orthogonal to all of the φ_k. For $\lambda \in \rho(A)$, the solution of (11.16) is given by (11.19).*

We see from all this that the operator (11.14) has many useful properties. Therefore, it would be desirable to determine conditions under which operators are guaranteed to be of that form. For this purpose, we note another property of A. It is expressed in terms of the Hilbert space adjoint of A.

Let H_1 and H_2 be Hilbert spaces, and let A be an operator in $B(H_1, H_2)$. For fixed $y \in H_2$, the expression $Fx = (Ax, y)$ is a bounded linear functional on H_1. By the Riesz representation theorem (Theorem 2.1), there is a $z \in H_1$ such that $Fx = (x, z)$ for all $x \in H_1$. Set $z = A^*y$. Then A^* is a linear operator from H_2 to H_1 satisfying

$$(Ax, y) = (x, A^*y). \tag{11.21}$$

A^* is called the *Hilbert space adjoint* of A. Note the difference between A^* and the operator A' defined in Section 3.1. As in the case of the operator A', we see that A^* is bounded and

$$\|A^*\| = \|A\|. \tag{11.22}$$

The proof is left as an exercise.

Returning to the operator A, we remove the assumption that each $\lambda_k \neq 0$ and note that

$$(Au, v) = \sum \lambda_k (u, \varphi_k)(\varphi_k, v) = (u, \sum \bar{\lambda}_k (v, \varphi_k) \varphi_k),$$

showing that

$$A^*v = \sum \bar{\lambda}_k(v, \varphi_k)\varphi_k. \tag{11.23}$$

(If H is a complex Hilbert space, then the complex conjugates $\overline{\lambda}_k$ of the λ_k are required. If H is a real Hilbert space, then the λ_k are real, and it does not matter.) Now, by Lemma 11.2, we see that each $\overline{\lambda}_k$ is an eigenvalue of A^* with φ_k a corresponding eigenvector. Note also that

$$\|A^*f\|^2 = \sum |\lambda_k|^2|(f, \varphi_k)|^2, \tag{11.24}$$

showing that

$$\|A^*f\| = \|Af\|, \quad f \in H. \tag{11.25}$$

An operator satisfying (11.25) is called *normal*. An important characterization is given by

Theorem 11.3. *An operator is normal and compact if and only if it is of the form (11.14) with $\{\varphi_k\}$ an orthonormal set and $\lambda_k \to 0$ as $k \to \infty$.*

In proving Theorem 11.3, we shall make use of the following properties of normal operators:

Lemma 11.4. *If A is normal, then*

$$\|(A^* - \bar{\lambda})u\| = \|(A - \lambda)u\|, \quad u \in H. \tag{11.26}$$

Corollary 11.5. *If A is normal and $A\varphi = \lambda\varphi$, then $A^*\varphi = \overline{\lambda}\varphi$.*

Lemma 11.6. *If A is normal and compact, then it has an eigenvalue λ such that $|\lambda| = \|A\|$.*

Let us assume these for the moment and give the proof of Theorem 11.3.

Proof. Let A be a normal compact operator on H. If $A = 0$, then the theorem is trivially true. Otherwise, by Lemma 11.6, there is an eigenvalue λ_0 such that $|\lambda_0| = \|A\| \neq 0$. Let φ_0 be a corresponding eigenvector with norm one, and let H_1 be the subspace of all elements of H orthogonal to φ_0. We note that A and A^* map H_1 into itself. In fact, if $v \in H_1$, then

$$(Av, \varphi_0) = (v, A^*\varphi_0) = \lambda_0(v, \varphi_0) = 0$$

by Corollary 11.5. A similar argument holds for A^*. Let A_1 be the restriction of A to H_1. For $u, v \in H_1$, we have

$$(u, A_1^*v) = (A_1u, v) = (Au, v) = (u, A^*v),$$

showing that $A_1^* = A^*$ on H_1. This implies that A_1 is normal as well as compact on the Hilbert space H_1. Now, if $A_1 = 0$, then

$$Au = \lambda_0(u, \varphi_0)\varphi_0, \quad u \in H,$$

and the proof is complete. Otherwise, by Lemma 11.6 there is a pair λ_1, φ_1 such that

$$|\lambda_1| = \|A_1\|, \ \varphi_1 \in H_1, \ \|\varphi_1\| = 1, \ A_1\varphi_1 = \lambda_1\varphi_1.$$

Let H_2 be the subspace orthogonal to both φ_0 and φ_1, and let A_2 be the restriction of A to H_2. Again, A_2 is normal and compact on H_2, and if $A_2 \neq 0$, there is a pair λ_2, φ_2 such that

$$|\lambda_2| = \|A_2\|, \ \varphi_2 \in H_2, \ \|\varphi_2\| = 1, \ A_2\varphi_2 = \lambda_2\varphi_2.$$

Continuing in this way, we get a sequence A_k of restrictions of A to subspaces H_k, and sequences $\{\lambda_k\}\{\varphi_k\}$ such that

$$|\lambda_k| = \|A_k\|, \ \varphi_k \in H_k, \ \|\varphi_k\| = 1, \ A_k\varphi_k = \lambda_k\varphi_k, \quad k = 1, 2, \cdots .$$

Moreover, from the way the φ_k were chosen we see that they form an orthonormal sequence. If none of the A_k equals 0, this is an infinite sequence. Since A_k is a restriction of A_{k-1} to a subspace, we have

$$|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots .$$

Moreover, by Theorem 6.2,

$$\lambda_k \to 0 \quad \text{as } k \to \infty. \tag{11.27}$$

Now, let u be any element of H. Set

$$u_n = u - \sum_0^n (u, \varphi_k)\varphi_k.$$

Then

$$(u_n, \varphi_k) = 0, \quad 0 \leq k \leq n.$$

This shows that $u_n \in H_{n+1}$. Hence,

$$\|Au_n\| \leq |\lambda_{n+1}| \cdot \|u_n\|, \tag{11.28}$$

while

$$\|u_n\|^2 = \|u\|^2 - 2\sum_0^n |(u, \varphi_k)|^2 + \sum_0^n |(u, \varphi_k)|^2 \leq \|u\|^2.$$

Hence, $Au_n \to 0$ as $n \to \infty$. This means that

$$Au = \sum_0^\infty \lambda_k(u, \varphi_k)\varphi_k, \tag{11.29}$$

and the theorem is proved in one direction. To prove it in the other direction, suppose A is of the form (11.14) with $\lambda_k \to 0$. Set

$$A_n u = \sum_1^n \lambda_k(u, \varphi_k)\varphi_k, \quad n = 1, 2, \cdots .$$

Then A_n is a linear operator of finite rank for each n. Now, for any $\varepsilon > 0$, there is an integer N such that $|\lambda_k| < \varepsilon$ for $k > N$. Thus

$$\|A_n u - Au\|^2 = \sum_{n+1}^{\infty} |\lambda_k|^2 |(u, \varphi_k)|^2 \leq \varepsilon^2 \|u\|^2.$$

This shows that

$$\|A_n - A\| \leq \varepsilon \quad \text{for} \;\; k > N,$$

which means that $A_n \to A$ in norm as $n \to \infty$. Hence, we see that A is compact by Theorem 4.11. That A is normal follows from (11.24). This completes the proof. $\square$

It remains to prove Lemmas 11.4 and 11.6. The former is very simple. In fact, one has

$$\begin{aligned} \|(A^* - \bar{\lambda})u\|^2 &= \|A^*u\|^2 - 2\,\Re e\,(\bar{\lambda}u, A^*u) + |\lambda|^2\|u\|^2 \\ &= \|Au\|^2 - 2\,\Re e\,(Au, \lambda u) + |\lambda|^2\|u\|^2 = \|(A-\lambda)u\|^2. \end{aligned}$$

Corollary 11.5 follows immediately. In proving Lemma 11.6 we shall make use of

Lemma 11.7. *If A is normal, then*

$$r_\sigma(A) = \|A\|. \tag{11.30}$$

Once this is known, it follows from Theorem 6.13 that there is a $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$. If $A = 0$, then clearly, 0 is an eigenvalue of A. Otherwise, $\lambda \neq 0$, and it must be an eigenvalue by Theorem 6.2.

Therefore, it remains to give the proof of Lemma 11.7.

Proof. Let us show that

$$\|A^n\| = \|A\|^n, \quad n = 1, 2, \cdots, \tag{11.31}$$

when A is normal. For this purpose we note that

$$\begin{aligned} \|A^k u\|^2 &= (A^k u, A^k u) \\ &= (A^* A^k u, A^{k-1} u) \\ &\leq \|A^* A^k u\| \cdot \|A^{k-1} u\| \\ &= \|A^{k+1} u\| \cdot \|A^{k-1} u\|, \end{aligned}$$

which implies

$$\|A^k\|^2 \leq \|A^{k+1}\| \cdot \|A^{k-1}\|, \quad k = 1, 2, \cdots. \tag{11.32}$$

Since $\|A^n\| \leq \|A\|^n$ for any operator, it suffices to prove

$$\|A\|^n \leq \|A^n\|, \quad n = 1, 2, \cdots. \tag{11.33}$$

Let k be any positive integer, and assume that (11.33) holds for all $n \le k$. Then by (11.32) we have

$$\|A\|^{2k} \le \|A^{k+1}\| \cdot \|A\|^{k-1},$$

which gives (11.33) for $n = k+1$. Since (11.31) holds for $n = 1$, the induction argument shows that it holds for all n. Once (11.31) is known, we have

$$r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \|A\|, \tag{11.34}$$

and the proof is complete. □

We also have

Corollary 11.8. *If A is a normal compact operator, then there is an orthonormal sequence $\{\varphi_k\}$ of eigenvectors of A such that every element u in H can be written in the form*

$$u = h + \sum (u, \varphi_k)\varphi_k, \tag{11.35}$$

where $h \in N(A)$. In particular, if $N(A)$ is separable, then A has a complete orthonormal set of eigenvectors.

Proof. By Theorem 11.3, A has an orthonormal set $\{\varphi_k\}$ of eigenvectors such that (11.14) holds. If u is any element of H, set

$$h = u - \sum (u, \varphi_k)\varphi_k.$$

Then, by (11.14), $Ah = 0$, showing that $h \in N(A)$. This proves (11.35). If $N(A)$ is separable, then we shall see that it has a complete orthonormal set $\{\psi_j\}$. Once we know this, we have

$$u = \sum (h, \psi_j)\psi_j + \sum (u, \varphi_k)\varphi_k,$$

by Theorem 1.6, and the fact that the φ_k are orthogonal to the ψ_j (Lemma 11.2). Hence,

$$u = \sum (u, \psi_j)\psi_j + \sum (u, \varphi_k)\varphi_k,$$

showing that $(\psi_1, \cdots, \varphi_1, \cdots)$ forms a complete orthonormal set. □

It remains only to show that $N(A)$ has a complete orthonormal set. This follows from

Lemma 11.9. *Every separable Hilbert space has a complete orthonormal sequence.*

Proof. Let H be a separable Hilbert space, and let $\{x_n\}$ be a dense sequence in H. Remove from this sequence any element which is a linear combination of the preceding x_j. Let N_n be the subspace spanned by $x_1, \cdots, x_n$, and let φ_n be an element of N_n having norm one and orthogonal to N_{n-1}. Such an element exists by Corollary 2.4. Clearly, the sequence $\{\varphi_n\}$ is orthonormal. It is complete, since the original sequence $\{x_n\}$ is contained in

$$W = \bigcup_1^\infty N_n,$$

and each element in W is a linear combination of a finite number of the φ_k. Thus the linear combinations of the φ_k are dense in H. This completes the proof. □

11.3. Approximation by operators of finite rank

In Section 4.3 we claimed that, for each compact operator K in a Hilbert space, one can find a sequence of operators of finite rank converging to K in norm. We are now in a position to prove this assertion.

First, assume that H is separable. Then, by Lemma 11.9, we know that H has a complete orthonormal sequence $\{\varphi_k\}$. Define the operator P_n by

$$P_n u = \sum_1^n (u, \varphi_k)\varphi_k. \tag{11.36}$$

Then by Theorem 1.6,

$$\|P_n\| \leq 1, \quad \|(I - P_n)\| \leq 1, \quad n = 1, 2, \cdots , \tag{11.37}$$

and

$$P_n u \to u \text{ as } n \to \infty, \quad u \in H. \tag{11.38}$$

Now, if the assertion were false, there would be an operator $K \in K(H)$ and a number $\delta > 0$ such that

$$\|K - F\| \geq \delta \tag{11.39}$$

for all operators F of finite rank. Now

$$F_n u = P_n K u = \sum_1^n (Ku, \varphi_k)\varphi_k$$

is an operator of finite rank. Hence, (11.39) implies that for each n there is a $u_n \in H$ satisfying

$$\|u_n\| = 1, \quad \|(K - F_n)u_n\| \geq \delta/2. \tag{11.40}$$

Since the sequence $\{u_n\}$ is bounded, it has a subsequence $\{v_j\}$ such that Kv_j converges to some element $w \in H$. But

$$\begin{aligned} \|(K - F_n)u_n\| &\leq \|(I - P_n)(Ku_n - w)\| + \|(I - P_n)w\| \\ &\leq \|Ku_n - w\| + \|(I - P_n)w\|. \end{aligned}$$

Now, for n sufficiently large, we have by (11.38)

$$\|(I - P_n)w\| < \delta/4,$$

and, for an infinite number of indices,

$$\|Ku_n - w\| < \delta/4.$$

This contradicts (11.40) and proves the assertion in the case when H is separable.

Now let H be any Hilbert space, and let K be any operator in $K(H)$. Set $A = K^*K$. Then

$$A^* = A. \tag{11.41}$$

Operators satisfying (11.41) are called *selfadjoint*. In particular, A is normal. By Theorem 11.3, A is of the form (11.14) for some orthonormal sequence $\{\varphi_k\}$. Let H_0 be the subspace of all linear combinations of elements of the form

$$K^n\varphi_k, \quad n = 0, 1, 2, \cdots ; \ k = 1, 2, \cdots .$$

Then $\overline{H}_0$ is a separable closed subspace of H, and K maps $\overline{H}_0$ into itself. By what we have already proved, there is a sequence $\{F_n\}$ of finite rank operators on $\overline{H}_0$ converging in norm to the restriction $\hat{K}$ of K to $\overline{H}_0$. Now every element $u \in H$ can be written in the form $u = v + w$, where $v \in \overline{H}_0$, and w is orthogonal to H_0. Observe that $Kw = 0$. To see this, note that w is orthogonal to each φ_k, and hence, is in $N(A)$ (Lemma 11.2). Consequently,

$$0 = (Aw, w) = (K^*Kw, w) = \|Kw\|^2,$$

showing that $Kw = 0$. Thus, if we define

$$G_n u = F_n v, \quad n = 1, 2, \cdots ,$$

then G_n is of finite rank, and $\|G_n - K\| \to 0$ as $n \to \infty$. This completes the proof.

11.4. Integral operators

Let us now describe an application of the theorems of Section 11.2 to some integral operators of the form

$$Ku(x) = \int_a^b K(x, y)u(y)dy, \tag{11.42}$$

where $-\infty \le a \le b \le \infty$ and the function $K(x, y)$ satisfies

$$\int_a^b \int_a^b |K(x, y)|^2 dx\, dy < \infty. \tag{11.43}$$

[In the terminology of Section 1.4, $K(x, y) \in L^2(Q)$, where Q is the square $\{(x, y) \in \mathbb{R}^2 : a \le x, \ y \le b\}$.] We now have

Lemma 11.10. *The operator K given by (11.42) is a compact operator on $L^2(a,b)$ if $K(x,y)$ satisfies (11.43).*

We shall save the proof of Lemma 11.10 until the end of this section. Let us see what conclusions can be drawn from it. By Theorem 11.5, we have

Theorem 11.11. *If*

$$K(x,y)\overline{K(x,z)} = K(z,x)\overline{K(y,x)}, \quad a \le x,\ y,\ z \le b, \tag{11.44}$$

then there exists an orthonormal sequence $\{\varphi_k\}$(finite or infinite) of functions in $L^2(a,b)$ such that

$$Ku(x) = \sum \lambda_k (u,\varphi_k)\varphi_k, \tag{11.45}$$

where the series converges in $L^2(a,b)$. Moreover,

$$\int_a^b \int_a^b |K(x,y) - \sum_{k<m} \lambda_k \varphi_k(x)\overline{\varphi_k(y)}|^2 dx\, dy = \sum_{k\ge m} |\lambda_k|^2, \quad m = 1,2,\cdots. \tag{11.46}$$

In particular, the sequence $\{\varphi_k\}$ is infinite unless $K(x,y)$ is of the form

$$K(x,y) = \sum_1^N \lambda_k \varphi_k(x)\overline{\varphi_k(y)}. \tag{11.47}$$

If

$$\int_a^b |K(x,y)|^2 dy \le M^2 < \infty, \quad a \le x \le b, \tag{11.48}$$

then the convergence in (11.45) is uniform.

Proof. In view of Lemma 11.10, we can prove (11.45) by verifying that the operator (11.42) is normal and applying Theorem 11.3. This is simple, since

$$K^*v(y) = \int \overline{K(x,y)} v(x)dx. \tag{11.49}$$

Hence,

$$KK^*u(z) = \int K(z,x)K^*u(x)dx = \iint K(z,x)\overline{K(y,x)}u(y)dx\,dy,$$

while

$$K^*Ku(z) = \int \overline{K(x,z)}Ku(x)dx = \iint \overline{K(x,z)}K(x,y)u(y)dx\,dy.$$

By (11.44), we see that

$$KK^* = K^*K, \tag{11.50}$$

which implies that K is normal, since

$$\|K^*u\|^2 = (KK^*u, u) = (K^*Ku, u) = \|Ku\|^2.$$

Thus, (11.45) holds. To prove (11.46), note that for $u, v \in L^2(a, b)$, we have

$$\iint [K(x, y) - \sum \lambda_k \varphi_k(x)\overline{\varphi_k(y)}]u(y)\overline{v(x)}dx\, dy$$

$$= (Ku, v) - \sum \lambda_k(u, \varphi_k)(\varphi_k, v) = 0$$

by (11.45). Since this is true for any $u, v \in L^2(a, b)$, it follows that

(11.51) $$K(x, y) = \sum \lambda_k \varphi_k(x)\overline{\varphi_k(x)},$$

where the series converges in $L^2(Q)$. This gives (11.46). The last statement of the theorem follows from the inequality

$$|Ku(x)|^2 = \left|\int K(x, y)u(y)dy\right|^2 \le \int |K(x, y)|^2 dy \int |u(y)|^2 dy \le M^2\|u\|^2.$$

Hence

(11.52) $$|Ku(x)| \le M\|u\|, \quad a \le x \le b.$$

In particular, if $u_k \to u$ in $L^2(a, b)$, then

(11.53) $$|Ku_k(x) - Ku(x)| \le M\|u_k - u\| \to 0,$$

showing that $Ku_k(x)$ converges uniformly to $Ku(x)$. This completes the proof. □

It remains to prove Lemma 11.10. We do this by showing that it is an easy consequence of

Lemma 11.12. *If $\{\varphi_k\}$ is a complete orthonormal sequence in a Hilbert space H, and K is an operator in $B(H)$ satisfying*

(11.54) $$\sum_1^\infty \|K\varphi_k\|^2 < \infty,$$

then K is in $K(H)$.

Let us show how Lemma 11.12 implies Lemma 11.10.

Proof. As we saw in Section 1.4, $L^2(a, b)$ has a complete orthonormal sequence $\{\varphi_k\}$. [Actually, there we took $a = 0$, $b = 2\pi$. If a and b are other finite values, a slight change of variables is needed. Otherwise, we can use the fact that (a, b) is the denumerable union of bounded intervals. The details are left as an exercise.] Thus, we have by Theorem 1.41,

$$\sum \|K\varphi_j\|^2 = \sum_{j,k} |(K\varphi_j, \varphi_k)|^2 \tag{11.55}$$

$$= \sum_{j,k} \left| \iint K(x,y)\varphi_j(y)\overline{\varphi_k(x)}dx\,dy \right|^2$$

$$\leq \iint |K(x,y)|^2 dx\,dy,$$

where we have made use of the fact that $\{\varphi_j(y)\overline{\varphi_k(x)}\}$ is an orthonormal set in $L^2(Q)$ and applied Bessel's inequality [see (1.60)]. This proves Lemma 11.10. $\square$

Now, we give the simple proof of Lemma 11.12.

Proof. First note that

$$Ku = \sum_1^\infty (u, \varphi_k)K\varphi_k, \quad u \in H. \tag{11.56}$$

In fact, since

$$\sum_1^n (u, \varphi_k)\varphi_k$$

converges to u, and K is continuous,

$$K_n u = \sum_1^n (u, \varphi_k)K\varphi_k \tag{11.57}$$

must converge to Ku. Let K_n be the operator of finite rank defined by (11.57). Then

$$\|Ku - K_n u\|^2 \leq \left(\sum_{n+1}^\infty |(u, \varphi_k)| \cdot \|K\varphi_k\| \right)^2$$

$$\leq \sum_{n+1}^\infty |(u, \varphi_k)|^2 \sum_{n+1}^\infty \|K\varphi_k\|^2$$

$$\leq \|u\|^2 \sum_{n+1}^\infty \|K\varphi_k\|^2.$$

Hence,

$$\|K - K_n\|^2 \leq \sum_{n+1}^\infty \|K\varphi_k\|^2 \to 0 \quad \text{as } n \to \infty. \tag{11.58}$$

This shows that K is the limit in norm of a sequence of operators of finite rank. Hence, K is compact, and the proof is complete. $\square$

11.5. Hyponormal operators

An operator A in $B(H)$ is called *hyponormal* if

$$\|A^*u\| \leq \|Au\|, \quad u \in H, \tag{11.59}$$

or, equivalently, if

$$([AA^* - A^*A]u, u) \leq 0, \quad u \in H. \tag{11.60}$$

Of course, a normal operator is hyponormal. An operator $A \in B(H)$ is called *seminormal* if either A or A^* is hyponormal. In analogy with Lemma 11.7, we have

Theorem 11.13. *If A is seminormal, then*

$$r_\sigma(A) = \|A\|. \tag{11.61}$$

Proof. First, assume that A is hyponormal. Then (11.31) holds. The proof is the same as that for normal operators with the exception that in this case we have

$$\|A^*A^k u\| \cdot \|A^{k-1}u\| \leq \|A^{k+1}u\| \cdot \|A^{k-1}u\|$$

in the proof of (11.32). Thus (11.34) holds.

If A^* is hyponormal, then the result just proved gives

$$r_\sigma(A^*) = \|A^*\|. \tag{11.62}$$

But in general we have

$$(A^*)^n = (A^n)^*, \quad n = 1, 2, \cdots . \tag{11.63}$$

(This follows immediately from the definition.) Thus, by (11.22) and (11.63), we have

$$\|(A^*)^n\| = \|(A^n)^*\| = \|A^n\|,$$

showing that

$$r_\sigma(A^*) = r_\sigma(A). \tag{11.64}$$

We now see that (11.61) follows from (11.22), (11.62) and (11.64). This completes the proof. $\square$

In Section 7.5 we defined the essential spectrum of an operator A to be

$$\sigma_e(A) = \bigcap_{K \in K(H)} \sigma(A + K). \tag{11.65}$$

It was shown that $\lambda \notin \sigma_e(A)$ if and only if $\lambda \in \Phi_A$ and $i(A - \lambda) = 0$ (Theorem 7.27). Let us show that we can be more specific in the case of seminormal operators.

Theorem 11.14. *If A is a seminormal operator, then $\lambda \in \sigma(A) \setminus \sigma_e(A)$ if and only if λ is an isolated eigenvalue with $r(A-\lambda) < \infty$ (see Section 5.5).*

Before proving Theorem 11.14, we note that some brief comments are in order. Let M be a subset of a Hilbert space H. By $M^\perp$ we denote the set of those elements of H that are orthogonal to M. There is a connection between this set and the set M^0 of annihilators of M (see Section 3.3). In fact, a functional is in M^0 if and only if the element given by the Riesz representation theorem (Theorem 2.1) is in $M^\perp$. In particular, if either of these sets is finite dimensional, then

$$\dim M^0 = \dim M^\perp. \tag{11.66}$$

Now we know that

$$N(A') = R(A)^0 \tag{11.67}$$

(see (3.12)). Similarly,

$$N(A^*) = R(A)^\perp. \tag{11.68}$$

Hence,

$$\beta(A) = \dim N(A^*). \tag{11.69}$$

Another fact we shall need is

Lemma 11.15. *If A is hyponormal, then so is $B = A - \lambda$ for any complex λ.*

Proof.

$$\begin{aligned}\|B^*u\|^2 &= ([AA^* - \lambda A^* - \overline{\lambda}A + |\lambda|^2]u, u)\\ &\le ([A^*A - \lambda A^* - \overline{\lambda}A - |\lambda|^2]u, u)\\ &= \|Bu\|^2.\end{aligned}$$

This gives the desired inequality. □

We shall also make use of

Lemma 11.16. *If A is hyponormal and maps a closed subspace M into itself, then the restriction of A to M is hyponormal.*

Proof. Let A_1 be the restriction of A to M. Then for $u, v \in M$, we have

$$(u, A^*v) = (Au, v) = (A_1u, v) = (u, A_1^*v). \tag{11.70}$$

In particular,

$$\|A_1^*u\|^2 = (A_1^*u, A_1^*u) = (A_1^*u, A^*u).$$

Hence,

$$\|A_1^*u\| \le \|A^*u\| \le \|Au\| = \|A_1u\|^2, \quad u \in M.$$

Thus, A_1 is hyponormal. $\square$

Now we are ready for the proof of Theorem 11.14.

Proof. Suppose $\lambda \in \sigma(A)$ is not in $\sigma_e(A)$. Set $B = A - \lambda$. Then $B \in \Phi(H)$, and $i(B) = 0$. By Lemma 11.15, B is also seminormal. If B is hyponormal, then

$$N(B) \subset N(B^*).$$

Since $i(B) = 0$, the dimensions of these subspaces are finite and equal [see (11.69)]. Hence,

$$N(B^*) = N(B). \tag{11.71}$$

If B^* is hyponormal, then

$$N(B^*) \subset N(B)$$

[here we have made use of the fact that $A^{**} = A$ for $A \in B(H)$]. Again, since they have the same finite dimension, (11.71) holds in this case as well. By (11.68), (11.71) and the fact that $R(B)$ is closed, we have

$$H = N(B) \oplus R(B). \tag{11.72}$$

We see from this that B is one-to-one on $R(B)$ and maps $R(B)$ into itself. Thus, if $B^2 v = 0$, we must have $Bv = 0$. Hence,

$$N(B^2) = N(B),$$

and consequently,

$$N(B^k) = N(B), \quad k = 1, 2, \cdots,$$

from which we see that

$$r(B) = \alpha(B) < \infty.$$

Since $i(B) = 0$, we also have $r'(B) < \infty$. It now follows from Theorem 6.23 that 0 is an isolated point of $\sigma(B)$. Hence, λ is an isolated point of $\sigma(A)$ with $r(A - \lambda) < \infty$. Since $\lambda \in \Phi_A$ with $i(A - \lambda) = 0$, λ must be an eigenvalue of A. This proves the theorem in one direction.

The proof of the rest of the theorem follows easily from

Lemma 11.17. *If B is hyponormal with 0 an isolated point of $\sigma(B)$ and either $\alpha(B)$ or $\beta(B)$ is finite, then $B \in \Phi(H)$ and $i(B) = 0$.*

We shall postpone the proof of Lemma 11.17 a moment and suppose that A is seminormal with λ an isolated eigenvalue such that $r(A - \lambda) < \infty$. Set $B = A - \lambda$. If A is hyponormal, so is B. Moreover, 0 is an isolated point of $\sigma(B)$ with $\alpha(B) < \infty$. Hence, by Lemma 11.17, we see that $B \in \Phi(H)$ with $i(B) = 0$. This is precisely what we want. If A^* is hyponormal, so is B^*. Moreover, 0 is an isolated point of $\sigma(B^*)$, and $\beta(B^*) = \alpha(B)$ is finite. Another application of Lemma 11.17 gives the desired result. Hence, Theorem 11.14 will be proved once we have given the proof of Lemma 11.17. □

We now give the proof of Lemma 11.17.

Proof. Set

$$P = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} (z - B)^{-1} dz,$$

where $\varepsilon > 0$ is so small that the points $0 < |z| < \varepsilon$ are in $\rho(B)$. Then by (6.30), we have

$$H = R(P) \oplus N(P), \tag{11.73}$$

and B maps both of these closed subspaces into themselves. Let B_1 and B_2 denote the restrictions of B to $R(P)$ and $N(P)$, respectively. Then $\sigma(B_1)$ consists precisely of the point 0 (Theorem 6.19). Hence, $r_\sigma(B_1) = 0$ (Theorem 6.12). But B_1 is hyponormal (Lemma 11.16), so that $r_\sigma(B_1) = \|B_1\|$ (Theorem 11.13). Hence, $B_1 = 0$, showing that $R(P) \subset N(B)$. But, in general, we have $N(B) \subset R(P)$ (see (6.32)). Hence,

$$R(P) = N(B). \tag{11.74}$$

Next, note that $0 \in \sigma(B_2)$ (Theorem 6.19). In particular, this gives

$$N(P) = R(B_2) \subset R(B). \tag{11.75}$$

Thus, we have

$$R(P) = N(B) \subset N(B^*) = R(B)^\perp \subset N(P)^\perp. \tag{11.76}$$

This, together with (11.73), shows that

$$R(P) = N(P)^\perp. \tag{11.77}$$

For if $u \in N(P)^\perp$, then $u = u_1 + u_2$, where $u_1 \in R(P),\ u_2 \in N(P)$. Now $u_1 \in N(P)^\perp$ by (11.76), showing that $u_2 \in N(P)^\perp$ as well. This can happen only if $u_2 = 0$, and this implies that $u = u_1 \in R(P)$. By (11.76) and (11.77), we have

$$N(B) = N(B^*), \tag{11.78}$$

and

$$R(B) \subset N(B^*)^\perp = R(P)^\perp = N(P).$$

This, together with (11.75), gives

$$R(B) = N(P). \tag{11.79}$$

From (11.79) we see that $R(B)$ is closed, and from (11.78) we see that $\alpha(B) = \beta(B)$. Since one of them is assumed finite, we see that $B \in \Phi(H)$ and $i(B) = 0$. This completes the proof. □

There is a simple consequence of Lemma 11.17.

Corollary 11.18. *If A is seminormal and λ is an isolated point of $\sigma(A)$, then λ is an eigenvalue of A.*

Proof. Set $B = A - \lambda$. If A is hyponormal, so is B (Lemma 11.15). If $\alpha(B) = 0$, then by Lemma 11.17, we have $B \in \Phi(H)$ and $i(B) = 0$. But then $R(B) = H$, showing that $\lambda \in \rho(A)$. Thus, we must have $\alpha(B) > 0$, which is what we want to show. If A^* is hyponormal, so is B^*. If $\alpha(B) = 0$, then $\beta(B^*) = 0$, and by Lemma 11.17, $B^* \in \Phi(H)$ and $i(B) = 0$. This implies $0 \in \rho(B^*)$, which is equivalent to $0 \in \rho(B)$. We again obtain a contradiction, showing that $\alpha(B) > 0$. This completes the proof. □

We also have the following:

Theorem 11.19. *Let A be a seminormal operator such that $\sigma(A)$ has no non-zero limit points. Then A is compact and normal. Thus, it is of the form (11.14) with the $\{\varphi_k\}$ orthonormal and $\lambda_k \to 0$.*

Proof. We follow the proof of Theorem 11.3. If $A = 0$, the theorem is trivially true. Thus, we may assume that $A \neq 0$. First, assume that A is hyponormal. By Theorem 11.13, there is a point $\lambda_0 \in \sigma(A)$ such that $|\lambda_0| = \|A\| \neq 0$ (see Theorem 6.13). By hypothesis, λ_0 is not a limit point of $\sigma(A)$. Hence, it is an eigenvalue (Corollary 11.18). Let φ_0 be an eigenvector with norm one, and let H_1 be the subspace of all the elements of H orthogonal to φ_0. We see that A maps H_1 into itself. This follows from the fact that $A - \lambda_0$ is hyponormal, showing that $\varphi_0 \in N(A^* - \bar{\lambda}_0)$. Hence, if $v \in H_1$, then

$$(Av, \varphi_0) = (v, A^*\varphi_0) = \bar{\lambda}_0(v, \varphi_0) = 0,$$

and consequently, $Av \in H_1$. Thus, the restriction A_1 of A to H_1 is hyponormal (Lemma 11.16). If $A_1 = 0$, we are finished. Otherwise, we repeat the process to find a pair λ_1, φ_1 such that

$$|\lambda_1| = \|A_1\|, \quad \varphi_1 \in H_1, \quad \|\varphi_1\| = 1, \quad A_1\varphi_1 = \lambda_1\varphi_1.$$

We let H_2 be a subspace orthogonal to both φ_0 and φ_1 and if $A_2 \neq 0$, there is a pair λ_2, φ_2 such that

$$|\lambda_2| = \|A_2\|, \quad \varphi_2 \in H_2, \quad \|\varphi_2\| = 1, \quad A_2\varphi_2 = \lambda_2\varphi_2.$$

We continue in this way, obtaining a sequence A_k of restrictions of A to subspaces H_k and sequences $\{\lambda_k\}\{\varphi_k\}$ such that

$$|\lambda_k| = \|A_k\|, \quad \varphi_k \in H_k, \quad \|\varphi_k\| = 1, \quad A_k\varphi_k = \lambda_k\varphi_k, \quad k = 1, 2, \cdots .$$

Since the λ_k are eigenvalues of A, we see that either there is a finite number of them or $\lambda_k \to 0$. Continuing as in the proof of Theorem 11.3, we see that A is of the form (11.14) with $\lambda_k \to 0$. Thus, A is normal and compact (Theorem 11.3).

If A^* is hyponormal, then we apply the same reasoning to A^*, making use of the fact that 0 is the only possible limit point of $\sigma(A^*)$. Thus, A^* is of the form (11.14) with $\lambda_k \to 0$. By (11.23) the same is true of $A = A^{**}$. This completes the proof. $\square$

Corollary 11.20. *If A is seminormal and compact, then it is normal.*

11.6. Problems

(1) Suppose A is a seminormal operator in $B(H)$ such that $\sigma(A)$ consists of a finite number of points. Show that A is of finite rank.

(2) Show that if A is hyponormal and $\sigma(A)$ has at most a finite number of limit points, then A is normal.

(3) If the operator K is defined by (11.42), show that

$$M \equiv \int_a^b \int_a^b |K(x,y)|^2 dx\, dy < 1$$

implies that $I - K$ has an inverse in $B(H)$. Do this by showing that $\|K\|^2 \leq M$.

(4) Define A^* for unbounded operators. Show that if A is closed and densely defined, then A^{**} exists and equals A.

(5) An operator A on H is said to have a singular sequence if there is a sequence $\{u_n\}$ of elements in $D(A)$ converging weakly to 0 such that $\|u_n\| = 1,\ Au_n \to 0$. If A is closed, show that it has a singular sequence if and only if either $R(A)$ is not closed or $\alpha(A) = \infty$. Thus A has a singular sequence if and only if it is not in $\Phi_+(H)$.

(6) Show that if A is normal, then $A - \lambda$ has a singular sequence if and only if $\lambda \in \sigma_e(A)$.

(7) If the operator A is given by (11.14), show that $0 \in \rho(A)$ if and only if the sequence $\{\varphi_k\}$ is complete.

(8) Prove (11.22).

(9) Prove that $L^2(-\infty, \infty)$ is separable.

(10) Show that if A is seminormal and $A^n = 0$ for some positive integer n, then $A = 0$.

(11) Prove the projection theorem and the Riesz representation theorem for complex Hilbert spaces.

(12) Show that A is normal if and only if $AA^* = A^*A$.

(13) Show that if $K(x, y) = \overline{K(y, x)}$, then the operator (11.42) is self-adjoint.

(14) Show that the operator A given by (11.14) has λ in its resolvent set if λ is not a limit point of the λ_k.

Chapter 12

BILINEAR FORMS

12.1. The numerical range

Let A be a bounded linear operator on a Hilbert space H. If $\lambda \in \sigma(A)$, then either $N(A-\lambda) \neq \{0\}$ or $N(A^* - \bar{\lambda}) \neq \{0\}$ or $R(A)$ is not closed in H [otherwise, the bounded inverse theorem would imply that $A-\lambda$ has an inverse in $B(H)$]. Thus, we have a $u \in H$ satisfying either

$$\|u\| = 1, \; Au = \lambda u, \tag{12.1}$$

or

$$\|u\| = 1, \; A^*u = \bar{\lambda}u, \tag{12.2}$$

or we have a sequence $\{u_k\}$ of elements of H satisfying

$$\|u_k\| = 1, \; (A-\lambda)u_k \to 0 \quad \text{as} \;\; k \to \infty. \tag{12.3}$$

In the case of (12.1) or (12.2), we have

$$(Au, u) = \lambda, \; \|u\| = 1, \tag{12.4}$$

while if (12.3) holds, we have

$$(Au_k, u_k) \to \lambda \quad \text{as} \;\; k \to \infty, \; \|u_k\| = 1. \tag{12.5}$$

To put this in a concise form, let $W(A)$ be the set of all scalars λ that equal (Au, u) for some $u \in H$ satisfying $\|u\| = 1$. The set $W(A)$ is called the *numerical range* of A. Now (12.4) says that $\lambda \in W(A)$, while (12.5) says that $\lambda \in \overline{W(A)}$. Hence, we have

Theorem 12.1. *If A is in $B(H)$, then $\sigma(A) \subset \overline{W(A)}$.*

If we try to extend the concept to unbounded operators, we must proceed cautiously. First, we can only define (Au, u) for $u \in D(A)$. Hence, $\lambda \in W(A)$ if there is a $u \in D(A)$ such that $\|u\| = 1$ and $(Au, u) = \lambda$. Second, we must define A^* for an unbounded operator A. As in the case of A', this can be done in a unique way only if $D(A)$ is dense in H (see Section 7.1). In this case, we say that $v \in D(A^*)$ if there is an $f \in H$ such that

$$(u, f) = (Au, v), \quad u \in D(A). \tag{12.6}$$

We then define A^*v to be f. Let us assume once and for all that $D(A)$ is dense in H. Then A^* exists.

Now for an unbounded closed operator we also have that $\lambda \in \sigma(A)$ implies that either $N(A - \lambda) \neq \{0\}$ or $N(A^* - \bar{\lambda}) \neq \{0\}$ or that $R(A)$ is not closed in H. The first possibility implies that there is a $u \in D(A)$ satisfying (12.4), while the third implies the existence of a sequence $\{u_k\}$ of elements in $D(A)$ satisfying (12.5). However, the second need not imply the existence of a $u \in D(A)$ satisfying (12.4). Thus, all we can conclude at the moment is

Theorem 12.2. *If A is a closed, densely defined operator on H and $\lambda \notin \overline{W(A)}$, then $\alpha(A - \lambda) = 0$ and $R(A - \lambda)$ is closed in H.*

In particular, it may very well happen that $\sigma(A)$ is not contained in $\overline{W(A)}$. As we see from Theorem 12.2, the fault would be in $R(A - \lambda)$ not being large enough. The question thus arises as to whether we can enlarge $D(A)$ in such a way to make $\sigma(A) \subset \overline{W(A)}$. We are going to see in Sections 12.5, 12.9, and 12.10 that this indeed can be done under very general circumstances.

In the considerations which follow it would be a nuisance to consider real Hilbert spaces. So as usual, we take the easy way out and assume throughout this chapter that we are dealing with a complex Hilbert space.

12.2. The associated operator

Let H be a Hilbert space. A *bilinear form* (or *sesquilinear functional*) $a(u, v)$ on H is an assignment of a scalar to each pair of vectors u, v of a subspace $D(a)$ of H in such a way that

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w), \quad u, v, w \in D(a), \tag{12.7}$$

$$a(u, \alpha v + \beta w) = \bar{\alpha} a(u, v) + \bar{\beta} a(u, w), \quad u, v, w \in D(a). \tag{12.8}$$

The subspace $D(a)$ is called the *domain* of the bilinear form. Examples of bilinear forms are the scalar product of H and an expression of the form (Au, v), where A is a linear operator on H. The domain of the scalar product is the whole of H, while that of (Au, v) is $D(A)$. When $v = u$ we shall write $a(u)$ in place of $a(u, u)$.

As in the case of an operator, we can define the numerical range $W(a)$ of a bilinear form $a(u,v)$ as the set of scalars λ which equal $a(u)$ for some $u \in D(a)$ satisfying $\|u\| = 1$.

With any densely defined bilinear form $a(u,v)$ we can associate a linear operator A on H as follows: We say that $u \in D(A)$ if $u \in D(a)$ and there is a constant C such that

$$|a(u,v)| \le C\|v\|, \quad v \in D(a). \tag{12.9}$$

Now by the Hahn-Banach theorem (Theorem 2.5) and the Riesz representation theorem (Theorem 2.1), inequality (12.9) implies that there is an $f \in H$ such that

$$a(u,v) = (f,v), \quad u \in D(a) \tag{12.10}$$

(you will have to take conjugates to apply them). Moreover, the density of $D(a)$ in H implies that f is unique. Define Au to be f. Clearly, A is a linear operator on H, and we shall call it the operator *associated* with the bilinear form $a(u,v)$. We shall prove the following:

Theorem 12.3. *Let $a(u,v)$ be a densely defined bilinear form with associated operator A. Then*

(a) If $\lambda \notin \overline{W(a)}$, then $A-\lambda$ is one-to-one and

$$\|u\| \le C\|(A-\lambda)u\|, \quad u \in D(A). \tag{12.11}$$

(b) If $\lambda \notin \overline{W(a)}$ and A is closed, then $R(A-\lambda)$ is closed in H.

Proof. (a) Since $\lambda \notin \overline{W(a)}$, there is a $\delta > 0$ such that

$$|a(u)-\lambda| \ge \delta, \quad \|u\| = 1,\ u \in D(a). \tag{12.12}$$

Thus

$$|a(u)-\lambda\|u\|^2| \ge \delta\|u\|^2, \quad u \in D(A). \tag{12.13}$$

Now if $u \in D(A)$ and $(A-\lambda)u = f$, then

$$a(u,v) - \lambda(u,v) = (f,v), \quad v \in D(a). \tag{12.14}$$

In particular,

$$a(u) - \lambda\|u\|^2 = (f,u),$$

and

$$|a(u)-\lambda\|u\|^2| \le \|f\| \cdot \|u\|. \tag{12.15}$$

Combining this with (12.13), we obtain

$$\|u\| \le \frac{\|f\|}{\delta},$$

which is clearly (12.11). Now, (12.11) implies that $A-\lambda$ is one-to-one. Thus, (a) is proved.

(b) Apply Theorem 3.12. $\square$

12.3. Symmetric forms

A bilinear form $a(u,v)$ is said to be *symmetric* if

$$a(v,u) = \overline{a(u,v)}. \tag{12.16}$$

An important property of symmetric forms is given by

Lemma 12.4. *Let $a(u,v)$ and $b(u,v)$ be symmetric bilinear forms satisfying*

$$|a(u)| \le Mb(u), \quad u \in D(a) \cap D(b). \tag{12.17}$$

Then

$$|a(u,v)|^2 \le M^2 b(u)b(v), \quad u,v \in D(a) \cap D(b). \tag{12.18}$$

Proof. Assume first that $a(u,v)$ is real. Then

$$a(u \pm v) = a(u) \pm 2a(u,v) + a(v), \tag{12.19}$$

and hence,

$$4a(u,v) = a(u+v) - a(u-v).$$

Thus

$$4|a(u,v)| \le M[b(u+v) + b(u-v)] = 2M[b(u) + b(v)]$$

by (12.17) and (12.19). Replacing u by αu and v by v/α, $\alpha \in \mathbb{R}$, we get

$$2|a(u,v)| \le M\left[\alpha^2 b(u) + \frac{b(v)}{\alpha^2}\right]. \tag{12.20}$$

If $b(u) = 0$, we let $\alpha \to \infty$, showing that $a(u,v) = 0$. In this case, (12.18) holds trivially. If $b(v) = 0$, we let $\alpha \to 0$. In this case as well, $a(u,v) = 0$, and (12.18) holds. If neither vanishes, set

$$\alpha^4 = \frac{b(v)}{b(u)}.$$

This gives

$$|a(u,v)| \le Mb(u)^{1/2}b(v)^{1/2}, \tag{12.21}$$

which is just (12.18). If $a(u,v)$ is not real, then $a(u,v) = e^{i\theta}|a(u,v)|$ for some θ. Hence $a(e^{-i\theta}u, v)$ is real. Applying (12.18) to this case, we have

$$|a(e^{-i\theta}u,v)|^2 \le M^2 b(e^{-i\theta}u)b(v).$$

This implies (12.18) for u and v. The proof is complete. $\square$

Corollary 12.5. *If $b(u,v)$ is symmetric but $a(u,v)$ is not and (12.17) holds, then*

$$|a(u,v)|^2 \le 4M^2 b(u)b(v), \quad u,v \in D(a)\cap D(b). \tag{12.22}$$

Proof. Set

$$a_1(u,v) = \frac{1}{2}[a(u,v) + \overline{a(v,u)}], \tag{12.23}$$

$$a_2(u,v) = \frac{1}{2i}[a(u,v) - \overline{a(v,u)}]. \tag{12.24}$$

Then a_1 and a_2 are symmetric bilinear forms, and

$$a(u,v) = a_1(u,v) + ia_2(u,v). \tag{12.25}$$

(The forms a_1 and a_2 are known as the *real* and *imaginary* parts of a, respectively. Note that, in general, they are not real valued.) Now, by (12.17),

$$|a_j(u)| \le Mb(u), \quad j=1,2,\ u \in D(a)\cap D(b).$$

Hence, by (12.21),

$$|a_j(u,v)| \le Mb(u)^{1/2}b(v)^{1/2},$$

which implies (12.22). □

Corollary 12.6. *If $b(u,v)$ is a symmetric bilinear form such that*

$$b(u) \ge 0, \quad u \in D(b), \tag{12.26}$$

then

$$|b(u,v)|^2 \le b(u)b(v), \quad u,v \in D(b), \tag{12.27}$$

and

$$b(u+v)^{1/2} \le b(u)^{1/2} + b(v)^{1/2}, \quad u,v \in D(b). \tag{12.28}$$

Proof. By (12.26), we have $|b(u)| = b(u)$. Setting $a(u,v) = b(u,v)$ in Lemma 12.4, we get (12.27). Inequality (12.28) follows from (12.27) in the usual fashion. In fact,

$$b(u+v) = b(u) + b(u,v) + b(v,u) + b(v)$$

$$\le b(u) + 2b(u)^{1/2}b(v)^{1/2} + b(v) = [b(u)^{1/2} + b(v)^{1/2}]^2.$$

This proves (12.28). □

The following criteria for recognizing a symmetric bilinear form are sometimes useful. As in the case with most other statements in this chapter, they are true only in a complex Hilbert space.

Theorem 12.7. *The following statements are equivalent for a bilinear form:*

(i) $a(u, v)$ *is symmetric;*

(ii) $\Im m\ a(u) = 0, \quad u \in D(a);$

(iii) $\Re e\ a(u, v) = \Re e\ a(v, u), \quad u, v \in D(a).$

Proof. That (i) implies (ii) is trivial. To show that (ii) implies (iii), note that

$$a(iu + v) = a(u) + ia(u, v) - ia(v, u) + a(v).$$

Taking the imaginary parts of both sides and using (ii), we get (iii). To prove that (iii) implies (i), observe that by (iii),

$$\Im m\ a(u, v) = \Im m\ (-i)a(iu, v) = -\Re e\ a(iu, v)$$

$$= -\Re e\ a(v, iu) = -\Re e\ (-i)a(v, u) = -\Im m\ a(v, u).$$

This, together with (iii), gives (i), and the proof is complete. □

12.4. Closed forms

A bilinear form $a(u, v)$ will be called *closed* if $\{u_n\} \subset D(a)$, $u_n \to u$ in H, $a(u_n - u_m) \to 0$ as $m, n \to \infty$ imply that $u \in D(a)$ and $a(u_n - u) \to 0$ as $n \to \infty$. The importance of closed bilinear forms may be seen from

Theorem 12.8. *Let* $a(u, v)$ *be a densely defined closed bilinear form with associated operator* A. *If* $\overline{W(a)}$ *is not the whole plane, a half-plane, a strip, or a line, then* A *is closed and*

$$\sigma(A) \subset \overline{W(a)} = \overline{W(A)}. \tag{12.29}$$

In proving Theorem 12.8, we shall make use of the following facts, some of which are of interest in their own right. They will be proved in Section 12.7.

Theorem 12.9. *The numerical range of a bilinear form is a convex set in the plane.*

Lemma 12.10. *If* W *is a closed convex set in the plane which is not the whole plane, a half-plane, a strip or a line, then* W *is contained in an angle of the form*

$$|\arg(z - z_0) - \theta_0| \le \theta < \frac{\pi}{2}. \tag{12.30}$$

Theorem 12.11. *Let $a(u,v)$ be a closed bilinear form such that $W(a)$ is not a half-plane, a strip, or a line, and such that $0 \notin \overline{W(a)}$. Then for each linear functional Fv on $D(a)$ satisfying*

$$|Fv|^2 \le C|a(v)|, \quad v \in D(a), \tag{12.31}$$

there are unique elements $w, u \in D(a)$ such that

$$Fv = a(v,w), \quad v \in D(a), \tag{12.32}$$

and

$$Fv = \overline{a(u,v)}, \quad v \in D(a). \tag{12.33}$$

Let us show how Theorems 12.9 and 12.11 together with Lemma 12.12 imply Theorem 12.8. Simple consequences of Theorem 12.9 and Lemma 12.8 are

Corollary 12.12. *If $\overline{W(a)}$ is not the whole plane, a half-plane, a strip, or a line, then there are constants γ, k, k_0 such that $|\gamma| = 1$, $k > 0$, k_0 is real, and*

$$|a(u)| \le k[\Re e\, \gamma a(u) + k_0\|u\|^2], \quad u \in D(a). \tag{12.34}$$

Theorem 12.13. *Let $a(u,v)$ be a bilinear form such that $W(a)$ is not the whole plane, a half-plane, a strip, or a line. Then there is a symmetric bilinear form $b(u,v)$ with $D(b) = D(a)$ such that there is a constant C satisfying*

$$C^{-1}|a(u)| \le b(u) \le |a(u)| + C\|u\|^2, \quad u \in D(a). \tag{12.35}$$

In particular, if

$$\{u_k\} \subset D(a),\ u \in D(a),\ u_k \to u \text{ and } a(u_k - u) \to 0,$$

then

$$a(u_k, v) \to a(u,v), \quad v \in D(a). \tag{12.36}$$

The proof of Corollary 12.12 and Theorem 12.13 will be given at the end of this section. Let us now show how they can be used to give the proof of Theorem 12.8.

Proof. To see that A is closed, suppose that there is a sequence $\{u_k\} \subset D(A)$ such that $u_k \to u$, $Au_k \to f$ in H. Then

$$\begin{aligned} |a(u_j - u_k)| &= |(Au_j - Au_k, u_j - u_k)| \\ &\le \|Au_j - Au_k\| \cdot \|u_j - u_k\| \to 0 \quad \text{as } j,k \to \infty. \end{aligned}$$

Since $a(u,v)$ is closed, this implies that $u \in D(a)$ and that $a(u_k - u) \to 0$. Thus, by Theorem 12.13,

$$a(u_k, v) \to a(u,v) \text{ as } k \to \infty, \quad v \in D(a). \tag{12.37}$$

Since

$$a(u_k, v) = (Au_k, v), \quad v \in D(a),$$

we have in the limit

$$a(u, v) = (f, v), \quad v \in D(a), \tag{12.38}$$

showing that $u \in D(A)$ and $Au = f$. Thus, A is a closed operator.

To show that $\sigma(A) \subset \overline{W(a)}$, let λ be any scalar not in $\overline{W(a)}$. Then by Theorem 12.3(a), $A - \lambda$ is one-to-one, and (12.11) holds. In particular, (12.13) holds. Let

$$a_\lambda(u, v) = a(u, v) - \lambda(u, v).$$

Then a_λ satisfies the hypotheses of Theorem 12.11. If f is any element of H, then (v, f) is a linear functional on $D(a_\lambda) = D(a)$, and

$$|(v, f)|^2 \leq \|v\|^2\|f\|^2 \leq C|a_\lambda(v)|$$

by (12.13). Hence, by Theorem 12.11, there is a $u \in H$ such that

$$a_\lambda(u, v) = (f, v), \quad v \in D(a). \tag{12.39}$$

This shows that $u \in D(A)$ and $(A - \lambda)u = f$. Since f was any element of H, we see that $R(A - \lambda) = H$, and consequently, $\lambda \in \rho(A)$.

Lastly, in order to prove $\overline{W(a)} = \overline{W(A)}$, we shall show that

$$W(A) \subset W(a) \subset \overline{W(A)}. \tag{12.40}$$

The first inclusion is obvious, since $u \in D(A)$ implies $(Au, u) = a(u)$. To prove the second, we want to show that, for each $u \in D(a)$, there is a sequence $\{u_k\} \subset D(A)$ such that $a(u_k) \to a(u)$. Let $b(u, v)$ be a symmetric bilinear form satisfying (12.35). Then by Corollary 12.5,

$$|a(v) - a(u)| \leq |a(v - u, v)| + |a(u, v - u)| \leq 2C[b(v)^{1/2} + b(u)^{1/2}]b(v - u)^{1/2}.$$

Thus, it suffices to show that for each $u \in D(a)$, there is a sequence $\{u_k\} \subset D(A)$ such that $b(u_k - u) \to 0$.

Now consider $D(a)$ as a vector space with scalar product $b(u, v) + (u, v)$. This makes $D(a)$ into a normed vector space X with norm $[b(u) + \|u\|^2]^{1/2}$. (Actually, X is a Hilbert space, as we shall see later.) We want to show that $D(A)$ is dense in X. If it were not, then there would be an element $w \in X$ having a positive distance from $D(A)$. By Theorem 2.9 there is a bounded linear functional $F \neq 0$ on X which annihilates $D(A)$. Let λ be any scalar not in $\overline{W(a)}$, and define a_λ as above. Then a_λ satisfies the hypotheses of Theorem 12.11, and by (12.13) and (12.35),

$$|Fv|^2 \leq K[b(v) + \|v\|^2] \leq K'|a_\lambda(v)|, \quad v \in X.$$

Thus, by Theorem 12.11, there is a $w \in X$ such that

$$Fv = a_\lambda(v, w), \quad v \in X. \tag{12.41}$$

Since F annihilates $D(A)$, we have

$$a_\lambda(v, w) = 0, \quad v \in D(A).$$

This is equivalent to

$$((A - \lambda)v, w) = 0, \quad v \in D(A).$$

But we have just shown that $R(A - \lambda) = H$, so that there is a $v \in D(A)$ such that $(A - \lambda)v = w$. This shows that $w = 0$, which, by (12.41), implies that $F = 0$, providing a contradiction. Hence, $D(A)$ is dense in X, and the proof of Theorem 12.8 is complete. $\square$

Let us now give the simple proof of Corollary 12.12.

Proof. By Theorem 12.9, $W(a)$ is a convex set. Hence, so is $\overline{W(a)}$. By Lemma 12.10, $W(a)$ must satisfy (12.30) for some z_0, θ_0, θ. Now, (12.30) is equivalent to

$$|\Im m\{e^{-i\theta_0}[a(u) - z_0]\}| \leq \tan\theta \,\Re e\,\{e^{-i\theta_0}[a(u) - z_0]\}. \tag{12.42}$$

Set $\gamma = e^{-i\theta_0}$. Inequality (12.42) implies

$$|\Im m\ \gamma a(u)| \leq \tan\theta\ [\Re e\ \gamma a(u) + k_0], \quad \|u\| = 1,\ u \in D(a),$$

where

$$k_0 = \frac{|\Re e\ \gamma z_0| + |\Im m\ \gamma z_0|}{\tan\theta}.$$

This implies (12.34) with $k = 1 + \tan\theta$. $\square$

We also give the proof of Theorem 12.13.

Proof. By Corollary 12.12, there are constants γ, k, k_0 such that (12.34) holds. Let $b_1(u, v)$ be the real part of the bilinear form $\gamma a(u, v)$ [see (12.23)], and set

$$b(u, v) = b_1(u, v) + k_0(u, v). \tag{12.43}$$

Then by (12.34),

$$|a(u)| \leq kb(u), \quad u \in D(a).$$

Moreover, by (12.43),

$$b(u) = \Re e\ \gamma a(u) + k_0\|u\|^2 \leq |a(u)| + |k_0| \cdot \|u\|^2.$$

Thus (12.35) holds. To prove (12.36), note that by Corollary 12.5,

$$\begin{aligned}|a(u_k, v) - a(u, v)|^2 &= |a(u_k - u, v)|^2 \le 4C^2 b(u_k - u) b(v) \\ &\le 4C^2 b(v)[|a(u_k - u)| \\ &+ C\|u_k - u\|^2] \to 0 \text{ as } k \to \infty.\end{aligned}$$

This completes the proof. □

The proofs of Theorems 12.9 and 12.11 and Lemma 12.10 will be given in Section 12.7.

12.5. Closed extensions

Let A be a linear operator on a Hilbert space H. As we saw in Section 12.1, it may not be true that $\sigma(A) \subset \overline{W(A)}$. In this section we shall concern ourselves with the question of when A can be extended to an operator having this property.

An operator B is said to be an *extension* of an operator A if $D(A) \subset D(B)$ and $Bx = Ax$ for $x \in D(A)$. We are going to prove

Theorem 12.14. *Let A be a densely defined linear operator on H such that $\overline{W(A)}$ is not the whole plane, a half-plane, a strip, or a line. Then A has a closed extension $\hat{A}$ such that*

$$\sigma(\hat{A}) \subset \overline{W(A)} = \overline{W(\hat{A})}. \tag{12.44}$$

The proof of Theorem 12.14 will be based on the following two theorems.

Theorem 12.15. *If A is a linear operator, then $W(A)$ is a convex set in the plane.*

Theorem 12.16. *Let $a(u, v)$ be a densely defined bilinear form such that $\overline{W(a)}$ is not the whole plane, a half-plane, a strip, or a line. Suppose that $\{u_k\} \subset D(a)$, $u_n \to 0$, $a(u_n - u_m) \to 0$ imply $a(u_n) \to 0$. Then $a(u, v)$ has a closed extension $\hat{a}(u, v)$ such that $D(a)$ is dense in $D(\hat{a})$ and*

$$W(a) \subset W(\hat{a}) \subset \overline{W(a)}.$$

A few words of explanation might be needed for Theorem 12.16. A bilinear form $b(u, v)$ is called an *extension* of a bilinear form $a(u, v)$ if $D(a) \subset D(b)$ and $b(u, v) = a(u, v)$ for $u, v \in D(a)$. A set U will be called *dense* in $D(a)$ if for each $w \in D(a)$ and each $\varepsilon > 0$ there is a $u \in U$ such that $a(w - u) < \varepsilon$ and $\|w - u\| < \varepsilon$.

Theorem 12.15 is an immediate consequence of Theorem 12.9. Theorem 12.16 will be proved at the end of this section. Now we shall show how they imply Theorem 12.14.

Proof. Let $a(u,v)$ be the bilinear form defined by

$$a(u,v) = (Au, v), \quad u, v \in D(A), \tag{12.45}$$

with $D(a) = D(A)$. Then $W(a) = W(A)$. By Theorem 12.13, there is a symmetric bilinear form $b(u,v)$ satisfying (12.35). Now suppose $\{u_n\} \subset D(a)$, $u_m \to 0$, and $a(u_n - u_m) \to 0$. Since

$$a(u_n) = a(u_n, u_n - u_m) + (Au_n, u_m),$$

we have, by Corollary 12.5,

$$|a(u_n)| \le 2Cb(u_n)^{1/2} b(u_n - u_m)^{1/2} + \|Au_n\| \cdot \|u_m\|. \tag{12.46}$$

Now by (12.35),

$$b(u_n - u_m) \le |a(u_n - u_m)| + C\|u_n - u_m\|^2 \to 0 \text{ as } m, n \to \infty.$$

Thus, there is a constant K such that

$$b(u_n) \le K^2, \quad n = 1, 2, \cdots. \tag{12.47}$$

Now let $\varepsilon > 0$ be given. Take N so large that

$$b(u_n - u_m) < \frac{\varepsilon^2}{4C^2K^2}, \quad m, n > N.$$

Thus

$$|a(u_n)| < \varepsilon + \|Au_n\| \cdot \|u_m\|, \quad m, n > N.$$

Letting $m \to \infty$, we obtain

$$|a(u_n)| \le \varepsilon, \quad n > N.$$

This means that $a(u_n) \to 0$ as $n \to \infty$. Thus, $a(u,v)$ satisfies the hypotheses of Theorem 12.16. Therefore, we conclude that $a(u,v)$ has a closed extension $\hat{a}(u,v)$ with $D(a)$ dense in $D(\hat{a})$. Let $\hat{A}$ be the operator associated with $\hat{a}(u,v)$. Then, by Theorem 12.8, $\hat{A}$ is closed, and

$$\sigma(\hat{A}) \subset \overline{W(\hat{a})} = \overline{W(\hat{A})}.$$

But, by Theorem 12.16,

$$\overline{W(\hat{a})} = \overline{W(a)} = \overline{W(A)}.$$

All that remains is the minor detail of verifying that $\hat{A}$ is an extension of A. So, suppose $u \in D(A)$. Then

$$a(u,v) = (Au, v), \quad v \in D(a) = D(A). \tag{12.48}$$

Since $\hat{a}(u,v)$ is an extension of $a(u,v)$,

$$\hat{a}(u,v) = (Au, v), \quad v \in D(a). \tag{12.49}$$

Now, we claim that (12.49) holds for all $v \in D(\hat{a})$. This follows from the fact that $D(a)$ is dense in $D(\hat{a})$. Thus, if $v \in D(\hat{a})$, there is a sequence $\{v_n\} \subset D(a)$ such that $\hat{a}(v_n - v) \to 0$ and $\|v_n - v\| \to 0$. Now, $\overline{W(\hat{a})}$ is not the whole plane, a half-plane, a strip, or a line. Thus, we may apply Theorem 12.13 to conclude that

$$\hat{a}(u, v_n) \to \hat{a}(u, v).$$

Since

$$\hat{a}(u, v_n) = (Au, v_n),$$

we have in the limit that (12.49) holds. Thus, $u \in D(\hat{A})$ and $\hat{A}u = Au$. Hence, $\hat{A}$ is an extension of A, and the proof is complete. $\square$

Let us now give the proof of Theorem 12.16.

Proof. Define $\hat{a}(u, v)$ as follows: $u \in D(\hat{a})$ if there is a sequence $\{u_n\} \subset D(a)$ such that $a(u_n - u_m) \to 0$ and $u_n \to u$ in H. If $\{v_n\}$ is such a sequence for v, then define

$$\hat{a}(u, v) = \lim_{n\to\infty} a(u_n, v_n). \tag{12.50}$$

This limit exists. To see this, note that

$$a(u_n, v_n) - a(u_m, v_m) = a(u_n, v_n - v_m) + a(u_n - u_m, v_m).$$

Now, by Theorem 12.13, there is a symmetric bilinear form $b(u, v)$ satisfying (12.35). Hence, by Corollary 12.5,

$$|a(u_n, v_n) - a(u_m, v_m)| \le 2C[b(u_n)^{1/2} b(v_n - v_m)^{1/2} + b(u_n - u_m)^{1/2} b(v_m)^{1/2}].$$

This converges to zero by (12.35). Moreover, the limit in (12.50) is unique (i.e., it does not depend on the particular sequences chosen). To see this, let $\{u'_n\}$ and $\{v'_n\}$ be other sequences for u and v, respectively. Set $u''_n = u'_n - u_n,\ v''_n = v'_n - v_n$. Then

$$b(u''_n - u''_m)^{1/2} \le b(u'_n - u'_m)^{1/2} + b(u_n - u_m)^{1/2} \to 0 \quad \text{as} \ m, n \to \infty,$$

by Theorem 12.13. Thus, $a(u''_n - u''_m) \to 0$, and similarly $a(v''_n - v''_m) \to 0$. Since $u''_n \to 0,\ v''_n \to 0$ in H, we may conclude, by hypothesis, that

$$a(u''_n) \to 0,\ a(v''_n) \to 0 \ \text{ as } \ n \to \infty,$$

which implies

$$b(u''_n) \to 0,\ b(v''_n) \to 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\begin{aligned} |a(u'_n, v'_n) - a(u_n, v_n)| &\le |a(u'_n, v''_n)| + |a(u''_n, v_n)| \\ &\le 2C[b(u'_n)^{1/2} b(v''_n)^{1/2} + b(u''_n)^{1/2} b(v_n)^{1/2}] \to 0. \end{aligned}$$

From the way $\hat{a}(u, v)$ was defined, it is obvious that

$$W(a) \subset W(\hat{a}) \subset \overline{W(a)}. \tag{12.51}$$

To show that $D(a)$ is dense in $D(\hat{a})$, note that if $u \in D(\hat{a})$, there is a sequence $\{u_n\} \subset D(a)$ such that $a(u_n - u_m) \to 0$, while $u_n \to u$ in H and $a(u_n) \to \hat{a}(u)$. In particular, for each n,

$$a(u_n - u_m) \to \hat{a}(u_n - u) \quad \text{as } m \to \infty.$$

Now, let $\varepsilon > 0$ be given, and take N so large that

$$|a(u_n - u_m)| < \varepsilon, \quad m, n > N.$$

Letting $m \to \infty$, we obtain

$$|\hat{a}(u_n - u)| \leq \varepsilon, \quad n > N,$$

which shows that

$$\hat{a}(u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{12.52}$$

Consequently, $D(a)$ is dense in $D(\hat{a})$.

It remains only to show that $\hat{a}$ is closed. In order to do this, we note that $\overline{W(\hat{a})}$ is not one of the sets mentioned in Theorem 12.13 [see (12.51)]. Thus, there is a symmetric bilinear form $\hat{b}(u, v)$ satisfying

$$C^{-1}|\hat{a}(u)| \leq \hat{b}(u) \leq |\hat{a}(u)| + C\|u\|^2, \quad u \in D(\hat{a}). \tag{12.53}$$

Now, suppose $\{u_n\} \subset D(\hat{a})$, $\hat{a}(u_n - u_m) \to 0$, and $u_n \to u$ in H. Then by (12.53),

$$\hat{b}(u_n - u_m) \to 0 \quad \text{as } n \to \infty.$$

By the density of $D(a)$ in $D(\hat{a})$, for each n there is a $v_n \in D(a)$ such that

$$|\hat{a}(u_n - v_n)| < \frac{1}{n^2}, \quad \|u_n - v_n\| < \frac{1}{n}.$$

Thus, by (12.53),

$$\hat{b}(u_n - v_n) < \frac{C+1}{n^2}. \tag{12.54}$$

Since $\hat{b}$ is a symmetric form,

$$\hat{b}(v_n - v_m)^{1/2} \leq \hat{b}(v_n - u_n)^{1/2} + \hat{b}(u_n - u_m)^{1/2} + \hat{b}(u_m - v_m)^{1/2} \to 0$$

as $m, n \to \infty$. Hence

$$a(v_n - v_m) = \hat{a}(v_n - v_m) \to 0 \quad \text{as } m, n \to \infty.$$

Since

$$\|v_n - u\| \leq \|v_n - u_n\| + \|u_n - u\| \to 0 \quad \text{as } n \to \infty,$$

we see that $u \in D(\hat{a})$ and

$$\hat{a}(v_n - u) \to 0 \quad \text{as } n \to \infty$$

by (12.52). Thus,

$$\hat{b}(v_n - u) \to 0 \quad \text{as } n \to \infty,$$

which implies, by (12.54), that

$$\hat{b}(u_n - u) \to 0 \quad \text{as } n \to \infty,$$

which, in turn, implies

$$\hat{a}(u_n - u) \to 0 \quad \text{as } n \to \infty.$$

Hence, $\hat{a}$ is closed, and the proof is complete. □

12.6. Closable operators

In the preceding section we gave sufficient conditions for a linear operator A on H to have a closed extension $\hat{A}$ satisfying $\sigma(\hat{A}) \subset \overline{W(A)}$. Suppose we are only interested in determining whether A has a closed extension. Then the condition can be weakened. In fact, we shall prove

Theorem 12.17. *If A is a densely defined linear operator on H such that $W(A)$ is not the whole complex plane, then A has a closed extension.*

Before we give the proof of Theorem 12.17, let us discuss closed extensions in general. Let A be a linear operator from a normed vector space X to a normed vector space Y. It is called *closable* (or *preclosed*) if $\{x_k\} \subset D(A)$, $x_k \to 0$, $Ax_k \to y$ imply that $y = 0$. Clearly, every closed operator is closable. We also have

Theorem 12.18. *A linear operator has a closed extension if and only if it is closable.*

We shall postpone the simple proof of Theorem 12.18 until the end of this section. By this theorem, we see that in order to prove Theorem 12.17, it suffices to show that a densely defined operator A on H is closable if $W(A)$ is not the whole plane. To do this, we make use of

Lemma 12.19. *A convex set in the plane which is not the whole plane is contained in a half-plane.*

From this lemma we have

Corollary 12.20. *If $a(u, v)$ is a bilinear form such that $W(a)$ is not the whole plane, then there are constants γ, k_0 with $|\gamma| = 1$ such that*

$$\Re e\,[\gamma a(u) + k_0 \|u\|^2] \ge 0, \quad u \in D(a). \tag{12.55}$$

We shall also postpone the proof of Lemma 12.19 until the end of this section. Corollary 12.20 follows easily from the lemma. In fact, we know that $W(a)$ is convex from Theorem 12.9. Hence, it must be contained in a half-plane by Lemma 12.19. But every half-plane is of the form

$$\Re e\,[\gamma z + k_0] \ge 0, \quad |\gamma| = 1.$$

Thus,

$$\Re e\,[\gamma a(u)+k_0]\geq 0,\quad u\in D(a),\ \|u\|=1,$$

which implies (12.55). A consequence of Corollary 12.20 is

Theorem 12.21. *Let $a(u,v)$ be a densely defined bilinear form such that $W(a)$ is not the whole plane. Let A be the operator associated with $a(u,v)$. If $D(A)$ is dense in H, then A is closable.*

Proof. By Corollary 12.20, there are constants γ, k_0 such that (12.55) holds. Set

$$b(u,v)=\gamma a(u,v)+k_0(u,v)$$

and

$$B=\gamma A+k_0.$$

Then B is the operator associated with $b(u,v)$. Moreover, A is closable if and only if B is. Hence, it suffices to show that B is closable. So suppose $\{u_n\}\subset D(A),\ u_n\to 0,\ Bu_n\to f$. Then for $\alpha>0$ and $w\in D(A)$, we have

$$\begin{aligned} b(u_n-\alpha w) &= b(u_n)-\alpha b(u_n,w)-\alpha b(w,u_n)+\alpha^2 b(w)\\ &= (Bu_n,u_n)-\alpha(Bu_n,w)-\alpha(Bw,u_n)+\alpha^2 b(w)\\ &\to -\alpha(f,w)+\alpha^2 b(w). \end{aligned}$$

Hence,

$$\Re e\,[-(f,w)+\alpha b(w)]\geq 0,\quad \alpha>0,\ w\in D(A).$$

Letting $\alpha\to 0$, we see that

$$\Re e\,(f,w)\leq 0,\quad w\in D(A). \tag{12.56}$$

Since $D(A)$ is dense in H, there is a sequence $\{v_n\}\subset D(A)$ such that $v_n\to f$ in H. Since $\Re e\,(f,v_n)\leq 0$, we have, in the limit, $\|f\|^2\leq 0$, which shows that $f=0$. Hence, A is closable, and the proof is complete. □

We can now give the proof of Theorem 12.17.

Proof. Set

$$a(u,v)=(Au,v),\quad u,v\in D(A).$$

Then $a(u,v)$ is a bilinear form with $D(a)=D(A)$ and $W(a)=W(A)$. Moreover, A is the operator associated with $a(u,v)$. Thus $a(u,v)$ satisfies all of the hypotheses of Theorem 12.21. Thus, we may conclude that A is closable, and the proof is complete. □

We are now in a position to give the proof of Theorem 12.18.

Proof. Suppose A has a closed extension $\hat{A}$, and let $\{x_n\}$ be a sequence in $D(A)$ such that $x_n \to 0$ in X while $Ax_n \to y$ in Y. Since $\hat{A}$ is an extension of A, $x_n \in D(\hat{A})$ and $\hat{A}x_n \to y$. Since $\hat{A}$ is closed, we have $\hat{A}0 = y$, showing that $y = 0$. Hence, A is closable. Conversely, assume that A is closable. Define the operator $\bar{A}$ as follows: An element $x \in X$ is in $D(\bar{A})$ if there is a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ in X and Ax_n converges in Y to some element y. Define $\bar{A}x$ to be y. This definition does not depend on the choice of the particular sequence $\{x_n\}$. This follows from the fact that if $\{z_n\} \subset D(A)$, $z_n \to x$ in X and $Az_n \to w$ in Y, then $x_n - z_n \to 0$ and $A(x_n - z_n) \to y - w$. Since A is closable, we see that $y = w$. Clearly, $\bar{A}$ is a linear extension of A, and we see that it is closed. To see this, suppose $\{x_n\} \subset D(\bar{A})$ and $x_n \to x$, $\bar{A}x_n \to y$. Then for each n there is a sequence $\{w_{nk}\} \subset D(A)$ such that $w_{nk} \to x_n$ and $Aw_{nk} \to \bar{A}x_n$. In particular, one can find a $z_n \in D(A)$ such that

$$\|z_n - x_n\| < \frac{1}{n}, \quad \|Az_n - \bar{A}x_n\| < \frac{1}{n}.$$

Therefore,

$$\|z_n - x\| \leq \|z_n - x_n\| + \|x_n - x\| \to 0,$$

and

$$\|Az_n - y\| \leq \|Az_n - \bar{A}x_n\| + \|\bar{A}x_n - y\| \to 0.$$

This shows that $x \in D(\bar{A})$ and $\bar{A}x = y$. Hence, $\bar{A}$ is closed, and the proof is complete. □

The operator $\bar{A}$ constructed in the proof of Theorem 12.18 is called the *closure* of A. It is the "smallest" closed extension of A (we leave this as a simple exercise).

Now we give the proof of Lemma 12.19.

Proof. Let W be a convex set in the plane, which is not the whole plane. Then $\overline{W}$ cannot be the whole plane either (this is left as an exercise). Now consider the complex plane as a Hilbert space Z with scalar product $(z_1, z_2) = z_1\bar{z}_2$. Let z_0 be a point not in $\overline{W}$. Then by Theorem 7.17, there is a bounded linear functional $f \neq 0$ on z such that

$$\Re e\, f(z_0) \geq \Re e\, f(z), \quad z \in \overline{W}. \tag{12.57}$$

By Theorem 2.1, there is a complex number $\gamma \neq 0$ such that

$$f(z) = z\bar{\gamma}, \quad z \in Z.$$

Hence, (12.57) becomes

$$\Re e\, z_0\bar{\gamma} \geq \Re e\, z\bar{\gamma}, \quad z \in \overline{W}. \tag{12.58}$$

We leave as an exercise the simple task of showing that the set of all complex z satisfying (12.58) is a half-plane. $\square$

We can also give a proof that does not make use of functional analysis.

Proof. Let Q be a point not in $\overline{W}$. If W is empty, there is nothing to prove. Otherwise, some ray from Q intersects $\overline{W}$. The first point P of $\overline{W}$ encountered by the ray is a boundary point of W. Let V be the collection of all rays from P which intersect $\overline{W}$ at a point not equal to P, and let M be the set of all points on these rays. Clearly, M is a convex set. To prove this, note that if S_1 and S_2 are points in M, then the rays PS_1 and PS_2 contain points T_1 and T_2 of $\overline{W}$, respectively. Since $\overline{W}$ is convex, the segment T_1T_2 is in $\overline{W}$, and, hence, all rays between PS_1 and PS_2 are in M. This includes the points on the segment S_1S_2. Now, a convex set of points consisting of rays from P is merely an angle θ with vertex at P (the verification is left as an exercise). Clearly, $\theta \leq \pi$; otherwise, one would have a line segment lying outside of M connecting points of M. If one extends either of the rays forming the sides of M, one obtains a half-plane free of $\overline{W}$. This completes the proof. $\square$

12.7. Some proofs

We now give the proofs that were deferred from Section 12.4. First, we give the proof of Theorem 12.9.

Proof. Let u and v be two elements of $D(a)$ satisfying $\|u\| = \|v\| = 1$. We want to show that for each θ satisfying $0 < \theta < 1$, we can find a $w \in D(a)$ such that $\|w\| = 1$, and

$$a(w) = (1-\theta)a(u) + \theta a(v). \tag{12.59}$$

If $a(u) = a(v)$, we can take $w = u$ and we are finished. Otherwise, there is a scalar γ such that $|\gamma| = 1$ and

$$\Im m\ \gamma a(u) = \Im m\ \gamma a(v) = d. \tag{12.60}$$

Let α, β, φ be real numbers, and set

$$\begin{aligned} g(\alpha,\beta) &= \gamma a(\alpha e^{i\varphi}u + \beta v) - id\|\alpha e^{i\varphi}u + \beta v\|^2 \\ &= \alpha^2[\gamma a(u) - id] + \alpha\beta e^{i\varphi}[\gamma a(u,v) - (u,v)id] \\ &\quad + \alpha\beta e^{-i\varphi}[\gamma a(v,u) - (v,u)id] + \beta^2[\gamma a(v) - id]. \end{aligned} \tag{12.61}$$

Now pick φ so that

$$\Im m\{e^{i\varphi}[\gamma a(u,v) - (u,v)id] + e^{-i\varphi}[\gamma a(v,u) - (v,u)id]\} = 0.$$

(This can always be done.) Thus, from (12.60) and (12.61), we see that $g(\alpha, \beta)$ is real. Now if

$$\alpha e^{i\varphi} u + \beta v = 0, \tag{12.62}$$

then we have

$$\|\alpha e^{i\varphi} u\| = \|\beta v\|$$

and

$$a(\alpha e^{i\varphi} u) = a(\beta v).$$

The first implies $\alpha^2 = \beta^2$, while the second gives

$$\alpha^2 a(u) = \beta^2 a(v).$$

Since we are assuming $a(u) \neq a(v)$, the only way (12.62) can hold is if $\alpha = \beta = 0$. From this, we see that the function

$$h(t) = \frac{g(t, 1-t)}{\|te^{i\varphi} u + (1-t)v\|^2}$$

is continuous and real valued in $0 \leq t \leq 1$. Moreover,

$$h(0) = \gamma a(v) - id, \quad h(1) = \gamma a(u) - id.$$

Hence, there is a value t_1 satisfying $0 < t_1 < 1$ such that

$$h(t_1) = \theta[\gamma a(v) - id] + (1-\theta)[\gamma a(u) - id] = \gamma[\theta a(v) + (1-\theta) a(u)] - id.$$

Set

$$w = \frac{t_1 e^{i\varphi} u + (1-t_1) v}{\|t_1 e^{i\theta} u + (1-t_1) v\|^2}.$$

Then, $\|w\| = 1$ and

$$h(t_1) = \gamma a(w) - id.$$

This gives (12.59), and the proof is complete. $\square$

We now give the proof of Lemma 12.10.

Proof. As we saw in the second proof of Lemma 12.19, there is a half-plane V containing W such that the boundary line L of V contains a point P of W. The line L is called a *support line* for W. Suppose $L \subset W$. If W is not a half-plane or a line, then the interior of V contains a boundary point Q of W (see the proof of Lemma 12.19). A support line L_1 through Q cannot intersect L, for then we would have points of W on both sides of L_1 (namely, those on L). Hence, L_1 must be parallel to L. Moreover, every point in the strip between L and L_1 is on a segment connecting Q and a point on L. This would mean that W is a strip, contrary to assumption. Hence, L must contain a point R not in W. Consequently, the ray emitted from R away from P must be free of W. For simplicity, assume that this ray is the positive real axis, R is the origin, P is the point $-c$, $c > 0$, and V is the upper half-plane. We can make the assertion that there is a $\delta > 0$ such that W is contained in the angle $\delta \le \theta \le \pi$. This would give us exactly what we want. If it were not true, then for each $\varepsilon > 0$, there would be a point $z \in W$ such that

$$0 < c\,\Im m\, z < \varepsilon\,\Re e\, z.$$

The segment connecting P and z must be in W. Moreover, the distance from this segment to the origin is less than

$$d = \frac{c\,\Im m\, z}{c + \Re e\, z} < \frac{\varepsilon\,\Re e\, z}{c + \Re e\, z} < \varepsilon.$$

(The quantity d is just the distance from the origin to the point where the segment intersects the imaginary axis. Use similar triangles.) Since this is true for any $\varepsilon > 0$, we see that there is a sequence of points of W converging to the origin, i.e., to R. Since W is closed, we must have $R \in W$. But we chose R not to be in W. This contradiction proves the lemma. $\square$

We now give the proof of Theorem 12.11.

Proof. Since $a(u, v)$ satisfies the hypotheses of Theorem 12.13, we know that there is a symmetric bilinear form $b(u, v)$ such that $D(b) = D(a)$ and (12.35) holds. Moreover, since $0 \notin \overline{W(a)}$, there is a $\delta > 0$ such that

$$|a(u)| \ge \delta > 0, \quad u \in D(a),\ \|u\| = 1. \tag{12.63}$$

Hence,

$$|a(u)| \ge \delta \|u\|^2, \quad u \in D(a). \tag{12.64}$$

Thus, there are constants M_1, M_2, M_3 such that

$$\|u\|^2 \le M_1 b(u) \le M_2 |a(u)| \le M_3 b(u), \quad u \in D(a). \tag{12.65}$$

Let X be the vector space $D(a)$ equipped with the scalar product $b(u, v)$. We can show that X is a Hilbert space. The only property that needs verification is completeness. Let $\{u_n\}$ be a Cauchy sequence in X; i.e., suppose $b(u_n - u_m) \to 0$ as $m, n \to \infty$. By (12.65), $\{u_n\}$ is a Cauchy sequence in H, and $a(u_n - u_m) \to 0$ as $m, n \to \infty$. Since H is complete, there is an element $u \in H$ such that $u_n \to u$ in H as $n \to \infty$. Since $a(u, v)$ is a closed bilinear form, we know that $u \in D(a)$ and $a(u_n - u) \to 0$ as $n \to \infty$. Thus $b(u_n - u) \to 0$ by (12.65), and X is complete.

Now, for each $w \in X$,

$$Gv = a(v, w), \quad v \in X, \tag{12.66}$$

is a linear functional on X. It is bounded by (12.65) and Corollary 12.5. Hence, there is an element $Sw \in X$ such that

$$Gv = b(v, Sw), \quad v \in X \tag{12.67}$$

(Theorem 2.1). As anyone can plainly see, S is a linear mapping of X into itself. It is bounded since

$$b(Sw)^2 = a(Sw, w)^2 \le 4M_3^2 b(Sw)b(w),$$

whence

$$b(Sw) \le 4M_3^2 b(w). \tag{12.68}$$

It is also one-to-one and has a closed range, since

$$M_1^2 b(w)^2 \le M_2^2 |a(w)|^2 = M_2^2 |b(w, Sw)|^2 \le M_2^2 b(w)b(Sw),$$

by (12.65), (12.66) and (12.67). This gives

$$b(w) \le M_2^2 b(Sw)/M_1^2. \tag{12.69}$$

We claim that $R(S) = X$. To see this, let h be any element of X orthogonal to $R(S)$, i.e., satisfying

$$b(h, Sw) = 0, \quad w \in X.$$

Then by (12.66) and (12.67),

$$a(h, w) = 0, \quad w \in X.$$

In particular, this holds when $w = h$, showing that $a(h) = 0$, which implies $b(h) = 0$ by (12.65). Hence, $h = 0$.

Now, let F be any linear functional in $D(a)$ satisfying (12.31). Then F is a bounded linear functional on X by (12.65). Hence, there is an element $f \in X$ such that

$$Fv = b(v, f), \quad v \in X.$$

Since S is one-to-one and onto, there is a $w \in X$ such that $Sw = f$. Hence,

$$Fv = b(v, f) = b(v, Sw) = a(v, w), \quad v \in X,$$

which proves (12.32). The proof of (12.33) is almost identical and is omitted. □

12.8. Some representation theorems

Theorem 12.11 is a representation theorem similar to the Riesz representation theorem (Theorem 2.1). There are some interesting consequences that follow from it.

Theorem 12.22. *Let $b(u, v)$ be a closed, symmetric bilinear form on a Hilbert space H satisfying*

$$\|u\|^2 \le Cb(u), \quad u \in D(b). \tag{12.70}$$

Suppose that $a(u, v)$ is a bilinear form with $D(b) \subset D(a)$ satisfying for $m > 0$

$$mb(u) \le |a(u)| \le Mb(u), \quad u \in D(b). \tag{12.71}$$

Then for each linear function F on $D(b)$ satisfying

$$|Fv|^2 \leq Kb(v), \quad v \in D(b), \tag{12.72}$$

there are elements $w, u \in D(b)$ such that

$$Fv = a(v, w), \quad v \in D(b), \tag{12.73}$$

$$Fv = \overline{a(u, v)}, \quad v \in D(b). \tag{12.74}$$

Proof. Let X be $D(b)$ with scalar product $b(u, v)$. Since $b(u, v)$ is a closed bilinear form, (12.70) and (12.71) imply that X is a Hilbert space (see the proof of Theorem 12.11). Now $a(u, v)$ is a bilinear form on X and

$$0 < m \leq |a(u)| \leq M, \quad u \in X, \ b(u) = 1. \tag{12.75}$$

Let $\tilde{a}$ be the restriction of a to X. Then by (12.71), $\tilde{a}$ is a closed bilinear form on X with $D(\tilde{a}) = X$. By (12.75) we see that $\overline{W(\tilde{a})}$ is bounded and does not contain the point 0. Hence, $\tilde{a}$ satisfies the hypotheses of Theorem 12.11. The conclusions follow immediately. □

An important case of Theorem 12.22 is

Corollary 12.23. *Let $a(u, v)$ be a bilinear form defined on the whole of H such that*

$$m\|u\|^2 \leq |a(u)| \leq M\|u\|^2, \quad u \in H \tag{12.76}$$

holds for positive m, M. Then for each bounded linear functional F on H, there are elements $w, u \in H$ such that

$$Fv = a(v, w), \quad v \in H, \tag{12.77}$$

$$Fv = \overline{a(u, v)}, \quad v \in H. \tag{12.78}$$

Proof. We merely take $b(u, v)$ to be the scalar product of H in Theorem 12.22. □

Corollary 12.23 is sometimes called the Lax-Milgram lemma.

12.9. Dissipative operators

In Section 12.5, we showed that if A is a linear operator on H such that $D(A)$ is dense in H and $\overline{W(A)}$ is not the whole plane, a half-plane, a strip, or a line, then A has a closed extension $\hat{A}$ satisfying

$$\sigma(\hat{A}) \subset \overline{W(A)} = \overline{W(\hat{A})}. \tag{12.79}$$

One of the questions we have deliberately deferred until now is: What happens when $\overline{W(A)}$ is one of these sets? We shall now discuss some of these cases. Further considerations are given in Section 12.10.

Suppose $\overline{W(A)}$ is the whole plane. Then (12.79) is vacuously true for all extensions of A. On the other hand, the existence of a closed extension of A depends on whether A is closable (Theorem 12.18). Hence, this case is not very interesting.

In all other cases, including that of Section 12.5, $W(A)$ is contained in a half-plane (Lemma 12.19). Thus, there are constants γ, k such that $|\gamma| = 1$ and

$$\Re e\, \gamma(Au, u) - k\|u\|^2 \le 0, \quad u \in D(A).$$

Set

$$B = \gamma A - k. \tag{12.80}$$

Then $D(B) = D(A)$ and

$$\Re e\, (Bu, u) \le 0, \quad u \in D(B). \tag{12.81}$$

An operator B satisfying (12.81) is called *dissipative*. We are going to prove

Theorem 12.24. *Let B be a dissipative operator on H with $D(B)$ dense in H. Then B has a closed dissipative extension $\hat{B}$ such that $\sigma(\hat{B})$ is contained in the half-plane $\Re e\, \lambda \le 0$.*

In particular, if $\overline{W(A)}$ is a half-plane, then Theorem 12.24 gives a closed extension $\hat{A}$ of A satisfying (12.79). In fact, all we need do is define B by (12.80) for appropriate γ, k. Then extend B to $\hat{B}$ by Theorem 12.24. The extension $\hat{A}$ defined by

$$\hat{A} = \frac{\hat{B} + k}{\gamma}$$

clearly has all of the desired properties. Hence, Theorem 12.24 implies

Theorem 12.25. *Let A be a densely defined linear operator on H such that $\overline{W(A)}$ is a half-plane. Then A has a closed extension $\hat{A}$ satisfying (12.79).*

If $W(A)$ is a line or a strip, then things are more complicated. We can use Theorem 12.24 to obtain a closed extension having one of the adjacent half-planes in its resolvent set as well. A study of the conditions under which this will be true is given in the next section.

Let us now give the proof of Theorem 12.24.

Proof. By (12.81),

$$\Re e\,([I-B]u,u) \geq \|u\|^2,$$

showing that $I-B$ is one-to-one. Hence, it has an inverse $(I-B)^{-1}$ defined on $R(I-B)$. Define

$$T = (I+B)(I-B)^{-1},$$

where $D(T) = R(I-B)$. We claim that

$$\|Tu\| \leq \|u\|, \quad u \in D(T). \tag{12.82}$$

In fact, if

$$v = (I-B)^{-1}u, \tag{12.83}$$

then

$$Tu = (I+B)v. \tag{12.84}$$

Hence,

$$\begin{aligned}\|Tu\|^2 &= \|v\|^2 + \|Bv\|^2 + 2\,\Re e\,(Bv,v)\\ &\leq \|v\|^2 + \|Bv\|^2 - 2\,\Re e\,(Bv,v)\\ &= \|(I-B)v\|^2 = \|u\|^2,\end{aligned}$$

by (12.81) and (12.83). Now, we can show that it is an easy matter to find an extension $\hat{T}$ of T in $B(H)$ such that

$$\|\hat{T}\| \leq 1. \tag{12.85}$$

[Remember that operators in $B(H)$ are defined on all of H.] To do this, first extend T to $\overline{D(T)}$. This we do in the usual way. If u is an element in $\overline{D(T)}$, then there is a sequence $\{u_n\}$ of elements of $D(T)$ converging to u in H. By (12.82), $\{Tu_n\}$ is a Cauchy sequence in H, and hence, has a limit z. Define $\bar{T}u$ to be z. As usual, we check that this definition is independent of the sequence chosen, and that $\bar{T}$ is an extension of T to $\overline{D(T)}$. Now, let u be any element of H. By the projection theorem (Theorem 2.3), $u = w + y$, where $w \in \overline{D(T)}$ and y is orthogonal to $D(T)$. Define

$$\hat{T}u = \bar{T}w.$$

Then

$$\|\hat{T}u\| = \|\bar{T}w\| \leq \|w\| \leq \|u\|.$$

Thus, $\hat{T} \in B(H)$, and (12.85) holds.

Now by (12.83) and (12.84)

$$u = v - Bv, \quad Tu = v + Bv, \tag{12.86}$$

or

$$2v = u + Tu, \quad 2Bv = Tu - u. \tag{12.87}$$

The first equation in (12.87) together with (12.83) shows that $I+T$ is one-to-one and that its range is $D(B)$. Hence, we have

$$B = (T-I)(I+T)^{-1}. \tag{12.88}$$

A candidate for the extension $\hat{B}$ is

$$\hat{B} = (\hat{T}-I)(I+\hat{T})^{-1}, \tag{12.89}$$

with $D(\hat{B}) = R(I+\hat{T})$. In order that (12.89) make sense, we must check that $I+\hat{T}$ is one-to-one. To see this, let u be an element of H such that

$$(I+\hat{T})u = 0. \tag{12.90}$$

Let v be any element of H, and let α be a positive real number. Set $g = (I+\hat{T})v$. Then, by (12.85),

$$\|g - v + \alpha u\|^2 \leq \|v - \alpha u\|^2. \tag{12.91}$$

Expanding (12.91) out, we get

$$\|g\|^2 - 2\,\Re e\ (g,v) + 2\,\alpha \Re e\ (g,u) \leq 0.$$

Divide by α and let $\alpha \to \infty$. This gives

$$\Re e\ (g,u) \leq 0, \quad g \in R(I+\hat{T}). \tag{12.92}$$

Since

$$R(I+\hat{T}) \supset R(I+T) = D(B),$$

we see that $R(I+\hat{T})$ is dense in H. Hence, there is a sequence $\{g_n\}$ of elements in $R(I+\hat{T})$ such that $g_n \to u$ in H. Thus, (12.92) implies

$$\Re e\ \|u\|^2 \leq 0,$$

which shows that $u = 0$. Thus, the operator $\hat{B}$ given by (12.89) is well defined. It is clearly an extension of B. We claim that it is closed. To see this, suppose $\{u_n\}$ is a sequence of elements in $D(\hat{B}) = R(I+\hat{T})$ such that

$$u_n \to u, \ \ Bu_n \to h \ \text{ in } \ H \ \text{ as } \ n \to \infty.$$

Since, $u_n \in R(I+\hat{T})$, there is a $w_n \in H$ such that

$$u_n = (I+\hat{T})w_n.$$

By (12.89),

$$\hat{B}u_n = (\hat{T}-I)w_n.$$

Hence,

$$2w_n = u_n - \hat{B}u_n \to u - h \ \text{ as } \ n \to \infty.$$

Since $\hat{T} \in B(H)$, this implies

$$2u_n = 2(I+\hat{T})w_n \to (I+\hat{T})(u-h),$$

$$2\hat{B}u_n = 2(\hat{T}-I)w_n \to (\hat{T}-I)(u-h),$$

from which we conclude

$$2u = (I + \hat{T})(u - h),\ 2h = (\hat{T} - I)(u - h).$$

In particular, we see that $u \in R(I + \hat{T}) = D(\hat{B})$ and

$$\hat{B}u = (\hat{T} - I)(I + \hat{T})^{-1}u = \frac{1}{2}(\hat{T} - I)(u - h) = h.$$

Hence, $\hat{B}$ is a closed operator.

We must also show that $\hat{B}$ is dissipative. This follows easily, since

$$(\hat{B}u, u) = ([\hat{T} - I]w, [I + \hat{T}]w) = \|\hat{T}w\|^2 - (w, \hat{T}w) + (\hat{T}w, w) - \|w\|^2,$$

where $w = (I + \hat{T})^{-1}u$. Hence,

$$\Re e\ (\hat{B}u, u) = \|\hat{T}w\|^2 - \|w\|^2 \leq 0 \tag{12.93}$$

by (12.85).

Finally, we must verify that $\lambda \in \rho(\hat{B})$ for $\Re e\ \lambda > 0$. Since $\hat{B}$ is dissipative, we have

$$\Re e\ ([\hat{B} - \lambda]u, u) \leq -\Re e\ \lambda \|u\|^2,$$

or

$$\Re e\ \lambda \|u\|^2 \leq -\Re e\ ([\hat{B} - \lambda]u, u) \leq \|(\hat{B} - \lambda)u\| \cdot \|u\|,$$

showing that $\hat{B} - \lambda$ is one-to-one for $\Re e\ \lambda > 0$. Thus, all we need show is that $R(\hat{B} - \lambda) = H$ for $\Re e\ \lambda > 0$. Now,

$$\hat{B} - \lambda = [(1 - \lambda)\hat{T} - (I + \lambda)](I + \hat{T})^{-1}.$$

Thus, one can solve

$$(\hat{B} - \lambda)u = f \tag{12.94}$$

if and only if one can solve

$$[(1 - \lambda)\hat{T} - (1 + \lambda)]w = f. \tag{12.95}$$

Note that (12.95) can be solved for all $f \in H$ when $\Re e\ \lambda > 0$. This is obvious for $\lambda = 1$. If $\lambda \neq 1$, all we need note is that

$$\left| \frac{1 + \lambda}{1 - \lambda} \right| > 1$$

for $\Re e\ \lambda > 0$. Since $\|\hat{T}\| \leq 1$, this shows that $(1 + \lambda)/(1 - \lambda)$ is in $\rho(\hat{T})$ (Lemma 6.5). Hence, (12.95) can be solved for all $f \in H$. This completes the proof. $\square$

12.10. The case of a line or a strip

Let us now outline briefly what one can do in case $\overline{W(A)}$ is a line or a strip. Of course we can consider a line as a strip of thickness 0. So suppose that $\overline{W(A)}$ is a strip of thickness $a-1$, where $a \geq 1$. As before, we can find an operator B of the form

$$B = \gamma A - k \tag{12.96}$$

such that $\overline{W(B)}$ is the strip $1-a \leq \Re e\ z \leq 0$. Thus,

$$1 - a \leq \Re e\ (Bu, u) \leq 0, \quad u \in D(B),\ \|u\| = 1. \tag{12.97}$$

In particular, B is dissipative.

Now suppose B has a closed extension $\hat{B}$ such that $\overline{W(\hat{B})}$ is the same strip and $\rho(\hat{B})$ contains the two complementary half-planes. Set

$$\hat{T} = (a + \hat{B})(a - \hat{B})^{-1}. \tag{12.98}$$

Let u be any element of H, and set

$$v = (a - \hat{B})^{-1} u. \tag{12.99}$$

Then

$$\hat{T} u = (a + \hat{B}) v. \tag{12.100}$$

Hence,

$$\|\hat{T} u\|^2 - \|u\|^2 = 4a(\hat{B} v, v), \tag{12.101}$$

showing that

$$4a(1-a)\|v\|^2 \leq \|\hat{T} u\|^2 - \|u\|^2 \leq 0. \tag{12.102}$$

By (12.99) and (12.100), we have

$$(\hat{T} + I)u = 2av, \quad (\hat{T} - I)u = 2\hat{B} v. \tag{12.103}$$

Thus (12.102) becomes

$$\frac{1-a}{a}\|(\hat{T} + I)u\|^2 \leq \|\hat{T} u\|^2 - \|u\|^2 \leq 0, \tag{12.104}$$

or

$$\|\hat{T} u\|^2 \leq \|u\|^2 \leq (2a-1)\|\hat{T} u\|^2 + 2(a-1)\,\Re e\ (\hat{T} u, u). \tag{12.105}$$

In short, $\hat{T}$ is an extension of

$$T = (a + B)(a - B)^{-1}, \quad D(T) = R(a - B), \tag{12.106}$$

which satisfies (12.105) and such that $R(\hat{T}) = H$. Conversely, if we can find such an extension $\hat{T}$ of T, then we can show that

$$\hat{B} = a(\hat{T} - I)(\hat{T} + I)^{-1} \tag{12.107}$$

is a closed extension of B with $\overline{W(\hat{B})} = \overline{W(B)} \supset \sigma(\hat{B})$. In fact, we have, by the reasoning of the last section, that $\hat{B}$ is closed and dissipative, while $\rho(\hat{B})$ contains the half-plane $\Re e\ \lambda > 0$. Moreover, (12.105) and (12.101) imply that $\overline{W(\hat{B})}$ is the strip $1 - a \leq \Re e\ z \leq 0$.

However, we also want the half-plane $\Re e\ \lambda < 1 - a$ to be in $\rho(\hat{B})$. We know that $\alpha(\hat{B} - \lambda) = 0$ and that $R(\hat{B} - \lambda)$ is closed for such points (Theorem 12.3). It suffices to show that the half-plane contains one point in $\rho(\hat{B})$. Then we can apply

Theorem 12.26. *Let A be a closed linear operator on a Banach space X. If λ is a boundary point of $\rho(A)$, then either $\alpha(A - \lambda) \neq 0$ or $R(A - \lambda)$ is not closed in X.*

We shall postpone the proof of the theorem until the end of this section. To continue our argument, if the half-plane $\Re e\ \lambda < 1 - a$ contains a point of $\rho(\hat{B})$, then the entire half-plane must be in $\rho(\hat{B})$. Otherwise, it would contain a boundary point λ of $\rho(\hat{B})$. But this would imply, by Theorem 12.26, that either $\alpha(\hat{B} - \lambda) \neq 0$ or $R(\hat{B} - \lambda)$ is not closed, contradicting the conclusion reached above.

To complete the argument, let us show that the point $\lambda = -a$ is, indeed, in $\rho(\hat{B})$. To see this, note that

$$a + \hat{B} = \hat{T}(a - \hat{B}), \tag{12.108}$$

and since $R(\hat{T}) = R(a - \hat{B}) = H$, it follows that $R(a + \hat{B}) = H$. This, coupled with the fact that $\alpha(a + \hat{B}) = 0$, shows that $-a$ is in $\rho(\hat{B})$.

So we must try to find an extension $\hat{T}$ of T satisfying (12.105) and such that $R(\hat{T}) = H$. Let us consider first the case $a = 1$ (i.e., the case of a line). Now (12.105) says

$$\|\hat{T}u\| = \|u\|, \quad u \in H. \tag{12.109}$$

A linear operator satisfying (12.109) is called an *isometry.* If an isometry maps a normed vector space onto itself, it is called *unitary.* Thus, we want T to have a unitary extension $\hat{T}$. By continuity, we can extend it to be an isometry $\bar{T}$ of $\overline{R(I - B)}$ onto $\overline{R(I + B)}$. Thus, to determine $\hat{T}$, we need only define it on $R(I+B)^{\perp}$. We can see that $\hat{T}$ would have to map $R(I-B)^{\perp}$ into $R(I + B)^{\perp}$. This follows from the general property of isometries on Hilbert spaces:

$$(\hat{T}u, \hat{T}w) = (u, w) \tag{12.110}$$

(see below). Moreover, $\hat{T}$ must map onto $R(I + B)^{\perp}$; otherwise, we could not have $R(\hat{T}) = H$. Thus, we have

Theorem 12.27. *Let B be a densely defined linear operator on H such that $W(B)$ is the line $\Re e\ \lambda = 0$. Then a necessary and sufficient condition that B have a closed extension $\hat{B}$ such that*

$$\sigma(\hat{B}) \subset W(\hat{B}) = W(B), \tag{12.111}$$

is that there exist an isometry from $R(I-B)^{\perp}$ onto $R(I+B)^{\perp}$. In particular, this is true if they both have the same finite dimension or if they are both separable and infinite dimensional.

The last statement follows from the fact that $R(I+B)^{\perp}$ and $R(I-B)^{\perp}$ have complete orthonormal sequences $\{\varphi_k\}$ and $\{\psi_k\}$, respectively (Lemma 11.9). Moreover, these sequences are either both infinite or have the same finite number of elements. In either case, we can define $\hat{T}$ by

$$\hat{T}\varphi_k = \psi_k, \quad k = 1, 2, \cdots . \tag{12.112}$$

If $a \neq 1$ (i.e., in the case of a strip), the situation is not so simple. It is necessary for $\hat{T}$ to map $R(a-B)^{\perp}$ onto a closed subspace M such that

$$H = \overline{R(a+B)} \oplus M$$

and in such a way that (12.105) holds. We shall just give a sufficient condition.

Theorem 12.28. *Let B be a densely defined linear operator on H such that $\overline{W(B)}$ is the strip $1-a \leq \Re e\ z \leq 0,\ a > 1$. If $\overline{R(a-B)} = \overline{R(a+B)}$, then B has a closed extension $\hat{B}$ satisfying*

$$\sigma(\hat{B}) \subset \overline{W(\hat{B})} = \overline{W(B)}. \tag{12.113}$$

Proof. On $R(a-B)^{\perp} = R(a+B)^{\perp}$ we define $\hat{T}$ to be $-I$. Then $\hat{T}$ is isometric on this set. Thus, (12.104) [and hence, (12.105)] holds for $u \in \overline{R(a-B)}$ and for $u \in R(a-B)^{\perp}$. For any $u \in H,\ u = u_1 + u_2$, where $u_1 \in \overline{R(a-B)}$ and $u_2 \in R(a-B)^{\perp}$. Thus

$$\frac{1-a}{a}\|(\hat{T}+I)u\|^2 = \frac{1-a}{a}\|(\bar{T}+I)u_1\|^2 \leq \|\bar{T}u_1\|^2 - \|u_1\|^2 \leq 0.$$

But

$$\|\bar{T}u_1\|^2 - \|u_1\|^2 = \|\bar{T}u_1 - u_2\|^2 - \|u_1+u_2\|^2 = \|\hat{T}u\|^2 - \|u\|^2.$$

Hence, (12.104) holds, and the proof is complete. □

To prove (12.110), expand both sides of

$$\|\hat{T}(u+w)\|^2 = \|u+w\|^2.$$

This gives

$$\Re e\ (\hat{T}u, \hat{T}w) = \Re e\ (u, w).$$

Now, substitute iu in place of u. Thus, (12.110) holds.

In proving Theorem 12.26, we shall make use of

Lemma 12.29. *Let A be a closed linear operator on a Banach space X. If λ is a boundary point of $\rho(A)$ and $\{\lambda_n\}$ is a sequence of points in $\rho(A)$ converging to λ, then $\|(A-\lambda_n)^{-1}\| \to \infty$.*

Proof. If the lemma were not true, there would be a sequence $\{\lambda_n\} \subset \rho(A)$ such that $\lambda_n \to \lambda$ as $n \to \infty$ while

$$\|(A-\lambda_n)^{-1}\| \le C. \tag{12.114}$$

Since

$$(A-\lambda_n)^{-1} - (A-\lambda_m)^{-1} = (A-\lambda_n)^{-1}(\lambda_n - \lambda_m)(A-\lambda_m)^{-1},$$

we have

$$\|(A-\lambda_n)^{-1} - (A-\lambda_m)^{-1}\| \le C^2|\lambda_m - \lambda_n| \to 0 \ \text{ as } \ m,n \to \infty.$$

Thus, $(A-\lambda_n)^{-1}$ converges to an operator $E \in B(X)$ as $n \to \infty$. Moreover, if x is any element in X, then

$$y_n = (A-\lambda_n)^{-1}x \to Ex \ \text{ as } \ n \to \infty.$$

But

$$Ay_n = (A-\lambda_n)y_n + \lambda_n y_n \to x + \lambda Ex \ \text{ as } \ n \to \infty.$$

Since A is closed, Ex is in $D(A)$ and $AEx = x + \lambda Ex$, whence,

$$(A-\lambda)Ex = x, \quad x \in X. \tag{12.115}$$

Similarly, if $x \in D(A)$, then

$$(A-\lambda_n)^{-1}(A-\lambda)x = x - (\lambda - \lambda_n)(A-\lambda_n)^{-1}x \to x \ \text{ as } \ n \to \infty.$$

Hence,

$$E(A-\lambda)x = x, \quad x \in D(A). \tag{12.116}$$

This shows that $\lambda \in \rho(A)$, contrary to assumption. The proof is complete. $\square$

We can now easily give the proof of Theorem 12.26.

Proof. If the theorem were not true, there would be a constant C such that

$$\|x\| \le C\|(A-\lambda)x\|, \quad x \in D(A) \tag{12.117}$$

(Theorem 3.12). Since λ is a boundary point of $\rho(A)$, there is a sequence $\{\lambda_n\}$ of points in $\rho(A)$ converging to λ. Set

$$B_n = (A-\lambda_n)^{-1}/\|(A-\lambda_n)^{-1}\|.$$

Then $\|B_n\| = 1$. In particular, for each n there is an element $x_n \in X$ such that

$$\|x_n\| = 1, \quad \|B_n x_n\| > \frac{1}{2}. \tag{12.118}$$

Now,

$$(A - \lambda)B_n = (A - \lambda_n)B_n + (\lambda_n - \lambda)B_n.$$

Hence

$$\|(A - \lambda)B_n\| \leq \|(A - \lambda_n)^{-1}\|^{-1} + |\lambda_n - \lambda|.$$

By Lemma 12.29, this tends to 0 as $n \to \infty$. In particular, the norm of $(A - \lambda)B_n$ can be made less than $1/3C$ for n sufficiently large. But by (12.117), we have

$$\frac{1}{2} < \|B_n x_n\| \leq C\|(A - \lambda)B_n x_n\| < \frac{1}{3}$$

for large n. This contradiction shows that (12.117) does not hold, and the proof is complete. □

12.11. Selfadjoint extensions

In Section 11.3, we defined a selfadjoint operator as one that satisfies $A^* = A$. In order for A^* to be defined, $D(A)$ must be dense. Moreover, A must be closed, since A^* is. Note also that $W(A)$ is a subset of the real axis, since

$$\overline{(Au, u)} = (u, Au) = (A^*u, u) = (Au, u).$$

Thus, $W(A)$ is either the whole real axis, a half-axis, or an interval. We claim that

$$\sigma(A) \subset \overline{W(A)} \tag{12.119}$$

when A is selfadjoint. This follows from the fact that, by Theorem 12.3, $\alpha(A - \lambda) = 0$ and $R(A - \lambda)$ is closed when $\lambda \notin \overline{W(A)}$. Moreover, if v is orthogonal to $R(A - \lambda)$, then

$$(v, (A - \lambda)u) = 0, \quad u \in D(A),$$

showing that $v \in D(A^*) = D(A)$ and

$$([A - \bar{\lambda}]v, u) = 0, \quad u \in D(A).$$

Since $D(A)$ is dense, we must have $(A - \bar{\lambda})v = 0$. Now, if $\lambda \notin \overline{W(A)}$, then $\bar{\lambda} \notin \overline{W(A)}$ as well. To see this, note that if λ is not real, then neither of them is in $\overline{W(A)}$. If λ is real, then $\lambda = \bar{\lambda}$. By what we have just said, $v = 0$. This shows that $R(A - \lambda)$ is dense as well as closed. Then $\lambda \in \rho(A - \lambda)$, and the assertion is proved.

Now, suppose A is a densely defined linear operator on H. Then $W(A)$ is a subset of the real axis if and only if A is *symmetric*, i.e., if

$$(Au, v) = (u, Av), \quad u, v \in D(A). \tag{12.120}$$

This follows immediately from Theorem 12.7. If A is symmetric and (12.119) holds, then we can show that A is selfadjoint. To see this, suppose that $v \in D(A^*)$. Let λ be any nonreal point. Then λ and $\bar{\lambda}$ are both in $\rho(A)$, since neither of them is in $\overline{W(A)}$. Hence, there is a $w \in D(A)$ such that

$$(A - \bar{\lambda})w = (A^* - \bar{\lambda})v.$$

Thus,

$$(v, (A - \lambda)u) = ([A^* - \bar{\lambda}]v, u) = ([A - \bar{\lambda}]w, u) = (w, (A - \lambda)u), \quad u \in D(A),$$

or

$$(v - w, (A - \lambda)u) = 0, \quad u \in D(A).$$

Since $\lambda \in \rho(A)$, this can happen only if $v = w$. Hence, we see that $v \in D(A)$ and $Av = A^*v$. In particular, since every bounded operator satisfies (12.119) (Theorem 12.1), we see that every bounded symmetric operator defined everywhere is selfadjoint.

Let us now pose and discuss the following question: Let A be a densely defined symmetric linear operator on a Hilbert space. Does it have a selfadjoint extension?

To examine the question, suppose first that $W(A)$ is not the whole real axis. Then, by Theorem 12.14, A has a closed extension $\hat{A}$ satisfying

$$\sigma(\hat{A}) \subset \overline{W(\hat{A})} = \overline{W(A)}. \tag{12.121}$$

In particular, $\hat{A}$ is symmetric and satisfies (12.119). Hence, $\hat{A}$ is selfadjoint.

On the other hand, if $W(A)$ is the whole real axis, then A has a closed extension $\hat{A}$ satisfying (12.121) if and only if there is an isometry of $R(i+A)^{\perp}$ onto $R(i-A)^{\perp}$ (just put $B = iA$ in Theorem 12.28). Since, in this case, an extension violating (12.121) could not be selfadjoint, the condition is both necessary and sufficient for A to have a selfadjoint extension. The problem is, therefore, solved.

12.12. Problems

(1) Let A be a unitary operator on H. Show that for any subspace $L \subset H$ one has $A(L^{\perp}) = A(L)^{\perp}$.

(2) Show that the graph of the closure of an operator is the closure of the graph of the operator.

(3) Show that a linear operator A is an isometry if and only if $A^*A = I$. Show that it is unitary if $A^*A = AA^* = I$.

(4) Show that if A is normal, then
$$\|A\| = \sup_{\|u\|=1} |(Au, u)|.$$

(5) Suppose A is a densely defined linear operator on H. Show that $D(A^*)$ is dense if and only if A is closed (see Problem 11.5).

(6) If A^{**} exists, show that it is the smallest closed extension of A.

(7) Show that if $a(u, v)$ is a bilinear form such that $W(a)$ consists of only the point 0, then $a(u, v) = 0$ for all $u, v \in D(a)$.

(8) Show that an operator $A \in B(H)$ is normal if and only if $A^*A = AA^*$.

(9) Show that an unbounded linear functional cannot be closable.

(10) Show that if either A or A' is densely defined, then the other is closable.

(11) Show that a bilinear form $a(u, v)$ is bounded if it is continuous in u for each fixed v and continuous in v for each fixed u.

(12) Show that in a complex Hilbert space, a symmetric bilinear form $a(u, v)$ is determined by $a(u)$. Is this true in a real Hilbert space?

(13) Show that $W \neq \mathbb{R}^2$ implies $\overline{W} \neq \mathbb{R}^2$ when W is convex.

(14) Show that the set of all $z \in \mathbb{R}^2$ satisfying (12.58) forms a half-plane.

(15) Show that a convex set of points consisting of rays from a point P is either an angle θ with vertex at P or the whole plane.

(16) Show that γ and d can be chosen so that (12.60) holds.

Chapter 13

SELFADJOINT OPERATORS

13.1. Orthogonal projections

Let H be a complex Hilbert space, and let M be a closed subspace of H. Then, by the projection theorem (Theorem 2.3), every element $u \in H$ can be written in the form

$$u = u' + u'', \quad u' \in M,\ u'' \in M^{\perp}. \tag{13.1}$$

Set

$$Eu = u'.$$

Then E is a linear operator on H. It is bounded, since

$$\|Eu\|^2 = \|u'\|^2 \leq \|u'\|^2 + \|u''\|^2 = \|u\|^2. \tag{13.2}$$

Since

$$Eu = u, \quad u \in M, \tag{13.3}$$

we have

$$\|E\| = 1. \tag{13.4}$$

We also have

$$E^2u = E(Eu) = Eu' = u' = Eu, \tag{13.5}$$

showing that E is a projection. It is called the *orthogonal projection onto M*. Let us describe some other properties of E.

(a) E is selfadjoint.

Proof.

$$(u, Ev) = (Eu, Ev) = (Eu, v).$$

□

(b) $R(E) = M$.

Proof. If $u \in M$, then $Eu = u$. If $u \in R(E)$, there is a $v \in H$ such that $Ev = u$. Thus, $Eu = E^2v = Ev = u$, by (13.5). □

(c) $u \in M$ if and only if $\|Eu\| = \|u\|$.

Proof. Since $u = Eu + (I - E)u$, we have

$$\|u\|^2 = \|Eu\|^2 + \|(I - E)u\|^2. \tag{13.6}$$

If $\|Eu\| = \|u\|$, then $(I - E)u = 0$, showing that $u \in M$. □

We also have

Lemma 13.1. *If F is a bounded linear selfadjoint projection on H, then $R(F)$ is closed and F is the orthogonal projection onto $R(F)$.*

Proof. Set $M = R(F)$. Then M is closed. This follows from the fact that if $Fu_n \to v$, then $F^2u_n \to Fv$. Thus, $v = Fv$, showing that $v \in M$. Now, if u is any element of H, then

$$u = Fu + (I - F)u.$$

Moreover, $Fu \in M$, while $(I - F)u \in M^\perp$, since

$$(Fv, (I - F)u) = ([F - F^2]v, u) = 0, \quad v \in H.$$

Thus F is the orthogonal projection onto M. □

Suppose $A \in B(H)$ and M is a closed subspace of H. We say that M *is invariant under* A if $A(M) \subset M$. If both M and $M^\perp$ are invariant under A, then M is said to *reduce* A.

Let M be a closed subspace of H, and let E be the orthogonal projection onto M. Then we have

Lemma 13.2. *If A is in $B(H)$, then*

(i) M is invariant under A if and only if $AE = EAE$;

(ii) M reduces A if and only if $AE = EA$.

Proof. (i) If M is invariant under A, then $AEu \in M$ for all $u \in H$. Hence, $EAEu = AEu$. Conversely, if $AE = EAE$ and $u \in M$, then $Au = AEu = EAEu = EAu$. Hence $Au \in M$.

(ii) Suppose M reduces A. We have

$$EA = EAE + EA(I - E).$$

Since M is invariant under A, $EAE = AE$. Moreover, for any $u \in H$, $(I - E)u \in M^\perp$ so that $A(I-E)u \in M^\perp$ by hypothesis. Hence, $EA(I-E)u = 0$, showing that the operator $EA(I - E) = 0$. Hence, $EA = EAE = AE$. Conversely, if $AE = EA$, then $EAE = AE^2 = AE$, showing that M is invariant under A. Moreover, if $u \in M^\perp$, then $Eu = 0$. Thus $EAu = AEu = 0$. This shows that $Au \in M^\perp$, and the proof is complete. □

13.2. Square roots of operators

Let us prove the following:

Theorem 13.3. *Let A be any real number satisfying $0 \le A \le 1$. Then there exists a real number $B \ge 0$ such that $B^2 = A$.*

Proof. Suppose B exists. Set $R = 1-A$, $S = 1-B$. Then $(1-S)^2 = 1-R$, or

$$S = \frac{1}{2}(R + S^2). \tag{13.7}$$

Conversely, if we can find a solution S of (13.7), then an easy calculation shows that $B = 1 - S$ satisfies $B^2 = A$. So all we need do is solve (13.7). To solve it, we use the method of iterations employed in Section 1.1. Set $S_0 = 0$ and

$$S_{n+1} = \frac{1}{2}(R + S_n^2), \quad n = 0, 1, \cdots. \tag{13.8}$$

If the sequence $\{S_n\}$ has a limit S, then S is clearly a solution of (13.7). Now we claim that

$$0 \le S_n \le 1, \quad n = 0, 1 \cdots. \tag{13.9}$$

This is true for $n = 0$, and if it is true for n, then

$$S_{n+1} = \frac{1}{2}(R + S_n^2) \le \frac{1}{2}(R + 1) \le 1,$$

showing that it is also true for $n+1$. Hence, (13.9) holds for all n by induction. Note also that S_n is a polynomial in R with nonnegative coefficients. Again, this is true for $n = 0$, while (13.8) shows that it is true for $n + 1$ whenever it is true for n.

Next we claim that

$$S_{n+1} \geq S_n, \quad n = 0, 1, \cdots . \tag{13.10}$$

This follows from the fact that

$$S_{n+1} - S_n = \frac{1}{2}(R + S_n^2) - \frac{1}{2}(R + S_{n-1}^2) = \frac{1}{2}(S_n + S_{n-1})(S_n - S_{n-1}),$$

which shows, in view of (13.9), that (13.10) holds for $n = k$ whenever it holds for $n = k - 1$. Since (13.10) holds for $n = 0$, it holds for all n by induction.

From (13.9) and (13.10) we see that $\{S_n\}$ is a nondecreasing sequence of real numbers bounded from above. By a theorem that should be well known to all of us, such a sequence has a limit. This is precisely what we wanted to show. The proof is complete. □

Note that the restriction $A \leq 1$ is not necessary. If $A > 1$, we let n be a positive integer satisfying $A < n^2$, and we take $B = A/n^2$. Then $0 < B < 1$, and consequently, B has a positive square root M. If we take $C = nM$, we see that $C > 0$ and $C^2 = n^2 B = A$.

Let us consider the following question: Given an operator A, does it have a square root? By a square root of an operator, we mean an operator B such that $B^2 = A$. Let us describe a class of operators having square roots. An operator $A \in B(H)$ is called *positive* if

$$(Au, u) \geq 0, \quad u \in H \tag{13.11}$$

[i.e., if $W(A)$ is contained in the nonnegative real axis]. Actually, such an operator should be called nonnegative. Still, no matter what we call it, we write $A \geq 0$ when A satisfies (13.11). Since we are in a complex Hilbert space, we know that such operators are selfadjoint (see Section 12.11). The expression $A \geq B$ will be used to mean $A - B \geq 0$.

An important observation is

Lemma 13.4. *If*

$$-MI \leq A \leq MI, \quad M \geq 0, \tag{13.12}$$

then

$$\|A\| \leq M. \tag{13.13}$$

Proof. By (13.12),

$$|(Au, u)| \leq M\|u\|^2, \quad u \in H. \tag{13.14}$$

Hence, by Lemma 12.4,

$$|(Au, v)| \leq M\|u\| \cdot \|v\|, \quad u, v \in H.$$

If we take $v = Au$, we obtain

$$\|Au\|^2 \leq M\|u\| \cdot \|Au\|,$$

which implies (13.13). □

Returning to square roots, we shall prove the following:

Theorem 13.5. *If A is a positive operator in $B(H)$, then there is a unique $B \geq 0$ such that $B^2 = A$. Moreover, B commutes with any $C \in B(H)$ which commutes with A.*

Proof. It suffices to consider the case

$$0 \leq A \leq I. \tag{13.15}$$

To see this, note that the operator $A/\|A\|$ always satisfies (13.15), and if we can find an operator G such that $G^2 = A/\|A\|$, then $B = \|A\|^{1/2}G$ satisfies $B^2 = A$.

Now, let us follow the proof of Theorem 13.3. Set $R = I - A$. Then $0 \leq R \leq I$. We wish to solve (13.7). Set $S_0 = 0$, and define S_{n+1} inductively by means of (13.8). Then

$$0 \leq S_n \leq I, \quad n = 0, 1, \cdots . \tag{13.16}$$

This is surely true for $n = 0$. Assume it true for n. Then

$$(S_{n+1}u, u) = \frac{1}{2}(Ru, u) + \frac{1}{2}\|S_n u\|^2, \tag{13.17}$$

which shows immediately that $S_{n+1} \geq 0$. Moreover, by (13.16) and Lemma 13.4, $\|S_n\| \leq 1$. Thus, (13.17) implies $S_{n+1} \leq I$, showing that (13.16) holds with n replaced by $n+1$. Consequently, (13.16) holds for all n by induction.

Next, we note that S_n is a polynomial in R with nonnegative coefficients. Again, this is true for $n = 0$, and if it is true for n, then (13.8) shows that it is true for $n + 1$. We can easily show that

$$S_{n+1} - S_n = \frac{1}{2}(S_n + S_{n-1})(S_n - S_{n-1}) \tag{13.18}$$

is a polynomial in R with nonnegative coefficients. This is true for $n = 0$, and (13.18) shows that it is true for $n = k$ if it is true for $n = k - 1$.

Now we can show that

$$R^k \geq 0, \quad k = 0, 1, \cdots . \tag{13.19}$$

If $k = 2j$, then

$$(R^k u, u) = \|R^j u\|^2 \geq 0,$$

while if $k = 2j + 1$, we have

$$(R^k u, u) = (RR^j u, R^j u) \geq 0.$$

From (13.19) and the fact that each $S_{n+1} - S_n$ is a polynomial in R with nonnegative coefficients, we see that

$$S_{n+1} \geq S_n, \quad n = 0, 1, \cdots . \tag{13.20}$$

To summarize, we see that the sequence $\{S_n\}$ satisfies

$$0 \leq S_n \leq S_{n+1} \leq I, \quad n = 0, 1, \cdots . \tag{13.21}$$

If the S_n were real numbers and not operators, we could conclude that the sequence approaches a limit, and the first part of the theorem would be proved. The fact that we can reach a similar conclusion in the present case follows from

Lemma 13.6. *If $\{S_n\}$ is a sequence of operators in $B(H)$ satisfying (13.21), then there is an operator S in $B(H)$ such that*

$$S_n u \to Su, \quad u \in H. \tag{13.22}$$

Assume Lemma 13.6 for the moment, and let us continue the proof of Theorem 13.5. By (13.22), we see that the operator S is a solution of (13.7). Thus, $B = I - S$ is a square root of A. Now, let $C \in B(H)$ be any operator that commutes with A. Then it commutes with $R = I - A$, and since each S_n is a polynomial in R, C must commute with each S_n. Then

$$CS_n u = S_n Cu, \quad u \in H.$$

Taking the limit as $n \to \infty$, we see that C commutes with S, and hence, with $B = I - S$.

Now, suppose $T \geq 0$ is another square root of A. Then T commutes with A, since

$$TA = TT^2 = T^2 T = AT.$$

Therefore, it commutes with B. Let u be any element of H, and set $v = (B - T)u$. Then

$$([B + T]v, v) = ([B^2 - T^2]u, v) = ([A - A]u, v) = 0.$$

But $B \geq 0$ and $T \geq 0$. Hence, we must have

$$(Bv, v) = (Tv, v) = 0.$$

Since $B \geq 0$, there is an $F \in B(H)$ such that $F^2 = B$. Consequently,

$$\|Fv\|^2 = (F^2 v, v) = (Bv, v) = 0,$$

showing that $Fv = 0$. But this implies

$$Bv = F^2 v = 0.$$

Similarly, we have $Tv = 0$. Thus,

$$\|(B - T)u\|^2 = ([B - T]^2 u, u) = ([B - T]v, u) = 0,$$

showing that

$$Bu = Tu$$

for all $u \in H$. Hence, B is unique.

The proof of Theorem 13.5 will be complete once we have given the proof of Lemma 13.6. $\square$

Proof. Set

$$S_{mn} = S_n - S_m, \quad m \leq n.$$

Then by (13.21),

$$0 \leq S_{mn} \leq I. \tag{13.23}$$

Hence,

$$\begin{aligned} \|S_{mn}u\|^4 &= (S_{mn}u, S_{mn}u)^2 \\ &\leq (S_{mn}u, u)(S_{mn}^2 u, S_{mn}u) \\ &\leq [(S_n u, u) - (S_m u, u)]\|u\|^2. \end{aligned} \tag{13.24}$$

This follows from Corollary 12.6 if we take $b(u) = (S_{mn}u, u)$. Now, for each $u \in H$ the sequence $\{(S_n u, u)\}$ is nondecreasing and bounded from above. Hence, it is convergent. Thus, inequality (13.24) implies that $\{S_n u\}$ is a Cauchy sequence in H. Consequently, the sequence converges to a limit which we denote by Su. Clearly, S is a linear operator on H. Moreover,

$$(Su, u) = \lim (S_n u, u),$$

showing that $0 \leq S \leq I$. Thus, $S \in B(H)$, and the proof is complete. $\square$

A consequence of Theorem 13.5 is

Corollary 13.7. *If $A \geq 0$, $B \geq 0$ and $AB = BA$, then $BA \geq 0$.*

Proof. By Theorem 13.5, A and B have square roots $A^{1/2} \geq 0$ and $B^{1/2} \geq 0$ which commute. Hence,

$$(ABu, u) = \|A^{1/2}B^{1/2}u\|^2 \geq 0.$$

$\square$

A fact that we shall need later is

Lemma 13.8. *Let M_1 and M_2 be closed subspaces of H, and let E_1 and E_2 be the orthogonal projections onto them, respectively. Then the following statements are equivalent:*

(a) $E_1 \leq E_2$;

(b) $\|E_1 u\| \leq \|E_2 u\|, \quad u \in H$;

(c) $M_1 \subset M_2$;

(d) $E_2 E_1 = E_1 E_2 = E_1$.

Proof. (a) implies (b). $\|E_1u\|^2 = (E_1u, E_1u) = (E_1^2u, u) = (E_1u, u) \leq (E_2u, u) = \|E_2u\|^2$.

(b) implies (c). If $u \in M_1$, then $\|u\| = \|E_1u\| \leq \|E_2u\| \leq \|u\|$ [cf. (c) of Section 13.1]. Hence, $\|u\| = \|E_2u\|$, showing that $u \in M_2$.

(c) implies (d). If u is any element of H, then $u = v + w$, where $v \in M_1$ and $w \in M_1^\perp$. Thus, $E_1u = v \in M_2$. Hence, $E_2E_1u = E_2v = v = E_1u$. Taking adjoints, we get $E_1E_2 = E_1$.

(d) implies (a). $E_2 - E_1 = E_2 - E_1E_2 = (I - E_1)E_2$. Since E_1 and E_2 commute and $I - E_1 \geq 0$, $E_2 \geq 0$, we see by Corollary 13.7 that $(I - E_1)E_2 \geq 0$. Hence, (a) holds, and the proof is complete. □

13.3. A decomposition of operators

Let us now show that a bounded, selfadjoint operator can be expressed as the difference of two positive operators. Suppose A is a selfadjoint operator in $B(H)$. Then the operator A^2 is positive. Hence, it has a square root that is positive and commutes with any operator commuting with A^2 (Theorem 13.5). Denote this square root by $|A|$. Since any operator commuting with A also commutes with A^2, we see that $|A|$ commutes with any operator that commutes with A.

Set

$$A^+ = \frac{1}{2}(|A| + A), \quad A^- = \frac{1}{2}(|A| - A). \tag{13.25}$$

These operators are selfadjoint and commute with any operator commuting with A. They satisfy

$$A = A^+ - A^-, \quad |A| = A^+ + A^-. \tag{13.26}$$

Moreover,

$$A^+A^- = \frac{1}{4}(|A| + A)(|A| - A) = \frac{1}{4}(A^2 - A^2) = 0. \tag{13.27}$$

Let E be the orthogonal projection onto $N(A^+)$ (see Section 13.1). Thus,

$$A^+E = 0. \tag{13.28}$$

Taking adjoints, we obtain

$$EA^+ = 0. \tag{13.29}$$

Now by (13.27), $R(A^-) \subset N(A^+)$. Hence,

$$EA^- = A^-, \tag{13.30}$$

and by adjoints

$$A^-E = A^-. \tag{13.31}$$

We see, therefore, that both A^+ and A^- commute with E. Consequently, so do $|A|$ and A [see (13.26)].

Note next that

$$EA = E(A^+ - A^-) = EA^+ - A^- = -A^-, \tag{13.32}$$

$$E|A| = E(A^+ + A^-) = A^-, \tag{13.33}$$

$$(I - E)A = A - EA = A + A^- = A^+, \tag{13.34}$$

and

$$(I - E)|A| = |A| - A^- = A^+. \tag{13.35}$$

Since E, $I - E$ and $|A|$ are positive operators which commute, we see from (13.33) and (13.35) that

$$A^+ \geq 0, \quad A^- \geq 0 \tag{13.36}$$

(Corollary 13.7). Hence, by (13.26),

$$|A| \geq A^+, \quad |A| \geq A^-. \tag{13.37}$$

Also,

$$A^+ - A = A^- \geq 0, \quad A^- + A = A^+ \geq 0.$$

Therefore,

$$A^+ \geq A, \quad A^- \geq -A. \tag{13.38}$$

We now have the following:

Lemma 13.9. *If $B \in B(H)$ commutes with A, then it commutes with E.*

Proof. As we mentioned above, B commutes with A^+. Thus, $BA^+ = A^+B$. This implies that $N(A^+)$ is invariant under B (see Section 13.1). Thus $BE = EBE$ (Lemma 13.2). Taking adjoints, we get $EB = EBE$, which implies $BE = EB$. □

We also have

Lemma 13.10. *Let B be an operator in $B(H)$ which commutes with A and satisfies $B \geq \pm A$. Then $B \geq |A|$. Thus, $|A|$ is the "smallest" operator having these properties.*

Proof. Since $B - A \geq 0$, $I - E \geq 0$ and they commute, we have

$$(I - E)(B - A) \geq 0$$

(Corollary 13.7). Thus by (13.34),

$$(I - E)B \geq (I - E)A = A^+. \tag{13.39}$$

Similarly, since $B + A \geq 0$ and $E \geq 0$, we have

$$E(B + A) \geq 0,$$

which implies

$$EB \geq -EA = A^- \tag{13.40}$$

by (13.32). Adding (13.39) and (13.40), we obtain

$$B \geq A^+ + A^- = |A|,$$

which proves the lemma. $\square$

In addition, we have

Lemma 13.11. *Let $B \geq 0$ be an operator in $B(H)$ which commutes with A and satisfies $B \geq A$. Then $B \geq A^+$.*

Proof. By (13.39),

$$B \geq A^+ + EB \geq A^+,$$

since $BE \geq 0$. $\square$

Also, we have

Lemma 13.12. *Let B be a positive operator in $B(H)$ which commutes with A and satisfies $B \geq -A$. Then $B \geq A^-$.*

Proof. By (13.40), $EB \geq A^-$. But $(I - E)B \geq 0$. Hence, $B \geq A^-$. $\square$

13.4. Spectral resolution

We saw in Chapter 6 that, in a Banach space X, we can define $f(A)$ for any $A \in B(X)$ provided $f(z)$ is a function analytic in a neighborhood of $\sigma(A)$. In this section, we shall show that we can do better in the case of selfadjoint operators.

To get an idea, let A be a compact, selfadjoint operator on H. Then by Theorem 11.3,

$$Au = \sum \lambda_k (u, \varphi_k)\varphi_k, \tag{13.41}$$

where $\{\varphi_k\}$ is an orthonormal sequence of eigenvectors and the λ_k are the corresponding eigenvalues of A. Now let $p(t)$ be a polynomial with real coefficients having no constant term

$$p(t) = \sum_1^m a_k t^k. \tag{13.42}$$

Then $p(A)$ is compact and selfadjoint. Let $\mu \neq 0$ be a point in $\sigma(p(A))$. Then $\mu = p(\lambda)$ for some $\lambda \in \sigma(A)$ (Theorem 6.8). Now $\lambda \neq 0$ (otherwise we would have $\mu = p(0) = 0$). Hence, it is an eigenvalue of A (see Section 6.1). If φ is a corresponding eigenvector, then

$$[p(A)-\mu]\varphi = \sum a_k A^k\varphi - \mu\varphi = \sum a_k \lambda^k \varphi - \mu\varphi = [p(\lambda)-\mu]\varphi = 0.$$

Thus μ is an eigenvalue of $p(A)$ and φ is a corresponding eigenvector. This shows that

$$p(A)u = \sum p(\lambda_k)(u,\varphi_k)\varphi_k. \tag{13.43}$$

Now, the right hand side of (13.43) makes sense if $p(t)$ is any function bounded on $\sigma(A)$ (see Section 11.2). Therefore it seems plausible to define $p(A)$ by means of (13.43). Of course, for such a definition to be useful, one would need certain relationships to hold. In particular, one would want $f(t)g(t) = h(t)$ to imply $f(A)g(A) = h(A)$. We shall discuss this a bit later.

If A is not compact, we cannot, in general, obtain an expansion in the form (13.41). However, we can obtain something similar. In fact, we have

Theorem 13.13. *Let A be a selfadjoint operator in $B(H)$. Set*

$$m = \inf_{\|u\|=1} (Au,u), \quad M = \sup_{\|u\|=1} (Au,u).$$

Then there is a family $\{E(\lambda)\}$ of orthogonal projection operators on H depending on a real parameter λ and such that:

(1) $E(\lambda_1) \le E(\lambda_2)$ for $\lambda_1 \le \lambda_2$;

(2) $E(\lambda)u \to E(\lambda_0)u$ as $\lambda_0 < \lambda \to \lambda_0, \quad u \in H$;

(3) $E(\lambda) = 0$ for $\lambda < m$, $E(\lambda) = I$ for $\lambda \ge M$;

(4) $AE(\lambda) = E(\lambda)A$;

(5) *if $a < m$, $b \ge M$ and $p(t)$ is any polynomial, then*

$$p(A) = \int_a^b p(\lambda)\,dE(\lambda). \tag{13.44}$$

This means the following. Let $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$ be any partition of $[a,b]$, and let λ_k' be any number satisfying $\lambda_{k-1} \le \lambda_k' \le \lambda_k$. Then

$$\sum_1^n p(\lambda_k')[E(\lambda_k) - E(\lambda_{k-1})] \to p(A) \tag{13.45}$$

in $B(H)$ as $\eta = \max(\lambda_k - \lambda_{k-1}) \to 0$.

Proof. Set $A(\lambda) = A - \lambda$. Then

$$A(\lambda_1) \geq A(\lambda_2) \text{ for } \lambda_1 \leq \lambda_2. \tag{13.46}$$

Let the operators $|A(\lambda)|$, $A^+(\lambda)$, $A^-(\lambda)$ be defined as in the preceding section. Then

$$A^+(\lambda_1) \geq A(\lambda_1) \geq A(\lambda_2) \text{ for } \lambda_1 \leq \lambda_2 \tag{13.47}$$

by (13.38), and hence

$$A^+(\lambda_1) \geq A^+(\lambda_2), \quad \lambda_1 \leq \lambda_2 \tag{13.48}$$

(Lemma 13.11). Thus,

$$A(\lambda) = |A(\lambda)| = A^+(\lambda), \quad \lambda \leq m \tag{13.49}$$

(Theorem 13.5). Similarly if $\lambda \geq M$, then $A(\lambda) \leq 0$, in which case, we have

$$A(\lambda) = -|A(\lambda)| = -A^-(\lambda), \quad \lambda \geq M. \tag{13.50}$$

In general, if $\lambda_1 \leq \lambda_2$, we have, by (13.48) and Corollary 13.7,

$$A^+(\lambda_2)[A^+(\lambda_1) - A^+(\lambda_2)] \geq 0.$$

Hence,

$$A^+(\lambda_2)A^+(\lambda_1) \geq A^+(\lambda_2)^2, \quad \lambda_1 \leq \lambda_2. \tag{13.51}$$

This implies

$$N[A^+(\lambda_1)] \subset N[A^+(\lambda_2)], \quad \lambda_1 \leq \lambda_2. \tag{13.52}$$

Let $E(\lambda)$ be the orthogonal projection onto $N[A^+(\lambda)]$. Then

$$E(\lambda_1) \leq E(\lambda_2), \quad \lambda_1 \leq \lambda_2, \tag{13.53}$$

by Lemma 13.8. Moreover, for $\lambda < m$, we have

$$([A - \lambda]u, u) \geq (m - \lambda)\|u\|^2,$$

showing that

$$N[A(\lambda)] = \{0\}, \quad \lambda < m.$$

Thus, by (13.49) we have

$$E(\lambda) = 0, \quad \lambda < m. \tag{13.54}$$

If $\lambda \geq M$, we have $A^+(\lambda) = 0$ by (13.50) so that

$$E(\lambda) = I, \quad \lambda \geq M. \tag{13.55}$$

Set

$$E(\lambda_1, \lambda_2) = E(\lambda_2) - E(\lambda_1).$$

Then $E(\lambda_1, \lambda_2) \geq 0$ for $\lambda_1 \leq \lambda_2$. Therefore,

$$E(\lambda_2)E(\lambda_1, \lambda_2) = E(\lambda_2) - E(\lambda_1) = E(\lambda_1, \lambda_2) \tag{13.56}$$

and

$$E(\lambda_1)E(\lambda_1,\lambda_2) = E(\lambda_1) - E(\lambda_1) = 0 \tag{13.57}$$

(Lemma 13.8). Hence,

$$A(\lambda_2)E(\lambda_1,\lambda_2) = A(\lambda_2)E(\lambda_2)E(\lambda_1,\lambda_2) = -A^-(\lambda_2)E(\lambda_1,\lambda_2) \leq 0$$

by (13.32). Also,

$$A(\lambda_1)E(\lambda_1,\lambda_2) = A(\lambda_1)[I - E(\lambda_1)]E(\lambda_1,\lambda_2) = A^+(\lambda_1)E(\lambda_1,\lambda_2) \geq 0$$

by (13.34). Combining these two inequalities, we obtain

$$\lambda_1 E(\lambda_1,\lambda_2) \leq AE(\lambda_1,\lambda_2) \leq \lambda_2 E(\lambda_1,\lambda_2), \quad \lambda_1 \leq \lambda_2. \tag{13.58}$$

Next, take $a < m,\ b \geq M$, and let

$$a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$$

be any partition of $[a,b]$. Then by (13.58),

$$\begin{aligned}\sum_1^n \lambda_{k-1}[E(\lambda_k) - E(\lambda_{k-1})] &\leq A\sum_1^n [E(\lambda_k) - E(\lambda_{k-1})] \\ &= A \leq \sum_1^n \lambda_k[E(\lambda_k) - E(\lambda_{k-1})].\end{aligned} \tag{13.59}$$

If λ_k' is any point satisfying $\lambda_{k-1} \leq \lambda_k' \leq \lambda_k$, then

$$\begin{aligned}A - \sum_1^n \lambda_k'[E(\lambda_k) - E(\lambda_{k-1})] &\leq \sum_1^n (\lambda_k - \lambda_{k-1})[E(\lambda_k) - E(\lambda_{k-1})] \\ &\leq \max(\lambda_k - \lambda_{k-1})I = \eta I.\end{aligned}$$

Similarly,

$$A - \sum_1^n \lambda_k'[E(\lambda_k) - E(\lambda_{k-1})] \geq -\eta I. \tag{13.60}$$

These inequalities imply, by Lemma 13.4, that

$$\|A - \sum_1^n \lambda_k'[E(\lambda_k) - E(\lambda_{k-1})]\| \leq \eta. \tag{13.61}$$

Hence,

$$A = \lim_{\eta\to 0} \sum_1^n \lambda_k'[E(\lambda_k) - E(\lambda_{k-1})] = \int_a^b \lambda dE(\lambda).$$

Next, we can show that for each integer $s \geq 0$,

$$\left(\sum_1^n \lambda_k'[E(\lambda_k) - E(\lambda_{k-1})]\right)^s = \sum_1^n {\lambda_k'}^s[E(\lambda_k) - E(\lambda_{k-1})]. \tag{13.62}$$

This follows from the fact that for $j < k$, we have

$$\begin{aligned} E(\lambda_{k-1}, \lambda_k)E(\lambda_{j-1}, \lambda_j) &= E(\lambda_k)E(\lambda_j) - E(\lambda_k)E(\lambda_{j-1}) \\ &- E(\lambda_{k-1})E(\lambda_j) + E(\lambda_{k-1})E(\lambda_{j-1}) \\ &= E(\lambda_j) - E(\lambda_{j-1}) - E(\lambda_j) + E(\lambda_{j-1}) = 0. \end{aligned}$$

But the left hand side of (13.62) approaches A^s as $\eta \to 0$, while the right hand side tends to

$$\int_a^b \lambda^s dE(\lambda).$$

Hence,

$$A^s = \int_a^b \lambda^s dE(\lambda), \quad s = 0, 1, \cdots, \tag{13.63}$$

from which we conclude that (13.44) holds.

We have proved (1), (3), (4) and (5). It remains to prove (2). Since

$$E(\lambda_1, \lambda) \geq E(\lambda_1, \mu), \quad \lambda > \mu,$$

we see that $E(\lambda_1, \lambda)$ is nondecreasing in λ. Hence, it approaches a limit $G(\lambda_1) \in B(H)$ as λ approaches λ_1 from above (Lemma 13.6). Statement (2) of the theorem is equivalent to the statement that $G(\lambda_1) = 0$. To see that this is indeed the case, note that

$$\lambda_1 E(\lambda_1, \lambda) \leq AE(\lambda_1, \lambda) \leq \lambda E(\lambda_1, \lambda), \quad \lambda \geq \lambda_1,$$

by (13.58). Letting $\lambda \to \lambda_1$, we obtain

$$\lambda_1 G(\lambda_1) \leq AG(\lambda_1) \leq \lambda_1 G(\lambda_1),$$

or

$$0 \leq A(\lambda_1)G(\lambda_1) \leq 0,$$

which implies, by Lemma 13.4, that

$$A(\lambda_1)G(\lambda_1) = 0.$$

Hence

$$A^+(\lambda_1)G(\lambda_1) = (I - E(\lambda_1))A(\lambda_1)G(\lambda_1) = 0$$

by (13.34). This shows that

$$R[G(\lambda_1)] \subset N[A^+(\lambda_1)],$$

which implies

$$E(\lambda_1)G(\lambda_1) = G(\lambda_1). \tag{13.64}$$

But

$$E(\lambda_1)E(\lambda_1, \lambda) = 0, \quad \lambda_1 \leq \lambda$$

by (13.57). This shows that

$$E(\lambda_1)G(\lambda_1) = 0.$$

This together with (13.64) implies $G(\lambda_1) = 0$, and the proof of Theorem 13.13 is complete. $\square$

The family $\{E(\lambda)\}$ is called the *resolution of the identity corresponding to* A. Theorem 13.13 is a form of the *spectral theorem.*

13.5. Some consequences

Let us see what consequences can be drawn from Theorem 13.13.

1. If $p(\lambda) \geq 0$ in $[a, b]$, then $p(A) \geq 0$.

Proof. This follows from the fact that

$$\sum_1^n p(\lambda_k')[E(\lambda_k) - E(\lambda_{k-1})] \geq 0.$$

$\square$

2. We can define $f(A)$ for any real function $f(\lambda)$ continuous in $[a, b]$.

Proof. We can do this as follows: If $f(\lambda)$ is continuous in $[a, b]$, one can find a sequence $\{p_j(\lambda)\}$ of polynomials converging uniformly to $f(\lambda)$ on $[a, b]$ (see Section 10.5). Thus, for each $\varepsilon > 0$, there is a number N such that

$$|p_j(\lambda) - p_i(\lambda)| < \varepsilon, \quad i, j > N,\ a \leq \lambda \leq b. \tag{13.65}$$

Thus, by our first remark,

$$-\varepsilon I \leq p_j(A) - p_i(A) \leq \varepsilon I, \quad i, j > N, \tag{13.66}$$

which implies, in view of Lemma 13.4,

$$\|p_j(A) - p_i(A)\| < \varepsilon, \quad i, j > N. \tag{13.67}$$

Consequently, the sequence $\{p_j(A)\}$ converges in $B(H)$ to an operator B. Moreover, it is easily verified that the limit B is independent of the particular choice of the polynomials $p_j(\lambda)$. We define $f(A)$ to be the operator B. $\square$

3. $f(A) = \int_a^b f(\lambda) dE(\lambda)$.

Proof. To see this, let $\{p_j(\lambda)\}$ be a sequence of polynomials converging uniformly to $f(\lambda)$ on $[a, b]$. Then for $\varepsilon > 0$, (13.65) and (13.67) hold. Letting $i \to \infty$, we have

$$|p_j(\lambda) - f(\lambda)| \leq \varepsilon, \quad j > N, \tag{13.68}$$

and

$$\|p_j(A) - f(A)\| \leq \varepsilon, \quad j > N. \tag{13.69}$$

Now,

$$f(A)-\sum_1^n f(\lambda_k')[E(\lambda_k)-E(\lambda_{k-1})]=\{f(A)-p_j(A)\}$$
$$+\left\{p_j(A)-\sum_{k=1}^n p_j(\lambda_k')[E(\lambda_k)-E(\lambda_{k-1})]\right\}$$
$$+\sum_{k=1}^n[p_j(\lambda_k')-f(\lambda_k')][E(\lambda_k)-E(\lambda_{k-1})]=Q_1+Q_2+Q_3.$$

Let j be any fixed integer greater than N. Then

$$-\varepsilon I\le Q_3\le\varepsilon I$$

by (13.68), while $\|Q_1\|\le\varepsilon$ by (13.69). Moreover, $\|Q_2\|\to 0$ as $\eta=\max(\lambda_k-\lambda_{k-1})\to 0$ (Theorem 13.13). Hence, we can take η so small that

$$\|f(A)-\sum_1^n f(\lambda_k')[E(\lambda_k)-E(\lambda_{k-1})]\|<3\varepsilon,$$

and the proof is complete. □

4. $f(\lambda)+g(\lambda)=h(\lambda)$ in $[a,b]$ implies $f(A)+g(A)=h(A)$.

5. $f(\lambda)g(\lambda)=h(\lambda)$ in $[a,b]$ implies $f(A)g(A)=h(A)$.

Proof. If $\{p_j(\lambda)\}$ converges uniformly to $f(\lambda)$ in $[a,b]$ and $\{q_j(\lambda)\}$ converges uniformly to $g(\lambda)$ there, then $\{p_j(\lambda)+q_j(\lambda)\}$ converges uniformly to $\{f(\lambda)+g(\lambda)\}$ and $\{p_j(\lambda)q_j(\lambda)\}$ converges uniformly to $\{f(\lambda)g(\lambda)\}$ there. Consequently, $p_j(A)+q_j(A)$ converges in $B(H)$ to $f(A)+g(A)$, and $p_j(A)q_j(A)$ converges in $B(H)$ to $f(A)g(A)$. □

6. $f(A)$ commutes with any operator in $B(H)$ commuting with A.

Proof. Any such operator commutes with each $E(\lambda)$ (Lemma 13.9). □

7. If $f(\lambda)\ge 0$ for $a\le\lambda\le b$, then $f(A)\ge 0$.

8. $\|f(A)u\|^2=\int_a^b f(\lambda)^2d(E(\lambda)u,u)$, where the right hand side is a Riemann-Stieltjes integral.

Proof. That the integral exists is obvious, since $(E(\lambda)u, u)$ is a nondecreasing function of λ. Now,

$$\begin{aligned}
&\|f(A)u\|^2 \\
&= \lim(\sum f(\lambda_k')[E(\lambda_k) - E(\lambda_{k-1})]u, \sum f(\lambda_j')[E(\lambda_j) - E(\lambda_{j-1})]u) \\
&= \lim \sum f(\lambda_k')^2([E(\lambda_k) - E(\lambda_{k-1})]u, u) \\
&= \int_a^b f(\lambda)^2 d(E(\lambda)u, w)
\end{aligned}$$

by the statement following (13.62). □

9.$\|f(A)\| \leq \max_{a \leq \lambda \leq b} |f(\lambda)|$.

Proof. By Remark 8,

$$\begin{aligned}
\|f(A)u\|^2 &\leq \max_{a \leq \lambda \leq b} f(\lambda)^2 \lim \sum([E(\lambda_k) - E(\lambda_{k-1})]u, u) \\
&\leq \left(\max_{a \leq \lambda \leq b} |f(\lambda)| \right)^2 \|u\|^2.
\end{aligned}$$

□

10. If $a \leq \alpha < \beta \leq b$ and $E(\alpha) = E(\beta)$, then

$$f(A) = \int_a^\alpha f(\lambda) dE(\lambda) + \int_\beta^b f(\lambda) dE(\lambda).$$

Proof. Just take α and β as partition points in the definition of the integrals. □

11. If $\alpha < \lambda_0 < \beta$ and $E(\alpha) = E(\beta)$, then $\lambda_0 \in \rho(A)$.

Proof. Choose $a < m$ and $b \geq M$ so that $a < \alpha < \lambda_0 < \beta < b$. Set $g(\lambda) = 1/(\lambda - \lambda_0)$ in $[a, \alpha]$ and $[\beta, b]$, and define it in $[\alpha, \beta]$ in such a way that it is continuous in $[a, b]$. Then $g(A) \in B(H)$ and

$$\begin{aligned}
g(A)(A - \lambda_0) &= \int_a^b g(\lambda)(\lambda - \lambda_0) dE(\lambda) \\
&= \int_a^\alpha dE(\lambda) + \int_\beta^b dE(\lambda) \\
&= \int_a^b dE(\lambda) = I.
\end{aligned}$$

□

12. If λ_0 is a real point in $\rho(A)$, then there is an $\alpha < \lambda_0$ and a $\beta > \lambda_0$ such that $E(\alpha) = E(\beta)$.

Proof. If the conclusion were not true, there would be sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\lambda_0 > \alpha_n \to \lambda_0$, $\lambda_0 < \beta_n \to \lambda_0$ and $E(\alpha_n) \neq E(\beta_n)$. Thus, there would be a $u_n \in H$ such that $\|u_n\| = 1$, $E(\alpha_n)u_n = 0$, $E(\beta_n)u_n = u_n$ (just take $u_n \in R[E(\beta_n)] \cap R[E(\alpha_n)]^\perp$). This would mean

$$\begin{aligned}\|(A - \lambda_0)u_n\|^2 &= \int_{\alpha_n}^{\beta_n} (\lambda - \lambda_0)^2 d(E(\lambda)u_n, u_n)\\ &\leq (\beta_n - \alpha_n)^2\|u_n\|^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

But this would imply that $\lambda_0 \in \sigma(A)$, contrary to assumption. This completes the proof. $\square$

13.6. Unbounded selfadjoint operators

The spectral theorem (Theorem 13.13) is very useful for bounded selfadjoint operators. Is there a counterpart for unbounded selfadjoint operators? What would it say? To guess, let us propose a candidate for such a "theorem" using Theorem 13.13 as a model. Firstly, we have to adjust the definitions of M and m to read

$$m = \inf_{u \in D(A), \|u\|=1} (Au, u), \quad M = \sup_{u \in D(A), \|u\|=1} (Au, u),$$

and we have to realize that we may have $m = -\infty$ or $M = \infty$, or both. Secondly, (4) of that theorem cannot hold if $D(A)$ is not the whole of H. Thirdly, the counterpart of (13.44) would be

$$p(A) = \int_{-\infty}^{\infty} p(\lambda) dE(\lambda),$$

and we have to define such an integral.

In dealing with a statement to replace (4) of Theorem 13.13, it is convenient to use the following notation. We say that an operator T is an *extension* of an operator S and write $S \subset T$ if $D(S) \subset D(T)$ and

$$Su = Tu, \quad u \in D(S).$$

While we cannot expect statement (4) to hold in general, we would like a statement such as

$$E(\lambda)A \subset AE(\lambda). \tag{13.70}$$

Even if we are willing to replace (4) with (13.70), it is not clear how to find such a family $\{E(\lambda)\}$. One attack is based on the following approach.

Suppose we can find a sequence $\{H_n\}$ of closed subspaces of H which are pairwise orthogonal and span the entire space H. By this we mean

$$(x, y) = 0, \quad x \in H_m,\ y \in H_n,\ m \neq n,$$

and for each $u \in H$ there is a sequence $\{u_1, u_2, \cdots\}$ with $u_j \in H_j$ and

$$\sum_{j=1}^{n} u_j \to u \text{ as } j \to \infty.$$

Note that the u_j are unique, for if $u_j' \in H_j$ and

$$\sum_{j=1}^{n} u_j' \to u \text{ as } n \to \infty,$$

then

$$\sum_{j=1}^{n} (u_j - u_j') \to 0 \text{ as } n \to \infty.$$

Since $u_j - u_j' \perp u_m - u_m'$ for $j \neq m$, we have, for each m,

$$\|u_m - u_m'\|^2 = \left(\sum_{j=1}^{n} (u_j - u_j'), u_m - u_m' \right) \to 0 \text{ as } n \to \infty,$$

showing that $u_m = u_m'$. Assume that for each n we are given a bounded, selfadjoint operator A_n on H_n. Then each A_n has a spectral resolution $\{E_n(\lambda)\}$ of the identity on H_n by Theorem 13.13. Define

$$Au = \sum_{i=1}^{\infty} A_i u_i \tag{13.71}$$

when

$$u = \sum_{i=1}^{\infty} u_i. \tag{13.72}$$

Without further assumptions, there is no reason to believe that the series (13.71) will converge in H. We define $D(A)$ to be the set of those $u \in H$ such that

$$\sum_{i=1}^{\infty} \|A_i u_i\|^2 < \infty. \tag{13.73}$$

Clearly, $D(A)$ is a linear subspace of H. It is dense since it contains all elements of the form

$$\sum_{i=1}^{n} u_i.$$

Moreover, the operator A is symmetric. To show this, we note that if

$$u = \sum_{i=1}^{\infty} u_i, \ v = \sum_{i=1}^{\infty} v_i \in D(A),$$

then

$$(Au, v) = \sum_{i=1}^{\infty} (A_i u_i, v_i) = \sum_{i=1}^{\infty} (u_i, A_i v_i) = (u, Av).$$

It is also selfadjoint. We show this by noting that if

$$u = \sum_{i=1}^{\infty} u_i, \ f = \sum_{i=1}^{\infty} f_i \in H$$

and

$$(u, Av) = (f, v), \quad v \in D(A),$$

then

$$\sum_{i=1}^{\infty} (u_i, A_i v_i) = \sum_{i=1}^{\infty} (f_i, v_i).$$

Take $v_i = 0$ for $i \neq m, \ v_m = v$. Then

$$(u_m, A_m v) = (f_m, v), \quad v \in H_m.$$

Since A_m is selfadjoint on H_m, we see that $A_m u_m = f_m$. Consequently,

$$\sum_{m=1}^{\infty} \|A_m u_m\|^2 = \sum_{m=1}^{\infty} \|f_m\|^2 < \infty,$$

and

$$Au = \sum_{i=1}^{\infty} A_i u_i = \sum_{i=1}^{\infty} f_i = f.$$

Note that A reduces to A_n on H_n. This follows from the fact that if $u_i = 0$ for $i \neq n$, then

$$Au = A_n u_n.$$

It is the only selfadjoint operator that does this. To see this, let $\tilde{A}$ be selfadjoint on H satisfying $\tilde{A} u_i = A_i u_i, \ u_i \in H_i$. If $u \in D(A)$ (i.e., it satisfies (13.72) and (13.73)), then

$$\| \sum_{m}^{n} A_i u_i \|^2 = \sum_{m}^{n} \|A_i u_i\|^2 \to 0 \ \text{ as } \ m, n \to \infty.$$

Consequently,

$$\tilde{u}_n = \sum_{i=1}^{n} u_i \to u, \quad \tilde{A} \tilde{u}_n = \sum_{i=1}^{n} A_i u_i \to Au.$$

Thus,

$$\begin{aligned}(\tilde{u}_n, \tilde{A}v) &= \sum_{i=1}^{n} (u_i, A_i v_i) \\ &= \sum_{i=1}^{n} (A_i u_i, v_i) \\ &= \left(\sum_{i=1}^{n} A_i u_i, v \right) \\ &= (\tilde{A}\tilde{u}_n, v)\end{aligned}$$

for all $v \in D(\tilde{A})$. In the limit, this gives

$$(u, \tilde{A}v) = (Au, v), \quad v \in D(\tilde{A}).$$

Since $\tilde{A}$ is selfadjoint, we have $u \in D(\tilde{A})$ and $\tilde{A}u = Au$. Thus $D(A) \subset D(\tilde{A})$ and

$$\tilde{A}u = Au, \quad u \in D(A).$$

But if $u \in D(\tilde{A})$, then

$$(u, \tilde{A}v) = (\tilde{A}u, v), \quad v \in D(\tilde{A}),$$

and consequently,

$$(u, Av) = (\tilde{A}u, v), \quad v \in D(A).$$

This implies that $u \in D(A)$ and $Au = \tilde{A}u$. Hence, $D(\tilde{A}) = D(A)$ and $\tilde{A} = A$.

Next, suppose A is a selfadjoint operator on H, and suppose that there is a sequence $\{H_n\}$ of pairwise orthogonal closed subspaces that span the whole of H such that $H_n \subset D(A)$ for each n, and the restriction A_n of A to H_n is a bounded, selfadjoint operator on H_n. Then each restriction A_n has a spectral family $\{E_n(\lambda)\}$ on H_n by Theorem 13.13. Since $E_n(\lambda)$ is selfadjoint on H_n for each fixed $\lambda \in \mathbb{R}$, it follows, by what we have shown, that for each λ there is a unique selfadjoint operator $E(\lambda)$ such that

$$E(\lambda) = \sum_{n=1}^{\infty} E_n(\lambda) u_n,$$

when u satisfies (13.72). From the fact that each $E_n(\lambda)$ is a projection and satisfies (1)-(3) of Theorem 13.13, it follows that $E(\lambda)$ has the same properties if we replace (3) by

$$(3') \qquad E(\lambda)u \to \begin{cases} 0 & \text{as } \lambda \to -\infty; \\ u & \text{as } \lambda \to \infty, \end{cases}$$

for $u \in D(E(\lambda))$. To show this, let $\varepsilon > 0$ be given. If $u \in D(E(\lambda))$ satisfies (13.72), take n so large that

$$\sum_n^\infty \|u_k\|^2 < \varepsilon^2,$$

and N so large that

$$\sum_1^n \|E_k(\lambda)u_k\| < \varepsilon, \quad \lambda < -N.$$

Then we have

$$\begin{aligned} \|E(\lambda)u\| &\leq \|\sum_1^{n-1} E_k(\lambda)u_k\| + \|\sum_n^\infty E_k(\lambda)u_k\| \\ &\leq \sum_1^{n-1} \|E_k(\lambda)u_k\| + \left(\sum_n^\infty \|u_k\|^2\right)^{1/2} < 2\varepsilon. \end{aligned}$$

This proves the first statement in (3′); the second is proved similarly.

We also note that (13.70) holds. If $u \in D(A)$ and satisfies (13.72), then

$$E(\lambda)Au_i = A_iE_i(\lambda)u_i,$$

and

$$\sum_1^\infty \|A_iE_i(\lambda)u_i\|^2 = \sum_1^\infty \|E_i(\lambda)A_iu_i\|^2 \leq \sum_1^\infty \|A_iu_i\|^2 < \infty.$$

Thus, $u \in D(AE(\lambda))$ and (13.70) holds.

The construction of the spectral family for an arbitrary selfadjoint operator A is based on the existence of a sequence $\{H_n\}$ of pairwise orthogonal closed subspaces that span the whole of H such that $H_n \subset D(A)$ for each n, and the restriction A_n of A to H_n is a bounded, selfadjoint operator on H_n. We now demonstrate the existence of such a sequence.

Let A be an arbitrary selfadjoint operator on H. We note that for each $f \in H$, there is a unique $x \in D(A)$ such that

$$(x, y) + (Ax, Ay) = (f, y), \quad y \in D(A). \tag{13.74}$$

To see this, let $[x, y]$ denote the left hand side of (13.74). It is a scalar product on $D(A)$ and converts it into a Hilbert space. Moreover, $F(y) = (y, f)$, $y \in D(A)$, is a bounded linear functional on this Hilbert space. Hence, there is a unique element $x \in D(A)$ such that $F(y) = [y, x]$. This gives (13.74). We write $x = Bf$ and note that $B \in B(H, D(A))$, since

$$[Bf, Bf] = (f, Bf) \leq \|f\| \cdot \|Bf\|,$$

and

$$\{\|Bf\|^2 + \|ABf\|^2\}^{\frac{1}{2}} \leq \|f\|, \quad f \in H.$$

Note also that B is symmetric on H, for if $x = Bf,\ y = Bg$, then

$$[x, y] = [Bf, Bg] = (f, Bg) = (Bf, g).$$

Also, if $x = Bf$, then

$$(Ax, Ay) = (f - x, y), \quad y \in D(A).$$

Consequently, $Ax \in D(A)$ and $A^2x = f - x$. Thus,

$$(I + A^2)B = I. \tag{13.75}$$

Conversely, if $x \in D(A^2)$ and

$$(I + A^2)x = f,$$

Then

$$[x, y] = (f, y), \quad y \in D(A).$$

This means that $x = Bf$. Hence,

$$B(I + A^2) \subset I. \tag{13.76}$$

Applying both sides to AB, we obtain

$$B(I + A^2)AB \subset AB.$$

But the left hand side equals

$$BA(I + A^2)B = BA,$$

in view of (13.75). Thus, we have

$$BA \subset AB. \tag{13.77}$$

Take $C = AB$. Then

$$BC = BAB \subset ABB = CB,$$

and since BC is defined everywhere, we obtain

$$BC = CB.$$

Since B is symmetric and bounded (and consequently selfadjoint) and satisfies

$$0 \leq B \leq I,$$

there is a spectral family $\{F(\lambda)\}$ such that

$$B = \int_0^1 \lambda dF(\lambda).$$

Since $B^{-1} \in B(H)$, we see that $F(\lambda)$ is continuous at $\lambda = 0$ and satisfies

$$F(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ I, & \lambda \geq 1. \end{cases}$$

We obtain a pairwise orthogonal sequence of subspaces as follows. Consider the sequence of projections

$$P_n = F(1/n) - F(1/(n+1)), \quad n = 1, 2, \cdots .$$

They are pairwise orthogonal and satisfy

$$\sum_{k=1}^{n} P_k \to I \text{ as } n \to \infty.$$

We take $H_n = R(P_n)$. Then $\{H_n\}$ is a pairwise orthogonal sequence of closed subspaces that span H. Let

$$s_n(\lambda) = \begin{cases} \dfrac{1}{\lambda}, & \dfrac{1}{n+1} < \lambda \leq \dfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\lambda s_n(\lambda) = 1$ when

$$\frac{1}{n+1} < \lambda \leq \frac{1}{n}$$

and $\lambda s_n(\lambda) = 0$ elsewhere, we see that

$$Bs_n(B) = s_n(B)B = P_n.$$

Moreover, since C is bounded and commutes with B, it commutes with $F(\lambda)$, P_n and $s_n(B)$. Thus,

$$AP_n = ABs_n(B) = Cs_n(B),$$

and

$$P_nA = s_n(B)BA \subset s_n(B)AB = s_n(B)C.$$

Hence, $H_n \subset D(A)$ for each n, and the restriction $A_n = AP_n$ of A to H_n is bounded, defined everywhere and selfadjoint on H_n. Once we know that such a sequence $\{H_n\}$ exists, we can construct the spectral family $\{E(\lambda)\}$ as above.

Let $\{\lambda_k\}, k = 0, \pm 1, \pm 2, \cdots$, be a sequence of numbers such that $\lambda_{k-1} < \lambda_k$ and

$$\lambda_k \to \pm\infty \text{ as } k \to \pm\infty.$$

Set

$$H_k = [E(\lambda_k) - E(\lambda_{k-1})]H$$

and

$$L_k = \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda)dE(\lambda),$$

where $p(t)$ is a polynomial with real coefficients. Then L_k is a bounded selfadjoint operator on H_k, and it coincides with $p(A)$ there. Thus, $L_k =$

$p(A_k)$, where A_k is the restriction of A to H_k. Let L denote the selfadjoint operator which reduces to L_k on H_k. Thus, $u \in D(L)$ if

$$u = \sum_{-\infty}^{\infty} u_k, \quad u_k = [E(\lambda_k) - E(\lambda_{k-1})]u \in H_k,$$

and

$$\sum_{-\infty}^{\infty} \|L_k u_k\|^2 < \infty.$$

This means that

$$\begin{aligned}\sum_{-\infty}^{\infty} \| \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda) dE(\lambda) u_k\|^2 &= \sum_{-\infty}^{\infty} \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda)^2 d\|E(\lambda)u\|^2 \\ &= \int_{-\infty}^{\infty} p(\lambda)^2 d\|E(\lambda)u\|^2 < \infty,\end{aligned}$$

since

$$E(\lambda)u_k = [E(\lambda) - E(\lambda_{k-1})]u, \quad \lambda_{k-1} \leq \lambda \leq \lambda_k.$$

Consequently,

$$\begin{aligned}Lu = \sum L_k u_k &= \sum \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda) dE(\lambda) u_k \\ &= \sum \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda) dE(\lambda) u \\ &= \int_{-\infty}^{\infty} p(\lambda) dE(\lambda).\end{aligned}$$

Finally, we note that $L = p(A)$. For they are both selfadjoint operators whose restrictions to each H_k are bounded and selfadjoint, and they coincide on each H_k. Since there can be only one selfadjoint operator with such properties, they must be equal.

In summary, we have

Theorem 13.14. *Let A be a selfadjoint operator on H. Then there is a family $\{E(\lambda)\}$ of orthogonal projection operators on H satisfying (1) and (2) of Theorem 13.13 and*

$$(3') \qquad E(\lambda) \to \begin{cases} 0 & \text{as } \lambda \to -\infty \\ I & \text{as } \lambda \to +\infty \end{cases}$$

$$(4') \qquad E(\lambda)A \subset AE(\lambda)$$

$$(5') \qquad p(A) = \int_{-\infty}^{\infty} p(\lambda) dE(\lambda)$$

for any polynomial $p(t)$.

13.7. Problems

(1) Let A be a closed, densely defined linear operator on H. Show that $-1 \in \rho(A^*A)$ and $\|(1+A^*A)^{-1}\| \leq 1$.

(2) Let A be a densely defined, selfadjoint operator on H. Show that

$$B = (A-i)(A+i)^{-1}$$

is unitary.

(3) Let $B \in B(H)$ be a unitary operator on H such that $B - I$ is one-to-one. Show that

$$A = i(I+B)(I-B)^{-1}$$

is selfadjoint.

(4) In the notation of Lemma 13.2.6, show that $M_1 \perp M_2$ if and only if $E_1E_2 = 0$.

(5) If E_1 and E_2 are orthogonal projections, show that $E_1 - E_2$ is an orthogonal projection if and only if $E_1 \geq E_2$.

(6) Let A be a selfadjoint operator in $B(H)$, and let m and M be defined as in Theorem 13.13. Show that both m and M are in $\sigma(A)$.

(7) Let $\{E(\lambda)\}$ be a resolution of the identity corresponding to a selfadjoint operator A. Show that for each $u, v \in H$, the function $(E(\lambda)u, v)$ is of bounded variation in the interval $[m, M]$.

(8) Let A be a normal operator in $B(H)$. Show that $A = SU = US$, where $S \geq 0$ and U is unitary.

(9) If $m > 0$, show that

$$A^{1/2} = \int_0^M \lambda^{1/2} dE(\lambda).$$

(10) If A is selfadjoint and $f(\lambda)$ is continuous, show that

$$\|f(A)\| = \max_{\lambda\in\sigma(A)} |f(\lambda)|.$$

(11) Prove the second statement in $(3')$.

(12) If A is a positive selfadjoint operator, show that

$$(A-I)(A+I)^{-1}$$

is a bounded selfadjoint operator satisfying $\|B\| \le 1$.

Chapter 14

MEASURES OF OPERATORS

We have encountered many instances in which a number associated with an operator will reveal important information concerning the operator. For instance, the norm of an operator will tell us much about it. The index of a Fredholm operator is very useful. We call any number associated with an operator a *measure* of that operator. In this chapter we shall discuss several measures associated with operators which provide much useful information with regard to them. Throughout this chapter, X, Y, Z will denote infinite dimensional Banach spaces, and M, N, V, W will denote infinite dimensional, closed subspaces, unless otherwise designated.

14.1. A seminorm

A very useful seminorm is given by

$$\|A\|_m = \inf_{\dim M^\circ < \infty} \|A|_M\|.$$

It is easily checked that it is indeed a seminorm. Clearly, it satisfies $\|A\|_m \leq \|A\|$. We call $\dim M^\circ$ the *codimension* of the subspace M. We also have

Lemma 14.1. *If $A \in B(X, Y)$ and $B \in B(Y, Z)$, then*

$$\|BA\|_m \leq \|B\| \cdot \|A\|_m. \tag{14.1}$$

Proof. Let $\varepsilon > 0$ be given. Then there is a subspace M of X having finite codimension such that

$$\|Ax\| \leq (\|A\|_m + \varepsilon)\|x\|, \quad x \in M.$$

Thus,

$$\|BAx\| \leq \|B\| \cdot \|Ax\| \leq (\|A\|_m + \varepsilon)\|x\|, \quad x \in M.$$

Hence,

$$\|BA\|_m \leq \|B\|(\|A\|_m + \varepsilon).$$

Since ε was arbitrary, the lemma is proved. $\square$

We also have

Theorem 14.2. $\|A\|_m = 0$ *if and only if* $A \in K(X,Y)$.

Proof. Assume $A \in K(X,Y)$, and let $\varepsilon > 0$ be given. Since the image of $\|x\| \leq 1$ is relatively compact, it is totally bounded (Theorem 4.17). Hence, there are elements $y_1, \cdots, y_n \in Y$ such that

$$\min_k \|Ax - y_k\| < \varepsilon, \quad \|x\| \leq 1. \tag{14.2}$$

Let $y_1', \cdots, y_n'$ be functionals in Y' such that

$$\|y_k'\| = 1,\ y_k'(y_k) = \|y_k\|, \quad 1 \leq k \leq n \tag{14.3}$$

(see Theorem 2.9), and let M be the set of all $x \in X$ that are annihilated by $A'y_1', \cdots, A'y_n'$, i.e., $M = {}^\circ[A'y_1', \cdots, A'y_n']$. The subspace M has finite codimension in X by definition. Let x be any element in M satisfying $\|x\| \leq 1$, and let y_k be an element of $y_1, \cdots, y_n$ closest to Ax. Since $x \in M$, we have

$$y_k'(Ax) = A'y_k'(x) = 0.$$

Consequently,

$$\|y_k\| = y_k'(y_k) = y_k'(y_k - Ax) \leq \|y_k - Ax\| < \varepsilon. \tag{14.4}$$

Since

$$\|Ax\| \leq \|Ax - y_k\| + \|y_k\| < 2\varepsilon,$$

we can conclude that

$$\|A\|_m \leq \|A|_M\| \leq 2\varepsilon.$$

Since ε was arbitrary, we see that $\|A\|_m = 0$.

Conversely, assume that $\|A\|_m = 0$. Then there is a subspace M of finite codimension such that

$$\|Ax\| \leq \varepsilon\|x\|, \quad x \in M. \tag{14.5}$$

Let P be a bounded projection onto M. Then $I - P$ is an operator of finite rank on X. Thus, for $x \in X$,

$$\begin{aligned} \|Ax\| &\leq \|APx\| + \|A(I-P)x\| \\ &\leq \varepsilon\|Px\| + \|A\| \cdot \|(I-P)x\| \\ &\leq \varepsilon\|x\| + (\|A\| + \varepsilon)\|(I-P)x\|. \end{aligned}$$

Since $I-P$ is compact, there are elements $x_1,\cdots,x_n$ of the set $\|x\|\le 1$ such that

$$\min_k \|(I-P)(x-x_k)\| < \varepsilon/(2\|A\|+\varepsilon), \quad \|x\|\le 1. \tag{14.6}$$

Now, let x be any element of X such that $\|x\|\le 1$, and let x_k be a member of $x_1,\cdots,x_n$ satisfying (14.6). Then by (14.6),

$$\|A(x-x_k)\| \le \varepsilon\|x-x_k\|+(2\|A\|+\varepsilon)\|(I-P)(x-x_k)\| \le \varepsilon\|x-x_k\|+\varepsilon \le 3\varepsilon.$$

Hence, the elements $Ax_1,\cdots,Ax_n$ form an ε-net for the image of $\|x\|\le 1$. By Theorem 4.17, this image is relatively compact. Consequently, $A\in K(X,Y)$. $\square$

Corollary 14.3. *For each A in $B(X,Y)$,*

$$\|A+K\|_m = \|A\|_m, \quad K\in K(X,Y). \tag{14.7}$$

Proof. Apply Theorem 14.2. $\square$

We also have

Theorem 14.4. *An operator A is in $\Phi_+(X,Y)$ with $i(A)\le 0$ if and only if there are an operator $K\in K(X,Y)$ and a constant C such that*

$$\|x\| \le C\|(A-K)x\|, \quad x\in D(A). \tag{14.8}$$

Proof. If (14.8) holds, then we know that $R(A-K)$ is closed in Y and that $A-K$ is one-to-one (Theorem 3.12). Thus $A-K\in\Phi_+(X,Y)$ with $\alpha(A-K)=0$. Since $\beta(A-K)\ge 0$, we see that $i(A-K)\le 0$. This implies that $A\in\Phi_+(X,Y)$ and $i(A)\le 0$ (Theorem 5.24).

Conversely, assume that $A\in\Phi_+(X,Y)$ and $i(A)\le 0$. Let $x_1,\cdots,x_n$ be a basis for $N(A)$, where $n=\alpha(A)$. Let $x_1',\cdots,x_n'$ be functionals in X' such that

$$x_j'(x_k)=\delta_{jk}, \quad 1\le j,k\le n$$

(cf. Lemma 4.14). Since $i(A)\le 0$, there is an n-dimensional subspace $Y_0\subset Y$ such that $R(A)\cap Y_0=\{0\}$ (Lemma 5.3). Let $y_1,\cdots,y_n$ be a basis for Y_0, and set

$$Kx=\sum_{j=1}^n x_j'(x)y_j.$$

Since K is bounded and of finite rank, it is compact. The operator $A-K$ is in $\Phi_+(X,Y)$. For $x\in N(A-K)$, we have

$$Ax=\sum_{j=1}^n x_j'(x)y_j.$$

This can happen only if $x \in N(A)$ and $x'_j(x) = 0$ for $1 \le j \le n$. But then

$$x = \sum_{k=1}^{n} \alpha_k x_k$$

and $x'_j(x) = \alpha_j = 0$, showing that $x = 0$. Thus, $\alpha(A-K) = 0$. Consequently, inequality (14.8) holds (Theorem 3.12). $\square$

Next we have

Theorem 14.5. *An operator A in $B(X,Y)$ is in $\Phi_+(X,Y)$ if and only if for each Banach space Z there is a constant C such that*

$$\|T\|_m \le C\|AT\|_m, \quad T \in B(Z,X). \tag{14.9}$$

Proof. If $A \in \Phi(X,Y)$, let A_0 be the operator satisfying Theorem 5.4. Then $F = A_0A - I$ is in $K(X)$. Thus, for $T \in B(Z,X)$, we have

$$T = A_0AT - FT.$$

In view of Theorem 14.2, this implies

$$\|T\|_m = \|A_0AT - FT\|_m = \|A_0AT\|_m \le \|A_0\| \cdot \|AT\|_m,$$

which is (14.9).

If $A \in \Phi_+(X,Y)$ but not in $\Phi(X,Y)$, then $\beta(A) = \infty$. In particular, $i(A) \le 0$. By Theorem 14.4, there is an operator $K \in K(X,Y)$ such that inequality (14.8) holds. Thus, if M is a subspace of Z with finite codimension, then

$$\|Tz\| \le C\|(A-K)Tz\|, \quad z \in M.$$

Now for any $\varepsilon > 0$, there is such a subspace satisfying

$$\|(A-K)Tz\| \le (\|(A-K)T\|_m + \varepsilon)\|z\|, \quad z \in M.$$

Consequently,

$$\|Tz\| \le C(\|(A-K)T\|_m + \varepsilon)\|z\|, \quad z \in M.$$

This implies

$$\|T\|_m \le C(\|(A-K)T\|_m + \varepsilon),$$

and since ε was arbitrary, we see that

$$\|T\|_m \le C\|(A-K)T\|_m = C\|AT\|_m$$

by Theorem 14.2. This proves (14.9).

Conversely, assume that for each Banach space Z, there is a constant C such that (14.9) holds. If $A \notin \Phi_+(X,Y)$, then there is a $K \in K(X,Y)$ such that $\alpha(A-K) = \infty$ (Theorem 9.43). Take $Z = N(A-K)$ with the norm of X, and let T be the imbedding of Z in X. Then the operator T is in $B(Z, X)$. Since Z is infinite dimensional, $\|T\|_m \neq 0$. But $(A - K)T = 0$, and consequently,

$$\|AT\|_m = \|(A - K)T\|_m = 0.$$

This clearly violates (14.9), and the proof is complete. □

It should be noted that the constant C in (14.9) depends only on A and not on the space Z. Thus, the proof of Theorem 14.5 implies that if for each Z there is a constant C depending also on Z, such that (14.9) holds, then there is a constant C independent of Z for which it holds.

14.2. Perturbation classes

Let X be a complex Banach space, and let S be a subset of X. We denote by $P(S)$ the set of elements of X that perturb S into itself, i.e., $P(S)$ is the set of those elements $a \in X$ such that $a + s \in S$ for all $s \in S$. We shall assume throughout that S satisfies

$$\alpha S \subset S, \quad \alpha \neq 0, \tag{14.10}$$

i.e., $\alpha s \in S$ whenever $s \in S$ and $\alpha \neq 0$. First we have

Lemma 14.6. *$P(S)$ is a linear subspace of X. If, in addition, S is an open subset of X, then $P(S)$ is closed.*

Proof. Suppose $s \in S,\ a, b \in P(S),\ \alpha \neq 0$. Then $\alpha a + s = \alpha(a + s/\alpha) \in S$ and $(a+b)+s = a+(b+s) \in S$. Thus, $P(S)$ is a subspace. Assume S open. Then for each $s \in S$, there is a $\delta > 0$ such that $\|q - s\| < \delta$ implies that $q \in S$. If $\{x_k\}$ is a sequence of elements in $P(S)$ converging to an element $x \in X$, then for k sufficiently large, $\|x_k - x\| < \delta$. Thus $s + x - x_k \in S$. Since $x_k \in P(S),\ s + x \in S$. Thus, $x \in P(S)$, and the lemma is proved. □

We also have

Lemma 14.7. *Let S, T be subsets of X which satisfy (14.10). Assume that S is open, that $S \subset T$ and that T does not contain any boundary points of S. Then $P(T) \subset P(S)$.*

Proof. Suppose $s \in S,\ b \in P(T)$. Then

$$\alpha b + s = \alpha(b + s/\alpha) \in T, \quad \alpha \neq 0.$$

Since S is open, $\alpha b + s \in S$ for $|\alpha|$ sufficiently small. It follows that $\alpha b + s \in S$ for all scalars α. Otherwise for some α_0 the element $\alpha_0 b + s$ would be a boundary point of S which is in T. Thus $b + s \in S$, and the proof is complete. □

Next, let B be a Banach algebra with identity e, and let G be the set of regular elements in B. A subspace M of B is called a *right ideal* if $xa \in M$ for $x \in M,\ a \in B$. It is called a *left ideal* if $ax \in M$ when $x \in M,\ a \in B$. If it is both a right and left ideal, it is called a *two-sided ideal* or merely an *ideal.* We have

Lemma 14.8. *If $GS \subset S$, then $P(S)$ is a left ideal. If $SG \subset S$, then $P(S)$ is a right ideal.*

Proof. Suppose $a \in G,\ b \in P(S),\ s \in S$. Then

$$ab + s = a(b + a^{-1}s) \in S.$$

Consequently, $ab \in P(S)$. Since every element of B is the sum of two elements of G and $P(S)$ is a subspace of B, the first statement follows. The second statement is proved in the same way. □

In summary, we have

Theorem 14.9. *If S is an open subset of B which satisfies*

$$GS \subset S,\ SG \subset S, \tag{14.11}$$

then $P(S)$ is a closed, two-sided ideal.

The *radical* R of B is defined as

$$R = \{b \in B : e + ab \in G \ \text{ for all } \ a \in G\}.$$

We have

Theorem 14.10. $P(G) = R$.

Proof. Suppose $b \in P(G)$ and $a \in G$. Then $a^{-1} + b \in G$. Hence, $e + ab = a(a^{-1} + b) \in G$. This means that $b \in R$. Conversely, suppose $b \in R$. If $a \in G$, then $e + a^{-1}b \in G$. Hence, $a + b = a(e + a^{-1}b) \in G$. Thus, $b \in P(G)$, and the proof is complete. □

Corollary 14.11. *R is a closed, two-sided ideal.*

An element $x \in B$ is called *right regular* if it has a *right inverse*, i.e., an element $y \in B$ such that $xy = e$. It is called *left regular* if it has a *left inverse*, i.e., an element $y \in B$ such that $yx = e$. If it is both right and left regular, then its right and left inverses coincide and it is regular in the sense defined above in Section 9.1. Let G_r denote the set of right regular elements of B, and let G_ℓ denote the set of left regular elements. We have

Theorem 14.12. $P(G_\ell) = P(G_r) = R$.

Proof. First we note that $G \subset G_\ell$ and that G_ℓ does not contain any boundary points of G. For if $a \in G_\ell$, $a_k \in G$ and $a_k \to a$, then $ba_k - e = ba_k - ba = b(a_k - a) \to 0$ as $k \to \infty$, where b is the left inverse of a. Then $ba_k = e + b(a_k - a) \in G$ for k large. Thus, $(ba_k)a_k^{-1} = b \in G$. In particular, $ab = b^{-1}bab = b^{-1}eb = e$, and $a \in G$ with $b = a^{-1}$. We can now apply Lemma 14.7 to conclude that $P(G_\ell) \subset P(G) = R$. Conversely, if $a \in G_\ell$, let b be its left inverse. If $s \in R$, then so is bs. Consequently, $e + bs \in G$, and $(e+bs)^{-1}b(a+s) = e$. Thus, $a+s \in G_\ell$, and $s \in P(G_\ell)$. This completes the proof for G_ℓ. The proof for G_r is similar. □

Let $\Phi_\ell(X)$ denote the set of those operators $A \in \Phi_+(x)$ such that $R(A)$ is complemented, and let $\Phi_r(X)$ denote the set of those $A \in \Phi_-(X)$ such that $N(A)$ is complemented. We now apply the concepts introduced above to the quotient space $B(X)/K(X)$ for a Banach space X. We have

Theorem 14.13. *$[A]$ is in G_ℓ if and only if $A \in \Phi_\ell(X)$. $[A]$ is in G_r if and only if $A \in \Phi_r(X)$.*

Proof. $[A] \in G_\ell$ if and only if there is a $A_0 \in B(X)$ such that $[A_0][A] = [I]$. Thus, $A_0A = I - K_1$, where $K_1 \in K(X)$. This is true if and only if $A \in \Phi_\ell(X)$ (Theorem 5.37). Similarly, $[A] \in G_r$ if and only if there is an $A_0 \in B(X)$ such that $[A][A_0] = [I]$. That is, $AA_0 = I - K_2$ for some $K_2 \in K(X)$. This is equivalent to $A \in \Phi_r(X)$ (Theorem 5.37). Thus, $[A] \in G$ if and only if $A \in \Phi_\ell(X) \cap \Phi_r(X) = \Phi(X)$. □

From this we get

Theorem 14.14. $P(\Phi) = P(\Phi_\ell) = P(\Phi_r) = F(X)$.

An interesting application can be given as follows. Let Z be any subset of the integers $\{0, \pm 1, \pm 2, \cdots, \pm\infty\}$, and let Φ_Z be the collection of operators $A \in \Phi_\ell \cup \Phi_r$ such that $i(A) \in Z$. We have

Theorem 14.15. *If Φ_Z is not empty, then*

$$P(\Phi_Z) = F(X). \tag{14.12}$$

Proof. Set $\tilde{\Phi} = \Phi_\ell \cup \Phi_r$. Note that Φ_Z is an open set in $B(X)$ and satisfies (14.10). Since $\Phi_Z \subset \tilde{\Phi}$ and $\tilde{\Phi}$ does not contain any boundary points of Φ_Z, we have, by Lemma 14.7, that

$$P(\Phi_Z) \supset P(\tilde{\Phi}) = F(X).$$

This proves (14.12) in one direction. Conversely, since Φ_Z is open and satisfies (14.11), we see from Theorem 14.9 that $P(\Phi_Z)$ is a closed, two-sided ideal. Let A be any operator in Φ, and let $A_0 \in B(X)$ be such that

$$[A_0A] = [AA_0] = [I].$$

By hypothesis, there is an operator in $A_1 \in \Phi_Z$. For definiteness, assume that $A_1 \in \Phi_\ell$. Let E be any operator in $P(\Phi_Z)$. Since $P(\Phi_Z)$ is an ideal, $\lambda A_1 A_0 E \in P(\Phi_Z)$ for each scalar λ. Thus,

$$A_1(I + \lambda A_0 E) = A_1 + \lambda A_1 A_0 E \in \Phi_Z \subset \tilde{\Phi}$$

for each λ. By the constancy of the index (Theorem 5.11), $A_1(I + \lambda A_0 E)$ must be in Φ_ℓ. Thus, for each λ there is a $B_\lambda \in B(X)$ such that

$$[B_\lambda A_1(I + \lambda A_0 E)] = [I].$$

This shows that $I + \lambda A_0 E \in \Phi_\ell$ for each λ. Again, by the constancy of the index, we see that $I + \lambda A_0 E \in \Phi$ for each λ. Since $A \in \Phi$, the same is true of $A(I + \lambda A_0 E) = A - \lambda KE + \lambda E$, showing that $A + E \in \Phi$. Thus, $E \in F(X)$, and the proof is complete. □

14.3. Related measures

Let X, Y be Banach spaces with $\dim X = \infty$, and let A be an operator in $B(X, Y)$. We define

$$\Gamma(A) = \inf_M \|A|_M\|,$$

$$\Delta(A) = \sup_M \inf_{N \subset M} \|A|_N\|,$$

$$\tau(A) = \sup_M \inf_{\substack{x \in M \\ \|x\|=1}} \|Ax\|,$$

$$\nu(A) = \sup_{\dim M^\circ < \infty} \inf_{\substack{x \in M \\ \|x\|=1}} \|Ax\|,$$

where M, N represent infinite dimensional, closed subspaces of X, and $A|_M$ denotes the restriction of A to M. We shall derive certain relationships among these quantities. Then we shall show how certain inequalities imply properties of A.

Each of the quantities defined above has a counterpart on a subspace. For instance, we let

$$\Gamma_M(A) = \inf_{N \subset M} \|A|_N\|, \quad \Gamma(A) = \Gamma_X(A),$$

$$\Delta_M(A) = \sup_{N \subset M} \Gamma_N(A), \quad \Delta(A) = \Delta_X(A).$$

From the definitions we check easily that

$$\Gamma_M(A) = \inf_{N \subset M} \Gamma_N(A) = \inf_{N \subset M} \Delta_N(A), \tag{14.13}$$

$$\Delta_M(A) = \sup_{N \subset M} \Delta_N(A), \tag{14.14}$$

and

$$\Gamma_M(A) = \Delta_M(A) \leq \|A|_M\|. \tag{14.15}$$

We also have

Theorem 14.16. $\Gamma(A+B) \leq \Delta(A) + \Gamma(B)$.

Proof. First we note that

$$\Gamma_N(A+B) \leq \Gamma_N(A) + \|B|_N\|. \tag{14.16}$$

To see this, let $\varepsilon > 0$ be given. By definition, there is an infinite dimensional subspace V of N such that $\|A|_V\| < \Gamma_N(A) + \varepsilon$. Thus,

$$\|(A+B)|_V\| \leq \|A|_V\| + \|B|_V\| \leq \Gamma_N(A) + \varepsilon + \|B|_N\|.$$

This implies

$$\Gamma_N(A+B) \leq \Gamma_N(A) + \varepsilon + \|B|_N\|.$$

Since this is true for every $\varepsilon > 0$, we obtain (14.16). By (14.16) and the definition of $\Delta(A)$, we have

$$\Gamma_N(A+B) \leq \Delta(A) + \|B|_N\|.$$

Thus

$$\inf_N \Gamma_N(A+B) \leq \Delta(A) + \inf_N \|B|_N\|,$$

from which the theorem follows. □

Next, we have

Theorem 14.17. $\Delta(A+B) \leq \Delta(A) + \Delta(B)$.

Proof. For any N, we have, by Theorem 14.16,

$$\Gamma_N(A+B) \leq \Delta_N(A) + \Gamma_N(B) \leq \Delta(A) + \Delta(B).$$

The theorem now follows from the definition. □

Before we continue, we shall state a few lemmas which will be useful in the sequel.

Lemma 14.18. *If M, N are subspaces of X such that* $\dim M^\circ < \infty$ *and* $\dim N = \infty$, *then* $\dim M \cap N = \infty$.

Proof. We know that $X = M \oplus L$, where $\dim L < \infty$ (Lemma 5.3). If $\dim M \cap N < \infty$, then $M = M \cap N \oplus M_1$ and $X = M_1 \oplus N_1$, where $N_1 = M \cap N \oplus L$ and $\dim N_1 < \infty$. Moreover, $M_1 \cap N = \{0\}$ since every element in $M \cap N$ is in N_1. By Lemma 5.12, $\dim N \leq \dim N_1 < \infty$, contrary to assumption. □

Lemma 14.19. *Let M be a subspace of X having infinite codimension. Then there is an infinite dimensional subspace W of X such that $W \cap M = \{0\}$.*

Proof. Let $\{x'_k\}$ be a sequence of linearly independent functionals in M°. Then there is a sequence $\{x_k\}$ of elements of X such that

$$x'_j(x_k) = \delta_{jk}, \quad j, k = 1, 2, \cdots$$

(Lemma 5.12). The x_k are linearly independent. This follows from the fact that

$$\sum_1^n \alpha_k x_k = 0$$

implies

$$x'_j\left(\sum_1^n \alpha_k x_k\right) = \alpha_j = 0, \quad j = 1, 2, \cdots .$$

Let W be the set of linear combinations of the x_k. Then $W \cap M = \{0\}$, and $\dim W = \infty$. □

Lemma 14.20. *If M and N are subspaces of X having finite codimension in X, then $M \cap N$ has finite codimension in X.*

Proof. By Lemma 5.3 there are finite dimensional subspaces $M_1,\ N_1 \subset X$ such that

$$X = M \oplus M_1, \quad X = N \oplus N_1.$$

If $M \cap N$ did not have finite codimension in M, then there would be an infinite dimensional subspace $W \subset M$ such that $W \cap (M \cap N) = \{0\}$. But then $W \cap N = \{0\}$, and we see from Lemma 5.12 that $\dim W \leq \dim N_1 < \infty$, providing a contradiction. Thus, $M \cap N$ has finite codimension in M. Since M has finite codimension in X, we see that $M \cap N$ has finite codimension in X. □

For an operator $T \in B(X,Y)$, we let TM denote the range of $T|_M$. We have

Theorem 14.21. *If $T \in B(X,Y)$ and $A \in B(Y,Z)$, then*

$$\Gamma_M(AT) \leq \Gamma_M(T)\Delta_{TM}(A) \tag{14.17}$$

when $\dim TM = \infty$. *If* $\dim TM < \infty$, *then* $\Gamma_M(AT) = 0$.

Proof. Let $\varepsilon > 0$ be given. Then there is an $N \subset M$ such that $\|T|_N\| < \Gamma_M(T) + \varepsilon$. Thus

$$\Gamma_N(AT) = \inf_{V \subset N} \|(AT)|_V\| \leq \inf_{V \subset N} \|A|_{TV}\| \cdot \|T|_V\| \leq \|T|_N\| \inf_{V \subset N} \|A|_{TV}\|.$$

Assume that $\dim TN = \infty$, and let W be any infinite dimensional subspace of TN. Let $U = N \cap T^{-1}W$. Then U consists of those $x \in N$ such that $Tx \in W$. Thus, $\dim U = \infty$ and $TU = W$. Hence,

$$\inf_{V \subset N} \|A|_{TV}\| = \inf_{W \subset TN} \|A|_W\| = \Gamma_{TN}(A).$$

Consequently,

$$\Gamma_N(AT) \leq \Gamma_{TN}(A)\|T|_N\|. \tag{14.18}$$

If $\dim TN < \infty$, then $AT|_N$ is a finite rank operator. In particular, it vanishes on an infinite dimensional subspace of N, and consequently, $\Gamma_N(AT) = 0$. Now, by (14.18) and the choice of N, we have

$$\Gamma_N(AT) \leq \Delta_{TN}(A)[\Gamma_M(T) + \varepsilon] \leq \Delta_{TM}(A)[\Gamma_M(T) + \varepsilon].$$

Since this is true for any $\varepsilon > 0$, the result follows. $\square$

Corollary 14.22. $\Delta_M(AT) \leq \Delta_M(T)\Delta(A)$.

Proof. For $N \subset M$, we have

$$\Gamma_N(AT) \leq \Gamma_N(T)\Delta(A) \leq \Delta_M(T)\Delta(A).$$

The corollary now follows from the definition. $\square$

Corollary 14.23. $\Delta(AT) \leq \Delta(A)\Delta(T)$.

Another quantity related to Γ and Δ is

$$\tau_M(A) = \sup_{N\subset M} \inf_{x\in N,\ \|x\|=1} \|Ax\|, \quad \tau(A) = \tau_X(A). \tag{14.19}$$

We have

Theorem 14.24. $\tau(A) \leq \Delta(A)$.

Proof. Suppose $\varepsilon > 0$. Then for each N there is a $V \subset N$ such that $\|A|_V\| < \Gamma_N(A) + \varepsilon$. Since

$$\inf_{x\in N,\ \|x\|=1} \|Ax\| \leq \|A|_V\| \leq \Delta(A) + \varepsilon,$$

we have $\tau(A) \leq \Delta(A) + \varepsilon$. Since this is true for any $\varepsilon > 0$, the result follows. $\square$

We also have

Lemma 14.25. $\Gamma(A)\tau(T) \leq \|AT\|$.

Proof. By definition, for each $\varepsilon > 0$ there is a W such that

$$\|Tz\| \geq [r(T) - \varepsilon]\|z\|, \quad z \in W.$$

If $\tau(T) = 0$, the lemma is trivial. Otherwise, pick $\varepsilon < \tau(T)$. Now for $z \in W$ we have

$$\|ATz\| \leq \|AT\| \cdot \|z\| \leq \|AT\| \cdot \|Tz\|/[\tau(T) - \varepsilon].$$

Set $M = TW$. Then $\dim M = \infty$ and

$$\|Ax\| \leq \|AT\| \cdot \|x\|/[\tau(T) - \varepsilon], \quad x \in M,$$

or

$$\|A|_M\| \leq \|AT\|/[\tau(T) - \varepsilon].$$

This gives

$$\Gamma(A) \leq \|AT\|/[\tau(T) - \varepsilon].$$

Letting $\varepsilon \to 0$, we obtain the desired result. □

Also, we have

Theorem 14.26.

$$\Gamma(A) = \inf_Z \inf_{T \in B(Z,X)} \frac{\|AT\|}{\tau(T)}. \tag{14.20}$$

Proof. By Lemma 14.25, $\Gamma(A) \leq$ the right hand side of (14.20). To show that it is equal, let $\varepsilon > 0$ be given. Then there is an $M \subset X$ such that

$$\|A|_M\| < \Gamma(A) + \varepsilon.$$

We may assume that M is closed. Take $Z = M$, and let T be the embedding of M into X. Then $\tau(T) = 1$, while $AT = A|_M$. Thus, the right hand side of (14.20) is $< \Gamma(A) + \varepsilon$. Since ε was arbitrary, the result follows. □

Corollary 14.27.

$$\Gamma(A) \geq \inf_Z \inf_{T \in B(Z,X)} \frac{\|AT\|}{\Delta(T)}. \tag{14.21}$$

Proof. Apply Theorems 14.24 and 14.26. □

As before, we define

$$\|A\|_m = \inf_{\dim M^\circ < \infty} \|A|_M\|.$$

We have

Theorem 14.28. $\Delta(A) \leq \|A\|_m$.

Proof. Let $\varepsilon > 0$ be given, and let V be a subspace having finite codimension such that

$$\|A|_V\| < \|A\|_m + \varepsilon.$$

Let M be given, and set $N = M \cap V$. Then $\dim N = \infty$ and $\|A|_N\| < \|A\|_m + \varepsilon$. Since $N \subset M$, we have

$$\Gamma_M(A) < \|A\|_m + \varepsilon.$$

Since this is true for all M and $\varepsilon > 0$, the result follows. □

We also have

Theorem 14.29. *$A \in \Phi_+(X, Y)$ if and only if $\Gamma(A) \neq 0$.*

Proof. If $A \in \Phi_+(X,Y)$, then there is a constant $c > 0$ such that

$$\|AT\|_m \geq c\|T\|_m, \quad T \in B(Z,X),$$

where c does not depend on Z (Theorem 14.5). Thus,

$$\|AT\| \geq c\Delta(T), \quad T \in B(Z,X)$$

(Theorem 14.28), and consequently, $\Gamma(A) \neq 0$ (Corollary 14.27). On the other hand, if $A \notin \Phi_+(X,Y)$, then for each $\varepsilon > 0$ there is a $K \in K(X,Y)$ such that $\|K\| < \varepsilon$ and $\alpha(A-K) = \infty$ (Theorem 14.41). Let $M = N(A-K)$. Then $\|A|_M\|_m = \|K|_M\|_m = 0$. This implies that for any $\varepsilon > 0$ there is an $N \subset M$ such that $\|A|_N\| < \varepsilon$. This implies that $\Gamma(A) < \varepsilon$. Hence, $\Gamma(A) = 0$, and the proof is complete. $\square$

We now have

Theorem 14.30. *If $\Delta(B) < \Gamma(A)$, then $A+B \in \Phi_+(X,Y)$ and $i(A+B) = i(A)$.*

Proof. By Theorem 14.16,

$$\Gamma(A + \lambda B) \geq \Gamma(A) - \lambda\Gamma(B) > 0$$

for $0 \leq \lambda \leq 1$. Thus $A + \lambda B \in \Phi_+(X,Y)$ for such λ (Theorem 14.29). The rest follows from the constancy of the index (Theorem 5.11). $\square$

It is clear from the definition that

$$\nu(A) \leq \tau(A). \tag{14.22}$$

Another consequence is

Lemma 14.31. *$A \in \Phi_+(X,Y)$ if and only if $\nu(A) \neq 0$.*

We also have

Theorem 14.32. *If $\tau(B) < \nu(A)$, then $A+B \in \Phi_+(X,Y)$ and $i(A+B) = i(A)$.*

Proof. If $A+B \notin \Phi_+(X,Y)$, then there are an infinite dimensional subspace M and a $K \in K(X,Y)$ such that $(A+B-K)|_M = 0$ (Theorem 9.43). Let $\varepsilon > 0$ be given. Then there is a subspace W having finite codimension such that $\|K|_W\| < \varepsilon$ (Theorem 14.2). Let V be any subspace having finite codimension, and set $N = V \cap M \cap W$. Then $\dim N = \infty$, and

$$\|Bx\| = \|(A-K)x\| \geq \|A\| - \varepsilon, \quad x \in N, \ \|x\| = 1.$$

Thus,

$$\inf_{x \in N, \|x\|=1} \|Bx\| \geq \inf_{x \in N, \|x\|=1} \|Ax\| - \varepsilon.$$

Since this is true for each subspace V having finite codimension, we have

$$\tau(B) \geq \nu(A) - \varepsilon.$$

Since this is true for each $\varepsilon > 0$, we have $\tau(B) \geq \nu(A)$. Thus $A + B \in \Phi_+(X, Y)$.

The same reasoning shows that $A+\lambda B \in \Phi_+(X, Y)$ for $0 \leq \lambda \leq 1$. Thus, the index is constant as before. □

Corollary 14.33. *If $X = Y$ and $\tau(A) < 1$, then $I - A \in \Phi(X, Y)$ with $i(I - A) = 0$.*

Proof. Clearly $\nu(I) = 1$. Apply Theorem 14.32 and the fact that $i(I) = 0$. □

We also have

Theorem 14.34. $\nu(A) \geq \Gamma(A)$.

Proof. Let $\varepsilon > 0$ be given. Then there is an M such that

$$\|A|_M\| < \Gamma(A) + \varepsilon.$$

Let W having finite codimension be given, and set $N = W \cap M$. Then $\dim N = \infty$, and

$$\inf_{x \in W, \|x\|=1} \|Ax\| \leq \sup_{x \in N, \|x\|=1} \|Ax\| = \|A|_N\| \leq \Gamma(A) + \varepsilon.$$

Since this is true for any subspace W having finite codimension, we have $\nu(A) \leq \Gamma(A)+\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain the desired result. □

In contrast with Theorem 14.5 we have

Theorem 14.35. *An operator A in $B(X, Y)$ is in $\Phi_+(X, Y)$ if and only if there is a constant C such that for any Banach space Z, one has*

$$\Delta(T) \leq C\Delta(AT), \quad T \in B(Z, X). \tag{14.23}$$

Proof. If (14.23) holds, then

$$\tau(T) \leq \Delta(T) \leq C\Delta(AT) \leq C\|AT\|$$

by Theorem 14.24. Thus, $\Gamma(A) \neq 0$ (Theorem 14.26). Consequently, $A \in \Phi_+(X, Y)$ (Theorem 14.29). Conversely, assume that $A \in \Phi_+(X, Y)$. If $A \in \Phi(X, Y)$, then there are operators $A_0 \in B(Y, X)$ and $F \in K(X)$ such that $A_0 A = I - F$ (Theorem 5.4). Thus,

$$T = A_0 AT - FT,$$

and consequently,

$$\Delta(T) = \Delta(A_0 AT - FT) \leq \Delta(A_0)\Delta(AT)$$

by Corollaries 14.22 and 14.23. If A is not in $\Phi(X,Y)$, then $i(A) = -\infty$. In this case, there are an operator $K \in K(X,Y)$ and a constant C such that $\|x\| \leq C\|(A-K)x\|$, $x \in X$ (Lemma 14.4). This implies

$$\|Tz\| \leq C\|(A-K)Tz\|, \quad z \in Z. \tag{14.24}$$

Let W be any infinite dimensional subspace of Z and let $\varepsilon > 0$ be given. Then there is a $V \subset W$ such that

$$\|[(A-K)T]|_V\| \leq \Delta[(A-K)T] + \varepsilon.$$

Thus, by (14.24) and Corollary 14.22,

$$\|T|_V\| \leq C[\Delta(AT) + \varepsilon].$$

This means that

$$\Gamma_W(T) \leq C[\Delta(AT) + \varepsilon].$$

Since W was arbitrary, the result folows. □

14.4. Measures of noncompactness

Let X be a Banach space. For a bounded subset $\Omega \subset X$ we let $q(\Omega)$ denote the infimum (greatest lower bound) of the set of numbers r such that Ω can be covered by a collection of open spheres of radius r. In particular, $q(\Omega) = 0$ if and only if Ω is totally bounded, i.e., if and only if its closure is compact. It is for this reason that $q(\Omega)$ is sometimes called the *measure of noncompactness* of Ω. For Ω and Ψ bounded subsets of X, we let $\Omega + \Psi$ denote the set of all sums $\omega + \psi$, $\omega \in \Omega$, $\psi \in \Psi$. If Ω can be covered by n ε-spheres and Ψ can be covered by m η-spheres, then $\Omega + \Psi$ can be covered by nm $(\varepsilon + \eta)$-spheres. Thus,

$$q(\Omega + \Psi) \leq q(\Omega) + q(\Psi). \tag{14.25}$$

Next, let X, Y be Banach spaces, and let S_X denote the closed unit ball in X, i.e.,

$$S_X = \{x \in X : \|x\| \leq 1\}.$$

We define

$$\|A\|_q = q[A(S_X)], \quad A \in B(X,Y). \tag{14.26}$$

One checks easily that this is a seminorm on $B(X,Y)$ and that $\|A\|_q = 0$ if and only if $A \in K(X,Y)$. Thus we can consider (14.26) as a measure of noncompactness of A. We also have

$$\|A\|_q \leq \|A\|, \tag{14.27}$$

$$\|A+K\|_q = \|A\|_q, \quad K \in K(X,Y). \tag{14.28}$$

Moreover, if Z is a Banach space and $B \in B(Y,Z)$, then

$$\|BA\|_q \leq \|B\| \cdot \|A\|_q. \tag{14.29}$$

A close relationship between the m-seminorm and the q-seminorm is given by

Theorem 14.36. *For A in $B(X,Y)$,*

$$\|A\|_q/2 \leq \|A\|_m \leq 2\|A\|_q. \tag{14.30}$$

Proof. Let $\varepsilon > 0$ be given. Then we can find elements $y_1, \cdots, y_n \in Y$ such that

$$\min_k \|Ax - y_k\| \leq \|A\|_q + \varepsilon, \quad x \in S_X. \tag{14.31}$$

Let $y_1', \cdots, y_n'$ be functionals in Y' such that

$$\|y_k'\| = 1, \;\; y_k'(y_k) = \|y_k\|, \quad 1 \leq k \leq n, \tag{14.32}$$

and let M be the set of all $x \in X$ that are annihilated by $A'y_1', \cdots, A'y_n'$. This subspace M has finite codimension in X. Let x be any element in $M \cap S_X$, and let y_k be one of the elements $y_1, \cdots, y_n$ which is closest to Ax. Since $x \in M$, we have

$$y_k'(Ax) = A'y_k'(x) = 0.$$

Consequently,

$$\|y_k\| = y_k'(y_k) = y_k'(y_k - Ax) \leq \|y_k - Ax\|. \tag{14.33}$$

Thus, by (14.31),

$$\|y_k\| \leq \|A\|_q + \varepsilon, \quad 1 \leq k \leq n. \tag{14.34}$$

Since

$$\|Ax\| \leq \|Ax - y_k\| + \|y_k\|,$$

this gives

$$\|Ax\| \leq 2(\|A\|_q + \varepsilon), \quad x \in M \cap S_X.$$

From the definition we conclude that

$$\|A\|_m \leq 2(\|A\|_q + \varepsilon),$$

and since ε was arbitrary, we obtain the right hand inequality in (14.30).

To prove the other half, again let $\varepsilon > 0$ be given. Then there is a subspace $M \subset X$ having finite codimension such that

$$\|Ax\| \leq (\|A\|_m + \varepsilon)\|x\|, \quad x \in M. \tag{14.35}$$

Let P be a bounded projection onto M. Then $I - P$ is an operator of finite rank on X. Thus, for $x \in X$,

$$\begin{aligned} \|Ax\| &\leq \|APx\| + \|A(I-P)x\| \\ &\leq (\|A\|_m + \varepsilon)\|Px\| + \|A\| \cdot \|(I-P)x\| \\ &\leq (\|A\|_m + \varepsilon)\|x\| + (2\|A\| + \varepsilon)\|(I-P)x\|, \end{aligned} \tag{14.36}$$

where we have used the fact that

$$\|Px\| \le \|x\| + \|(I-P)x\|$$

and

$$\|A\|_m \le \|A\|. \tag{14.37}$$

Since $I - P$ is compact, there are elements $x_1, \cdots, x_n \in S_X$ such that

$$\min_k \|(I-P)(x-x_k)\| < \varepsilon/(2\|A\| + \varepsilon), \quad x \in S_X. \tag{14.38}$$

Now, let x be any element of S_X, and let x_k be a member of $x_1, \cdots, x_n$ satisfying (14.38). Then by (14.36)

$$\begin{aligned}\|A(x-x_k)\| &\le (\|A\|_m + \varepsilon)\|x - x_k\| + (2\|A\| + \varepsilon)\|(I-P)(x-x_k)\| \\ &\le 2(\|A\|_m + \varepsilon) + \varepsilon.\end{aligned}$$

Consequently,

$$\|A\|_q \le 2\|A\|_m + 3\varepsilon.$$

Since ε was arbitrary, we obtain the left hand inequality in (14.30), and the proof is complete. □

14.5. The quotient space

Another measure of noncompactness, which is more widely used, is

$$\|A\|_K = \inf_{K \in K(X,Y)} \|A - K\|. \tag{14.39}$$

It is the norm of the quotient space $B(X,Y)/K(X,Y)$, which is complete (cf. Section 3.5). By (14.27), (14.28), (14.37) and (14.7), it follows that

$$\|A\|_q \le \|A\|_K, \quad \|A\|_m \le \|A\|_K. \tag{14.40}$$

If the quotient space $B(X,Y)/K(X,Y)$ is complete with respect to the norm $\|A\|_q$ or $\|A\|_m$, then it follows from the closed graph theorem that these three norms are equivalent.

A Banach space X will be said to have the *compact approximation property with constant* C if for each $\varepsilon > 0$ and finite set of points $x_1, \cdots, x_n \in X$, there is an operator $K \in K(X)$ such that $\|I - K\| \le C$ and

$$\|x_k - Kx_k\| < \varepsilon, \quad 1 \le k \le n. \tag{14.41}$$

We have

Theorem 14.37. *If Y has the compact approximation property with constant C, then*

$$\|A\|_K \le C\|A\|_q, \quad A \in B(X,Y). \tag{14.42}$$

Proof. Let $\varepsilon > 0$ be given. Then there exist elements $y_1, \cdots, y_n \in Y$ such that (14.31) holds. By hypothesis, there exists an operator $K \in K(Y)$ such that $\|I - K\| \le C$ and

$$\|y_k - Ky_k\| < \varepsilon, \quad 1 \le k \le n. \tag{14.43}$$

Let $x \in S_X$ be given. Then there is a y_k among the $y_1, \cdots, y_n$ such that

$$\|Ax - y_k\| \le \|A\|_q + \varepsilon. \tag{14.44}$$

Thus,

$$\|(I-K)Ax\| \le \|(I-K)(Ax-y_k)\| + \|(I-K)y_k\|$$
$$\le C\|Ax - y_k\| + \varepsilon \le C(\|A\|_q + \varepsilon) + \varepsilon.$$

Since ε was arbitrary, we obtain (14.42). The proof is complete. □

Corollary 14.38. *If Y has the compact approximation property, then the space $B(X,Y)/K(X,Y)$ is complete with respect to the norms induced by $\|A\|_q$ and $\|A\|_m$.*

14.6. Strictly singular operators

An operator $S \in B(X,Y)$ is called *strictly singular* if it does not have a bounded inverse on any infinite dimensional subspace of X. Important examples of strictly singular operators are the compact operators. We have

Theorem 14.39. *Compact operators are strictly singular.*

Proof. Let K be an operator in $K(X,Y)$, and suppose it has a bounded inverse B on a subspace $M \subset X$. If $\{x_n\}$ is a bounded sequence in M, then $\{Kx_n\}$ has a convergent subsequence (cf. Section 4.3). Hence, $x_n = BKx_n$ has a convergent subsequence in M, showing that M is finite dimensional (Theorem 4.6). □

Before we proceed, we shall need a few simple lemmas.

Lemma 14.40. *$A \in \Phi_+(X,Y)$ if and only if it has a bounded inverse on some subspace having finite codimension.*

Proof. Lemma 9.40 and Theorem 9.41. □

Theorem 14.41. *For A in $B(X,Y)$, if $A \notin \Phi_+(X,Y)$, then for every $\varepsilon > 0$ there is a $K \in K(X,Y)$ such that $\|K\| \le \varepsilon$ and $\alpha(A-K) = \infty$. Thus there is an infinite dimensional subspace M such that the restriction of A to M is compact and has norm $\le \varepsilon$.*

Proof. By Theorem 9.42 there are sequences $\{x_k\} \subset X$, $\{x'_k\} \subset X'$ such that (9.47) holds. Let $\varepsilon > 0$ be given, and take n so large that $2^{1-n} < \varepsilon$. Let M be the closed subspace spanned by $\{x_n, x_{n+1}, \cdots\}$. Define

$$Kx = \sum_{k=n}^{\infty} x'_k(x) A x_k.$$

As in the proof of Theorem 9.42 one verifies that $K \in K(X,Y)$ and

$$\begin{aligned} \|Kx\| &\leq \sum_{k=n}^{\infty} \|x'_k\| \cdot \|Ax_k\| \cdot \|x\| \leq \sum_{k=n}^{\infty} 2^{-k}\|x\| \\ &\leq 2^{-n} \sum_{k=n}^{\infty} 2^{n-k}\|x\| = 2^{-n} \sum_{j=0}^{\infty} 2^{-j}\|x\| = 2^{1-n}\|x\| \leq \varepsilon\|x\|. \end{aligned}$$

Now

$$Kx_k = Ax_k, \quad n \leq k < \infty,$$

and by continuity

$$Kx = Ax, \quad x \in M.$$

This completes the proof. □

Corollary 14.42. *If S is strictly singular, then for each $\varepsilon > 0$ there is an infinite dimensional subspace M such that the restriction of S to M is compact with norm $\leq \varepsilon$.*

Proof. We merely note that S is not in $\Phi_+(X,Y)$ by Lemma 14.40 and apply Theorem 14.41. □

Corollary 14.43. *Strictly singular operators are in $F_\pm(X,Y)$.*

Proof. If the strictly singular operator S were not in $F_+(X,Y)$, then there would be an operator $A \in \Phi_+(X,Y)$ such that $\alpha(A - S) = \infty$ (Theorem 9.46). Let $M = N(A-S)$. Then $A|_M = S|_M$. Now, A has a bounded inverse on some subspace X_0 having finite codimension (Lemma 14.40). Moreover, $X_0 \cap M$ is infinite dimensional (Lemma (14.18)). By Lemma 14.42, for each $\varepsilon > 0$ there is an infinite dimensional subspace $N \subset X_0 \cap M$ such that $S|_N$ is compact and has norm $\leq \varepsilon$. Since $A|_N = S|_N$, the same is true of $A|_N$. But this contradicts the fact that $A \in \Phi_+(X,Y)$ (Theorem 14.41). Similar reasoning applies to the set $F_-(X,Y)$. □

We continue with some important properties of strictly singular operators.

Theorem 14.44. *An operator S is strictly singular if and only if $\Gamma_M(S) = 0$ for all $M \subset X$.*

Proof. If S is strictly singular from X to Y, then it is strictly singular from M to Y for each infinite dimensional subspace $M \subset X$. Thus by Corollary 14.42, for each $\varepsilon > 0$ there is an infinite dimensional subspace $N \subset M$ such that $S|_N$ has norm $\leq \varepsilon$. Thus, $\Gamma_M(S) \leq \varepsilon$. Since this is true for any $\varepsilon > 0$, we have $\Gamma_M(S) = 0$. Conversely, assume that S is not strictly singular. Then there is an infinite dimensional subspace $M \subset X$ on which S has a bounded inverse. Thus, there is a constant C_0 such that

$$\|x\| \leq C_0 \|Sx\|, \quad x \in M.$$

If $\Gamma_M(S) = 0$, then for $0 < C_0\varepsilon < 1/2$, there is an infinite dimensional subspace $N \subset M$ such that $S|_N$ has norm $\leq \varepsilon$. Consequently,

$$\|x\| \leq C_0 \|Sx\| \leq C_0 \varepsilon \|x\| \leq \frac{1}{2}\|x\|, \quad x \in N,$$

providing a contradiction. This completes the proof. $\square$

Corollary 14.45. *T in $B(X, Y)$ is strictly singular, if and only if $\Delta(T) = 0$.*

Theorem 14.46. *T in $B(X, Y)$ is strictly singular if and only if*

$$\Gamma_M(A + T) = \Gamma_M(A), \quad A \in B(X, Y). \tag{14.45}$$

Proof. If T is strictly singular, then

$$\Gamma_M(A + T) \leq \Gamma_M(A) + \Delta_M(T) = \Gamma_M(A)$$

for any $A \in B(X, Y)$ by Theorem 14.16 and Corollary 14.45. For the same reasons,

$$\Gamma_M(A) \leq \Gamma_M(A + T) + \Delta_M(T) = \Gamma_M(A + T).$$

Thus (14.45) holds. Conversely, if (14.45) holds, then

$$\Gamma_M(T) = \Gamma_M(0 + T) = \Gamma_M(0) = 0$$

for each M. Thus, T is strictly singular by Theorem 14.44. $\square$

Corollary 14.47. *T in $B(X, Y)$ is strictly singular if and only if*

$$\Delta(A + T) = \Delta(A), \quad A \in B(X, Y). \tag{14.46}$$

Proof. Clearly, (14.45) implies (14.46). Thus, if T is strictly singular, (14.46) holds by Theorem 14.46. On the other hand, if (14.46) holds, then $\Delta(T) = \Delta(0 + T) = \Delta(0) = 0$. We now apply Corollary 14.45. $\square$

We also have

Theorem 14.48. *A is strictly singular if and only if $\tau(A) = 0$.*

Proof. If A is strictly singular, let $\varepsilon > 0$ be given. Then every M contains an N such that $\|A|_N\| < \varepsilon$ (Corollary 14.42). Thus

$$\inf_{x\in M,\|x\|=1} \|Ax\| < \varepsilon.$$

This gives $\tau(A) \leq \varepsilon$. Since ε was arbitrary, we have $\tau(A) = 0$. Conversely, if $\tau(A) = 0$, then

$$\inf_{x\in M,\|x\|=1} \|Ax\| = 0$$

for each M. This means that A cannot have a bounded inverse on any M. Hence, A is strictly singular. □

This should be contrasted with

Theorem 14.49. *The operator $T \in B(X, Y)$ is strictly singular if and only if*

$$\tau(A + T) = \tau(A), \quad A \in B(X, Y). \tag{14.47}$$

Proof. Suppose T is strictly singular, and let $\varepsilon > 0$ be given. For $A \in B(X, Y)$ there is an M such that

$$\|Ax\| \geq [\tau(A) - \varepsilon]\|x\|, \quad x \in M.$$

Moreover, there is an $N \subset M$ such that $\|T|_N\| < \varepsilon$. Thus,

$$\|(A + T)x\| \geq [\tau(A) - 2\varepsilon]\|x\|, \quad x \in N.$$

Hence, $\tau(A + T) \geq \tau(A) - 2\varepsilon$. Since ε was arbitrary, we have

$$\tau(A + T) \geq \tau(A).$$

Since A was arbitrary and $-T$ is strictly singular,

$$\tau(A) = \tau(A + T - T) \geq \tau(A + T).$$

Thus, (14.47) holds. Conversely, if (14.47) holds, then

$$\tau(T) = \tau(0 + T) = \tau(0) = 0.$$

Hence, T is strictly singular (Theorem 14.48). This completes the proof. □

14.7. Norm perturbations

Suppose A is a Φ_+ operator. Does there exist a quantity $q(A)$ such that $\|T\| < q(A)$ implies that $A - T \in \Phi_+$? The answer is yes, as we shall see in this section. If $A \in B(X, Y)$, where X, Y are Banach spaces, define

$$\mu_0(A) = \begin{cases} \displaystyle\inf_{\alpha(A-T)>\alpha(A)} \|T\|, & \alpha(A) < \infty, \\ 0, & \alpha(A) = \infty \end{cases} \tag{14.48}$$

and

$$\mu(A) = \inf_{\alpha(A-T)=\infty} \|T\|. \tag{14.49}$$

We shall prove

Theorem 14.50. *If T is in $B(X,Y)$ and*

$$\|T\| < \mu(A), \tag{14.50}$$

then $A - T \in \Phi_+(X,Y)$ and

$$i(A-T) = i(A). \tag{14.51}$$

If

$$\|T\| < \mu_0(A), \tag{14.52}$$

then

$$\alpha(A-T) \le \alpha(A). \tag{14.53}$$

Proof. Let $\varepsilon > 0$ be such that $\|T\| + \varepsilon < \mu(A)$. If $A - T \notin \Phi_+(X,Y)$, then there is an operator $K \in K(X,Y)$ such that $\|K\| < \varepsilon$ and $\alpha(A-T-K) = \infty$ (Theorem 14.41). Consequently,

$$\mu(A) \le \|T+K\| \le \|T\| + \|K\| \le \|T\| + \varepsilon < \mu(A),$$

providing a contradiction. Thus $A - T \in \Phi_+(X,Y)$. The same is true for $A - \theta T$ for $0 \le \theta \le 1$. Thus, (14.51) follows from the constancy of the index (Theorem 5.11). If (14.52) holds, the same must be true of (14.53). Otherwise, we would have to have $\mu_0(A) \le \|T\|$. This completes the proof. $\square$

As we saw in Section 3.5, $R(A)$ is closed in Y if and only if

$$\gamma(A) = \inf_{x \notin N(A)} \frac{\|Ax\|}{d(x, N(A))} \tag{14.54}$$

is positive. There is a connection between this constant and those given by (14.48) and (14.49). In fact, we have

Theorem 14.51. *If $\alpha(A) < \infty$, then*

$$\gamma(A) \le \mu_0(A) \le \mu(A). \tag{14.55}$$

Corollary 14.52. *If $\|T\| < \gamma(A)$, then $A - T \in \Phi_+(X,Y)$ and (14.51) and (14.53) hold.*

It is surprising that the proof of Theorem 14.51 involves a lot of preparation. It is based upon the following lemma due to Borsuk.

Lemma 14.53. *Let M, N be subspaces of a normed vector space X such that* $\dim N < \dim M$. *(In particular,* $\dim N < \infty$.*) If $G(x)$ is a continuous map of*

$$\partial B_R \cap M = \{x \in M : \|x\| = R\}$$

into N, then there is an element $x_0 \in \partial B_R \cap M$ such that $G(-x_0) = G(x_0)$.

The proof of Lemma 14.53 would take us too far afield, so we shall content ourselves with showing how it can be used to prove Theorem 14.51. A not too obvious consequence of Lemma 14.53 is

Lemma 14.54. *Under the hypotheses of Lemma 14.53 there is an element $u \in M$ such that*

$$\|u\| = d(u, N) = 1. \tag{14.56}$$

Once we know Lemma 14.54, we can easily give the proof of Theorem 14.51.

Proof. Suppose $T \in B(X, Y)$ is such that $\alpha(A - T) > \alpha(A)$. Then by Lemma 14.54 there is an element $u \in N(A - T)$ such that

$$\|u\| = d(u, N(A)) = 1.$$

Consequently,

$$\gamma(A) \leq \frac{\|Au\|}{\|u\|} = \frac{\|Tu\|}{\|u\|} \leq \|T\|.$$

Since this is true for any such T, we must have $\gamma(A) \leq \mu_0(A)$. This is the first inequality (14.55). The second is obvious. □

It remains to show how Lemma 14.53 implies Lemma 14.54.

Proof. First, we note that we may assume that M is finite dimensional (just consider a subspace). We can then assume that X is finite dimensional (same reason). But we have no right to assume that X is *strictly convex.* This requires

$$\|u + v\| < \|u\| + \|v\|, \quad u, v \in X,$$

unless u, v are linearly dependent. But we will assume it and pay the consequences later. In this case, for each $u \in X$ there is an element $w = G(u) \in N$ such that

$$\|u - w\| = d(u, N).$$

To see this, note that there is a sequence $\{w_k\} \subset N$ such that $\|u - w_k\| \to d(u, N)$. The sequence $\{w_k\}$ is bounded, and the finite dimensionality of N implies that there is a renamed subsequence converging to an element $w \in N$. Thus, $\|u - w_k\| \to \|u - w\| = d(u, N)$. Next, we claim that

(a) $w = G(u)$ is unique;

(b) $G(u)$ is continuous;

(c) $G(-u) = -G(u)$.

Once (a) - (c) are known, we merely apply Lemma 14.53 to conclude that there is an element $u_0 \in M$ such that $\|u_0\| = 1$ and

$$G(u_0) = G(-u_0) = -G(u_0).$$

Consequently, $G(u_0) = 0$. But

$$\|u_0 - G(u_0)\| = d(u_0, N).$$

Hence, u_0 satisfies the conclusion of Lemma 14.54.

Let us now turn to the proof of (a) - (c).

To prove (a), let $\tilde{w}$ be another point in N satisfying

$$\|u - \tilde{w}\| = \|u - w\| = d(u, N),$$

and let $0 \leq s \leq 1$. Then

$$\begin{aligned} \|u - s\tilde{w} - (1-s)w\| &= \|s(u - \tilde{w}) + (1-s)(u - w)\| \\ &< s\|u - \tilde{w}\| + (1-s)\|u - w\| = d(u, N), \end{aligned}$$

since $u - \tilde{w}$ and $u - w$ are not linearly dependent. This violates the definition of $d(u, N)$ and proves (a).

To prove (b), we first note that $d(u, N)$ is continuous in u. This follows from

$$\|u - v\| \leq \|u - u_0\| + \|u_0 - v\|.$$

Thus,

$$d(u, N) \leq \|u - u_0\| + \|u_0 - v\|, \quad v \in N.$$

This implies

$$d(u, N) \leq \|u - u_0\| + d(u_0, N),$$

from which we obtain

$$|d(u, N) - d(u_0, N)| \leq \|u - u_0\|.$$

Hence, if $u_k \to u$ in X, then

$$\|u_k - G(u_k)\| = d(u_k, N) \to d(u, N).$$

This shows that the $G(u_k)$ are bounded, and the finite dimensionality of N provides us with a renamed subsequence converging to some element $w \in N$. Thus,

$$\|u_k - G(u_k)\| \to \|u - w\| = d(u, N),$$

showing that $w = G(u)$. Moreover, the whole sequence must converge to w. Otherwise, there would be a subsequence converging to something else, violating (a). Hence, $u_k \to u$ implies $G(u_k) \to G(u)$. This proves (b).

To prove (c), we note that

$$d(-u, N) = \inf_{v \in N} \| - u - v\| = \inf_{v \in N} \|u + v\| = d(u, N).$$

If $w = G(u),\ h = G(-u)$, then

$$\|u - w\| = d(u, N) = d(-u, N) = \| - u - h\| = \|u + h\|,$$

from which we conclude that $h = -w$.

Now we have to pay the penalty for assuming that X is strictly convex. Let $x_1, \cdots, x_m$ be a basis for X. If

$$u = \sum_1^m \alpha_k x_k,\ y = \sum_1^m \beta_k x_k,$$

define

$$(u, v)_0 = \sum_1^m \alpha_k \beta_k.$$

Then $(u, v)_0$ is a scalar product on X. For each integer $n > 0$, define

$$\|u\|_n^2 = \|u\|^2 + \frac{1}{n}\|u\|_0^2,$$

where $\|\cdot\|_0$ is the norm corresponding to $(\cdot,\cdot)_0$. We note that the norm $\|\cdot\|_n$ is strictly convex, since

$$\begin{aligned}
\|u + v\|_n^2 &= \|u + v\|^2 + \frac{1}{n}[\|u\|_0^2 + \|v\|_0^2 + 2(u, v)_0] \\
&\leq (\|u\| + \|v\|)^2 + \frac{1}{n}(\|u\|_0 + \|v\|_0)^2 + \frac{2}{n}[(u, v)_0 - \|u\|_0\|v\|_0] \\
&\leq (\|u\|_n + \|v\|_n)^2 + \frac{2}{n}[(u, v)_0 - \|u\|_0\|u\|_0].
\end{aligned}$$

This shows that the only time we can have $\|u + v\|_n = \|u\|_n + \|v\|_n$ is when $(u, v)_0 = \|u\|_0\|v\|_0$, i.e., when u, v are linearly dependent. Thus, for each n the norm $\|\cdot\|_n$ is strictly convex. By what we have already shown, there is an element $u_n \in M$ such that $\|u_n\|_n = d_n(u_n, N) = 1$, where d_n is the distance as measured by the $\|\cdot\|_n$ norm. Since $\|u_n\| \leq \|u_n\|_n = 1$, there is a renamed subsequence such that $u_n \to u$ in X. Since

$$\|u_n - u\|_n \to \|u - v\|, \quad v \in N,$$

we have

$$1 = d_n(u_n, N) \to d(u, N), \quad 1 = \|u_n\|_n \to \|u\|,$$

and the proof is complete. $\square$

14.8. Perturbation functions

Let $m(T, A)$ be a real valued function depending on two operators, $T, A \in B(X, Y)$. Assume

$$m(\lambda T, A) = |\lambda| m(T, A). \tag{14.57}$$

We shall call $m(T, A)$ a Φ *perturbation function* if $m(T, A) < 1$ for $A \in \Phi$ implies that $A - T \in \Phi$. We shall call it a Φ_+ *perturbation function* if $m(T, A) < 1$ for $A \in \Phi_+$ implies $A - T \in \Phi_+$. Finally, we shall call it a Φ_α *perturbation function* if $m(T, A) < 1$ for $A \in \Phi_+$ implies that $A - T \in \Phi_+$ and $\alpha(A - T) \leq \alpha(A)$.

In our discussions of perturbation functions, we shall assume that X, Y are Banach spaces. By Theorems 14.50 and 14.51, we see that $\|T\|/\mu(A)$ is a Φ_+ perturbation function, while $\|T\|/\mu_0(A)$ and $\|T\|/\gamma(A)$ are Φ_α perturbation functions. The following lemma shows that a Φ_+ perturbation function is also a Φ perturbation function.

Lemma 14.55. *If $m(T, A)$ is a Φ_+ perturbation function and $A \in \Phi_+$, then $m(T, A) < 1$ implies that*

$$i(A - T) = i(A). \tag{14.58}$$

Proof. If $m(T, A) < 1$, then (14.57) shows that $A - \theta T \in \Phi_+$ for $0 \leq \theta \leq 1$. By the constancy of the index (Theorem 5.11), we see that (14.58) holds. $\square$

The *injection modulus* $j(A)$ of an operator $A \in B(X, Y)$ is defined as

$$j(A) = \sup\{\lambda : \lambda \|x\| \leq \|Ax\|, \quad x \in X\}. \tag{14.59}$$

If A is one-to-one, then $j(A) = \gamma(A)$. Otherwise, $j(A) = 0$.

By Theorems 14.30 and 14.32, the quantities $\Delta(T)/\Gamma(A)$ and $\tau(B)/\nu(T)$ are Φ_+ perturbation functions. We can improve on them by introducing

$$\rho(T, A) = \inf_{\dim M^\circ < \infty} \sup_{N \subset M} \inf_{x \in N} \frac{\|Tx\|}{\|Ax\|}. \tag{14.60}$$

We have

Theorem 14.56. *The function $\rho(T, A)$ is a Φ_+ perturbation function and satisfies*

$$\rho(T, A) \leq \Delta(T)/\Gamma(A), \tag{14.61}$$

and

$$\rho(T, A) \leq \tau(T)/\nu(A). \tag{14.62}$$

Proof. Suppose that $\rho(T, A) < a < 1$. Then there is a subspace M of finite codimension, on which A is one-to-one, such that for each subspace $N \subset M$ there is a element $x' \in N$ with $\|Tx'\|/\|Ax'\| < a$. We may assume that $\|x'\| = 1$. Assume also that $A - T$ is not in Φ_+ and, therefore, that its restriction to M is not in Φ_+ as well. Thus, for each $\varepsilon > 0$, there is a subspace $N \subset M$ such that

$$\|Ax - Tx\| \le \varepsilon\|x\|, \quad x \in N.$$

Hence,

$$\|Ax\| \le \|Tx\| + \varepsilon\|x\|, \quad x \in N.$$

For x' chosen as above, we have

$$\|Ax'\| \le a\|Ax'\| + \varepsilon.$$

But $\gamma(A|_M) > 0$, and

$$\gamma(A|_M) \le \|Ax'\| \le \varepsilon/(1 - a),$$

which is impossible if ε is small enough. Thus, $A - T \in \Phi_+$, and therefore, $\rho(T, A)$ is a perturbation function for Φ_+.

Given N, set

$$\frac{1}{c} = \inf_{x \in N} \frac{\|Ax\|}{\|Tx\|},$$

which may be $+\infty$, so that

$$c\|Ax\| \le \|Tx\|, \quad x \in N. \tag{14.63}$$

Then $c\|T|_L\| \le \|B|_L\|$ for $L \subset N$, and so $c\Gamma(T|_L) \le \Gamma(B|_L)$. Then

$$\rho(T, A) \le \inf_{\dim M^\circ < \infty} \sup_{N \subset M} \frac{\Gamma(T|_N)}{\Gamma(A|_N)} \le \inf_{\dim M^\circ < \infty} \frac{\Delta(T|_M)}{\Gamma(A|_M)} \le \frac{\Delta(T)}{\Gamma(A)}.$$

Inequality (14.63) implies that $cj(A|_N) \le j(T|_N)$, and therefore

$$\rho(T, A) \le \inf_{\dim M^\circ < \infty} \sup_{N \subset M} \frac{j(T|_N)}{j(A|_N)} \le \inf_{\dim M^\circ < \infty} \frac{\tau(T|_M)}{j(A|_M)} \le \frac{\tau(T)}{\nu(A)}.$$

This completes the proof. □

Since $\rho(T, A)$ is smaller than the perturbation function in (14.61) and (14.62), it is better in the most obvious sense. For perturbation functions, smaller is better and smallest is best. The following theorem shows this.

Theorem 14.57. *There is a smallest perturbation function.*

(i): *The smallest Φ_+ perturbation function is*

$$m_1(T, A) = \max\{|\lambda| : \lambda A - T \notin \Phi_+\} \tag{14.64}$$

(ii): *The smallest Φ perturbation function is*

$$m_2(T, A) = \max\{|\lambda| : \lambda A - T \notin \Phi\}. \tag{14.65}$$

(iii): *The smallest* Φ_α *perturbation function is*

(14.66)

$$m_3(T,A) = \max\{|\lambda| : \text{ either } \lambda A - T \notin \Phi_+ \text{ or } \alpha(\lambda A - T) > \alpha(A)\}.$$

Proof. It is immediate from(14.64), (14.65) and (14.66) that m_1, m_2 and m_3 are perturbation functions. If $m(T,A)$ is a Φ_+ perturbation function and $m(T,A) < m_1(T,A)$ for some T and some $A \in \Phi_+$, then by (14.64) there is a λ' satisfying $m(T,A) < |\lambda'|$ with $\lambda' A + T$ not a Φ_+ operator. But this is a contradiction because $m(T/\lambda', A) < 1$, so that $A - (T/\lambda')$, and therefore $\lambda' A - T$, is a Φ_+ operator.

Similar arguments show that m_2 and m_3 are minimal. □

The point of Theorem 14.57 is that there are minimal perturbation functions. The formulas (14.64), (14.65) and (14.66) beg the question of whether, for given A and T, the operator $A + T$ has the desired property. Therefore, while it is not possible to have, say, a better Φ perturbation function than m_2, it is certainly possible to have a formula for m_2 which is more informative than (14.65).

Recall that an operator A is Fredholm if and only if there is an operator A_0 such that $A_0 A = I + K_1$ on X and $AA_0 = I + K_2$ on Y, where K_1, K_2 are compact operators (Theorems 5.4). Let $[A]$ denote the coset containing the operator A in the quotient space $B(X)/K(X)$ (cf. Section 3.5). We have

Theorem 14.58. *Let* A *be a Fredholm operator on* X, *and let* A_0 *be as above. Then, setting* $C = A_0 T$, *the best Fredholm perturbation function* m_2 *is given by*

$$m_2(T,A) = \lim(\tau(C^n))^{1/n}, \tag{14.67}$$

$$m_2(T,A) = \lim(\Delta(C^n))^{1/n}, \tag{14.68}$$

$$m_2(T,A) = r_\sigma([C]). \tag{14.69}$$

Proof. Since A_0 is also Fredholm, $\lambda A - T \in \Phi$ if and only if $A_0(\lambda A - T) = \lambda - A_0 T + \lambda K_1 \in \Phi$ if and only if $\lambda + A_0 T \in \Phi$. Thus, (14.69) follows from the equation

$$r_\sigma([C]) = \sup\{|\lambda| : \lambda - C \notin \Phi\}$$

(Theorem 6.13). Since

$$r_\sigma([C]) = \lim(\tau(C^n))^{1/n} = \lim(\Delta(C^n))^{1/n}, \quad n \to \infty$$

(Theorem 6.13), we see that (14.67) and (14.68) hold as well. □

A technical detail bars the extension of Theorem 14.58 to Φ_+ perturbations and m_1. Namely, an operator $A \in \Phi_+$ may not have a complemented range. One must consider, instead of Φ_+ operators, the class Φ_ℓ of those operators in Φ_+ which have complemented ranges and use the fact that A belongs to Φ_ℓ if and only if there is a bounded operator A_0 such that $A_0 A \in \Phi$ (Theorem 5.37).

Note that if $m(T, A)$ is a Fredholm perturbation function and $A \in \Phi$ with A_0 an operator as in Theorem 14.58, then $m'(T, A) = m(A_0 T, I)$ is a Fredholm perturbation function.

We note also that the function $\rho(T, A)$ is not the smallest Φ_+ perturbation function. To see this, let $X = Y = \ell_2$, and define T for

$$a = \sum \alpha_j e_j$$

by

$$Ta = \alpha_2 e_3 + \alpha_4 e_5 + \cdots .$$

Let N be the closed subspace spanned by $\{e_2, e_4, \cdots\}$. For M a subspace of finite codimension, $M' = N \cap M$ is infinite-dimensional, and $j(T|_{M'}) = \|T|_{M'}\| = 1$, so that $\rho(T, I) = 1$. However, $(\lambda + T)(\lambda - T) = \lambda^2$ since $T^2 = 0$, and thus, $m_3(T, I) = m_2(T, I) = m_1(T, I) = 0$. Note that since $\Delta(T) = \tau(T) = 1$, the perturbation functions $\Delta(T)/\gamma(I)$ and $\tau(T)/\nu(I)$ are both 1.

Next we have

Lemma 14.59. *Let H be a Hilbert space, and suppose that $A \in B(H)$, M is a subspace and $\nu(A|_M) > 0$. Then for each $\varepsilon > 0$, there is an $N \subset M$ such that $\|A|_N\| < \nu(A|_M) + \varepsilon$.*

Proof. To simplify the notation, assume that $M = H$. Then by definition, for each $\varepsilon > 0$ there is a subspace M' of finite codimension with $\nu(A|_{M'}) > 0$, and therefore $A \in \Phi_+$, and there is a norm one element $x_1 \in M'$ with $\|Ax_1\| < j(A|_{M'}) + \varepsilon \le \gamma(A) + \varepsilon$. Let N be the orthogonal complement of the span of x_1, let N' be the orthogonal complement of the span of Ax_1, and set $M'' = N \cap A^{-1}(N') \cap M'$, which has finite codimension in M'. There is a norm one element $x_2 \in M''$ with $\|Ax_2\| < j(A|_{M''}) + \varepsilon \le \nu(A) + \varepsilon$. Since $x_2 \in M''$, x_2 is orthogonal to x_1 and Ax_2 is orthogonal to Ax_1.

Continuing in this way we obtain an orthonormal sequence $\{x_n\}$ and an orthogonal sequence $\{Ax_n\}$ with $\|Ax_n\| < \nu(A) + \varepsilon$. On the closed span of the $\{x_n\}$, we have

$$\|A \sum \alpha_n x_n\|^2 = \sum |\alpha_n|^2 \|Ax_n\|^2 < (\nu(A) + \varepsilon) \| \sum \alpha_n x_n\|^2.$$

$\square$

We also have

Theorem 14.60. *On a Hilbert space, $\tau(T) = \Delta(T)$ and $\nu(A) = \Gamma(A)$. Consequently, the two perturbation functions $\Delta(T)/\Gamma(A)$ and $\tau(T)/\nu(A)$ are equal.*

Proof. To show that $\Delta(T) \le \tau(A)$, choose an M with $\Gamma(T|_M) > 0$. Then $T|_M \in \Phi_+$, and consequently, $\nu(T|_M) > 0$ (Lemma 14.31). Given an $\varepsilon > 0$ there is an $N \subset M$ such that $\|T|_N\| < \nu(T|_M) + \varepsilon \le \tau(T|_M) + \varepsilon \le \tau(T) + \varepsilon$ (Lemma 14.59). Hence, $\Gamma(T|_M) \le \tau(T) + \varepsilon$, and consequently, $\Delta(T) \le \tau(T)$. The reverse inequality follows from Theorem 14.24.

It also follows from Lemma 14.59 that $\Gamma(T) \ge \nu(T)$, and the reverse inequality follows from Theorem 14.34. □

14.9. Factored perturbation functions

Given a Φ_+ operator A and an operator T, the computation of $\|T\|/\gamma(A)$ involves the computation of the two useful numbers $\|T\|$ and $\gamma(A)$. After this computation, $\gamma(A)$ can, of course, be used in studying the perturbation of A by other operators, and $\|T\|$ can be used in studying the perturbation of other Φ_+ operators by T. In contrast, the computation of, say, $\rho(T, A)$ must be completely redone if either T or A changes. This observation leads to the following definition.

Definition. A perturbation function m is *factored* if it can be written in the form

$$m(T, A) = m_1(T)/m_2(A). \tag{14.70}$$

We have

Theorem 14.61. *Let the factored perturbation function m be given by (14.70). For each numerator m_1 there is a best (i.e., largest) denominator $d(A, m_1)$ for which $m_1(T)/d(A, m_1)$ is a perturbation function of the same type as m (Φ, Φ_+ or Φ_α). For each denominator m_2, there is a best (i.e., smallest) numerator $n(T, m_2)$ for which $n(T, m_2)/m_2(A)$ is a perturbation function of the same type as m.*

Proof. Suppose that m is a Φ_+ perturbation function. Set

$$d(A) = \inf\{m_1(T) : A - T \notin \Phi_+\}. \tag{14.71}$$

We claim that $m_1(T)/d(A)$ is the minimal Φ_+ perturbation function with numerator $m_1(T)$. First, if $m_1(T)/d(A) < 1$, then by the definition of d in (14.71), $A + T$ is in Φ_+. Second, suppose that $m_1(T)/d'(A)$ is a Φ_+ perturbation function. Let a Φ_+ operator A be given. For each $\varepsilon > 0$ there is a T with $d(T) + \varepsilon > m_1(T)$ and $A + T \notin \Phi_+$. This implies that

$m_1(T)/d'(A) \geq 1$, and consequently that $d'(A) < d(A) + \varepsilon$. Hence, for each $A \in \Phi_+$, we have $d'(A) \leq d(A)$. Note, in particular, that $d(A) \geq m_2(A) > 0$.

To define $d(A)$ for a Fredholm perturbation function, replace Φ_+ by Φ in (14.71). To define $d(A)$ for a Φ_α perturbation function, set

$$d(A) = \inf\{m_1(T) : \text{ either } A - T \notin \Phi_+ \text{ or } \alpha(A-T) > \alpha(A)\}. \tag{14.72}$$

In either case, proceed as above to see that the quantity d so defined is maximal.

Suppose again that m is a Φ_+ perturbation function. To show that for each denominator m_2 there exists a best numerator, let N be the set of all those numerators n' for which $n'(T)/m_2(A)$ is a Φ_+ perturbation function, and set

$$n(T) = \inf\{n'(T) : n' \in N\}. \tag{14.73}$$

If $n(T)/m_2(A) < 1$, then there is an $n' \in N$ such that $n'(T)/m_2(A) < 1$. Hence, $A + T \in \Phi_+$. That $n(T)$ is the smallest member of N and that $n(\lambda T) = |\lambda| n(T)$ follow from (14.73).

Similar definitions and proofs establish the results for Φ and Φ_α perturbation functions. □

Note that one could alternatively define $d(A)$ analogous to (14.73) and $n(T)$ analogous to (14.71).

Next, we have

Lemma 14.62. *Suppose that $m_1(T)/m_2(A)$ is a Φ_+ perturbation function. Let $n(T) = n(T, m_2)$ be the best numerator for a given denominator m_2, and let $d(A) = d(A, m_1)$ be the best denominator for a given numerator m_1, as described in Theorem 14.61. Then:*

- **(i):** $n(\lambda T) = |\lambda| n(T)$.
- **(ii):** $d(\lambda A) = |\lambda| d(A)$.
- **(iii):** $n(T + S) = n(T)$ for $S \in F_+$.
- **(iv):** $d(A + S) = d(A)$ for $S \in F_+$.

Proof. Property (i) is immediate. To see that (ii) holds, note that

$$\begin{aligned} d(\lambda A) &= \inf\{m_1(T) : \lambda A + T \notin \Phi_+\} \\ &= \inf\{m(\lambda T/\lambda) : A + T/\lambda \notin \Phi_+\} \\ &= |\lambda| d(A). \end{aligned}$$

Define

$$n'(T) = \inf\{n(T + S) : S \in F_+\}.$$

If $n'(T)/m_2(A) < 1$, then there is an $S \in F_+$ such that $A + T + S \in \Phi_+$ and so $A + T \in \Phi_+$ (Corollary 9.52). Since $n'(\lambda T) = |\lambda| n'(T)$, we see that $n'(T)/m_2(A)$ is a Φ_+ perturbation function, and therefore, by the minimality of n, $n' = n$.

Define

$$d'(A) = \sup\{d(A + S) : S \in F_+\}.$$

As above, $m_1(T)/d'(A) \in \Phi_+$, and by the maximality of d, we have $d' = d$. This completes the proof. □

There is a corresponding theorem for Φ perturbation functions with F_+ replaced by F in (iii) and (iv).

This leads to

Corollary 14.63. *Given the numerator $\|T\|$:*

- **(i):** *$\mu_0(A)$ is the best denominator for a Φ_α perturbation function;*
- **(ii):** *$\mu(A)$ is the best denominator for a Φ_+ perturbation function;*
- **(iii):** *$\mu(A)$ is also the best denominator for a Φ perturbation function.*

Proof. Statements (i) and (ii) follow from Theorem 14.50 and Theorem 14.61. By Lemma 14.55, $\|T\|/\mu(A)$ is also a Φ perturbation function, and since $\mu(A)$ is the best demoninator for a Φ_+ perturbation function, it is also the best denominator for a Φ perturbation function. □

We shall say that a Φ_+ perturbation function $m(T, A)$ is A-exact if, given any operator $A \in \Phi_+$ and $\varepsilon > 0$, there is a T with $m(T, A) < 1 + \varepsilon$, yet $A + T \notin \Phi_+$. A Fredholm perturbation function $m(T, A)$ is A-exact if, given any Fredholm operator A and $\varepsilon > 0$, there is an operator T with $m(T, A) < 1 + \varepsilon$, yet $A + T \notin \Phi$. A Φ_α perturbation function $m(T, A)$ is A-exact if, given any $A \in \Phi_+$ and $\varepsilon > 0$, there is a T with $m(T, A) < 1 + \varepsilon$, and yet either $A + T \notin \Phi_+$ or $\alpha(A + T) > \alpha(A)$.

We have

Lemma 14.64. *If $m_1(T)/m_2(A)$ is an A-exact perturbation function, then the best denominator $d(A, m_1)$ for m_1 is $m_2(A)$.*

Proof. See the proof of Theorem 14.61. □

A Φ_+ perturbation function m which has $m(A, A) = 1$ for each operator $A \in \Phi_+$ is A-exact, since $T = -A$ then satisfies $m(T, A) = 1$ and $\alpha(A+T) = \infty$.

14.10. Problems

(1) Show that the expression (14.26) is a seminorm on $B(X,Y)$.

(2) Show that $\|A\|_q = 0$ if and only if $A \in K(X,Y)$.

(3) Prove (14.28).

(4) Prove the second half of Lemma 14.8.

(5) Prove: $P(G_r) = R$.

(6) Prove (14.13), (14.14) and (14.15).

(7) If X is a normed vector space, V is a subspace of finite codimension, and W is an infinite dimensional subspace, show that $\dim V \cap W = \infty$.

(8) Prove: If $\dim M_j^\circ < \infty,\ j = 1,2$, then $\dim (M_1 \cap M_2)^\circ < \infty$.

(9) Prove (14.25).

(10) Show that $\|A\|_q$ is a seminorm.

(11) Prove (14.28).

(12) Prove (14.29).

Chapter 15

EXAMPLES AND APPLICATIONS

15.1. A few remarks

Throughout these chapters, we have avoided involved applications. The primary reason for this is that we wanted to stress the beauty of the subject itself and to show that it deserves study in its own right. How successful we have been remains to be seen. However, lest we leave you with the wrong impression, we want to point out that functional analysis has many useful applications scattered throughout mathematics, science, and engineering. Indeed, a large part of its development came from these applications.

Therefore, we think it appropriate to give some examples to illustrate some of the applications. In this connection, one must concede the fact that the more meaningful and useful the application, the more involved the details and technicalities. Since only a minimal background in mathematics has been assumed for this book, the choice of applications that need no further preparation is extremely limited. Moreover, our intentions at the moment are not to teach you other branches of mathematics, and certainly not to expound upon other branches of science.

Faced with this dilemma, we have chosen a few applications that do not need much in the way of technical knowledge. It is, therefore, to be expected that these will not be the most interesting of the possible choices. Moreover, due to time and space limitations, it will not be possible to motivate the importance of those applications arising from other branches of science or the mathematical tools that are used in their connection.

15.2. A differential operator

Consider the Hilbert space $L^2 = L^2(-\infty, \infty)$ of functions $u(t)$ square integrable on the whole real line (see Section 1.4). Consider the operator

$$Au = \frac{du(t)}{dt}$$

on L^2, with $D(A)$ consisting of those $u \in L^2$ having continuous first derivatives in L^2. At this point, one might question why we took L^2 to be the space in which A is to act, rather than some other space that might seem more appropriate. The reason is that we want to consider an oversimplification of a problem which has considerable significance in physics.

It will be more convenient for us to consider complex valued functions $u(t)$. Thus, L^2 will be a complex Hilbert space.

It is easy to see that A is not a bounded operator on L^2. In fact, let $\varphi(t) \not\equiv 0$ be any nonnegative continuously differentiable function which vanishes outside the interval $|t| < 1$. For instance, we can take

$$\varphi(t) = \begin{cases} a \exp\left(\frac{1}{t^2-1}\right), & |t| < 1, \\ 0, & |t| \geq 1. \end{cases} \tag{15.1}$$

By multiplying by a suitable constant [e.g., the constant a in (15.1)], we may assume that

$$\int_{-\infty}^{\infty} \varphi(t)\, dt = 1. \tag{15.2}$$

Set

$$\varphi_n = n\varphi(nt), \quad n = 1, 2, \cdots . \tag{15.3}$$

Then $\varphi_n \in D(A)$ and

$$\|\varphi_n\|^2 = n^2 \int_{-\infty}^{\infty} |\varphi(nt)|^2 dt = n \int_{-\infty}^{\infty} |\varphi(\tau)|^2 d\tau = n\|\varphi\|^2. \tag{15.4}$$

But

$$\|A\varphi_n\|^2 = n^4 \int_{-\infty}^{\infty} |\varphi'(nt)|^2 dt = n^3\|A\varphi\|^2.$$

Note that $A\varphi \neq 0$; otherwise, φ would be a constant, and the only constant in L^2 is 0. Hence, $\|A\varphi_n\|/\|\varphi_n\| = n\|A\varphi\|/\|\varphi\| \to \infty$ as $n \to \infty$. This shows that A is not bounded.

The next question we might ask about A is whether it is closed. Again, the answer is negative. To show this, we again make use of the functions (15.3), where $\varphi(t)$ is given by (15.1). For any $u \in L^2$, set

$$J_n u(x) = \int_{-\infty}^{\infty} \varphi_n(x - t)u(t)\, dt. \tag{15.5}$$

We claim that $J_n u \in L^2$. In fact,

$$\|J_n u\| \leq \|u\|. \tag{15.6}$$

Moreover, for any $u \in L^2$, we have

$$\|u - J_n u\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{15.7}$$

Assume (15.6) and (15.7) for the moment, and let $u(t)$ be the function in L^2 given by

$$u(t) = \begin{cases} 1+t, & -1 \leq t \leq 0, \\ 1-t, & 0 < t \leq 1, \\ 0, & |t| > 1. \end{cases}$$

Clearly, $u(t)$ is continuous in $(-\infty, \infty)$, but its derivative is not. Thus, u is not in $D(A)$. However, $J_n u$ is in $D(A)$ for each n. This follows from the fact that the integral in (15.5) really extends only from -1 to 1, and $u(t)$ is continuous. Thus, we can differentiate under the integral sign and obtain a continuous function. Since $J_n u$ vanishes for $|t| > 2$, the same is true for its derivative. This shows that $J_n u \in D(A)$ for each n. Now $u'(t)$ may not be continuous, but it is in L^2, since

$$u'(t) = \begin{cases} 1, & -1 < t < 0, \\ -1, & 0 < t < 1, \\ 0, & |t| > 1. \end{cases}$$

Moreover,

$$\begin{aligned} AJ_n u(x) &= \int_{-1}^{1} \varphi_n'(x-t)u(t)\,dt \\ &= \int_{-1}^{0} \varphi_n'(x-t)(1+t)\,dt + \int_{0}^{1} \varphi_n'(x-t)(1-t)\,dt \\ &= \int_{-1}^{0} \varphi_n(x-t)\,dt - \int_{0}^{1} \varphi_n(x-t)\,dt \\ &= \int_{-1}^{1} \varphi_n(x-t)u'(t)\,dt = J_n u'. \end{aligned}$$

Now, if we are to believe (15.7), we have

$$\|u - J_n u\| \longrightarrow 0, \quad \|u' - AJ_n u\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

But if A were a closed operator, then it would follow that $u \in D(A)$, which we know to be false.

It remains to prove (15.6) and (15.7). Applying the Cauchy-Schwarz inequality to (15.5), we get

$$|J_n u(x)|^2 \leq \int_{-\infty}^{\infty} \varphi_n(x-t)\,dt \int_{-\infty}^{\infty} \varphi_n(x-t)|u(t)|^2 dt.$$

By (15.2),

$$\int_{-\infty}^{\infty} \varphi_n(x-t)\,dt = 1. \tag{15.8}$$

Thus,

$$\begin{aligned}\int_a^b |J_n u(x)|^2 dx &\le \int_a^b \int_{-\infty}^{\infty} \varphi_n(x-t)|u(t)|^2 dt\,dx \\ &= \int_{-\infty}^{\infty} \int_a^b \varphi_n(x-t)|u(t)|^2 dx\,dt.\end{aligned}$$

(The order of integration can be interchanged because the double integral is absolutely convergent.) Since

$$\int_a^b \varphi_n(x-t)\,dx \le \int_{-\infty}^{\infty} \varphi_n(x-t)\,dx = 1,$$

we have

$$\int_a^b |J_n u(x)|^2 dx \le \int_{-\infty}^{\infty} |u(t)|^2 dt = \|u\|^2.$$

Letting $a \to -\infty$ and $b \to \infty$, we see that $J_n u \in L^2$ and that (15.6) holds.

Turning to (15.7), we note that it suffices to prove it for continuous $u \in L^2$. The reason is that such functions are dense in L^2 (in fact, so far as we are concerned, L^2 is the completion of such functions with respect to the norm of L^2). Moreover, it follows from Lemma 10.2 that once we know that (15.6) holds for all $u \in L^2$, and (15.7) is proved for a set of u dense in L^2, then (15.7) holds for all $u \in L^2$.

To prove (15.7) for continuous $u \in L^2$, let $\varepsilon > 0$ and u be given. Take R so large that

$$\int_{|t|>R-1} |u(t)|^2 dt < \frac{\varepsilon}{16}.$$

Since $u(t)$ is uniformly continuous on bounded intervals, there is a $\delta > 0$ such that

$$|u(s+t) - u(t)|^2 < \frac{\varepsilon}{4R}, \quad |t| < R,\ |s| \le \delta.$$

Thus,

$$\int_{-R}^{R} |u(s+t) - u(t)|^2 dt < \frac{\varepsilon}{2}, \quad |s| < \delta.$$

We may assume $\delta < 1$. Then

$$\begin{aligned}\int_R^{\infty} |u(s+t) - u(t)|^2 dt &\le 2\int_R^{\infty} (|u(s+t)|^2 + |u(t)|^2) dt \\ &= 2\int_{R+s}^{\infty} |u(r)|^2 dr + 2\int_R^{\infty} |u(t)|^2 dt < \frac{\varepsilon}{4},\end{aligned}$$

with a similar inequality holding for the interval $(-\infty, -R)$. Combining the inequalities, we have

$$\int_{-\infty}^{\infty} |u(s+t) - u(t)|^2 dt < \varepsilon, \quad |s| \le \delta. \tag{15.9}$$

Now, by (15.5),

$$J_n u(x) = \int_{-\infty}^{\infty} \varphi_n(s) u(x-s)\, ds, \tag{15.10}$$

so that by (15.8), we have

$$J_n u(x) - u(x) = \int_{-\infty}^{\infty} \varphi_n(s)[u(x-s) - u(x)]\, ds. \tag{15.11}$$

Applying Schwarz's inequality and integrating with respect to x, we obtain

$$\int_{-\infty}^{\infty} |J_n u - u|^2 dx \le \int_{-\infty}^{\infty} \varphi_n(s) \int_{-\infty}^{\infty} |u(x-s) - u(x)|^2 dx\, ds. \tag{15.12}$$

Now, by (15.1) and (15.3), $\varphi_n(s)$ vanishes for $|ns| > 1$. Thus, the integration with respect to s in (15.12) is only over the interval $|s| < 1/n$. If we take $n > 1/\delta$, then (15.9) will hold for s in this interval. This will give

$$\|J_n u - u\|^2 < \varepsilon \int_{-\infty}^{\infty} \varphi_n(s)\, ds = \varepsilon,$$

and the proof is complete. $\square$

15.3. Does A have a closed extension?

We have shown that the operator A given in the preceding section is unbounded and not a closed operator in L^2. We now ask whether A has a closed extension. By Theorem 12.18, this is equivalent to asking if A is closable. For a change, this question has an affirmative answer. To prove that A is closable, we must show that whenever $\{u_n\}$ is a sequence of functions in $D(A)$ such that $u_n \to 0$ and $Au_n \to f$ in L^2, then $f = 0$. To see this, let v be any function in $D(A)$ that vanishes for $|t|$ large. Then, by integration by parts, we have, for each n,

$$(Au_n, v) = -(u_n, v'). \tag{15.13}$$

(Note that no boundary terms appear, because v vanishes outside some finite interval.) Taking the limit as $n \to \infty$, we get

$$(f, v) = 0 \tag{15.14}$$

for all such v. We maintain that this implies that $f = 0$.

We show this as follows: For each $R > 0$, let f_R be defined by

$$f_R = \begin{cases} f, & |t| \leq R, \\ 0, & |t| > R. \end{cases} \tag{15.15}$$

Since $f \in L^2$, the same is true of f_R. Now, by (15.7), for each $\varepsilon > 0$, we can make

$$\|f - J_n f\| < \frac{\varepsilon}{2} \tag{15.16}$$

by taking n sufficiently large. Moreover, we can take R so large that

$$\|f - f_R\|^2 = \int_{|t|>R} |f|^2 dt < \frac{\varepsilon^2}{4}. \tag{15.17}$$

In view of (15.6), this gives

$$\|J_n f - J_n f_R\| \leq \|f - f_R\| < \frac{\varepsilon}{2}.$$

Thus,

$$\|f - J_n f_R\| < \varepsilon. \tag{15.18}$$

Now, for each n and R, the function $J_n f_R$ vanishes for $|t|$ large. Moreover, if we use the function φ given by (15.1), then $J_n f_R$ is infinitely differentiable (just differentiate repeatedly under the integral sign). In particular, $J_n f_R \in D(A)$ for each n and R so that

$$(f, J_n f_R) = 0.$$

Thus, by (15.18),

$$\|f\|^2 = (f, f - J_n f_R) \leq \|f\| \cdot \|f - J_n f_R\|$$

for all n and R. By (15.18) this implies

$$\|f\| < \varepsilon.$$

Since ε was arbitrary, we must have $f = 0$, and the proof is complete. □

A function $\varphi(t)$ which vanishes for $|t|$ large is said to have *compact support.* Let C_0^∞ denote the class of infinitely differentiable functions with compact supports. We have just shown that C_0^∞ is dense in L^2.

15.4. The closure of A

We know that A has at least one closed extension. In particular, it has a closure $\bar{A}$ (see the proof of Theorem 12.18). By definition, $D(\bar{A})$ consists of those $u \in L^2$ for which there is a sequence $\{u_n\}$ of functions in $D(A)$ such that $u_n \to u$ in L^2 and $\{Au_n\}$ converges in L^2 to some function f. $\bar{A}u$ is defined to be f. Let us examine $\bar{A}$ a bit more closely. In particular, we want to determine $\rho(\bar{A})$. The problem of deciding which values of λ are such that

$$(\bar{A} - \lambda)u = f \tag{15.19}$$

has a unique solution $u \in D(\bar{A})$ for each $f \in L^2$ does not appear to be easy at all, since $\bar{A}$ is defined by a limiting procedure, and it is not easy to "put our hands on it."

To feel our way, assume that u and f are smooth functions satisfying (15.19). In this case, $\bar{A}$ reduces to A, and (15.19) becomes

$$u' - \lambda u = f. \tag{15.20}$$

This is a differential equation that most of us have come across at one time or another. At any rate, if we multiply both sides of (15.20) by $e^{-\lambda t}$ and integrate between 0 and x, we get

$$e^{-\lambda x}u(x) = u(0) + \int_0^x e^{-\lambda t} f(t)\, dt. \tag{15.21}$$

Suppose $\lambda = \alpha + i\beta$. We must consider several cases depending on λ. In general, we have

$$|u(x)| = e^{\alpha x} \left| u(0) + \int_0^x e^{-\lambda t} f(t)\, dt \right|. \tag{15.22}$$

If $\alpha > 0$, we will have $|u(x)| \to \infty$ as $x \to \infty$ unless

$$u(0) + \int_0^\infty e^{-\lambda t} f(t)\, dt = 0. \tag{15.23}$$

Thus, the only way u can be in L^2 for $\alpha > 0$ is when (15.23) holds. In this case,

$$u(x) = -\int_x^\infty e^{\lambda(x-t)} f(t)\, dt. \tag{15.24}$$

This function is, indeed, in L^2 since

$$|u(x)|^2 \le \int_x^\infty e^{\alpha(x-t)} dt \int_x^\infty e^{\alpha(x-t)} |f(t)|^2 dt$$

by the Cauchy-Schwarz inequality. Since

$$\int_x^\infty e^{\alpha(x-t)} dt = \frac{1}{\alpha},$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty}|u(x)|^2dx &\le \frac{1}{\alpha}\int_{-\infty}^{\infty}\int_{x}^{\infty}e^{\alpha(x-t)}\,|f(t)|^2dt\,dx\\ &= \frac{1}{\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{t}e^{\alpha(x-t)}dx|f(t)|^2dt\\ &= \frac{1}{\alpha^2}\|f\|^2.\end{aligned} \tag{15.25}$$

Looking back, we note the following: If $\alpha > 0$, then for each continuous function $f \in L^2$, there is a unique continuously differentiable solution $u \in L^2$ of (15.20). This solution is given by (15.24) and satisfies

$$\|u\| \le \frac{\|f\|}{\alpha}. \tag{15.26}$$

Now, suppose f is any function in L^2. Then, by the results of the preceding section, there is a sequence $\{f_n\}$ of functions in C_0^∞ converging to f in L^2. For each f_n, there is a solution $u_n \in D(A)$ of

$$(A-\lambda)u_n = f_n. \tag{15.27}$$

Moreover, by (15.26),

$$\|u_n - u_m\| \le \frac{\|f_n - f_m\|}{\alpha} \longrightarrow 0 \quad \text{as} \quad m,n \longrightarrow \infty, \tag{15.28}$$

showing that $\{u_n\}$ is a Cauchy sequence in L^2. Since L^2 is complete, there is a $u \in L^2$ such that $u_n \to u$. By definition, $u \in D(\bar{A})$ and $(\bar{A}-\lambda)u = f$. Since $\|u_n\| \le \|f_n\|/\alpha$ for each n, we see that u and f satisfy (15.26).

This shows that when $\alpha > 0$, equation (15.19) can be solved for each $f \in L^2$. The same is true for $\alpha < 0$. In this case, (15.22) shows that $|u(x)| \to \infty$ as $x \to -\infty$ unless

$$u(0) = \int_{-\infty}^{0} e^{-\lambda t} f(t)dt. \tag{15.29}$$

Thus, the desired solution of (15.20) is

$$u(x) = \int_{-\infty}^{x} e^{\lambda(x-t)} f(t)\,dt. \tag{15.30}$$

Simple applications of Schwarz's inequality now show that u satisfies

$$\|u\| \le \frac{\|f\|}{|\alpha|}. \tag{15.31}$$

Applying the limiting process as before, we see that (15.19) can be solved for all $f \in L^2$ in this case as well.

What about uniqueness? The fact that the solutions of (15.20) are unique when they exist does not imply that the same is true for (15.19), since solutions of (15.20) are assumed continuously differentiable.

To tackle this problem, suppose $\alpha \neq 0$ and that for some $f \in L^2$, equation (15.19) had two distinct solutions. Then their difference u would be a solution of

$$(\bar{A} - \lambda)u = 0. \tag{15.32}$$

Since $u \in D(\bar{A})$, there is a sequence $\{u_n\}$ of functions in $D(A)$ such that $u_n \to u$ while $Au_n \to \lambda u$. Let v be any function in $D(A)$, and assume that (15.13) holds. Assuming this, we have in the limit,

$$\lambda(u, v) = -(u, v').$$

Whence,

$$(u, v' + \bar{\lambda} v) = 0$$

for all $v \in D(A)$. Since $\Re e(-\bar{\lambda}) = -\Re e\lambda = -\alpha \neq 0$, we know, by the argument above, that for each $g \in C_0^\infty$, there is a $v \in D(A)$ such that

$$v' + \bar{\lambda} v = g.$$

Hence,

$$(u, g) = 0$$

for all $g \in C_0^\infty$. Since C_0^∞ is dense in L^2 (cf. Section 15.3), it follows that $u = 0$.

It remains to prove (15.13) for $v \in D(\bar{A})$. Since (15.13) holds whenever $v \in C_0^\infty$, it suffices to show that for each $v \in D(A)$, there is a sequence $\{\varphi_k\}$ of functions in C_0^∞ such that

$$\|\varphi_k - v\| \longrightarrow 0, \quad \|\varphi_k' - v'\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \tag{15.33}$$

To prove this, we remark that for $\psi \in C_0^\infty$ and $w \in L^2$, we have

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \psi(x-t) w(t)\, dt \right|^2 dx$$

$$\leq \int_{-\infty}^{\infty} |\psi(\tau)|\, d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x-t)| dx\, |w(t)|^2 dt$$

$$= \left(\int_{-\infty}^{\infty} |\psi(\tau)| d\tau \right)^2 \|w\|^2.$$

In particular, if we take $\psi(t) = \varphi_n'(t)$ and $w = u - u_R$, we see that for each n and each $u \in L^2$,

$$AJ_n u_R \longrightarrow AJ_n u \quad \text{as} \quad R \longrightarrow \infty. \tag{15.34}$$

Now, let v be any function in $D(A)$. Let $\varepsilon > 0$ be given, and pick n so large that

$$\|v - J_n v\| < \frac{\varepsilon}{2}, \quad \|v' - J_n v'\| < \frac{\varepsilon}{2}.$$

Once n is fixed, we can take R so large that

$$\|J_n(v_R - v)\| < \frac{\varepsilon}{2}, \quad \|AJ_n(v_R - v)\| < \frac{\varepsilon}{2}.$$

Since $J_n v_R \in C_0^\infty$ and $AJ_n v = J_n v'$, (15.33) is proved.

We have seen that the half-planes $\Re e\lambda > 0$ and $\Re e\lambda < 0$ are in $\rho(\bar{A})$. What about the case $\alpha = 0$? A hint that things are not so nice comes from examining (15.22). From this we see that u can be in L^2 only if (15.23) and (15.29) both hold. In this case, we must have

$$\int_{-\infty}^{\infty} e^{i\beta t} f(t)\, dt = 0. \tag{15.35}$$

Thus, (15.35) is a necessary condition that (15.20) have a solution. To make matters worse, if

$$f(t) = \frac{e^{i\beta t}}{t}, \quad t \geq a,$$

for some $a > 0$, then (15.21) gives

$$\begin{aligned} e^{-i\beta x} u(x) &= u(0) + \int_0^a e^{-i\beta t} f(t)\, dt + \int_a^x \frac{dt}{t} \\ &= C + \log \frac{x}{a} \end{aligned}$$

for $x > a$. This is clearly not in L^2, even though f is continuous and in L^2.

However, the fact that we cannot solve (15.20) for all continuous $f \in L^2$ when $\lambda = i\beta$ does not prove that $i\beta$ is in $\sigma(\bar{A})$. But this, indeed, is the case. To show it, let $\psi(t)$ be any function in C_0^∞ such that $\|\psi\| = 1$. For $\varepsilon > 0$ set

$$\psi_\varepsilon(t) = \sqrt{\varepsilon} e^{i\beta t} \psi(\varepsilon t). \tag{15.36}$$

Then

$$A\psi_\varepsilon = i\beta\psi_\varepsilon + \varepsilon^{3/2} e^{i\beta t} \psi'(\varepsilon t),$$

which shows that

$$\|(A - i\beta)\psi_\varepsilon\|^2 = \varepsilon^3 \int_{-\infty}^{\infty} |\psi'(\varepsilon t)|^2 dt = \varepsilon^2 \|\psi'\|^2.$$

But

$$\|\psi_\varepsilon\|^2 = \varepsilon \int_{-\infty}^{\infty} |\psi(\varepsilon t)|^2 dt = \|\psi\|^2 = 1.$$

Letting $\varepsilon \to 0$, we see that

$$\|\psi_\varepsilon\| = 1, \quad \|(A - i\beta)\psi_\varepsilon\| \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0, \tag{15.37}$$

which shows that $i\beta$ cannot be in $\rho(\bar{A})$.

We have therefore seen that $\lambda \in \sigma(\bar{A})$ if and only if $\Re e\lambda = 0$.

15.5. Another approach

We have just seen that $\sigma(\bar{A})$ consists of the imaginary axis. Most people would stop at this point since the original problem is solved. Some, however, may not be satisfied. Such people might be wondering what it was in the nature of the operator that caused its spectrum to be this particular set. In other words, is there a way of looking at the operator and telling what its spectrum is? In this section, we shall introduce a method that sheds much light on the matter. It employs the Fourier transform defined by

$$Fu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} u(t)\, dt. \tag{15.38}$$

The function Fu is bounded and continuous so long as $u \in L^1$, i.e., so long as

$$\int_{-\infty}^{\infty} |u(t)|\, dt < \infty. \tag{15.39}$$

However, we shall find it convenient to introduce a class S of functions having properties suitable for our purposes. A function $u(t)$ is said to be in S if u is infinitely differentiable in $(-\infty, \infty)$ and

$$\frac{t^j d^k u(t)}{dt^k}$$

is a bounded function for any pair of nonnegative integers j, k. Clearly, any function in C_0^∞ is also in S, and since C_0^∞ is dense in L^2 (Section 15.3), the same is true of S. We leave as a simple exercise the fact that $u(t)/P(t)$ is in S whenever $u \in S$ and P is a polynomial that has no real roots.

If $u \in S$, then $t^k u(t)$ is in L^1 for each k. Hence, we may differentiate (15.38) under the integral sign as much as we like, obtaining

$$D_x^k Fu(x) = (-i)^k F[t^k u(t)], \quad k = 1, 2, 3, \cdots, \tag{15.40}$$

where

$$D_x^k = \frac{d^k}{dx^k}.$$

Moreover, if $u \in S$, so is u'. Thus,

$$Fu' = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-ixt} u'(t)\, dt.$$

Integrating by parts, we obtain

$$\int_{-R}^{R} e^{-ixt} u'(t)\, dt = ix \int_{-R}^{R} e^{-ixt} u(t)\, dt + e^{-ixR} u(R) - e^{ixR} u(-R).$$

Since $u \in S$, we must have $u(R) \to 0,\ u(-R) \to 0$ as $R \to \infty$. Thus,

$$Fu' = ixFu. \tag{15.41}$$

Repeated applications of (15.41) give

$$F[D_t^k u] = (ix)^k Fu, \quad k = 1, 2, \cdots . \tag{15.42}$$

Combining (15.40) and (15.42), we get

$$x^j D_x^k Fu = (-i)^{j+k} F[D_t^j (t^k u)], \quad j, k = 1, 2, \cdots , \tag{15.43}$$

which shows that $Fu \in S$ when $u \in S$.

We still need two important properties of the Fourier transform. The first is the inversion formula

$$u(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} Fu(x)\, dx, \quad u \in S, \tag{15.44}$$

and the second is Parseval's identity

$$(u, v) = (Fu, Fv). \tag{15.45}$$

Let me assume these for the moment (they will be proved in the next section) and show how they can be used in our situation. Note that we can put (15.44) in the following forms:

$$F[Fu(-x)](\xi) = u(\xi), \quad u \in S. \tag{15.46}$$

$$F[Fu(x)](-\xi) = u(\xi), \quad u \in S. \tag{15.47}$$

These show that for each function $w(x) \in S$, there is a $u(t) \in S$ such that $w = Fu$.

Now, suppose that $f \in L^2$ and that $u \in L^2$ is a solution of

$$(\bar{A} - \lambda) u = f. \tag{15.48}$$

By definition, there is a sequence $\{u_n\}$ of functions in $D(A)$ such that

$$u_n \longrightarrow u, \quad (A - \lambda) u_n \longrightarrow f \quad \text{as} \quad n \longrightarrow \infty$$

in L^2. Let v be an arbitrary function in S. Then, by (15.13) and (15.33),

$$((A - \lambda) u_n, v) = -(u_n, v' + \bar{\lambda} v).$$

Thus, in the limit, we get

$$(f, v) = -(u, v' + \bar{\lambda} v).$$

By (15.45) and (15.41), this becomes

$$\begin{aligned} (Ff, Fv) &= -(Fu, F[v' + \bar{\lambda} v]) \\ &= -(Fu, [ix + \bar{\lambda}] Fv) \\ &= ([ix - \lambda] Fu, Fv). \end{aligned}$$

This is true for all $v \in S$. Moreover, we remarked above that for each $w \in S$, there is a $v \in S$ such that $w = Fv$. Hence,

$$(Ff - [ix - \lambda]Fu, w) = 0$$

for all $w \in S$. Since S is dense in L^2, this implies that

$$Fu = \frac{Ff}{ix - \lambda}. \tag{15.49}$$

Now, let us examine (15.49). If $\alpha \neq 0$, then $ix - \lambda$ is a polynomial that does not vanish on the real axis. Hence, if $f \in S$, the same is true of the right-hand side of (15.49). Thus, we can solve (15.49) for u, obtaining a function in S. Clearly, this function is a solution of (15.20). Moreover,

$$|Fu| \leq \frac{|Ff|}{|\alpha|}, \tag{15.50}$$

which implies

$$\|Fu\| \leq \frac{\|Ff\|}{|\alpha|},$$

which, in turn, implies, via (15.45),

$$\|u\| \leq \frac{\|f\|}{|\alpha|}. \tag{15.51}$$

The reasoning of the last section now shows that $\lambda \in \rho(\bar{A})$.

On the other hand, if $\alpha = 0$, then $ix - \lambda$ vanishes for $x = \beta$. If $f \in S$, then the only way we can have $u \in L^2$ is if $Ff(\beta) = 0$. This is precisely (15.35). Thus, (15.49) cannot be solved for all f, and consequently, $\lambda \in \sigma(\bar{A})$.

Now we can "see" why the imaginary axis forms the spectrum of A. If we write our original operator in the form

$$A = P(D_t), \tag{15.52}$$

then $P(\xi) = \xi$. The denominator appearing in the right-hand side of (15.49) is $P(ix) - \lambda$. If this polynomial has no real roots, then $\lambda \in \rho(\bar{A})$. Otherwise, it is in $\sigma(\bar{A})$.

Thus, we can make the following conjecture: Let $P(\xi)$ be any polynomial

$$P(\xi) = \sum_0^m a_k \xi^k.$$

Consider the differential operator A given by (15.52):

$$A = \sum_0^m a_k D_t^k$$

defined on a suitable class of functions in L^2 (say, S). If $\bar{A}$ is the closure of A, then $\lambda \in \rho(\bar{A})$ if and only if the polynomial $P(ix) - \lambda$ has no real roots. We shall see, in Section 15.8, that this is, indeed, the case.

We want to make a remark concerning the nature of the spectrum of $\bar{A}$. It has no eigenvalues. For if $f = 0$, then (15.49) implies that $Fu = 0$ and consequently, $u = 0$. Since there is a sequence ψ_ε satisfying (15.37), it therefore follows that the range of $\bar{A} - i\beta$ is not closed for all real β. In particular, the spectrum of $\bar{A}$ coincides with its essential spectrum (see Section 7.5).

15.6. The Fourier transform

We now give proofs of (15.44) and (15.45). To prove the former, set

$$G_R(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{i\xi x} Fu(x)\, dx.$$

We wish to show that $G_R \to u$ in L^2 as $R \to \infty$. Now,

$$\begin{aligned} G_R(\xi) &= \frac{1}{2\pi} \int_{-R}^{R} e^{i\xi x} \left[\int_{-\infty}^{\infty} e^{-ixt} u(t)\, dt \right] dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(t) \left[\int_{-R}^{R} e^{i(\xi - t)} dx \right] dt. \end{aligned}$$

(We were able to interchange the order of integration because the double integral is absolutely convergent.) Now,

$$\int_{-R}^{R} e^{-i\xi s} ds = 2s^{-1} \sin Rs. \tag{15.53}$$

Moreover, we know that

$$\int_{-\infty}^{\infty} \frac{\sin Rs\, ds}{s} = \pi. \tag{15.54}$$

Hence,

$$\begin{aligned} G_R(\xi) - u(\xi) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t) - u(\xi)}{t - \xi} \sin R(t - \xi)\, dt \\ &= \frac{1}{\pi R} \int_{-\infty}^{\infty} \cos R(t - \xi) \frac{d}{dt} \left(\frac{u(t) - u(\xi)}{t - \xi} \right) dt \end{aligned} \tag{15.55}$$

by integrating by parts. Set

$$H(t, \xi) = \frac{d}{dt} \left[\frac{u(t) - u(\xi)}{t - \xi} \right] = \frac{(t - \xi)u'(t) - u(t) + u(\xi)}{(t - \xi)^2} \tag{15.56}$$

and

$$g(t) = (t - \xi)u'(t) - u(t) + u(\xi). \tag{15.57}$$

Expanding $g(t)$ in a Taylor series with remainder, we get

$$g(t) = g(\xi) + (t-\xi)g'(\xi) + \frac{1}{2}(t-\xi)^2 g''(\xi_1), \tag{15.58}$$

where ξ_1 is between ξ and t. Since

$$g'(t) = (t-\xi)u''(t) \tag{15.59}$$

and

$$g''(t) = (t-\xi)u'''(t) + u''(t), \tag{15.60}$$

we have

$$g(t) = \frac{1}{2}(t-\xi)^2[(\xi_1-\xi)u'''(\xi_1) + u''(\xi_1)]. \tag{15.61}$$

From (15.61) we see that $H(t,\xi)$ is continuous in t and ξ together. Moreover, since $u \in S$, we see by (15.59) and (15.61) that

$$|g(t)| \leq C_1(1+|\xi|) \tag{15.62}$$

$$|g(t)| \leq C_2|t-\xi|^2(1+|\xi|). \tag{15.63}$$

Substituting into (15.55), we get

$$|G_R(\xi) - u(\xi)| \leq \frac{1}{\pi R}\int_{-\infty}^{\infty} |H(t,\xi)|dt$$

$$\leq \frac{1}{\pi R}\left[\int_{|t-\xi|<1} |H(t,\xi)|dt + \int_{|t-\xi|>1} |H(t,\xi)|dt\right]$$

$$\leq \frac{1+|\xi|}{\pi R}\left[2C_1 + 2C_2\int_1^{\infty} \frac{ds}{s^2}\right].$$

Letting $R \to \infty$, we see that $G_R(\xi) \to u(\xi)$. Moreover, we see that this convergence is uniform on any bounded interval.

To prove (15.45), note that

$$\int_{-R}^{R} u(t)\overline{v(t)}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\left[\int_{-\infty}^{\infty} e^{itx}Fu(x)\,dx\right]\overline{v(t)}dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-R}^{R} e^{itx}\overline{v(t)}\,dt\right] Fu(x)\,dx.$$

Since

$$\int_{-\infty}^{\infty} e^{itx}\overline{v(t)}\,dt = \overline{Fv},$$

we have

$$\left|\int_{-R}^{R} u(t)\overline{v(t)}\,dt - (Fu, Fv)\right| \leq \int_{-\infty}^{\infty}\left|\int_{|t|>R} e^{itx}\overline{v(t)}\,dt\right| \cdot |Fu(x)|dx$$

$$\leq \int_{-\infty}^{\infty} |Fu(x)| dx \int_{|t|>R} |v(t)| dt \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

This completes the proof. □

15.7. Multiplication by a function

Let $q(t)$ be a function defined on $(-\infty, \infty)$. We can define an operator Q on L^2 corresponding to q as follows: A function $u \in L^2$ is in $D(Q)$ if $qu \in L^2$. We then set $Qu = qu$.

We are interested in the following problem: We want to find conditions on q which will ensure that Q will be $\bar{A}$-compact (see Section 7.2). One particular benefit we derive from such a situation is that it implies

$$\sigma_e(\bar{A} + Q) = \sigma_e(\bar{A}) \tag{15.64}$$

(see Theorem 7.28). But

$$\sigma_e(\bar{A}) = \sigma(\bar{A}), \tag{15.65}$$

and $\sigma(\bar{A})$ is known to be the imaginary axis. It will therefore follow that $\sigma_e(\bar{A} + Q)$ consists of the imaginary axis.

In order that Q be $\bar{A}$-compact we must have $D(Q) \supset D(\bar{A})$. Moreover, if $\{u_n\}$ is a sequence of functions in $D(\bar{A})$ such that

$$\|u_n\| + \|\bar{A}u_n\| \leq C, \tag{15.66}$$

this must imply that $\{Qu_n\}$ has a convergent subsequence. This can be true only if

$$\|qu\| \leq C_1(\|u\| + \|\bar{A}u\|), \quad u \in D(\bar{A}). \tag{15.67}$$

If (15.67) did not hold, there would be a sequence $\{u_n\}$ of functions in $D(\bar{A})$ such that (15.66) holds, while $\|qu_n\| \to \infty$. In this case, Q could not be $\bar{A}$-compact. A weaker form of (15.67) is

$$\|qu\| \leq C_1(\|u\| + \|Au\|), \quad u \in D(A). \tag{15.68}$$

By what we have just said, it is a necessary condition for Q to be $\bar{A}$-compact.

Let us see what (15.68) implies concerning the function q. Let φ be a function in C_0^∞ which satisfies

$$\varphi(t) \geq 0, \quad -\infty < t < \infty,$$

$$\varphi \geq 1, \quad 0 < t < 1.$$

For each real a, the function $\psi_a(t) = \varphi(t - a)$ is in C_0^∞. Such a function is certainly in $D(A)$. Hence, in order that (15.68) hold, we must have

$$\int_{-\infty}^{\infty} |q\psi_a|^2 dt \leq C_1(\|\psi_a\| + \|\psi_a'\|),$$

which gives

$$\int_a^{a+1} |q(t)|^2 dt \leq C_1(\|\varphi\| + \|\varphi'\|) = C_1 C_2.$$

Thus, we have shown that in order for (15.68) to hold, we must have

$$\int_a^{a+1} |q(t)|^2 dt \leq M < \infty, \quad a \text{ real.} \tag{15.69}$$

Next, I claim that for Q to be $\bar{A}$-compact, one needs in addition to (15.69) the fact that

$$\int_a^{a+1} |q(t)|^2 dt \longrightarrow 0 \quad \text{as } |a| \longrightarrow \infty. \tag{15.70}$$

Suppose (15.70) did not hold. Then there would be a sequence $\{a_k\}$ such that $|a_k| \to \infty$ while

$$\int_{a_k}^{a_k+1} |q(t)|^2 dt \geq \delta > 0, \quad k = 1, 2, \cdots. \tag{15.71}$$

Let $\varphi(t)$ be defined as above, and set

$$v_k(t) = \varphi(t - a_k), \quad k = 1, 2, \cdots. \tag{15.72}$$

Clearly,

$$\|v_k\| + \|Av_k\| = \|\varphi\| + \|\varphi'\| = C_2.$$

If Q were $\bar{A}$-compact, there would be a subsequence (again denoted by $\{v_k\}$) such that qv_k converges in L^2 to some function f. By (15.71),

$$\int_{-\infty}^{\infty} |qv_k|^2 dt \geq \int_{a_k}^{a_k+1} |q(t)|^2 dt \geq \delta,$$

which implies

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \geq \delta. \tag{15.73}$$

On the other hand, for any finite interval I, we have $v_k(t) \equiv 0$ on I for k sufficiently large. This shows that

$$\int_I |f(t)|^2 dt = 0$$

for each bounded interval I. This clearly contradicts (15.73), showing that (15.70) holds.

Now let us show that (15.69) and (15.70) are sufficient for Q to be $\bar{A}$-compact. Let us first prove (15.68) from (15.69). It clearly suffices to do so for q, u real valued. Let I be any interval of length one and let x and x' be two points in it. For $u \in D(A)$ we have

$$u^2(x) - u^2(x') = \int_{x'}^{x} \frac{d}{dt}[u^2(t)]dt = 2\int_{x'}^{x} u'(t)u(t)\, dt.$$

Hence,

$$|u^2(x) - u^2(x')| \leq 2\int_I |u'(t)u(t)|dt \leq \int_I [(u')^2 + u^2]dt. \tag{15.74}$$

By the mean value theorem, we can pick x' so that

$$u^2(x') = \int_I u^2 dt.$$

Hence,

$$u^2(x) \leq \int_I [(u')^2 + 2u^2]dt. \tag{15.75}$$

Multiplying both sides by $q^2(x)$ and integrating over I, we have

$$\int_I q^2(x)u^2(x)dx \leq \int_I q^2(x)dx \int_I [(u')^2 + 2u^2]dt \leq M\int_I [(u')^2 + 2u^2]dt \tag{15.76}$$

by (15.69). Summing over intervals of unit length, we obtain

$$\int_{-\infty}^{\infty} q^2(x)u^2(x)dx \leq M\int_{-\infty}^{\infty} [(u')^2 + 2u^2]dt, \tag{15.77}$$

which implies (15.68).

Next consider the function q_R [see (15.15)]. Denote the corresponding operator by Q_R. We claim that, for $R < \infty$, the operator Q_R is $\bar{A}$-compact. To see this, we note that for $u \in D(A)$, I a bounded interval and $x, x' \in I$, we have

$$u(x) - u(x') = \int_{x'}^{x} u'(t)\, dt.$$

Thus,

$$|u(x) - u(x')|^2 \leq |x - x'| \int_{x'}^{x} |u'(t)|^2 dt \leq |x - x'| \cdot \|Au\|^2. \tag{15.78}$$

Moreover, by taking x' to satisfy

$$u(x') = \int_I u(t)\, dt$$

(which we can do by the theorem of the mean), we have, when I is of length one,

$$|u(x')|^2 \leq \int_I |u(t)|^2 dt \leq \|u\|^2. \tag{15.79}$$

Combining (15.78) and (15.79), one obtains

$$|u(x)| \leq \|u\| + \|Au\|. \tag{15.80}$$

Now, we claim that functions $u \in D(\bar{A})$ are continuous and satisfy

$$|u(x)| \leq \|u\| + \|\bar{A}u\|, \quad u \in D(\bar{A}), \tag{15.81}$$

(15.82) $$|u(x)-u(x')|^2 \le |x-x'|\cdot\|\bar{A}u\|^2, \quad u \in D(\bar{A}).$$

In fact, if $u \in D(\bar{A})$, there is a sequence $\{u_n\}$ of functions in $D(A)$ such that $u_n \to u$, $Au_n \to \bar{A}u$ in L^2. By (15.80),

$$|u_n(x)-u_m(x)| \le \|u_n-u_m\| + \|Au_n - Au_m\|,$$

which shows that the sequence $\{u_n(x)\}$ of continuous functions converges uniformly on any bounded interval. Since the limit must be u, we see that u is the uniform limit of continuous functions and consequently, is continuous. Moreover, by (15.78),

$$|u_n(x)-u_n(x')|^2 \le |x-x'|\cdot\|Au_n\|^2$$

for each n. Taking the limit as $n\to\infty$, we get (15.82).

To show that Q_R is $\bar{A}$-compact, suppose that $\{u_n\}$ is a sequence of functions in $D(\bar{A})$ satisfying (15.66). Then, by (15.81) and (15.82),

$$|u_n(x)| \le C, \quad |u_n(x)-u_n(x')|^2 \le C^2|x-x'|.$$

The first inequality says that the u_n are uniformly bounded, while the second implies that they are equicontinuous (i.e., one modulus of continuity works for all of them). We can now make use of the Arzela-Ascoli theorem, which says that on a finite interval, a uniformly bounded equicontinuous sequence has a uniformly convergent subsequence. Let us pick the interval to be $[-R,R]$ and denote the subsequence again by $\{u_n\}$. Now,

$$\int_{-\infty}^{\infty} |q_R|^2|u_n-u_m|^2 dt = \int_{-R}^{R} |q|^2|u_n-u_m|^2 dt$$
$$\le \max_{|x|\le R} |u_n(x)-u_m(x)|^2 \int_{-R}^{R} |q|^2 dt$$
$$\le 2(R+1)M \max_{|x|\le R} |u_n(x)-u_m(x)|^2 \longrightarrow 0 \quad \text{as } m,n\longrightarrow\infty.$$

This shows that $\{q_R u_n\}$ converges in L^2. Thus, Q_R is $\bar{A}$-compact.

We can now show that Q is $\bar{A}$-compact. First of all, (15.77) implies

(15.83) $$\|qu\| \le C_3 M(\|u\| + \|Au\|), \quad u \in D(A),$$

where the constant C_3 does not depend on u or q, and M satisfies (15.69). We claim that (15.83) implies that $D(Q) \supset D(\bar{A})$ and that

(15.84) $$\|qu\| \le C_3 M(\|u\| + \|\bar{A}u\|), \quad u \in D(\bar{A}).$$

This is a consequence of the following reasoning. If $u \in D(\bar{A})$, there is a sequence $\{u_n\} \subset D(A)$ such that $u_n \to u$, $Au_n \to \bar{A}u$ in L^2. By what we have just shown, u_n converges pointwise to u. By (15.83), $\{qu_n\}$ is a Cauchy sequence in L^2 and, hence, converges to a function $h \in L^2$. But qu_n converges pointwise to qu. Hence, $h = qu$, showing that $u \in D(Q)$. Applying (15.83) to u_n and taking the limit, we get (15.84).

Since $\bar{A}$ is closed, we can make $D(\bar{A})$ into a Banach space by introducing the graph norm on it. Inequality (15.84) states that Q is a bounded operator from $D(\bar{A})$ to L^2 with norm $C_3 M$ or less. If we replace Q by $Q - Q_R$, we see that $Q - Q_R$ is a bounded operator from $D(\bar{A})$ to L^2 with

$$\|Q - Q_R\| \le C_3 \sup \int_a^{a+1} |q(t) - q_R(t)|^2 dt \le C_3 \sup_{|a| \ge R-1} \int_a^{a+1} |q(t)|^2 dt.$$

Thus, $\|Q - Q_R\| \to 0$ as $R \to \infty$ by (15.70). But we have shown that Q_R is a compact operator from $D(\bar{A})$ to L^2 for each $R < \infty$. We now merely apply Theorem 4.11 to conclude that Q is a compact operator from $D(\bar{A})$ to L^2. This completes the proof. □

Finally, we want to point out that if

$$\int_a^{a+1} |q(t)|^2 dt < \infty, \quad a \text{ real}, \tag{15.85}$$

then (15.70) implies (15.69). To see this, suppose (15.70) and (15.85) held, but (15.69) did not. Then there would be a sequence $\{a_k\}$ such that

$$\int_{a_k}^{a_k+1} |q(t)|^2 dt \longrightarrow \infty.$$

By (15.70), the sequence $\{a_k\}$ is bounded. Hence, it has a subsequence (also denoted by $\{a_k\}$) such that $a_k \to a$, where a is a finite number. In particular,

$$|a_k - a| < 1$$

for k sufficiently large. But for such k,

$$\int_{a-1}^{a+2} |q(t)|^2 dt \ge \int_{a_k}^{a_k+1} |q(t)|^2 dt \longrightarrow \infty,$$

which shows that (15.85) is violated.

Thus, we may conclude that Q is $\bar{A}$-compact if and only if (15.70) and (15.85) hold.

15.8. More general operators

Consider the differential operator

$$Bu = a_m A^m u + a_{m-1} A^{m-1} u + \cdots + a_0 u, \quad a_m \ne 0, \tag{15.86}$$

where $A = D_t$ is the operator defined in Section 15.2. Thus, $A^k = d^k/dt^k$ and $A^0 = 1$. We can write B symbolically in the form

$$B = P(A), \tag{15.87}$$

where

$$P(z) = a_m z^m + \cdots + a_0. \tag{15.88}$$

We take $D(B)$ to be the set of those $u \in L^2$ having continuous derivatives up to order m in L^2. The argument used in Section 15.3 in connection with A shows that B is closable. Let $\bar{B}$ denote its closure. We are interested in determining $\sigma(\bar{B})$.

Let us employ the method of Section 15.5. Applying the Fourier transform to the equation

$$(\bar{B} - \lambda)u = f, \tag{15.89}$$

we obtain, by the argument of that section,

$$Fu = \frac{Ff}{P(ix) - \lambda}. \tag{15.90}$$

Assume first that $P(ix) \neq \lambda$ for all real x. Then there exists a positive constant c such that

$$|P(ix) - \lambda| \geq c, \quad x \text{ real}. \tag{15.91}$$

(We leave this as an exercise.) In this case

$$|Fu| \leq \frac{|Ff|}{c},$$

and consequently,

$$\|u\| \leq \frac{\|f\|}{c}. \tag{15.92}$$

Now, if f is any function in S, the same is true of Ff and of $Ff/[P(ix) - \lambda]$. Hence, there is a function $u \in S$ satisfying (15.90). Moreover, an application of (15.44) to both sides of

$$[P(ix) - \lambda]Fu = Ff$$

shows that u is a solution of

$$(B - \lambda)u = f. \tag{15.93}$$

Thus, (15.93) can be solved when $f \in S$. The solution is in S and satisfies (15.92). Moreover, if f is any function in L^2, there is a sequence of functions in S such that $f_n \to f$ in L^2. For each n there is a solution $u_n \in S$ of

$$(B - \lambda)u_n = f_n.$$

By (15.92),

$$\|u_n - u_m\| \leq \frac{\|f_n - f_m\|}{c} \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty.$$

This shows that there is a $u \in L^2$ such that $u_n \to u$. Since $\bar{B}$ is a closed operator, it follows that $u \in D(B)$ and satisfies (15.89). All of this goes to show that $P(ix) \neq \lambda$ for all real x implies that $\lambda \in \rho(\bar{B})$.

Next, suppose that there is a real β such that $P(i\beta) = \lambda$. In order for Fu to be in L^2, we see by (15.90) that we must have $Ff(\beta) = 0$. Thus, (15.35) must hold. This shows that $\lambda \in \sigma(\bar{B})$. However, λ is not an eigenvalue since $f = 0$ implies $u = 0$ from (15.90).

Moreover, we can show that $R(\bar{B} - \lambda)$ is not closed. To this end, we use the function ψ_ε defined by (15.36). To compute $P(D_t)\psi_\varepsilon$, we make use of the formula

$$P(D_t)(vw) = \sum_0^m P^{(k)}(D_t)vD_t^k \frac{w}{k!}, \tag{15.94}$$

where

$$P^{(k)}(x) = \frac{d^k P(x)}{dx^k}.$$

We shall prove (15.94) in a moment. Meanwhile, let us use it. Thus,

$$\begin{aligned} P(D_t)\psi_\varepsilon &= \sqrt{\varepsilon} \sum_0^m P^{(k)}(D_t)e^{i\beta t} D_t^k \frac{\psi(\varepsilon t)}{k!} \\ &= \sqrt{\varepsilon} e^{i\beta t} \sum_0^m P^{(k)}(i\beta) D_t^k \frac{\psi(\varepsilon t)}{k!}. \end{aligned}$$

Since $P^{(0)}(i\beta) = P(i\beta) = \lambda$, this gives

$$[P(D_t) - \lambda]\psi_\varepsilon = \sqrt{\varepsilon} e^{i\beta t} \sum_1^m P^{(k)}(i\beta) D_t^k \frac{\psi(\varepsilon t)}{k!}.$$

Now,

$$\varepsilon \int_{-\infty}^{\infty} |D_t^k \psi(\varepsilon t)|^2 dt = \varepsilon^{2k} \int_{-\infty}^{\infty} |D_y^k \psi(y)|^2 dy.$$

Hence,

$$\|[P(D_t) - \lambda]\psi_\varepsilon\| \le \sum_1^m |P^{(k)}(i\beta)| \varepsilon^k \frac{\|D_t^k \psi\|}{k!} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Since $\|\psi_\varepsilon\| = 1$, we see that $R(\bar{B} - \lambda)$ cannot be closed.

To summarize, $\sigma(\bar{B})$ consists of those complex λ which equal $P(ix)$ for some real x. Moreover, $\sigma_e(\bar{B}) = \sigma(\bar{B})$.

It remains to prove (15.94). We know that $P(D_t)(vw)$ is the sum of terms of the form

$$a_{jk} D_t^j v D_t^k w,$$

where the a_{jk} are constants and $j + k \le m$. Combining the coefficients of $D_t^k w$, we get

$$P(D_t)(vw) = \sum_0^m T_k(D_t) v D_t^k w,$$

where the $T_k(z)$ are polynomials. Now take $v = e^{\alpha t}$ and $w = e^{\beta t}$. This gives

$$P(\alpha+\beta) = \sum_0^m T_k(\alpha)\beta^k. \tag{15.95}$$

Taking the derivative of order k with respect to β and then putting $\beta = 0$ shows that

$$T_k(\alpha) = \frac{P^{(k)}(\alpha)}{k!}.$$

This completes the proof. $\square$

We have shown that $\sigma(\bar{A})$ consists of the imaginary axis, while $\sigma(\bar{B})$ consists of all scalars of the form $P(ix)$, x real. Thus, we have the following variation of the spectral mapping theorem

$$\sigma[\overline{P(A)}] = P[\sigma(\bar{A})]. \tag{15.96}$$

15.9. B-Compactness

We now ask when Q is $\bar{B}$-compact. The answer, which may seem a bit surprising, is that a necessary and sufficient condition that Q be $\bar{B}$-compact is that (15.69) and (15.70) hold. This follows from the inequality

$$\|Au\| \leq C_4(\|u\| + \|Bu\|), \quad u \in D(B), \tag{15.97}$$

which we shall verify shortly.

First, (15.97) implies that $D(\bar{B}) \subset D(\bar{A})$. To see this, note that if $u \in D(\bar{B})$, then there is a sequence $\{u_n\}$ of functions in $D(B)$ such that $u_n \to u$, $Bu_n \to \bar{B}u$ in L^2. By (15.97), Au_n converges as well. Hence, $u \in D(\bar{A})$ and $Au_n \to \bar{A}u$. Applying (15.97) to u_n and taking the limit, we have

$$\|\bar{A}u\| \leq C_4(\|u\| + \|\bar{B}u\|), \quad u \in D(\bar{B}). \tag{15.98}$$

Now (15.69) and (15.70) imply that Q is $\bar{A}$-compact. Thus $D(Q) \supset D(\bar{A}) \supset D(\bar{B})$. Moreover, if $\{u_n\}$ is a sequence of functions in $D(\bar{B})$ satisfying

$$\|u_n\| + \|\bar{B}u_n\| \leq C_5, \tag{15.99}$$

then, by (15.98), the u_n satisfy

$$\|u_n\| + \|\bar{A}u_n\| \leq C_5(1 + C_4).$$

Since Q is $\bar{A}$-compact, it follows that $\{Qu_n\}$ has a convergent subsequence. Thus, Q is $\bar{B}$-compact.

Conversely, if Q is $\bar{B}$-compact, then we must have

$$\|qu\| \leq C_6(\|u\| + \|Bu\|), \quad u \in D(B) \tag{15.100}$$

[See (15.68)]. Making use of the functions $\psi_a(t)$ as in the derivation of (15.69), we have

$$\int_{-\infty}^{\infty} |q\psi_a|^2 dt \le C_6(\|\psi_a\| + \|B\psi_a\|).$$

Thus,

$$\int_a^{a+1} |q(t)|^2 dt \le C_6(\|\psi\| + \|B\psi\|) = C_6 C_7.$$

This gives (15.69). Similarly, if (15.70) did not hold, there would be a sequence $\{a_k\}$ such that $|a_k| \to \infty$ and (15.71) holds. Making use of the sequence $\{v_k\}$ defined by (15.72), we prove a contradiction as in Section 15.7.

It remains to prove (15.97). It is equivalent to

$$\|Au\|^2 \le C_8(\|u\|^2 + \|Bu\|^2),$$

which is equivalent to

$$\int_{-\infty}^{\infty} x^2 |Fu(x)|^2 dx \le C_8 \int_{-\infty}^{\infty} (1 + |P(ix)|^2)|Fu(x)|^2 dx \tag{15.101}$$

in view of (15.42) and (15.44). But (15.101) is a simple consequence of

$$x^2 \le C_8(1 + |P(ix)|^2). \tag{15.102}$$

The verification of (15.102) for any polynomial of degree 1 or more is left as an exercise.

Next, consider the operator

$$Eu = \sum_0^{m-1} q_k(t) A^k u, \tag{15.103}$$

where each of the functions $q_k(t)$ satisfies (15.69) and (15.70). We take $D(E) = D(B)$. We claim that

$$\|Eu\| \le C_9(\|u\| + \|Bu\|), \quad u \in D(B). \tag{15.104}$$

This shows that E is B-closable. This means that $u_n \to 0, \quad Bu_n \to 0$ implies $Eu_n \to 0$. By using a limiting process, we obtain a unique extension $\tilde{E}$ of E defined on $D(\bar{B})$. Now we can show that $\tilde{E}$ is $\bar{B}$-compact. To see this, note that by (15.68),

$$\|q_k A^k u\| \le C_{10}(\|A^k u\| + \|A^{k+1} u\|), \quad u \in D(B). \tag{15.105}$$

Thus,

$$\|Eu\| \le C_{11} \sum_0^m \|A^k u\|. \tag{15.106}$$

If we can show that

$$\sum_0^m \|A^k u\| \le C_{12}(\|u\| + \|Bu\|), \quad u \in D(B), \tag{15.107}$$

it will not only follow that (15.104) holds, but also that each term of E is $\bar{B}$-closable and its extension to $D(B)$ is $\bar{B}$-compact. Moreover, $\tilde{E}$ is the sum of the extension of its terms, so that $\tilde{E}$ is $\bar{B}$-compact as well.

Thus, it remains to prove (15.107). It is equivalent to

$$\int_{-\infty}^{\infty}(1 + x^2 + \cdots + x^{2m})|Fu(x)|^2 dx \le C_{13}\int_{-\infty}^{\infty}(1 + |P(ix)|^2)|Fu(x)|^2 dx,$$

which follows from

$$1 + x^2 + \cdots + x^{2m} \le C_{13}(1 + |P(ix)|^2). \tag{15.108}$$

Again the verification of (15.108) is left as an exercise.

Since $\tilde{E}$ is $\bar{B}$-compact, we can conclude that

$$\sigma_e(\bar{B} + \tilde{E}) = \sigma_e(\bar{B}) = \sigma(\bar{B}) = \{P(ix) : \ x \ \text{real}\}. \tag{15.109}$$

15.10. The adjoint of $\bar{A}$

Since $D(\bar{A})$ is dense in L^2, we know that $\bar{A}$ has a Hilbert space adjoint $\bar{A}^*$ (See Section 7.1 and Problem 4 of Chapter 11.) Let us try to determine $\bar{A}^*$.

If u and v are in $D(\bar{A})$, then

$$(\bar{A}u, v) = -(u, \bar{A}v). \tag{15.110}$$

To verify (15.110), note that it holds for $u \in D(A)$ and $v \in C_0^\infty$ by mere integration by parts [see (15.13)]. Next suppose u and v are in $D(A)$. By (15.33), there is a sequence $\{\varphi_k\}$ of functions in C_0^∞ such that $\varphi_k \to v, \ A\varphi_k \to Av$ in L^2. Since

$$(Au, \varphi_k) = -(u, A\varphi_k),$$

we have (15.110) in the limit. Finally, if u and v are in $D(\bar{A})$, then there are sequences $\{u_n\}$ and $\{v_n\}$ in $D(A)$ such that $u_n \to u, \ Au_n \to \bar{A}u, \ v_n \to v, \ Av_n \to \bar{A}v$ in L^2. Since

$$(Au_n, v_n) = -(u_n, Av_n),$$

we have (15.110) in the limit.

From (15.110) we expect $\bar{A}^*$ to be an extension of $-\bar{A}$. In fact, we can show that

$$\bar{A}^* = -\bar{A}. \tag{15.111}$$

To see this, let v be any function in $D(\bar{A}^*)$. Let $\lambda \neq 0$ be a real number. Then $\pm\lambda \in \rho(\bar{A})$. In particular, there is a $w \in D(\bar{A})$ such that

$$-(\bar{A}+\lambda)w = (\bar{A}^* - \lambda)v. \tag{15.112}$$

Now, if $u \in D(A)$, we have

$$((\bar{A}-\lambda)u, v) = (u, (\bar{A}^* - \lambda)v) = -(u, (\bar{A}+\lambda)w) = ((\bar{A}-\lambda)u, w)$$

by (15.110). Hence,

$$((\bar{A}-\lambda)u, v-w) = 0$$

for all $u \in D(\bar{A})$. Since $\lambda \in \rho(\bar{A})$, $R(\bar{A}-\lambda)$ is the whole of L^2. Hence, $v = w \in D(\bar{A})$. By (15.112), we have $\bar{A}^*v = -\bar{A}v$. This proves (15.111).

By (15.111), we have

$$(i\bar{A})^* = (-i)(-\bar{A}) = i\bar{A}, \tag{15.113}$$

which shows that the operator $i\bar{A}$ is selfadjoint. Since $\sigma(\bar{A})$ is the imaginary axis, $\sigma(i\bar{A})$ consists of the real axis.

Now, suppose $q(t)$ is a *real* function satisfying (15.69) and (15.70). Then Q is $i\bar{A}$-compact and

$$\sigma_e(i\bar{A}+Q) = \sigma_e(i\bar{A}) = \sigma(i\bar{A}) \tag{15.114}$$

consists of the real axis. All other points λ of the complex plane are in $\Phi_{i\bar{A}+Q}$ with $i\bar{A}+Q$ having index zero. Thus, a point λ that is not real can be in $\sigma(i\bar{A}+Q)$ only if it is an eigenvalue. But nonreal points λ cannot be eigenvalues of $i\bar{A}+Q$ since $(i\bar{A}u, u)$ and (Qu, u) are both real, the former because $i\bar{A}$ is selfadjoint (see Theorem 12.7) and the latter because q is real valued. Thus, if

$$(i\bar{A}+Q-\lambda)u = 0,$$

then we have

$$\Im m\ ((i\bar{A}+Q-\lambda)u, u) = -\Im m\ \lambda\|u\|^2 = 0.$$

This shows that either λ is real or $u = 0$. Thus, the operator $i\bar{A}+Q$ can have no nonreal eigenvalues. By what we have just said, this means that all nonreal points are in $\rho(i\bar{A}+Q)$. Consequently, $\sigma(i\bar{A}+Q)$ consists of the real axis. A repetition of the argument that proved (15.111) now shows that $i\bar{A}+Q$ is selfadjoint.

15.11. An integral operator

Let us now consider the operator

$$Vu(x) = \int_0^x u(t)\,dt \tag{15.115}$$

defined on those continuous functions $u(t)$ such that u and Vu are both in L^2. As before, it is easily checked that V is closable. In fact, we have for $u \in D(V)$ and $v(x) \in C_0^\infty$,

$$(Vu, v) = \int_{-\infty}^{\infty} \left[\int_0^x u(t)\, dt \right] \overline{v(x)}\, dx$$

$$= \int_0^{\infty} \left[\int_t^{\infty} \overline{v(x)}\, dx \right] u(t)\, dt - \int_{-\infty}^{0} \left[\int_{-\infty}^{t} \overline{v(x)}\, dx \right] u(t)\, dt. \tag{15.116}$$

Let $\bar{V}$ denote the closure of V, and let us try to determine $\sigma(\bar{V})$.

If f and u are smooth functions, then the equation

$$(\bar{V} - \mu)u = f \tag{15.117}$$

is equivalent to

$$u - \mu u' = f', \quad \mu u(0) + f(0) = 0.$$

If $\mu \neq 0$ and we set $\lambda = 1/\mu$, this becomes

$$u' - \lambda u = -\lambda f', \quad u(0) + \lambda f(0) = 0. \tag{15.118}$$

We have already solved equations of the type (15.118) for $\alpha = \Re e\, \lambda \neq 0$ (see Section 15.4). This will be the case when $\Re e\, \mu \neq 0$. Assuming $\alpha \neq 0$ for the moment, we have by (15.21) and integration by parts

$$e^{-\lambda x} u(x) = u(0) - \lambda \int_0^x e^{-\lambda t} f(t)\, dt. \tag{15.119}$$

This gives

$$u(x) = -\lambda f(x) - \lambda^2 \int_0^x e^{\lambda(x-t)} f(t) dt \tag{15.120}$$

in view of the second member of (15.118). This gives a candidate for a solution of (15.118). Is it in L^2? It will be only if

$$g(x) = \int_0^x e^{\lambda(x-t)} f(t)\, dt \tag{15.121}$$

is. But

$$|g(x)| = e^{\alpha x} \left| \int_0^x e^{-\lambda t} f(t)\, dt \right|. \tag{15.122}$$

If $\alpha > 0$ (which is the case when $\Re e\, \mu > 0$), we will have $|g(x)| \to \infty$ as $x \to \infty$ unless

$$\int_0^{\infty} e^{-\lambda t} f(t)\, dt = 0. \tag{15.123}$$

This immediately shows that we cannot solve (15.118) for all f. Let us assume that f satisfies (15.123). Then (15.120) is equivalent to

$$u(x) = -\lambda f(x) + \lambda^2 \int_x^\infty e^{\lambda(x-t)} f(t)\, dt. \tag{15.124}$$

This is easily seen to be in L^2. In fact, the reasoning in Section 15.4 gives

$$\|u\| \le \left(\frac{|\lambda| + |\lambda|^2}{\alpha^2} \right) \|f\|. \tag{15.125}$$

If $\alpha < 0$ (which is the case when $\Re e\ \mu < 0$), then (15.122) shows us that $|g(x)| \to \infty$ as $x \to -\infty$ unless

$$\int_{-\infty}^0 e^{-\lambda t} f(t)\, dt = 0. \tag{15.126}$$

On the other hand, if f satisfies (15.126), then (15.120) becomes

$$u(x) = -\lambda f(x) - \lambda^2 \int_{-\infty}^x e^{\lambda(x-t)} f(t)\, dt, \tag{15.127}$$

which again is seen to be in L^2 and satisfy (15.125). Moreover, if we apply the operator V to (15.120), we have

$$Vu(y) = -\lambda V f(y) - \lambda^2 \int_0^y \left[\int_t^y e^{\lambda(x-t)} dx \right] f(t)\, dt$$

$$= -\lambda \int_0^y e^{\lambda(y-t)} f(t)\, dt, \tag{15.128}$$

which shows that (15.120) is a solution of (15.117). If $\alpha > 0$ and (15.123) holds, then

$$Vu(y) = \lambda \int_y^\infty e^{\lambda(y-t)} f(t) dt, \tag{15.129}$$

which shows that $Vu \in L^2$. Similarly, if $\alpha < 0$ and (15.126) holds, then

$$Vu(y) = -\lambda \int_{-\infty}^y e^{\lambda(y-t)} f(t)\, dt, \tag{15.130}$$

showing that $Vu \in L^2$ in this case as well.

Thus, for $f \in L^2$ and having a continuous first derivative in L^2, we can solve (15.117) for $\alpha \ne 0$ provided f satisfies (15.123) or (15.126) depending on whether $\alpha > 0$ or $\alpha < 0$. Moreover, the solution u given by (15.120) is continuously differentiable and satisfies (15.125).

Now, suppose $\alpha > 0$ and $f(t)$ is any function in L^2 satisfying (15.123). Define the function $h_1(t)$ by

$$h_1(t) = \begin{cases} e^{-\lambda t}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

Then $h_1 \in L^2$. (If this is not obvious, then consider the functions $J_n h_{1R}$, which are in C_0^∞ and converge to h_1 in L^2. See Section 15.3.) Moreover, (15.123) merely states that

$$(f, h_1) = 0. \tag{15.131}$$

Now, we know that there is a sequence $\{f_n\}$ of functions in C_0^∞ which converge to f in L^2. Let ψ be any function in C_0^∞ such that $(\psi, h_1) \neq 0$. Let

$$g_n = f_n - \frac{(f_n, h_1)\psi}{(\psi, h_1)}.$$

Then $g_n \in C_0^\infty$ and

$$(g_n, h_1) = 0. \tag{15.132}$$

In addition

$$\|g_n - f_n\| \leq |(f_n, h_1)| \frac{\|\psi\|}{|(\psi, h_1)|}.$$

But

$$|(f_n, h_1)| = |(f_n - f, h_1)| \leq \|f_n - f\| \cdot \|h_1\|.$$

These last two inequalities show that $g_n \to f$ in L^2. Since $g \in C_0^\infty$ and it satisfies (15.132), we know that there is a solution of

$$(V - \lambda)u_n = g_n. \tag{15.133}$$

Moreover, by (15.125), we see that the u_n converge in L^2 to some function u. Thus, $u \in D(\bar{V})$ and it is a solution of (15.117).

Thus, we have shown that for $\alpha > 0$, (15.117) has a solution for $f \in L^2$ if and only if (15.123) [or (15.131)] holds. Similarly, for $\alpha < 0$, equation (15.117) has a solution for $f \in L^2$ if and only if (15.126) holds. In particular, we see that $R(\bar{V} - \mu)$ is closed in L^2 for $\Re e\ \mu \neq 0$.

How about uniqueness? To get an idea, suppose $u \in D(V)$ and $v(x)$ is a continuous function in $[0, \infty]$ which vanishes for x large. Then

$$\int_0^\infty V u \bar{v}\, dx = \int_0^\infty \left[\int_0^x u(t)\, dt\right] \overline{v(x)}\, dx = \int_0^\infty \left[\int_0^\infty \overline{v(x)}\, dx\right] u(t)\, dt. \tag{15.134}$$

Now, if $u \in D(\bar{V})$, we can apply (15.134) to a sequence $\{u_n\}$ of functions in $D(V)$ which converges to u in L^2 and such that $Vu_n \to \bar{V}u$. This gives

$$\int_0^\infty (\bar{V}u - \mu u)\bar{v} dx = \int_0^\infty \left[\int_t^\infty \overline{v(x)}\, dx - \mu\overline{v(t)}\right] u(t)\, dt. \tag{15.135}$$

Now, if $\bar{V}u - \mu u = 0$, we have

$$\int_0^\infty \left[\int_t^\infty \overline{v(x)}\, dx - \mu\overline{v(t)}\right] u(t)\, dt = 0 \tag{15.136}$$

for all such v. But we can show that for $\mu \neq 0$ and $\psi \in C_0^\infty$ there is a $v(x)$ continuous in $[0, \infty]$, vanishing for x large and satisfying

$$\int_t^\infty v(x)\,dx - \mu v(t) = \psi(t), \quad t \geq 0. \tag{15.137}$$

In fact, the method used above gives

$$v(x) = -\bar{\lambda}\psi(x) - \bar{\lambda}^2 \int_x^\infty e^{\bar{\lambda}(t-x)}\psi(t)\,dt. \tag{15.138}$$

Clearly, the function v given by (15.138) is of the type mentioned. We leave the simple task of verifying that it is a solution of (15.137) as an exercise. Since C_0^∞ is dense in L^2, it follows that $u(t) = 0$ for $t \geq 0$. A similar argument holds for $t \leq 0$.

For $\mu = 0$, it is even easier. We merely take $v(x) = -\psi'(x)$. This clearly has the desired properties and is a solution of (15.137). Thus, $\alpha(\bar{V} - \mu) = 0$ for all μ.

Before we go further, we must take a look at $D(V)$. In the case of A and B we know that $D(A)$ and $D(B)$ are dense in L^2 because they contain C_0^∞. But a moment's reflection shows that this is not the case for V. In fact, a function $\psi \in C_0^\infty$ is in $D(V)$ if and only if

$$\int_{-\infty}^0 \psi(t)\,dt = \int_0^\infty \psi(t)\,dt = 0. \tag{15.139}$$

If (15.139) holds, then $V\psi(x)$ vanishes for $|x|$ large, and hence, $V\psi \in C_0^\infty$. On the other hand, if, say,

$$c = \int_0^\infty \psi(t)\,dt \neq 0,$$

then $V\psi(x) \to c \neq 0$ as $x \to \infty$, showing that $V\psi$ cannot be in L^2. It is, therefore, far from apparent that $D(V)$ is dense in L^2.

However, this indeed is the case. We shall prove it by showing that the set of functions $\psi \in C_0^\infty$ which satisfy (15.139) is dense in L^2. To do so, it suffices to show for each $\varepsilon > 0$ and each $\varphi \in C_0^\infty$, there is a $\psi \in C_0^\infty$ satisfying (15.139) such that

$$\|\varphi - \psi\| < \varepsilon. \tag{15.140}$$

So suppose $\varepsilon > 0$ and $\varphi \in C_0^\infty$ are given. Let $g(t)$ be any function in C_0^∞ which vanishes for $t \leq 0$ and such that $g(t) \geq 0$ and

$$\int_0^\infty g(t)\,dt = 1. \tag{15.141}$$

Set

$$h(t) = \varphi(t) - \varepsilon c_1 g(-\varepsilon t) - \varepsilon c_2 g(\varepsilon t),$$

where

$$c_1 = \int_{-\infty}^{0} \varphi(t)\,dt, \quad c_2 = \int_{0}^{\infty} \varphi(t)\,dt.$$

Clearly, $h \in C_0^\infty$. Moreover,

$$\int_{-\infty}^{0} h(t)dt = c_1 - \varepsilon c_1 \int_{-\infty}^{0} g(-\varepsilon t)\,dt = 0,$$

and similarly,

$$\int_{0}^{\infty} h(t)\,dt = 0.$$

Thus, h satisfies (15.139). Moreover,

$$\begin{aligned}\|h - \varphi\|^2 &= \varepsilon^2 c_1^2 \int_{-\infty}^{0} g^2(-\varepsilon t)dt + \varepsilon^2 c_2^2 \int_{0}^{\infty} g^2(\varepsilon t)\,dt \\ &= \varepsilon^2(c_1^2 + c_2^2) \int_{0}^{\infty} g^2(r)\,dr.\end{aligned}$$

Thus, h can be made as close as desired to φ in the L^2 norm by taking ε sufficiently small. This shows that $D(V)$ is dense in L^2.

We now have the complete picture for $\Re e\ \mu \neq 0$. In this case, $\alpha(\bar{V} - \mu) = 0$ and $R(\bar{V} - \mu)$ is closed in L^2. Moreover, $R(\bar{V} - \mu)$ consists of those $f \in L^2$ which are orthogonal to the function h_1. This means that the annihilators of $R(\bar{V} - \mu)$ form a one-dimensional subspace. By (3.12), we see that $\bar{V} - \mu$ is a Fredholm operator with index equal to -1.

It remains to consider the case $\Re e\ \mu = 0$. This means that $\lambda = i\beta$. From (15.122) we see that in order for $f \in C_0^\infty$ to be in the range of $V - \mu$, it is necessary that

$$\int_{-\infty}^{0} e^{-i\beta t} f(t)\,dt = \int_{0}^{\infty} e^{-i\beta t} f(t)\,dt = 0. \tag{15.142}$$

However, if $f \in C_0^\infty$ does satisfy (15.142), one checks easily that (15.120) is a solution of

$$(V - \mu)u = f. \tag{15.143}$$

But the set of those $f \in C_0^\infty$ satisfying (15.142) is dense in L^2. Thus, $R(V - \mu)$ is dense in L^2 for $\Re e\ \mu = 0$. However, $R(\bar{V} - \mu)$ cannot be the whole of L^2. If it were, the fact that $\alpha(\bar{V} - \mu) = 0$ would imply that $\mu \in \rho(\bar{V})$. But this would imply that $\nu \in \rho(\bar{V})$ for ν close to μ. However, we know that this is not the case.

Thus, $\sigma(\bar{V})$ consists of the whole complex plane. However, $\Phi_{\bar{V}}$ consists of all points not on the imaginary axis. We also know that

$$i(\bar{V} - \mu) = -1, \quad u \in \Phi_{\bar{V}}.$$

15.12. Problems

(1) Verify that the function defined by (15.1) is infinitely differentiable in $(-\infty, \infty)$.

(2) If $P(t)$ is a polynomial having no real roots, show that there is a positive constant c such that $|P(t)| \geq c$ for real t.

(3) Show that if $u \in S$ and $P(t)$ is a polynomial having no real roots, then u/P is in S.

(4) Show that the operator Q of Section 15.7 is closed. [Hint: Use the fact that every sequence of functions converging in L^2 has a subsequence that converges pointwise almost everywhere. Moreover, functions that agree almost everywhere are considered the same function in L^2.]

(5) Show that (15.68) holds if and only if the real and imaginary parts of q satisfy it for real $u \in D(A)$.

(6) For what values of α is the function $q(t) = |t|^\alpha$ $\bar{A}$-compact?

(7) Prove (15.108) for any polynomial P of degree $m \geq 1$. Supply the details in the reasoning following (15.107).

(8) Find $\bar{B}^*$. Under what conditions is $\bar{B}$ selfadjoint?

(9) If $\bar{B}$ is selfadjoint and q is a real valued function satisfying (15.69) and (15.70), is $\bar{B} + Q$ selfadjoint? Can you determine $\sigma(\bar{B} + Q)$?

(10) Show that the operator V of Section 15.11 is closable.

(11) Prove (15.125).

(12) Show that the function given by (15.138) is a solution of (15.137).

(13) Prove that a solution of $(V-\mu)u=0$ vanishes in $(-\infty,0]$ when $\mu\neq 0$.

(14) Using the fact that the set of those $\psi\in C_0^\infty$ satisfying (15.139) is dense in L^2, show that the set of those $f\in C_0^\infty$ satisfying (15.141) is dense in L^2.

TVSLB″O

Glossary

Adjoint operator. If A maps X into Y and $D(A)$ is dense in X, we say that $y' \in D(A')$ if there is an $x' \in X'$ such that

$$x'(x) = y'(Ax), \quad x \in D(A). \tag{A.1}$$

Then we define $A'y'$ to be x'.

$\alpha(A) = \dim N(A)$.

Almost commuting operators. For any two operators $A, B \in B(X)$, we shall write $A \smile B$ to mean that $AB - BA \in K(X)$. The reason for this notation is that $[A]\,[B] = [B]\,[A]$ in this case. Such operators will be said to "almost commute."

Annihilators. For S a subset of a normed vector space X, a functional $x' \in X'$ is called an *annihilator* of S if

$$x'(x) = 0, \quad x \in S.$$

The set of all annihilators of S is denoted by S°. For any subset T of X', we call $x \in X$ an annihilator of T if

$$x'(x) = 0, \quad x' \in T.$$

We denote the set of such annihilators of T by ${}^\circ T$.

Associated operator. With any densely defined bilinear form $a(u, v)$ we can associate a linear operator A on H as follows: We say that $u \in D(A)$ if $u \in D(a)$ and there is an $f \in H$ such that

$$a(u, v) = (f, v), \quad u \in D(a). \tag{A.2}$$

The density of $D(a)$ in H implies that f is unique. We define Au to be f. A is a linear operator on H; it is called the operator *associated* with the bilinear form $a(u, v)$.

B = B[a, b]. The set of bounded functions on an interval $[a, b]$. The norm is

$$\|\varphi\| = \sup_{a \leq x \leq b} |\varphi(x)|.$$

$\beta(A) = \dim N(A')$.

B(X,Y). The set of bounded linear operators from X to Y.

Banach algebra. A Banach space X having a "multiplication" satisfying:

$$\text{If } A, B \in X, \text{ then } AB \in X, \tag{A.3}$$

$$\|AB\| \leq \|A\| \cdot \|B\|, \tag{A.4}$$

$$(\alpha A + \beta B)C = \alpha(AC) + \beta(BC), \tag{A.5}$$

and

$$C(\alpha A + \beta B) = \alpha(CA) + \beta(CB). \tag{A.6}$$

Banach space. A complete normed vector space (cf. p. 7).

Bessel's identity.

$$\|f - \sum_1^n \alpha_i \varphi_i\|^2 = \|f\|^2 - 2\sum_1^n \alpha_i (f, \varphi_i) + \sum_1^n \alpha_i^2 = \|f\|^2 - \sum_1^n \alpha_i^2. \tag{A.7}$$

Bessel's inequality.

$$\sum_1^\infty \alpha_i^2 \leq \|f\|^2. \tag{A.8}$$

Bilinear form. A *bilinear form* (or *sesquilinear functional*) $a(u, v)$ on a Hilbert space H is an assignment of a scalar to each pair of vectors u, v of a subspace $D(a)$ of H in such a way that

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w), \quad u, v, w \in D(a), \tag{A.9}$$

$$a(u, \alpha v + \beta w) = \bar{\alpha} a(u, v) + \bar{\beta} a(u, w), \quad u, v, w \in D(a). \tag{A.10}$$

The subspace $D(a)$ is called the *domain* of the bilinear form. When $v = u$ we write $a(u)$ in place of $a(u,u)$.

Bounded linear functional. A linear functional satisfying

$$|F(x)| \leq M\,\|x\|, \quad x \in X. \tag{A.11}$$

Bounded linear operator. An operator A is called *bounded* if $D(A) = X$ and there is a constant M such that

$$\|Ax\| \leq M\|x\|, \quad x \in X. \tag{A.12}$$

The norm of such an operator is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}. \tag{A.13}$$

Bounded variation. A function defined on an interval $[a,b]$ satisfying

$$\sum_1^n |g(t_i) - g(t_{i-1})| \leq C \tag{A.14}$$

for any partition $a = t_0 < t_1 < \cdots < t_n = b$ is said to be of *bounded variation.*

C=C[a,b]. The set of functions continuous on a closed interval $[a,b]$. The norm is

$$\|\varphi\| = \max_{a\leq x\leq b} |\varphi(x)|.$$

Cauchy sequence. A sequence satisfying

$$\|h_n - h_m\| \to 0 \text{ as } m,n \to \infty.$$

Cauchy-Schwarz inequalities. Inequalities (1.26) and (1.28) on p. 10.

Closable operator. Let A be a linear operator from a normed vector space X to a normed vector space Y. It is called *closable* (or *preclosed*) if $\{x_k\} \subset D(A),\ x_k \to 0,\ Ax_k \to y$ imply that $y = 0$. Every closed operator is closable.

Closed bilinear form. A bilinear form $a(u,v)$ will be called *closed* if $\{u_n\} \subset D(a),\ u_n \to u$ in $H,\ a(u_n - u_m) \to 0$ as $m,n \to \infty$ imply that $u \in D(a)$ and $a(u_n - u) \to 0$ as $n \to \infty$.

Closed operator. An operator A is called *closed* if whenever $\{x_n\} \subset D(A)$ is a sequence satisfying

$$x_n \longrightarrow x \text{ in } X, \quad Ax_n \longrightarrow y \text{ in } Y, \tag{A.15}$$

then $x \in D(A)$ and $Ax = y$.

Closed set. A subset U of a normed vector space X is called *closed* if for every sequence $\{x_n\}$ of elements in U having a limit in X, the limit is actually in U.

Commutative Banach algebra. A Banach algebra B is called *commutative* (or *abelian*) if

$$ab = ba, \quad a, b \in B. \tag{A.16}$$

Compact approximation property. A Banach space X has the *compact approximation property with constant* C if for each $\varepsilon > 0$ and finite set of points $x_1, \cdots, x_n \in X$, there is an operator $K \in K(X)$ such that $\|I - K\| \le C$ and

$$\|x_k - Kx_k\| < \varepsilon, \quad 1 \le k \le n. \tag{A.17}$$

Compact operator. Let X, Y be normed vector spaces. A linear operator K from X to Y is called *compact* (or *completely continuous*) if $D(K) = X$ and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \le C$, the sequence $\{Kx_n\}$ has a subsequence which converges in Y. The set of all compact operators from X to Y is denoted by $K(X, Y)$.

Complete orthonormal sequence. An orthonormal sequence $\{\varphi_j\}$ in a Hilbert space H is called *complete* if sums of the form

$$S = \sum_1^n \alpha_i \varphi_i \tag{A.18}$$

are dense in H, i.e., if for each $f \in H$ and $\varepsilon > 0$, there is a sum S of this form such that $\|f - S\| < \varepsilon$.

Completeness. Property (14) on p. 21. The property that every Cauchy sequence converges to a limit in the space.

Dissipative operator. An operator B satisfying

$$\Re e\,(Bu, u) \le 0, \quad u \in D(B). \tag{A.19}$$

Dual space. The set of bounded linear functionals on a normed vector space X. It is denoted by X'.

Essential Spectrum.

$$\sigma_e(A) = \bigcap_{K \in K(X)} \sigma(A+K).$$

It consists of those points of $\sigma(A)$ which cannot be removed from the spectrum by the addition to A of a compact operator.

Euclidean n-dimensional real space, $\mathbb{R}^n$. (p. 9) A vector space consisting of sequences of n real numbers

$$f = (\alpha_1, \cdots, \alpha_n), \quad g = (\beta_1, \cdots, \beta_n),$$

where addition is defined by

$$f + g = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n),$$

and multiplication by a scalar is defined by

$$\gamma f = (\gamma\alpha_1, \cdots, \gamma\alpha_n).$$

Extension of a bilinear form. A bilinear form $b(u, v)$ is called an *extension* of a bilinear form $a(u, v)$ if $D(a) \subset D(b)$ and $b(u, v) = a(u, v)$ for $u, v \in D(a)$.

Extension of an operator. An operator B is an *extension* of an operator A if $D(A) \subset D(B)$ and $Bx = Ax$ for $x \in D(A)$.

Fredholm operator. Let X, Y be Banach spaces. Then the set $\Phi(X, Y)$ consists of linear operators from X to Y such that

(1) $D(A)$ is dense in X,

(2) A is closed,

(3) $\alpha(A) < \infty$,

(4) $R(A)$ is closed in Y,

(5) $\beta(A) < \infty$.

Fredholm perturbation. An operator $E \in B(X)$ is called a *Fredholm perturbation* if $A + E \in \Phi$ for all $A \in \Phi$. We designate the set of Fredholm perturbations by $F(X)$.

Functional. An assignment F of a number to each element x of a vector space and denoted by $F(x)$.

Hilbert space. A vector space which has a scalar product and is complete with respect to the induced norm.

Hilbert space adjoint. Let H_1 and H_2 be Hilbert spaces, and let A be an operator in $B(H_1, H_2)$. For fixed $y \in H_2$, the expression $Fx = (Ax, y)$ is a bounded linear functional on H_1. By the Riesz representation theorem (Theorem 2.1), there is a $z \in H_1$ such that $Fx = (x, z)$ for all $x \in H_1$. Set $z = A^*y$. Then A^* is a linear operator from H_2 to H_1 satisfying

$$(Ax, y) = (x, A^*y). \tag{A.20}$$

A^* is called the *Hilbert space adjoint* of A.

Hölder's inequality.

$$\sum_1^\infty x_i z_i \le \|x\|_p \|z\|_q. \tag{A.21}$$

Hyponormal operator. An operator A in $B(H)$ is called *hyponormal* if

$$\|A^*u\| \le \|Au\|, \quad u \in H, \tag{A.22}$$

or, equivalently, if

$$([AA^* - A^*A]u, u) \le 0, \quad u \in H. \tag{A.23}$$

Of course, a normal operator is hyponormal.

Ideal. Let B be a Banach algebra with identity e. A subspace M of B is called a *right ideal* if $xa \in M$ for $x \in M,\ a \in B$. It is called a *left ideal* if $ax \in M$ when $x \in M,\ a \in B$. If it is both a right and left ideal, it is called a *two-sided ideal* or merely an *ideal.*

Index. $i(A) = \alpha(A) - \beta(A)$.

Infinitesimal generator. A linear operator A on X is said to be an *infinitesimal generator* of a semigroup $\{E_t\}$ if it is closed, $D(A)$ is dense in X and consists of those $x \in X$ for which $(E_t x)'$ exists, $t \ge 0$, and

$$(E_t x)' = AE_t x, \quad x \in D(A). \tag{A.24}$$

Linear functional. A functional satisfying

$$F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2) \tag{A.25}$$

for α_1, α_2 scalars.

Linear operator. An operator A is called *linear* if a) $D(A)$ is a subspace of X and b) $A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$ for all scalars α_1, α_2 and all elements x_1, $x_2 \in D(A)$.

l_p, where p is a real number satisfying $1 \leq p < \infty$. It is the set of all infinite sequences $x = (x_1, \cdots, x_j, \cdots)$ such that

$$\sum_1^\infty |x_j|^p < \infty.$$

l_∞. A vector space consisting of infinite sequences of real numbers

$$f = (\alpha_1, \cdots, \alpha_n, \cdots) \tag{A.26}$$

for which

$$\sup_i |\alpha_i| < \infty.$$

Maximal element. An element x_0 is said to be *maximal* for a partially ordered set S if $x_0 \prec x$ implies $x = x_0$.

Maximal ideal. An ideal N in a Banach algebra B such that $N \neq B$, and every element of $a \in B$ can be written in the form $a = a_1 + \lambda e$, where $a_1 \in N$ and E is the unit element of B.

Measure of noncompactness. Let X be a Banach space. For a bounded subset $\Omega \subset X$ we let $q(\Omega)$ denote the infimum (greatest lower bound) of the set of numbers r such that Ω can be covered by a collection of open spheres of radius r. In particular, $q(\Omega) = 0$ if and only if Ω is totally bounded, i.e., if and only if its closure is compact. It is for this reason that $q(\Omega)$ is sometimes called the *measure of noncompactness* of Ω.

Measures of operators. Let X, Y be Banach spaces with $\dim X = \infty$, and let A be an operator in $B(X,Y)$. We define

$$\begin{aligned}
\Gamma(A) &= \inf_M \|A|_M\|,\\
\Delta(A) &= \sup_M \inf_{N\subset M} \|A|_N\|,\\
\tau(A) &= \sup_M \inf_{\substack{x\in M\\ \|x\|=1}} \|Ax\|,\\
\nu(A) &= \sup_{\dim M^\circ<\infty} \inf_{\substack{x\in M\\ \|x\|=1}} \|Ax\|,\\
\|A\|_m &= \inf_{\dim M^\circ<\infty} \|A|_M\|,\\
\|A\|_q &= q[A(S_X)],\\
\|A\|_K &= \inf_{K\in K(X,Y)} \|A-K\|,\\
\mu_0(A) &= \begin{cases} \inf\limits_{\alpha(A-T)>\alpha(A)} \|T\|, & \alpha(A)<\infty,\\ 0, & \alpha(A)=\infty,\end{cases}\\
\mu(A) &= \inf_{\alpha(A-T)=\infty} \|T\|,\\
\gamma(A) &= \inf_{x\notin N(A)} \frac{\|Ax\|}{d(x,N(A))},
\end{aligned}$$

where M, N represent infinite dimensional, closed subspaces of X, $A|_M$ denotes the restriction of A to M, and S_X denotes the closed unit ball in X, i.e.,

$$S_X = \{x\in X : \|x\|\le 1\}.$$

Multiplicative functional. A linear functional m on a commutative Banach algebra B is called *multiplicative* if $m\neq 0$ and

$$m(ab) = m(a)m(b), \quad a,\ b\in B. \tag{A.27}$$

Norm. A real number assigned to elements of a vector space and having properties (11)-(13) on p. 7.

Normal operator. An operator satisfying

$$\|A^*f\| = \|Af\|, \quad f\in H. \tag{A.28}$$

is called *normal.*

Normed vector space. A set of objects satisfying statements (1)–(13) on pp. 6, 7.

Numerical range. The numerical range $W(A)$ of an operator A on a Hilbert space H is the set of all scalars λ that equal (Au, u) for some $u \in D(A)$ satisfying $\|u\| = 1$. The numerical range $W(a)$ of a bilinear form $a(u, v)$ is the set of scalars λ which equal $a(u)$ for some $u \in D(a)$ satisfying $\|u\| = 1$.

Operator. A mapping A which assigns to each element x of a set $D(A) \subset X$ a unique element $y \in Y$. The set $D(A)$ is called its domain. The set $R(A)$ of all $y \in Y$ for which there is an $x \in X$ such that $Ax = y$, is called its range.

Orthogonal projection. Let H be a complex Hilbert space, and let M be a closed subspace of H. Then, by the projection theorem (Theorem 2.3), every element $u \in H$ can be written in the form

$$u = u' + u'', \quad u' \in M,\ u'' \in M^{\perp}. \tag{A.29}$$

Set

$$Eu = u'.$$

It is called the *orthogonal projection onto M.*

Parseval's equality.

$$\|f\|^2 = \sum_1^{\infty} (f, \varphi_i)^2. \tag{A.30}$$

Partially ordered set. A set S is called *partially ordered* if for some pairs of elements $x,\ y \in S$ there is an ordering relation $x \prec y$ such that

(1) $x \prec x, \quad x \in S,$

(2) $x \prec y,\ y \prec x \implies x = y,$

(3) $x \prec y,\ y \prec z \implies x \prec z.$

Positive operator. An operator $A \in B(H)$ is called *positive* if

$$(Au, u) \geq 0, \quad u \in H \tag{A.31}$$

[i.e., if $W(A)$ is contained in the nonnegative real axis].

Reflexive Banach space. A Banach space Y for which the embedding of Y into Y'' given by $J \in B(Y, Y'')$ such that

$$Jy(y') = y'(y), \quad y \in Y,\ y' \in Y' \tag{A.32}$$

is onto.

Relatively compact subset. A set $V \subset X$ is called *relatively compact* if every sequence of elements of V has a convergent subsequence. The limit of this subsequence, however, need not be in V.

Riesz operator. For a Banach space X, we call an operator $E \in B(X)$ a *Riesz operator* if $E - \lambda \in \Phi(X)$ for all scalars $\lambda \neq 0$. We denote the set of Riesz operators on X by $R(X)$.

Saturated subspace. A closed subspace $W \in X'$ having the property that for each $x' \in X' \setminus W$ there is an $x \in {}^{\circ}W$ such that $x'(x) \neq 0$.

Scalar product. A functional (f, g) defined for pairs of elements of a vector space satisfying

i. $(\alpha f, g) = \alpha(f, g)$
ii. $(f + g, h) = (f, h) + (g, h)$
iii. $(f, g) = (g, f)$
iv. $(f, f) > 0$ unless $f = 0$.

Schwarz's inequalities. Inequalities (1.26) and (1.28) on p. 10.

Semigroup. A one-parameter family $\{E_t\}$ of operators in $B(X)$, $t \geq 0$, with the following properties:

$$\text{(a)} \qquad E_s E_t = E_{s+t}, \quad s \geq 0,\ t \geq 0,$$
$$\text{(b)} \qquad E_0 = I.$$

Semi-Fredholm operators. $\Phi_+(X, Y)$ denotes the set of all closed linear operators from X to Y, such that $D(A)$ is dense in X, $R(A)$ is closed in Y and $\alpha(A) < \infty$. $\Phi_-(X, Y)$ denotes the set of all closed linear operators from X to Y, such that $D(A)$ is dense in X, $R(A)$ is closed in Y and $\beta(A) < \infty$. Operators in either set are called semi-Fredholm.

Semi-Fredholm perturbations. $F_+(X)$ denotes the set of all $E \in B(X)$ such that $A + E \in \Phi_+ \quad \forall A \in \Phi_+$. $F_-(X)$ denotes the set of all $E \in B(X)$ such that $A + E \in \Phi_- \ \forall\, A \in \Phi_-$. They are called semi-Fredholm perturbations.

Seminormal operator. An operator $A \in B(H)$ is called *seminormal* if either A or A^* is hyponormal.

Separable. A normed vector space is called *separable* if it has a dense subset that is denumerable.

Strictly singular operators. An operator $S \in B(X, Y)$ is called *strictly singular* if it does not have a bounded inverse on any infinite dimensional subspace of X.

Strongly continuous semigroup. A one-parameter family $\{E_t\}, t \geq 0$, of operators in $B(X)$ is called a *semigroup* if

$$E_s E_t = E_{s+t}, \quad s \geq 0,\ t \geq 0. \tag{A.33}$$

It is called *strongly continuous* if

$$E_t x \quad \text{is continuous in} \quad t \geq 0 \quad \text{for each} \quad x \in X. \tag{A.34}$$

Sublinear functional. A functional $p(x)$ on a vector space V is called *sublinear* if

$$p(x + y) \leq p(x) + p(y), \quad x, y \in V, \tag{A.35}$$

$$p(\alpha x) = \alpha p(x), \quad x \in V,\ \alpha > 0. \tag{A.36}$$

Subspace. A subset U of a vector space V such that $\alpha_1 x_1 + \alpha_2 x_2$ is in U whenever x_1, x_2 are in U and α_1, α_2 are scalars.

Supremum. The sup of any set of real numbers is the least upper bound, i.e., the smallest number, which may be $+\infty$, that is an upper bound for the set. (An important property of the real numbers is that every set of real numbers has a least upper bound.)

Symmetric bilinear form. A bilinear form $a(u, v)$ is said to be *symmetric* if

$$a(v, u) = \overline{a(u, v)}. \tag{A.37}$$

Total subset. A subset of X' is called total if the only element of X that annihilates it is 0.

Total variation. The total variation of a function g on an interval $[a, b]$ is defined as

$$V(g) = \sup \sum_1^n |g(t_i) - g(t_{i-1})|,$$

where the supremum (least upper bound) is taken over all partitions of $[a, b]$.

Totally bounded subset. Let $\varepsilon > 0$ be given. A set of points $W \subset X$ is called an ε-net for a set $U \subset X$ if for every $x \in U$ there is a $z \in W$ such that $\|x - z\| < \varepsilon$. A subset $U \subset X$ is called *totally bounded* if for every $\varepsilon > 0$ there is a finite set of points $W \subset X$ which is an ε-net for U.

Totally ordered set. The set S is called *totally ordered* if for each pair x, y of elements of S, one has either $x \prec y$ or $y \prec x$ (or both).

Upper bound. A subset T of a partially ordered set S is said to have the element $x_0 \in S$ as an *upper bound* if $x \prec x_0$ for all $x \in T$.

Vector space. A collection of objects which satisfies statements (1)–(9), (15) on pp. 6, 7.

Weak convergence. A sequence $\{x_k\}$ of elements of a Banach space X is said to converge *weakly* to an element $x \in X$ if

$$x'(x_k) \longrightarrow x'(x) \text{ as } k \longrightarrow \infty \tag{A.38}$$

for each $x' \in X'$.

Weak* closed. A subset W of X' is called *weak** (pronounced "weak star") closed if $x' \in W$ whenever it has the property that for each $x \in X$, there is a sequence $\{x'_n\}$ of members of W such that

$$x'_k(x) \longrightarrow x'(x). \tag{A.39}$$

Weak* convergence. A sequence $\{x'_n\}$ of elements in X' is said to be *weak* convergent* to an element $x' \in X'$ if

$$x'_n(x) \longrightarrow x'(x) \text{ as } n \longrightarrow \infty, \quad x \in X. \tag{A.40}$$

Major Theorems

We list here some of the important theorems that are proved in the text. For each we give the page number where it can be found.

Theorem 1.1. *(p. 7) Let X be a Banach space, and assume that K is an operator on X (i.e., maps X into itself) such that*

a) $K(v+w) = Kv + Kw$,
b) $K(-v) = -Kv$,
c) $\|Kv\| \leq M\|v\|$,
d) $\sum_0^\infty \|K^n v\| < \infty$

for all $v, w \in X$. Then for each $u \in X$ there is a unique $f \in X$ such that

$$f = u + Kf. \tag{B.1}$$

Theorem 1.4. *(p. 22) There is a one-to-one correspondence between l_2 and L^2 such that if $(\alpha_0, \alpha_1, \cdots)$ corresponds to f, then*

$$\|\sum_0^n \alpha_j \varphi_j - f\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{B.2}$$

and

$$\|f\|^2 = \sum_0^\infty \alpha_j^2, \quad \alpha_j = (f, \varphi_j). \tag{B.3}$$

Theorem 1.5. *(p. 23) Let* $(\alpha_1, \alpha_2, \cdots)$ *be a sequence of real numbers, and let* $\{\varphi_n\}$ *be an orthonormal sequence in* H*. Then*

$$\sum_1^n \alpha_i \varphi_i$$

converges in H *as* $n \to \infty$ *if, and only if,*

$$\sum_1^\infty \alpha_i^2 < \infty.$$

Theorem 1.6. *(p. 24) If* $\{\varphi_n\}$ *is complete, then for each* $f \in H$

$$f = \sum_1^\infty (f, \varphi_i)\varphi_i$$

and

$$\|f\|^2 = \sum_1^\infty (f, \varphi_i)^2. \tag{B.4}$$

Theorem 2.1. (Riesz Representation Theorem) *(p. 29) For every bounded linear functional* F *on a Hilbert space* H *there is a unique element* $y \in H$ *such that*

$$F(x) = (x, y) \text{ for all } x \in H. \tag{B.5}$$

Moreover,

$$\|y\| = \sup_{x \in H,\, x \neq 0} \frac{|F(x)|}{\|x\|}. \tag{B.6}$$

Theorem 2.2. *(p. 31) Let* N *be a closed subspace of a Hilbert space* H*, and let* x *be an element of* H *which is not in* N*. Set*

$$d = \inf_{z \in N} \|x - z\|. \tag{B.7}$$

Then there is an element $z \in N$ *such that* $\|x - z\| = d$*.*

Theorem 2.3. (Projection Theorem) *(p. 32) Let* N *be a closed subspace of a Hilbert space* H*. Then for each* $x \in H$*, there are a* $v \in N$ *and a* w *orthogonal to* N *such that* $x = v + w$*. This decomposition is unique.*

Theorem 2.5. (Hahn-Banach Theorem) *(p. 33) Let* V *be a vector space, and let* $p(x)$ *be a sublinear functional on* V*. Let* M *be a subspace of* V*, and let* $f(x)$ *be a linear functional on* M *satisfying*

$$f(x) \le p(x), \quad x \in M. \tag{B.8}$$

Then there is a linear functional $F(x)$ on the whole of V such that

$$F(x) = f(x), \quad x \in M, \tag{B.9}$$

$$F(x) \le p(x), \quad x \in V. \tag{B.10}$$

Theorem 2.6. *(p. 34) Let M be a subspace of a normed vector space X, and suppose that $f(x)$ is a bounded linear functional on M. Set*

$$\|f\| = \sup_{x \in M,\, x \neq 0} \frac{|f(x)|}{\|x\|}.$$

Then there is a bounded linear functional $F(x)$ on the whole of X such that

$$F(x) = f(x), \quad x \in M, \tag{B.11}$$

$$\|F\| = \|f\|. \tag{B.12}$$

Theorem 2.7. *(p. 36) Let X be a normed vector space and let $x_0 \neq 0$ be an element of X. Then there is a bounded linear functional $F(x)$ on X such that*

$$\|F\| = 1, \quad F(x_0) = \|x_0\|. \tag{B.13}$$

Theorem 2.9. *(p. 37) Let M be a subspace of a normed vector space X, and suppose x_0 is an element of X satisfying*

$$d = d(x_0, M) = \inf_{x \in M} \|x_0 - x\| > 0. \tag{B.14}$$

Then there is a bounded linear functional F on X such that $\|F\| = 1$, $F(x_0) = d$, and $F(x) = 0$ for $x \in M$.

Theorem 2.10. *(p. 38) X' is a Banach space whether or not X is.*

Theorem 2.11. *(p. 41) $l_p' = l_q$, where $q = p/(p-1)$.*

Theorem 2.12. *(p. 43) If $f \in l_p'$, then there is a $z \in l_q$ such that*

$$f(x) = \sum_1^\infty x_i z_i, \quad x \in l_p,$$

and

$$\|f\| = \|z\|_q. \tag{B.15}$$

Theorem 2.14. *(p. 50) For each bounded linear functional f on $C[a,b]$ there is a unique normalized function $\hat{g}$ of bounded variation such that*

$$f(x) = \int_a^b x(t)\, d\hat{g}(t), \quad x \in C[a,b], \tag{B.16}$$

and

$$V(\hat{g}) = \|f\|. \tag{B.17}$$

Conversely, every such normalized $\hat{g}$ gives a bounded linear functional on $C[a,b]$ satisfying (B.16) and (B.17).

Theorem 3.10. (Closed Graph Theorem) *(p. 62) If X, Y are Banach spaces, and A is a closed linear operator from X to Y, with $D(A) = X$, then $A \in B(X,Y)$.*

Theorem 3.12. *(p. 67) If X, Y are Banach spaces, and A is a closed linear operator from X to Y, then $R(A)$ is closed in Y if, and only if, there is a constant C such that*

$$d(x, N(A)) \leq C\|Ax\|, \quad x \in D(A). \tag{B.18}$$

Theorem 3.17. (Banach-Steinhaus Theorem) *(p.71) Let X be a Banach space, and let Y be a normed vector space. Let W be any subset of $B(X,Y)$ such that for each $x \in X$,*

$$\sup_{A \in W} \|Ax\| < \infty.$$

Then there is a finite constant M such that $\|A\| \leq M$ for all $A \in W$.

Theorem 3.18. (Open Mapping Theorem) *(p. 71) Let A be a closed operator from a Banach space X to a Banach space Y such that $R(A) = Y$. If Q is any open subset of $D(A)$, then the image $A(Q)$ of Q is open in Y.*

Theorem 4.12. *(p. 90) Let X be a Banach space and let K be an operator in $K(X)$. Set $A = I - K$. Then, $R(A)$ is closed in X and $\dim N(A) = \dim N(A')$ is finite. In particular, either $R(A) = X$ and $N(A) = \{0\}$, or $R(A) \neq X$ and $N(A) \neq \{0\}$.*

Theorem 4.17. *(p. 96) If a set $U \subset X$ is relatively compact, then it is totally bounded. If X is complete and U is totally bounded, then U is relatively compact.*

Theorem 6.2. *(p. 129) Under the hypothesis of Theorem 6.1, $\alpha(K-\lambda)=0$ except for, at most, a denumerable set S of values of λ. The set S depends on K and has 0 as its only possible limit point. Moreover, if $\lambda \neq 0$ and $\lambda \notin S$, then $\alpha(K-\lambda)=0,\ R(K-\lambda)=X$ and $K-\lambda$ has an inverse in $B(X)$.*

Theorem 6.17. *(p. 139) If $f(z)$ is analytic in a neighborhood of $\sigma(A)$, then*

$$\sigma(f(A)) = f(\sigma(A)), \tag{B.19}$$

i.e., $\mu \in \sigma(f(A))$ if and only if $\mu = f(\lambda)$ for some $\lambda \in \sigma(A)$.

Theorem 6.26. *(p. 148) Let V be a complex vector space, and let p be a real valued functional on V such that*

(i) $p(u+v) \leq p(u)+p(v), \quad u,v \in V,$

(ii) $p(\alpha u) = |\alpha| p(u), \quad \alpha$ complex, $u \in V.$

Suppose that there is a linear subspace M of V and a linear (complex valued) functional f on M such that

$$\Re e\, f(u) \leq p(u), \quad u \in M. \tag{B.20}$$

Then there is a linear functional F on the whole of V such that

$$F(u) = f(u), \quad u \in M, \tag{B.21}$$

$$|F(u)| \leq p(u), \quad u \in V. \tag{B.22}$$

Theorem 7.1. *(p. 157) If $A \in \Phi(X,Y)$, then there is an $A_0 \in B(Y,X)$ such that*

(a) $N(A_0) = Y_0,$

(b) $R(A_0) = X_0 \cap D(A),$

(c) $A_0 A = I$ on $X_0 \cap D(A),$

(d) $A A_0 = I$ on $R(A).$

Moreover, there are operators $F_1 \in B(X),\ F_2 \in B(Y)$ such that

(e) $A_0 A = I - F_1$ on $D(A),$

(f) $AA_0 = I - F_2$ on Y,

(g) $R(F_1) = N(A), \quad N(F_1) = X_0$,

(h) $R(F_2) = Y_0, \quad N(F_2) = R(A)$.

Theorem 7.2. *(p. 157) Let A be a densely defined closed linear operator from X to Y. Suppose there are operators $A_1, A_2 \in B(Y, X)$, $K_1 \in K(X)$, $K_2 \in K(Y)$ such that*

$$A_1 A = I - K_1 \text{ on } D(A), \tag{B.23}$$

and

$$AA_2 = I - K_2 \text{ on } Y. \tag{B.24}$$

Then $A \in \Phi(X, Y)$.

Theorem 7.3. *(p. 157) If $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$, then $BA \in \Phi(X, Z)$ and*

$$i(BA) = i(A) + i(B). \tag{B.25}$$

Theorem 7.8. *(p. 161) If $A \in \Phi(X, Y)$ and K is in $K(X, Y)$, then $A + K \in \Phi(X, Y)$ and*

$$i(A + K) = i(A). \tag{B.26}$$

Theorem 7.9. *(p. 161) For $A \in \Phi(X, Y)$, there is an $\eta > 0$ such that for every T in $B(X, Y)$ satisfying $\|T\| < \eta$, one has $A + T \in \Phi(X, Y)$,*

$$i(A + T) = i(A), \tag{B.27}$$

and

$$\alpha(A + T) \leq \alpha(A). \tag{B.28}$$

Theorem 7.10. *(p. 162) If $A \in \Phi(X, Y)$ and B is a densely defined closed linear operator from Y to Z such that $BA \in \Phi(X, Z)$, then $B \in \Phi(Y, Z)$.*

Theorem 7.15. *(p. 164) Let A be a densely defined closed linear operator from X to Y. If $R(A)$ is closed in Y, then $R(A') = N(A)^\circ$, and hence is closed in X'.*

Theorem 7.16. *(p. 165) If A is a densely defined closed linear operator from X to Y and $R(A')$ is closed in X', then $R(A) = {}^{\circ}N(A')$, and hence, is closed in Y.*

Theorem 7.17. *(p. 166) If U is a closed, convex subset of a normed vector space X and $x_0 \in X$ is not in U, then there is an $x' \in X'$ such that*

$$\Re e\, x'(x_0) \geq \Re e\, x'(x), \quad x \in U, \tag{B.29}$$

and $\Re e\, x'(x_0) \neq \Re e\, x'(x_1)$ for some $x_1 \in U$.

Theorem 7.19. *(p. 169) If A is in $B(X,Y)$, then $A \in \Phi(X,Y)$ if and only if $A' \in \Phi(Y',X')$.*

Theorem 7.20. *(p. 169) Let A be a closed linear operator from X to Y with $D(A)$ dense in X. Then $D(A')$ is total in Y'.*

Theorem 7.22. *(p. 170) If $A \in \Phi(X,Y)$ and Y is reflexive, then $A' \in \Phi(Y',X')$ and $i(A') = -i(A)$.*

Theorem 7.23. *(p. 170) Let A be a closed linear operator from X to Y with $D(A)$ dense in X. If $A' \in \Phi(Y',X')$, then $A \in \Phi(X,Y)$ with $i(A) = -i(A')$.*

Theorem 7.27. *(p. 172) $\lambda \notin \sigma_e(A)$ if and only if $\lambda \in \Phi_A$ and $i(A-\lambda) = 0$.*

Theorem 7.29. *(p. 173) Let A be a closed linear operator from X to Y with domain $D(A)$ dense in X. Then $A \in \Phi_+(X,Y)$ if and only if there is a seminorm $|\cdot|$ defined on $D(A)$, which is compact relative to the graph norm of A, such that*

$$\|x\| \leq C\|Ax\| + |x|, \quad x \in D(A). \tag{B.30}$$

Theorem 7.35. *(p. 178) Let X, Y, Z be Banach spaces, and assume that A is a densely defined, closed linear operator from X to Y such that $R(A)$ is closed in Y and $\beta(A) < \infty$ (i.e., $A \in \Phi_-(X,Y)$). Let B be a densely defined linear operator from Y to Z. Then $(BA)'$ exists and*

$$(BA)' = A'B'. \tag{B.31}$$

Theorem 8.1. *(p. 184) X is reflexive if and only if X' is.*

Theorem 8.2. *(p. 184) Every closed subspace of a reflexive Banach space is reflexive.*

Theorem 8.3. *(p. 185) A subspace W of X' is saturated if and only if $W = M^\circ$ for some subset M of X.*

Theorem 8.5. *(p. 186) A subspace W of X' is saturated if and only if it is weak* closed.*

Theorem 8.7. *(p. 186) A Banach space X is reflexive if and only if every closed subspace of X' is saturated.*

Theorem 8.11. *(p. 188) If X' is separable, so is X.*

Theorem 8.13. *(p. 189) If X is separable, then every bounded sequence in X' has a weak* convergent subsequence.*

Theorem 8.16. *(p. 191) If X is reflexive, then every bounded sequence has a weakly convergent subsequence.*

Theorem 8.18. *(p. 195) If X is a Banach space such that every total subspace of X' is dense in X', then X is reflexive.*

Theorem 9.5. (Spectral Mapping Theorem) *(p. 203) If $p(t)$ is a polynomial, then*

$$\sigma[p(a)] = p[\sigma(a)]. \tag{B.32}$$

More generally, if $f(z)$ is analytic in a neighborhood Ω of $\sigma(a)$, define

$$f(a) = \frac{1}{2\pi i}\oint_{\partial\omega} f(z)(ze-a)^{-1}dz, \tag{B.33}$$

where ω is an open set containing $\sigma(a)$ such that $\overline{\omega} \subset \Omega$, and $\partial\omega$ consists of a finite number of simple closed curves which do not intersect. Then

$$\sigma[f(a)] = f[\sigma(a)]. \tag{B.34}$$

Theorem 9.10. *(p. 207) A complex number λ is in $\sigma(a)$ if and only if there is a multiplicative linear functional m on B such that $m(a) = \lambda$.*

Theorem 9.11. *(p. 207) Every ideal H which is not the whole of B is contained in a maximal ideal.*

Theorem 9.16. *(p. 210) An ideal N is maximal if and only if the only ideal L satisfying $B \neq L \supset N$ is $L = N$.*

Theorem 9.26. *(p. 214) If $A \in \Phi(X)$, $E \in R(X)$ and $A \smile E$, then $A + E \in \Phi(X)$.*

Theorem 9.29. *(p. 214) The operator E in B(X) is in R(X) if and only if $A + E \in \Phi(X)$ for all $A \in \Phi(X)$ such that $A \smile E$.*

Theorem 9.30. *(p. 214) If $E_1, E_2 \in R(X)$ and $E_1 \smile E_2$, then $E_1 + E_2 \in R(X)$.*

Theorem 9.39. *(p. 216) F(X) is a closed two-sided ideal.*

Theorem 9.43. *(p. 217) A in B(X) is in Φ_+ if and only if $\alpha(A-K) < \infty$ for all $K \in K(X)$.*

Theorem 9.44. *(p. 218) $A \in \Phi$ if and only if $\alpha(A-K) < \infty$ and $\beta(A-K) < \infty$ for all $K \in K(X)$.*

Theorem 9.46. *(p. 218) $E \in F_+(X)$ if and only if $\alpha(A-E) < \infty$ for all $A \in \Phi_+$.*

Theorem 9.47. *(p. 218) E is in F(X) if and only if $\alpha(A-E) < \infty$ for all $A \in \Phi$.*

Theorem 9.52. *(p. 219) $F_+(X)$ is a closed two-sided ideal.*

Theorem 9.54. *(p. 220) A in B(X) is in Φ_- if and only if $\beta(A-K) < \infty$ for all K in K(X).*

Theorem 9.56. *(p. 221) $E \in F_-(X)$ if and only if $\beta(A-E) < \infty$ for all $A \in \Phi_-$.*

Theorem 9.57. *(p. 221) E is in F(X) if and only if $\beta(A-E) < \infty$ for all $A \in \Phi$.*

Theorem 9.62. *(p. 222) $F_-(X)$ is a closed two-sided ideal.*

Theorem 10.1. *(p. 230) Let A be a closed linear operator with dense domain $D(A)$ on X having the interval $[b, \infty)$ in its resolvent set $\rho(A)$, where $b \geq 0$, and such that there is a constant a satisfying*

$$\|(\lambda - A)^{-1}\| \leq (a + \lambda)^{-1}, \quad \lambda \geq b. \tag{B.35}$$

Then there is a family $\{E_t\}$ of operators in $B(X)$, $t \geq 0$, with the following properties:

(a) $E_s E_t = E_{s+t}, \quad s \geq 0,\ t \geq 0,$

(b) $E_0 = I,$

(c) $\|E_t\| \leq e^{-at}, \quad t \geq 0,$

(d) *$E_t x$ is continuous in $t \geq 0$ for each $x \in X$,*

(e) *$E_t x$ is differentiable in $t \geq 0$ for each $x \in D(A)$, and*

$$\frac{dE_t x}{dt} = AE_t x, \tag{B.36}$$

(f) $E_t(\lambda - A)^{-1} = (\lambda - A)^{-1} E_t, \quad \lambda \geq b,\ t \geq 0.$

Theorem 10.3. *(p. 235) Every strongly continuous, one-parameter semigroup $\{E_t\}$ of operators in $B(X)$ has an infinitesimal generator.*

Theorem 11.3. *(p. 248) An operator is normal and compact if and only if it is of the form*

$$Af = \sum \lambda_k (f, \varphi_k) \varphi_k. \tag{B.37}$$

with $\{\varphi_k\}$ an orthonormal set and $\lambda_k \to 0$ as $k \to \infty$.

Lemma 11.9. *(p. 251) Every separable Hilbert space has a complete orthonormal sequence.*

Theorem 11.19. *(p. 261) Let A be a seminormal operator such that $\sigma(A)$ has no non-zero limit points. Then A is compact and normal.*

Theorem 12.2. *(p. 266) If A is a closed, densely defined operator on H and $\lambda \notin \overline{W(A)}$, then $\alpha(A - \lambda) = 0$ and $R(A - \lambda)$ is closed in H.*

Theorem 12.3. *(p. 267) Let $a(u, v)$ be a densely defined bilinear form with associated operator A. Then*

(a) If $\lambda \notin \overline{W(a)}$, then $A - \lambda$ is one-to-one and

$$\|u\| \leq C\|(A - \lambda)u\|, \quad u \in D(A). \tag{B.38}$$

(b) If $\lambda \notin \overline{W(a)}$ and A is closed, then $R(A - \lambda)$ is closed in H.

Theorem 12.8. *(p. 270) Let $a(u, v)$ be a densely defined closed bilinear form with associated operator A. If $\overline{W(a)}$ is not the whole plane, a half-plane, a strip, or a line, then A is closed and*

$$\sigma(A) \subset \overline{W(a)} = \overline{W(A)}. \tag{B.39}$$

Theorem 12.9. *(p. 270) The numerical range of a bilinear form is a convex set in the plane.*

Theorem 12.14. *(p. 274) Let A be a densely defined linear operator on H such that $\overline{W(A)}$ is not the whole plane, a half-plane, a strip, or a line. Then A has a closed extension $\hat{A}$ such that*

$$\sigma(\hat{A}) \subset \overline{W(A)} = \overline{W(\hat{A})}. \tag{B.40}$$

Theorem 12.17. *(p. 278) If A is a densely defined linear operator on H such that $W(A)$ is not the whole complex plane, then A has a closed extension.*

Theorem 12.18. *(p. 278) A linear operator has a closed extension if and only if it is closable.*

Theorem 12.24. *(p. 286) Let B be a dissipative operator on H with $D(B)$ dense in H. Then B has a closed dissipative extension $\hat{B}$ such that $\sigma(\hat{B})$ is contained in the half-plane $\Re e\, \lambda \leq 0$.*

Theorem 12.25. *(p. 286) Let A be a densely defined linear operator on H such that $\overline{W(A)}$ is a half-plane. Then A has a closed extension $\hat{A}$ satisfying*

$$\sigma(\hat{A}) \subset \overline{W(A)} = \overline{W(\hat{A})}. \tag{B.41}$$

Theorem 12.28. *(p. 292) Let B be a densely defined linear operator on H such that $W(B)$ is the line $\Re e\, \lambda = 0$. Then a necessary and sufficient condition that B have a closed extension $\hat{B}$ such that*

$$\sigma(\hat{B}) \subset W(\hat{B}) = W(B), \tag{B.42}$$

is that there exist an isometry from $R(I-B)^{\perp}$ onto $R(I+B)^{\perp}$. In particular, this is true if they both have the same finite dimension or if they are both separable and infinite dimensional.

Theorem 13.5. *(p. 301) If A is a positive operator in $B(H)$, then there is a unique $B \geq 0$ such that $B^2 = A$. Moreover, B commutes with any $C \in B(H)$ which commutes with A.*

Theorem 13.4. *(p. 307) Let A be a selfadjoint operator in $B(H)$. Set*

$$m = \inf_{\|u\|=1} (Au, u), \quad M = \sup_{\|u\|=1} (Au, u).$$

Then there is a family $\{E(\lambda)\}$ of orthogonal projection operators on H depending on a real parameter λ and such that:

(1) $E(\lambda_1) \leq E(\lambda_2)$ for $\lambda_1 \leq \lambda_2$;

(2) $E(\lambda)u \to E(\lambda_0)u$ as $\lambda_0 < \lambda \to \lambda_0, \quad u \in H$;

(3) $E(\lambda) = 0$ for $\lambda < m$, $E(\lambda) = I$ for $\lambda \geq M$;

(4) $AE(\lambda) = E(\lambda)A$;

(5) *if $a < m$, $b \geq M$ and $p(t)$ is any polynomial, then*

$$p(A) = \int_a^b p(\lambda)dE(\lambda). \tag{B.43}$$

This means the following. Let $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$ be any partition of $[a, b]$, and let λ'_k be any number satisfying $\lambda_{k-1} \leq \lambda'_k \leq \lambda_k$. Then

$$\sum_1^n p(\lambda'_k)[E(\lambda_k) - E(\lambda_{k-1})] \to p(A) \tag{B.44}$$

in $B(H)$ as $\eta = \max(\lambda_k - \lambda_{k-1}) \to 0$.

Theorem 13.14. *(p. 321) Let A be a selfadjoint operator on H. Then there is a family $\{E(\lambda)\}$ of orthogonal projection operators on H satisfying (1) and (2) of Theorem 13.4 and*

$$E(\lambda) \to \begin{cases} 0 & \text{as } \lambda \to -\infty \\ I & \text{as } \lambda \to +\infty \end{cases} \tag{3'}$$

$$E(\lambda)A \subset AE(\lambda) \tag{4'}$$

$$p(A) = \int_{-\infty}^{\infty} p(\lambda)dE(\lambda) \tag{5'}$$

for any polynomial $p(t)$.

Theorem 14.16. *(p. 333)* $\Gamma(A + B) \leq \Delta(A) + \Gamma(B)$.

Theorem 14.17. *(p. 333)* $\Delta(A + B) \leq \Delta(A) + \Delta(B)$.

Corollary 14.23. *(p. 335)* $\Delta(AT) \leq \Delta(A)\Delta(T)$.

Theorem 14.24. *(p. 335)* $\tau(A) \leq \Delta(A)$.

Theorem 14.26. *(p. 336)*

$$\Gamma(A) = \inf_{Z} \inf_{T \in B(Z,X)} \frac{\|AT\|}{\tau(T)}. \tag{B.45}$$

Theorem 14.28. *(p. 336)* $\Delta(A) \leq \|A\|_m$.

Theorem 14.29. *(p. 336)* $A \in \Phi_+(X,Y)$ *if and only if* $\Gamma(A) \neq 0$.

Theorem 14.30. *(p. 337) If* $\Delta(B) < \Gamma(A)$, *then* $A + B \in \Phi_+(X,Y)$ *and* $i(A+B) = i(A)$.

Theorem 14.32. *(p. 337) If* $\tau(B) < \nu(A)$, *then* $A + B \in \Phi_+(X,Y)$ *and* $i(A+B) = i(A)$.

Theorem 13.36. *(p. 340) For A in B(X,Y),*

$$\|A\|_q/2 \leq \|A\|_m \leq 2\|A\|_q. \tag{B.46}$$

Theorem 14.37. *(p. 341) If Y has the compact approximation property with constant C, then*

$$\|A\|_K \leq C\|A\|_q, \quad A \in B(X,Y). \tag{B.47}$$

Theorem 14.39. *(p. 342) Compact operators are strictly singular.*

Theorem 14.41. *(p. 342) For A in B(X,Y), if* $A \notin \Phi_+(X,Y)$, *then for every* $\varepsilon > 0$ *there is a* $K \in K(X,Y)$ *such that* $\|K\| \leq \varepsilon$ *and* $\alpha(A-K) = \infty$. *Thus there is an infinite dimensional subspace M such that the restriction of A to M is compact and has norm* $\leq \varepsilon$.

Corollary 14.42. *(p. 343) If S is strictly singular, then for each* $\varepsilon > 0$ *there is an infinite dimensional subspace M such that the restriction of S to M is compact with norm* $\leq \varepsilon$.

Corollary 14.43. *(p. 343) Strictly singular operators are in* $F_{\pm}(X,Y)$.

Corollary 14.45. *(p. 344) T in* $B(X,Y)$ *is strictly singular, if and only if* $\Delta(T) = 0$.

Theorem 14.48. *(p. 344) A is strictly singular if and only if* $\tau(A) = 0$.

Theorem 14.51. *(p. 346) If* $\alpha(A) < \infty$, *then*

$$\gamma(A) \leq \mu_0(A) \leq \mu(A). \tag{B.48}$$

Corollary 14.52. *(p. 346) If* $\|T\| < \gamma(A)$, *then* $A - T \in \Phi_+(X, Y)$, $i(A - T) = i(A)$, *and* $\alpha(A - T) \leq \alpha(A)$.

$TVSLB''O$

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