Some important Remarks and Results on Hamiltonian graphs :

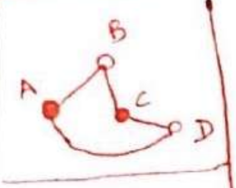
Remarks : ① The cycle graph C_n is hamiltonian graph $\forall n$.

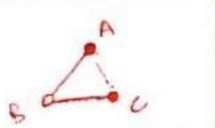
② The complete graph K_n is hamiltonian $\forall\ n \geq 3$

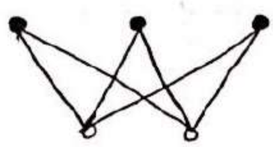
③ Since a tree doesn't contain a cycle. Therefore the only tree that is hamiltonian is the trivial tree having 1 vertex and no edge.

Result : A bipartite graph with an odd number of vertices is not Hamiltonian.

Proof: We know that in a bipartite graph the vertex set can be split into 2 sets A and B such that each edge has one end vertex in A and the other in B.







A is the set with black vertices & B is the set with white vertices.

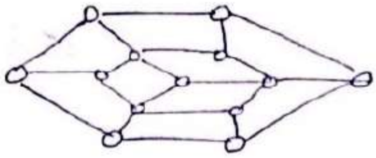
Thus, any hamiltonian cycle must alternate b/w A and B, ending in the same set as it started. This is possible only when A & B have the same no. of vertices.

$\therefore$ total no. of vertices cannot be odd.

Note : A bipartite graph with even number of vertices may or may not be Hamiltonian.

Assignment

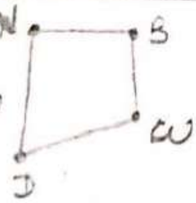
Q-1 Use the above result to prove that the following graph is not Hamiltonian.



$n = 4$

$\deg v + \deg w = 4 \geq 4$

Yes



Theorem 5 : (Ore's Theorem)

Let G be a simple connected graph having 'n' vertices, $n \geq 3$ and $\deg v + \deg w \geq n$ for each pair of non-adjacent vertices v & w. Then G is Hamiltonian.

⊛ Converse of above theorem doesn't hold

Counter eg : C_n, $n \geq 5$

Reason : C_n is a 2-regular graph.

$\therefore$ Sum of degrees of any pair of non-adjacent vertices $= 2 + 2 = 4$.

$\therefore$ When $n \geq 5$, the conditions of Ore's thm are not being satisfied but C_n is Hamiltonian.

Result: Let G be a simple connected graph with n vertices, $n \geq 3$ and $\deg v \geq \frac{n}{2}$ for each vertex v.

Show that G is Hamiltonian.

Proof: As $\deg v \geq \frac{n}{2}$ $\forall$ vertices v

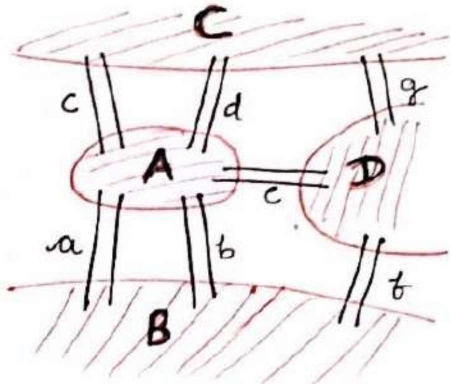
Then $\boxed{\deg v + \deg w \geq n}$ for each pair of vertices whether adjacent or not.

$\therefore$ Result follows from Ore's theorem.

Lecture-8
(Graph Theory)

Konigsberg Bridge Problem :

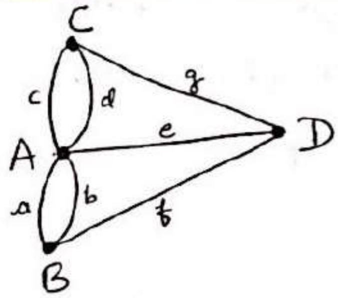
A, B, C and D are land areas.
a, b, c, d, e, f, g are the seven bridges that connect these land areas.
Is it possible to find a route that crosses each of these seven bridges and returns to the starting point ?



Solution

We represent the land areas by 4 vertices of a graph and the seven bridges as edges joining the corresponding pair of vertices.

(*) The problem is to find an Eulerian trail in the graph.

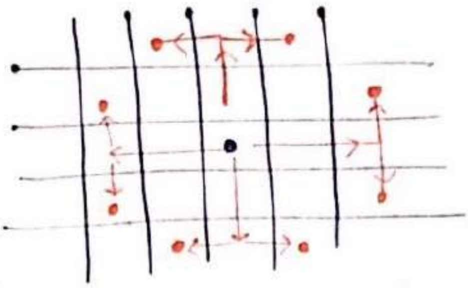


Clearly, the graph is not Eulerian.
Therefore, there doesn't exist any route of the desired kind.

Theorem 2
Theorem 3
(In Eulerian Section)

Knight's Tour Problem :

<u>Review</u> : On a chessboard, a knight always moves 2 squares in a horizontal or vertical direction and one square in a perpendicular direction, as illustrated below.



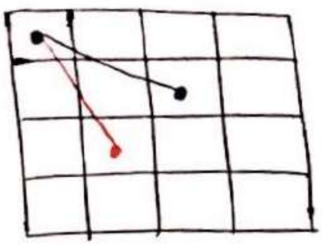
<u>Knight's Tour Problem</u> : Can a knight visit each square of a chessboard just once by a sequence of ~~kt~~ knight's moves and finish on the same square as it began?

<u>Note</u> : If we represent the board as a graph in which each vertex corresponds to a square and each edge corresponds to a pair of squares connected by a knight's move, then finding a knight's tour is equivalent to finding a hamiltonian cycle in the associated graph.

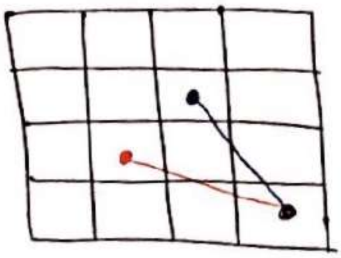
Problems based on Knight's Tour :

Problem 1 : Show that there is no knight's tour on a 4×4 chessboard.

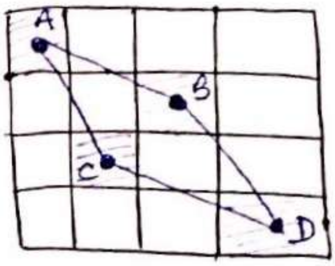
Solution : The only way to include the top left square is to include the 2 moves shown in (a) and similarly the only way to include the lower right square is to include the 2 moves in (b).
Combining these, the tour has to include all 4 moves in (c) which already form a cycle and thus it is not possible to include them as a part of the full tour.



(a)



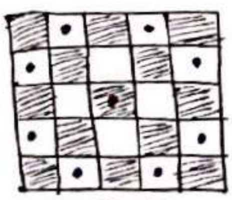
(b)



(c)

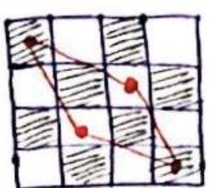
Problem 2 : Show that there is no knight's tour on a chessboard having odd number of squares.
(Result)

Solution : A knight's move always takes a knight to a square of different colour. Thus the graph associated with any chessboard is a bipartite graph. (Since the vertex set can be partitioned into 2 sets, one having all the white squares and the other set having all black squares).



5×5 chessboard

Now, if the chessboard has an odd number of squares then we get a bipartite graph having odd no. of vertices



(c) part (bipartite graph)

which cannot be Hamiltonian (proved earlier)
Therefore, there can be no knight's tour on a chessboard having odd no. of squares.

<u>Problem 3</u> : Show that there is no knight's tour on a 5×5 chessboard.

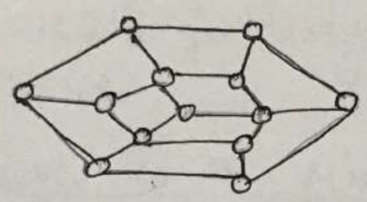
<u>Solution</u> : Same as problem 2 <u>solution</u>.

8x8 CHESSBOARD

0	59	38	33	30	17	8	63
37	34	31	60	9	62	29	16
58	1	36	39	32	27	18	7
35	48	41	26	61	10	15	28
42	57	2	49	40	23	6	19
47	50	45	54	25	20	11	14
56	43	52	3	22	13	24	5
51	46	55	44	53	4	21	12

Assignment-5

Q-1) Use the theorem [A bipartite graph with an odd no. of vertices is not Hamiltonian] to prove that the following graph is not Hamiltonian.



Q-2) Check whether the given graph is Hamiltonian or/and semi-hamiltonian. In case of any, give one suitable example of hamiltonian cycle / path from the graph.