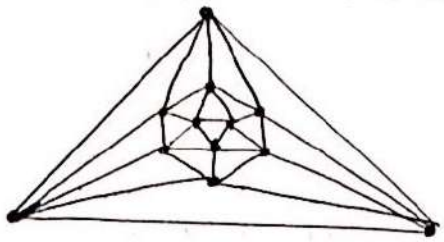
<u>Thm</u> : Let G be an <u>r-regular graph</u> having n vertices. Then G has $\frac{nr}{2}$ edges.

<u>Proof</u> : No. of vertices of $G = n$.
degree of each vertex $= r$
$\therefore$ sum of ~~eac~~ degrees of all vertices $= nr$
$\therefore$ By handshaking lemma, no. of edges $= \frac{nr}{2}$
(i.e. $\sum_{v \in V} \deg v = 2|E|$)

<u>Q-3)</u> Find the no. of edges in this 5-regular graph.



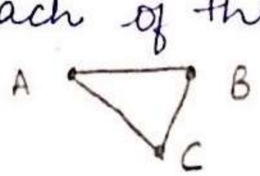
<u>Q4</u> Prove that there are no 3 regular graphs with 7 vertices.

<u>Solⁿ</u> By above theorem, 3 regular graph with 7 vertices has $\frac{3 \times 7}{2} = \frac{21}{2}$ edges, which is not possible.

(Assignment)
<u>Q5</u> Prove that there are no r-regular graphs having n vertices when n & r both are odd.

<u>Some Important classes of regular graphs</u>

① <u>Complete Graphs (K_n)</u> : A complete graph is a regular graph in which each vertex is joined to each of the others by exactly one edge.



$\boxed{K_3}$ (2-regular graph)

* A complete graph with n vertices is denoted by K_n where $n \in \mathbb{Z}^+$

Example :

K_4 i.e. 3-regular, K_1 i.e. 0-regular, K_2 i.e. 1-regular, K_3 i.e. 2-regular, K_5 i.e. 4-regular

Note : ① K_n is regular of degree $(n-1)$ & thus has $\frac{n(n-1)}{2}$ edges. (i.e. by thm $\boxed{\frac{nr}{2}}$ edges)

② multiple edges and loops are not permitted in complete graphs

② **Null graphs** (N_n) : A null graph is a regular graph with no edges.

* A null graph with 'n' vertices is denoted by N_n

Eg :

N_3 i.e. 0-regular graph

* The graph N_n is regular of degree zero.

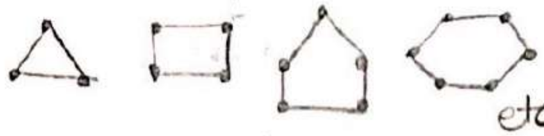
③ **Cycle Graphs** (C_n) : A cycle graph is a regular graph consisting of a single cycle of vertices & edges.

* The cycle graph with n vertices is denoted by C_n.

Eg :

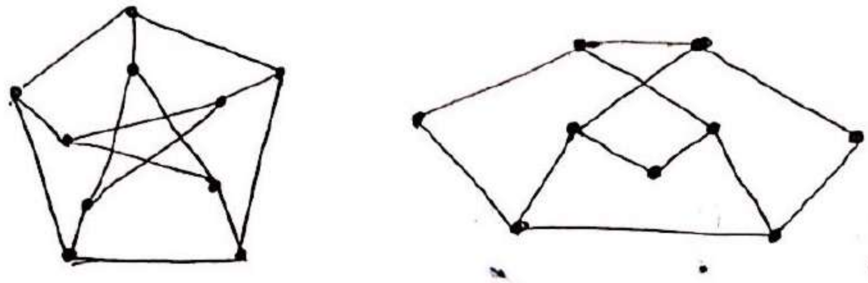
C_1 C_5 i.e. both are 2-regular graphs

Note: (1) C_n is regular of degree 2 & has $\frac{n \times 2}{2} = n$ edges. (ie by using thm; $\boxed{\frac{nr}{2}}$ edges)

(2) For $n \geq 3$, C_n can be drawn as a regular polygon.  etc

(4) The Peterson Graph: It is a 3-regular graph with 10 vertices & 15 edges (ie $\frac{3 \times 10}{2}$)

It can be drawn in various ways, two of which are:



(5) Platonic Graphs: (will study in Planar Graphs)

x ———————— x

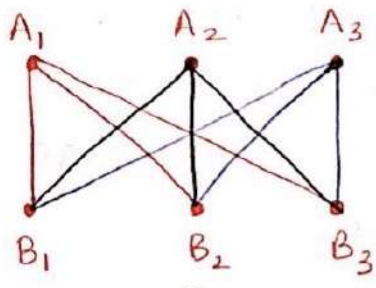
(Graph Theory):

Bipartite Graphs :

* These graphs are important because of various applications based on it.

A bipartite graph is a graph whose set of vertices can be partitioned into 2 subsets (disjoint) A & B (called bipartition sets) in such a way that every edge joins a vertex in A and a vertex in B.

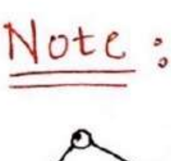
Example :



G

$A = \{A_1, A_2, A_3\}$
$B = \{B_1, B_2, B_3\}$
Total vertices : 6

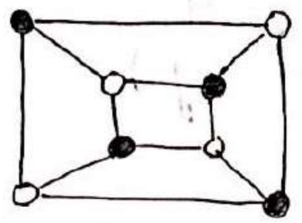
Note : 1) In a bipartite graph, there will be no edge b/w 2 vertices of the same set.



2) A graph is bipartite iff its vertices can be coloured with 2 colours such that every edge has ends of different colour.

We can distinguish b/w vertices of A and B by drawing one set in white and the other in black, then each edge of the graph will be incident with a black vertex and a white vertex.

Example :



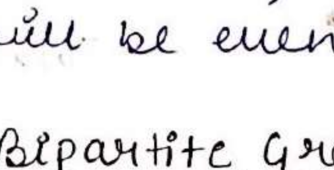
Q-1) Prove that in a bipartite graph, every cycle has an even number of edges.

A-1) Let G be a bipartite graph. Colour the vertices of G black & white such that no 2 black vertices and no 2 white vertices are adjacent to each other. We see that every cycle in G alternate b/w these 2 colours. Thus, no. of edges in every cycle will be even.

Complete Bipartite Graph :- It is a graph in which **each** vertex in set A is joined to every vertex in set B by just one edge.

The complete bipartite graph having r vertices in A and s vertices in B is denoted by $K_{r,s}$

Example :



$K_{2,4}$

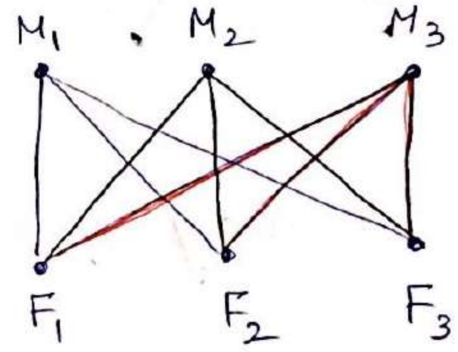
not a complete bipartite graph.

Application of ~~Bipartite~~ Complete Bipartite Graph

let $M = \{M_1, M_2, M_3\}$ and
$\downarrow$
Member 1

$F = \{F_1, F_2, F_3\}$
$F_1 \to$ Grocery store, $F_2 \to$ Parking, $F_3 \to$ gym

Q Check that each member of the society must get all the facilities



Observation: Each element of set M
① is connected to all the elts of set F

② Every edge consist of 2 vertices s.t. one vertex belongs to set M & other vertex belongs to set F i.e. bipartite graph.

Note : (1) $K_{r,s}$ is same as $K_{s,r}$

(2) $K_{r,s}$ has $(r+s)$ vertices in all, r vertices of degree s and s vertices of degree r.

(3) $K_{r,s}$ has $(r \cdot s)$ edges.

Remark : $K_{r,s}$ is a regular graph when $r = s$.

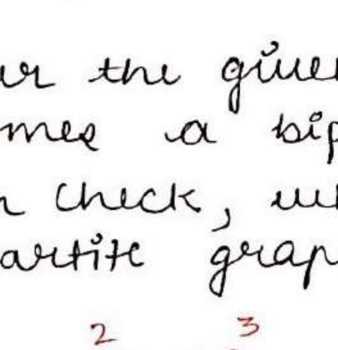
eg: $K_{2,2}$ 

Yes, it is a regular graph

(Assignment)

(Q-2) Draw the graph $K_{4,4}$ and $K_{1,7}$. How many vertices & edges does each have?

(Q-3) Colour the given graph such that it becomes a bipartite graph. Then check, whether it is a complete bipartite graph or not.



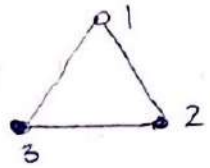
Result : A graph that contains a triangle cannot be bipartite graph.

Proof : Atleast 2 of the vertices of the Δ must lie in one of the bipartition sets. Since these 2 are joined by an edge, the graph cannot be bipartite.

$A = \{1, 2\}$
$B = \{3\}$

OR

$A = \{2, 3\}$
$B = \{1\}$



OR

$A = \{1, 3\}$
$B = \{2\}$

A = Black set
B = White set

Note: If no. of vertices is odd in a graph, it cannot be a bipartite graph.

Eg: Triangle (3), Pentagon (5), etc.

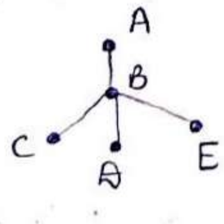
x ———— x

LECTURE -5
(Graph Theory)

One of the most important classes of bipartite graphs is the class of trees.

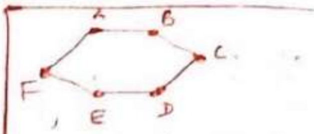
Trees: A tree is a connected graph with no cycles.

Eg:



← Path b/w A&E i.e. ABE

Note: A tree is a bipartite graph.



connected graph

Remark: In a tree, there is just one path b/w each pair of vertices.

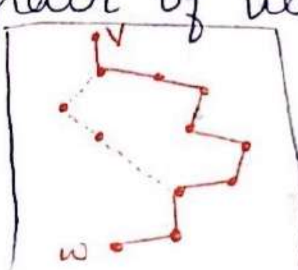
Reason: Since a tree is connected, therefore there exist atleast one path b/w each pair of vertices.

By contradiction:

If possible, suppose there are 2 vertices in the tree say v & w, that are joined by two paths. Then these paths would create a cycle (that includes all the edges of these 2 paths or only some of them) which contradicts the defn of a tree. (∵ we see that cycle is formed)

Thus, there exists exactly one path b/w each pair of vertices.

Graph 1 ←



→ Graph 2

Q-1) Draw all unlabelled trees with 5 or fewer vertices.

Sol^n 1)

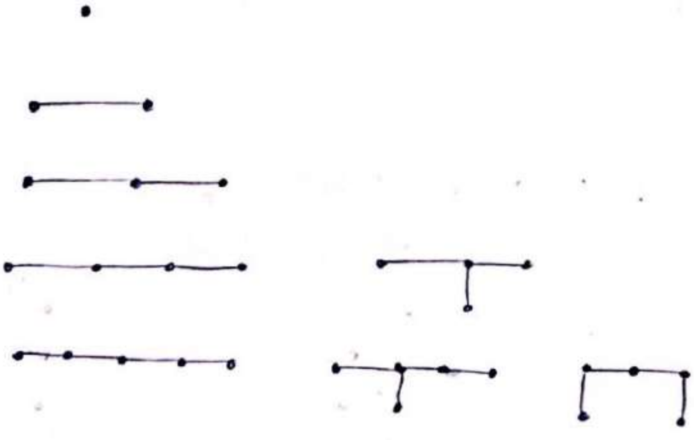
1 vertex

2 vertices

3 vertices

4 vertices

5 vertices

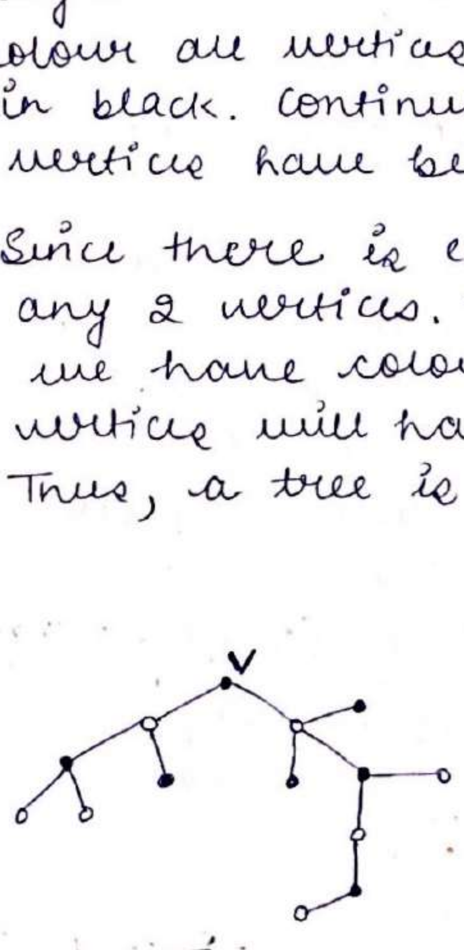


Q-2) Why is a tree a bipartite graph?

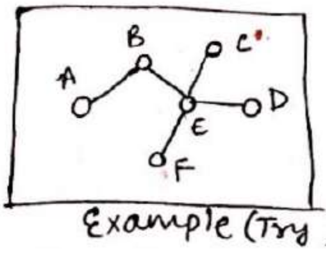
A-2) Choose any vertex v in the tree and colour it black. Colour all vertices adjacent to v in white colour... Next, colour all vertices adjacent to those in black. Continue this process till all vertices have been coloured.

Since there is exactly one path b/w any 2 vertices. Therefore, the way we have coloured, no two adjacent vertices will have the same colour. Thus, a tree is bipartite.

Tree	Bipartite
① Connected ② No cycles	① Edges should have one vertex in one set & other vertex in other set.



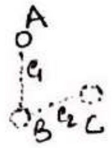
Tree



Example (Try)

Q-3) Why does a tree with n vertices have $(n-1)$ edges?

A-3) Every tree can be built up from a single vertex by successively adding an edge and a new vertex. Each time we increase the no. of edges by one and the no. of vertices at one.

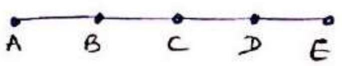


Since we started with 1 vertex and zero edge. Therefore, after n steps we will end with n vertices & $(n-1)$ edges.

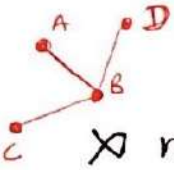
Path Graphs : A path graph is a tree consisting of a single path through all its vertices.

* Path graph with n vertices is denoted by P_n.

Eg :

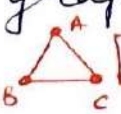


P_5



✗ not a path graph

Note : P_n has $(n-1)$ edges & can be obtained from the cycle graph C_n by removing an edge. Eg:



$\boxed{C_3}$

$\boxed{P_3}$

Remark : Path graph is a tree and hence a bipartite graph.

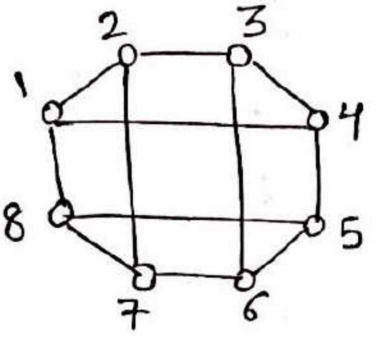
i.e. $\boxed{\text{Path graph}} \rightarrow \boxed{\text{Tree}} \rightarrow \boxed{\text{Bipartite graph}}$

Hence, Pathgraph $\rightarrow$ Bipartitegraph.

Assignment-3

Q-1) Draw the graph $K_{4,4}$ and $K_{1,7}$.
How many vertices and edges does each have? (4 Marks)

Q-2) Colour the given graph such that it becomes a bipartite graph. (1 Mark)



Check whether it is a complete bipartite graph or not.

Q-3) Give an example of a bipartite graph which is also a tree. (1 Mark)