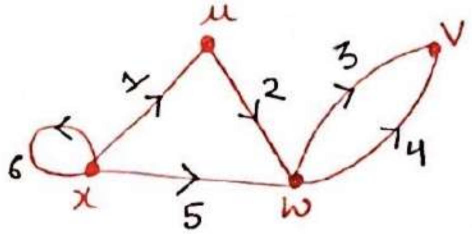
Lecture 9
(Graph Theory)

Digraphs

A digraph consists of a set of elements called vertices and a set of elements called arcs. Each arc joins 2 vertices in a specified direction.



vertices $\{u, v, w, x\}$
arcs $\{1, 2, 3, 4, 5, 6\}$

$\because$ arc 1 joins x to u.
$\therefore$ $\boxed{\text{arc } 1}$ is denoted by $\boxed{xu}$.

x —— u $\quad xu = ux$
$x \longrightarrow u$ $\quad xu \neq ux$

Similarly, $\boxed{\text{arc } 2}$ is denoted by $\boxed{uw}$
$\boxed{\text{arc } 3}$ is denoted by $\boxed{wV}$, etc.

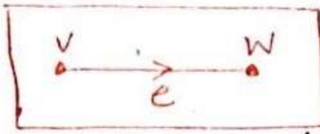
Definitions :

(1) Multiple arcs : In a digraph, 2 or more arcs joining the same pair of vertices in the same direction are called multiple arcs.

(2) Loop : An arc joining a vertex to itself is a loop.

(3) Simple digraph : A digraph with no multiple arcs or loops is a simple digraph.

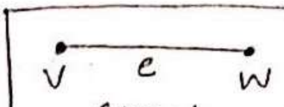
Adjacency and Incidence



Digraph

① The vertices v & w of a digraph are <u>adjacent vertices</u> if they are joined (in either direction) by an arc e.

② An arc e that joins v to w is <u>incident from v</u> & <u>incident to w</u>. The vertex v is <u>incident to</u> e & w is <u>incident from</u> e.



Graph

e is incident to v and w.

v is incident to e & w is incident to e

Q-1) In the given graph, check:

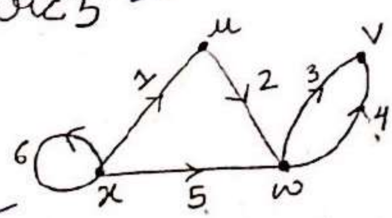
ⓐ u & x are adjacent ✓

ⓑ w is incident from arc 2 & arc 5 ✓

ⓒ x is incident to arc 5 ✓

ⓓ w is incident to arc 5 ✗

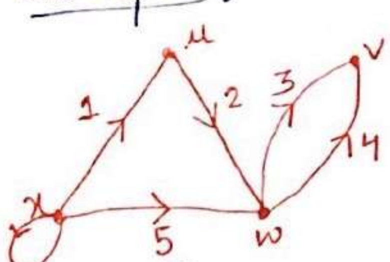
ⓔ w is incident to arcs 3 & 4 ✓



Subdigraph:

A subdigraph of a digraph D is a digraph all of ~~which~~ whose vertices are vertices of D & all of whose arcs are arcs of D.

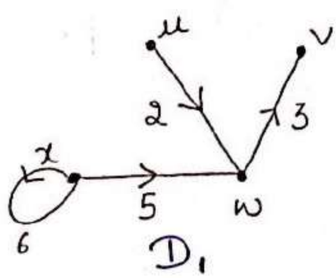
Example:



D

vertices: u, v, w, x

arcs: 1, 2, 3, 4, 5, 6



D_1

vertices: u, v, w, x

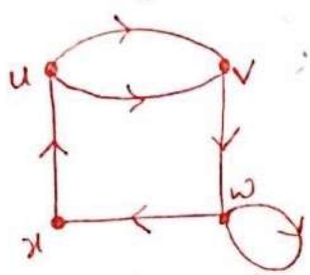
arcs: 2, 3, 5, 6

Check: ① vertices of $D_1 \subseteq D$

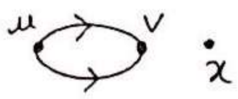
② arcs of $D_1 \subseteq D$

~~③ Direction of arcs~~

Q-2) Check which of the following are subdigraphs of the given digraph.

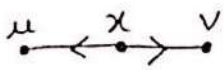


D



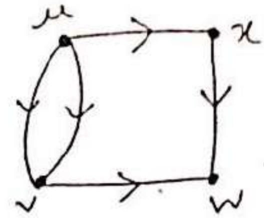
D_1

✓



D_2

×



D_3

×

$V = \{u, v, x, w\}$

$A = \{uv, uv, vw, ww, wx, xu\}$

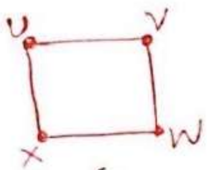
[∵ vertices of D_1 is contained in D and arcs of D_1 are there in D]

① $\{u, v\} \subseteq \{u, v, x, w\} = V$

② $\{uv, uv\} \subseteq \{uv, uv, vw, ww, wx, xu\} = A$

Difference :-

In case of subgraphs :



G

$V = \{u, v, w, x\}$

$E = \{uv, vw, wx, xu\}$

G_1

✓ $V_1 = \{x, w\}$

$E_1 = \{xw\}$

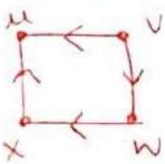
G_1 is subgraph of G because:

① $V_1 \subseteq V$

② $E_1 \subseteq E$

Note : $xw = wx$

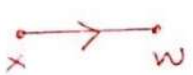
In case of digraph :



D

$V = \{u, v, x, w\}$

$A = \{xu, vu, vw, wx\}$



D_1

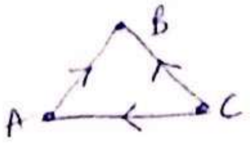
×

D_1 is not a ~~digraph~~ subdigraph of D because:

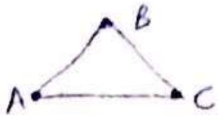
① arc in D_1 i.e. 'xw' is not in D.

Note : $xw \neq wx$ (∵ direction changes)

Underlying graph : The underlying graph of a digraph D is the graph obtained by replacing each arc of D by the corresponding undirected edges.

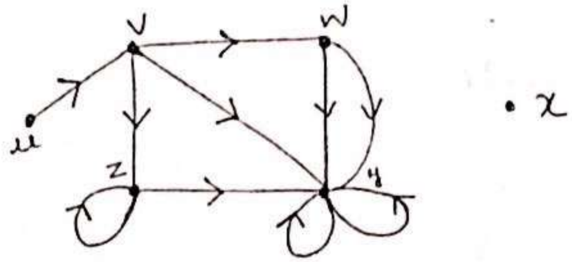
Example : 

Digraph

$\Longrightarrow$ 

Underlying Graph

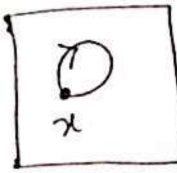
Vertex Degrees



Digraph 1

outdeg u = 1   outdeg v = 3
indeg u = 0   indeg v = 1

outdeg x = 0   outdeg z = 2
indeg x = 0   indeg z = 2



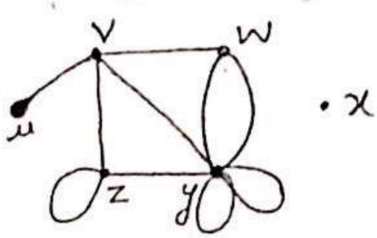
Digraph 2

$\Rightarrow$ outdeg x = 1
indeg x = 1

i.e. each loop contributes 1 to indegree & 1 to outdegree.

Definition : In a digraph, the out-degree of a vertex, say v, is the number of arcs incident from v and is denoted by outdeg v.
The in-degree of vertex v is the number of arcs incident to v and is denoted by indeg v

Review :



Graph 1

degree of u =
$\boxed{\text{deg } u = 1}$

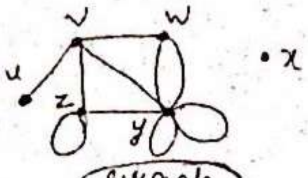
degree of v = 4

More
Definitions :

① **Out-degree sequence** : The out-degree sequence of a digraph D is the sequence obtained by listing the out-degree of D in increasing order, with repeats as necessary.

② **In-degree sequence** : The in-degree sequence of a digraph D is the seq obtained by listing the in-degree of D in increasing order, with repeats as necessary.

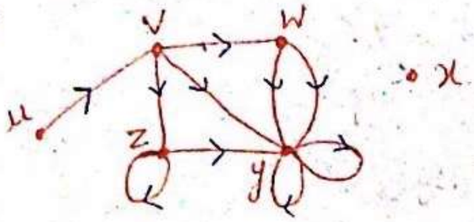
Review



graph

deg sequence
$= \{0, 1, 2, 4, 4, 8\}$

Example :



Digraph

$\boxed{\text{out deg } u = 1}$
indeg $u = 0$

$\boxed{\text{outdeg } v = 3}$
indeg $v = 1$

$\boxed{\text{outdeg } w = 2}$
indeg $w = 1$

$\boxed{\text{outdeg } x = 0}$
indeg $x = 0$

$\boxed{\text{outdeg } y = 2}$
indeg $y = 6$

$\boxed{\text{outdeg } z = 2}$
indeg $z = 2$

Outdegree sequence $= (0, 1, 2, 2, 2, 3)$
Indegree sequence $= (0, 0, 1, 1, 2, 6)$

- - - - - - - - - - - - - - - - - - - -

Theorem : (Handshaking Dilemma)

In any digraph, the sum of all the out-degrees and the sum of all the in-degrees are both equal to the number of arcs.

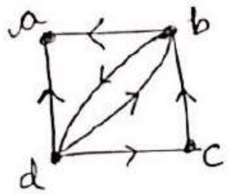
i.e. $\sum_{v \in V} \text{outdeg } v = \text{Total arcs}$

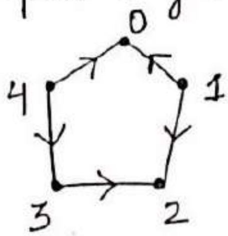
and

$\sum_{v \in V} \text{indeg } v = \text{Total arcs}$

Assignment

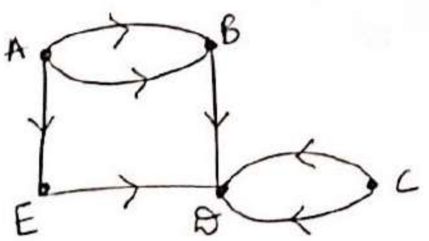
Q-1) Write down the vertices and arcs of each of the following digraphs. Are these digraphs simple digraphs?



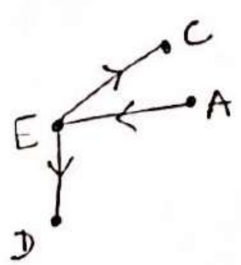


Q-2) Draw a digraph with vertices $= \{u, v, w, x\}$ and arcs : $\{vw, wu, wv, wx, xu\}$

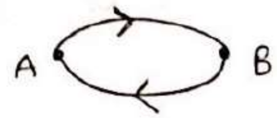
Q-3) Check whether D_1 & D_2 are subdigraphs of D or not.



D



D_1



D_2

Lecture 11

(Graph Theory)

Walks, Trail & Path in Digraph

(1) Walk : In a digraph, a walk is a succession of arcs of the form uv, vw, wx, ... yz. This walk is denoted by uvwx... yz & is referred as a walk from u to z.

(*) If we say walk is of 'K' length. It means it has 'K' ~~ed~~ arcs in that walk.

(2) Trail : A trail is a walk in which all the arcs, but not necessarity all the vertices, are different.

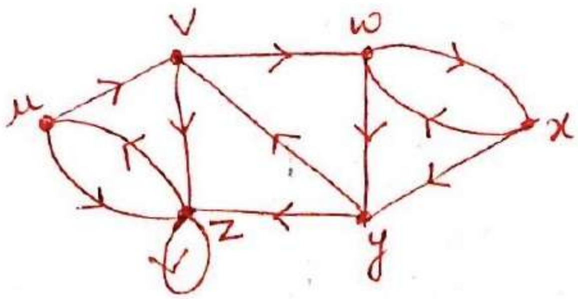
(3) Path : It is a walk in which all the arcs & all the vertices are different.

(4) Closed Walk : A closed walk in a digraph is a succession of arcs of the form uv, vw, wx, ... yz, zu i.e. first & last vertex is same.

(5) Closed Trail : It is a closed walk in which all the arcs are different.

(6) Closed Path/ Cycle : It is a closed walk in which all the arcs are different & all the intermediate vertices are different.

Example:



Digraph

Walk : vwxyvwyzzu length 9

Trail : uvwyvz length 5 Note : [not a path]

Path : vwxyz length 4

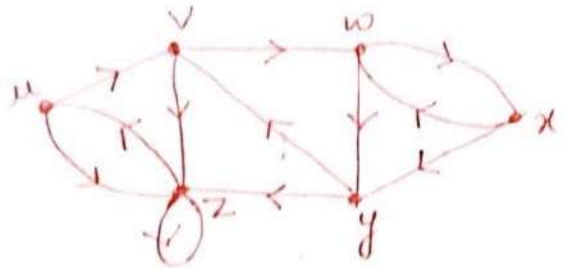
closed walk : uvzu

closed Trail : uvwyvzu

closed Path/cycle : uvwxyzu

vwyzv is a closed trail?

Example:



Digraph

Walk : vwxyvwyzzu length 9

Trail : uvwyvz length 5 Note: [not a path]

Path : vwxyz length 4

Closed Walk : uvzu

Closed Trail : uvwyvzu

Closed Path/cycle : uvwxyzu

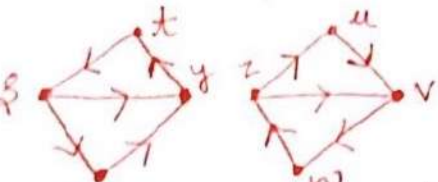
vwyzv is a closed trail?

Some More Definitions :

① Connected Digraph : A digraph is connected if its underlying graph is a connected graph.

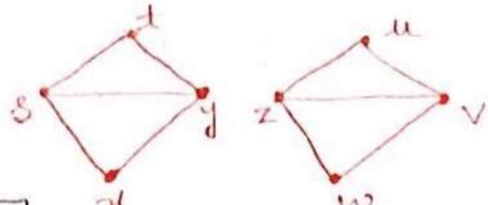
② Disconnected Digraph : A digraph ~~which~~ is dis-connected if its underlying graph is not connected.

③ Strongly connected Digraph : A digraph is strongly connected if there is a path b/w each pair of vertices.

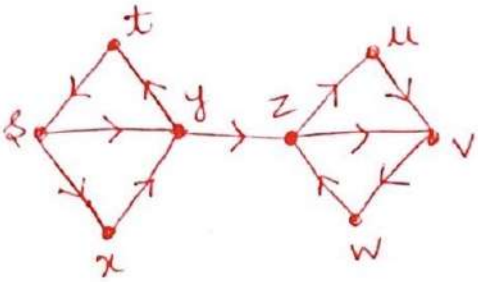


Digraph 1

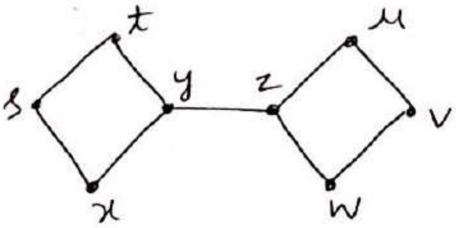
Disconnected



Underlying graph



Digraph2



underlying graph

connected

Digraph2 is not strongly connected (why?)

[∵ No path from z to y]

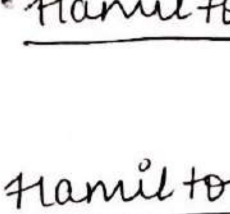
Eulerian and Hamiltonian Digraphs

① Eulerian trail : A closed trail that includes every arc of the digraph.

② Eulerian digraph : A connected digraph is Eulerian if it contains an ~~closed~~ eulerian trail.

③ Hamiltonian Cycle : A cycle that includes every vertex of the digraph.

④ Hamiltonian digraph : A connected digraph is hamiltonian if it contains a hamiltonian cycle.



Digraph

It is an eulerian digraph because we can find an eulerian trail ie. bcgfegb.

But, It is not a hamiltonian digraph because we cannot find even one hamiltonian cycle.