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WSP72 Systems Architecture

Essential Mathematics: *Propositional Calculus & Predicate Calculus*

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Lecture 14

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A Fundamental Principle

- **Principle of Definition**
  - ***One needs both a formal definition of a design for precision, and a prose definition for comprehensibility.****
- Levels of Precision
  - Formalisms (e.g. the Predicate Calculus) are the most precise.
  - Graphical languages can be more intuitive but may not be formal.
- Models
  - In software and systems engineering practices, a model is generally considered to be an abstraction that represents a system.
  - But the term *model* can also be given formal definitions.

*Brooks, *Mythical Man-Month*, p.234, 6.3. This is taken as the Third Principle of Architecture and Systems Engineering in EA&PSE. Refer to p.xxi.

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The Significance for MBSE

- INCOSE regards the aim of MBSE to be a *formalised* application of modelling to support
  - System Requirements
  - Analysis and Design
  - Verification & Validation
- The extent of the formalisation in the application is limited by how formal the supporting models are!
- Any approach to MBSE in which the models are not founded on formal models will fall short of the aim.
- Models using logic and set theory are formal.

MBSE: Model Based Systems Engineering
Formal: having a recognized form, structure, or set of rules (see OED)

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Overview

Key Concepts

The need for both precision and comprehension

Logic as a formal language

Logic as the basis for the concept of Model

Key Topics

- What is a Good Model?
- Propositional Calculus
- Circuit Example
  - Series
  - Parallel
- Predicate Calculus

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Models in Science and Logic

- Stephen Hawking attributes a good model to be
  - Simple
  - Mathematically correct
  - Experimentally verifiable
- By analogy, a logical model of a natural language sentence[1] should be
  - Simple
  - Logically well-formed and consistent
  - Verifiable through logical interpretation[2]

[1]The concept of logical modelling presented here is motivated by the meaning of models using the predicate calculus [see, e.g. J. L. Bell, *Models and Ultraproducts*]. Refer to the logical model of *system* in Lecture 24-02. See also ISO/IEC 24707:2018 – Common Logic.

[2] In mathematics, this is called the first order model theory of Tarski.

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Languages and Methods of Mathematics, Science, and Engineering

- Mathematics uses
  - Formal logic (the predicate calculus) for description
  - Logical methods and mathematical induction for reasoning
- Science uses
  - Mathematics for description (but not as a formal language)
  - Models and experimental methods for reasoning
- Engineering uses
  - Science, mathematics for description and reasoning
  - Tools, prototypes, tests as methods for design and decision

A mathematical formalisation of systems engineering was consistently sought in the late 20th century.[1]

[1] Refer to *EA&PSE* 1.1.1 for a brief history.

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First Order Formal Languages of Logic

- The logical languages of mathematical logic are the
  - Propositional Calculus[1]
  - Predicate Calculus
- The term *calculus* derives from the Latin for calculation.
  - In logic, *Calculus* refers to the calculation of truth.
  - In the *Calculus* of Newton and Leibnitz, it refers to the calculation of limits.

- *Propositions* are declarative statements and are represented by [abstract] propositional variables.
- *Predicates* are statements of relationships between variables and are represented by [abstract] predicate letters and [abstract] individual variables.

[1] The Propositional Calculus is also called the Sentence Calculus. It is the syntax of first order logic.

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What is the Propositional Calculus?

- A formal language built from
  - Propositional variables
  - Logical connectives
  - Punctuation symbols
- Formalises that part of logic in which validity (truth)
  - Depends only on the truth values of the variables *and*
  - How propositions are assembled using connectives
  - Not on the internal meaning of the propositions

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Propositional Formulae

- Propositional variables are denoted by p, q, …
- The logical connectives are
  - Not: $\neg$
  - And: $\wedge$
  - Or[1]: $\vee$
- Punctuation symbols: ( )
- Examples of propositional formulae:

$p \qquad q \qquad \neg p \qquad p \wedge q \qquad p \vee q$

[1] This is the 'inclusive' Or (and is not exclusive).

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Interpretation: Truth Values
Boolean rules are the basis of control logic

- Propositional formula are completely interpreted by
  - The truth values of the propositions involved
  - Boolean rules of logic
- The truth table to the right is sufficient to determine the truth value of any propositional formula.

p	q	$\neg$ p	p $\wedge$q	p $\vee$q
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

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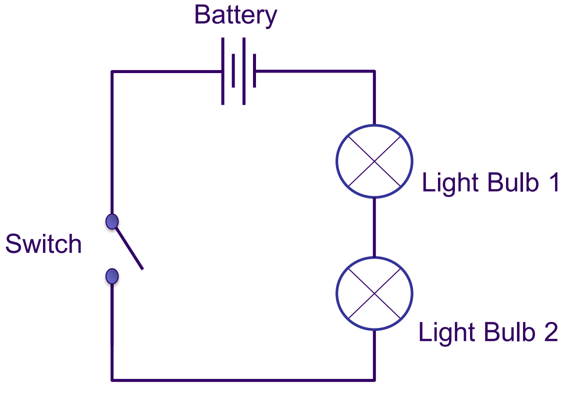
Series Circuits: Light Bulbs in Series



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Class Exercise on Series Circuit

- Write a sentence in the propositional calculus that expresses the state of the circuit in terms of the state of the light bulbs.
- Use the sentence to reason about the circuit reliability in terms of the light bulb reliability.

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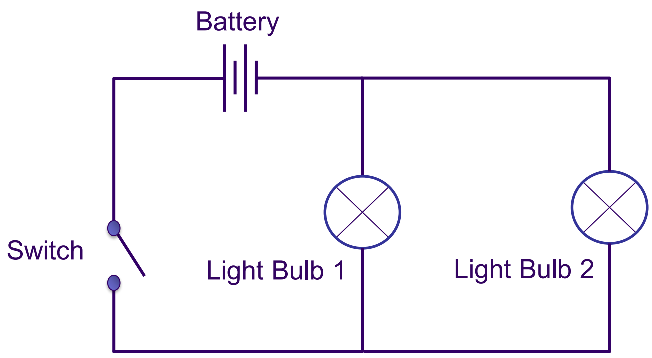
Parallel Circuits: Light Bulbs in Parallel



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Class Exercise on Parallel Circuit

- Write a sentence in the propositional calculus that expresses the state of the circuit in terms of the state of the light bulbs.
- Use the sentence to reason about the circuit reliability in terms of the light bulb reliability.

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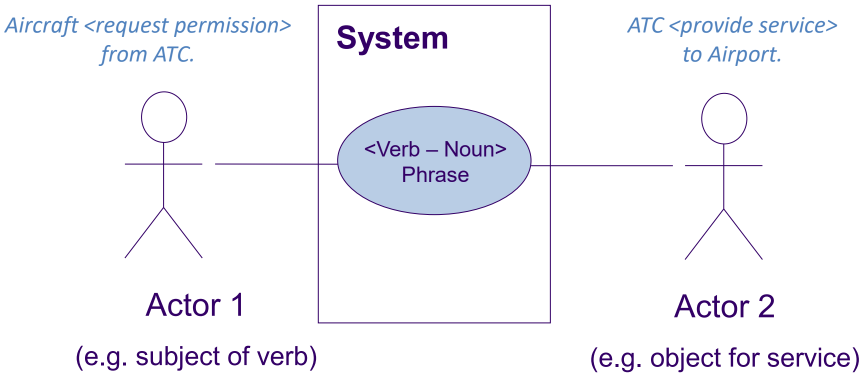
What is a Predicate?

Example of use cases for a system

Note that the System and <verb - noun> phrase always form a sentence.



A predicate is that part of a sentence containing a verb and stating something about the subject.

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What is the Predicate Calculus?

- A formal language built from
  - Individual variables (denoted by lower case letters)
  - Symbols for constants (e.g., c)
  - Predicate letters (denoted by capital letters)
  - Logical connectives (to include equality, =)
  - Punctuation symbols
- Predicate letters are interpreted as relations
- It formalises that part of logic in which validity
  - Depends on how predicates are assembled
  - As well as on the internal meaning of the predicates (i.e. their interpretations)

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Predicate Formulae

- Predicate variables are denoted by u_1, u_2 … , v, …
- Constants are denoted as c_1, c_2 … , c, etc.
- Predicate letters are denoted by P, Q, etc.
- The logical connectives are
  - Not: $\neg$
  - And: $\wedge$ Or: $\vee$
- Quantifiers: Universal, Existential (see 'sentences')
- Examples of predicate formulae:

$P(u_1, u_2)$ $\quad Q(v, c)$ $\quad P(u_1, u_2) \wedge Q(v, c)$ $\quad \neg P(u_1, u_2) \vee Q(v, c)$

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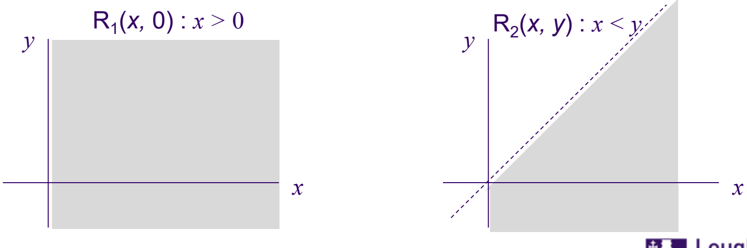
Predicates, Relations and Relational Structures

- The predicates $P_1(u_1, c)$ and $P_2(u_1, u_2)$ represent relations;

for example, the relations R_1 and R_2 where $u_1 \rightarrow x$, and $u_2 \rightarrow y$.

- The set of relations $\{R_1, R_2\}$ is a relational structure in the *xy*- plane
  - $x > 0$ is a relation in the *xy*- plane and $x < y$ is another
  - The intersection is the region below $y = x$ in Quadrants I, IV*
- The intersection is an *interpretation* of: $P_1(u_1, 0) \wedge P_2(u_1, u_2)$



***Note: this is how a feasible region is constructed in system design.**

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Sentences, Structures, and Models

- A *model* is typically considered to be an abstraction that represents something; *but in logic*…
  - There is a distinction between models and representations

    e.g., in UML, diagrams are *representations* of models; in the same way that the symbol 2 represents the concept of two
- In Tarski Model Theory* the concept of *Model* is precise:

  A *model* is a relational structure for which the interpretation of a sentence becomes valid.

The *relational structure* is referred to as a *model of the sentence*.

- In the previous slide, the feasible region in the graph is a *model of the sentence*: for each u_1, u_2 : $P_1(u_1, 0) \wedge P_2(u_1, u_2)$

* This is also called First Order Model Theory because the predicates represent first order relations. It is adopted in ISO/IEC:24707.

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Interpretation and Scope of Predicates

- If *x* were selected from the counting numbers (i.e. the values of *x* could be 1, 2, ...) then $R_1(x, 0)$ would be true.
- If *x* were selected from the positive real numbers and y from the negative ones then $R_2(x, y)$ would be true.
- But if *x* and *y* were both selected from the positive real numbers; the truth of $R_2(x, y)$ could only be decided based on specific values (instances) of *x, y*
- The truth of $R_1(x, 0)$ and $R_2(x, y)$ also depends on whether
  - *Every* value (instance) of *x, y* is considered or
  - Only *some* values (instances) of *x, y* are considered
  - This is why sentences in the Predicate Calculus use the universal and existential scope operators, i.e. for decidability

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Questions?

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