LECTURE - 6
(Graph Theory)

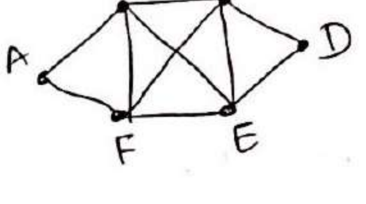
Explorer's Problem: An explorer wishes to find a tour that traverses each road exactly once and returns to the starting point.

Consider the road map as a connected graph whose vertices correspond to the cities and edges correspond to the roads.

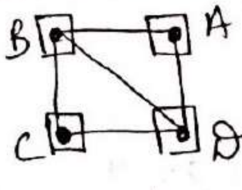
⊛ Thus the explorer's problem is to find a closed trail that includes all the edges of the graph.

Traveller's Problem: A traveller wishes to find a tour that visits each city exactly once and returns to the starting point.

⊛ Thus the traveller's problem is to find a cycle that includes every vertex of the graph.



ABCEFA → Closed Trail

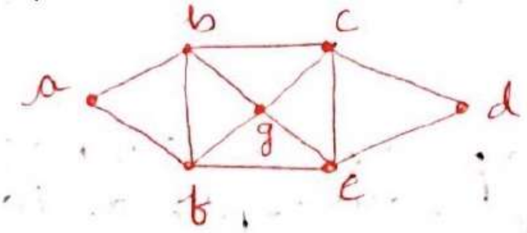


ABCDA → tour

ABCDECFBEF A → closed Trail in which all edges are covered.
(tour)

Definitions

① **Eulerian trail** : A closed trail that includes every edge of the graph.



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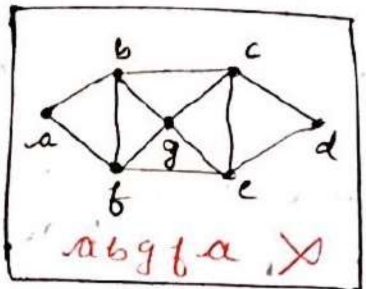
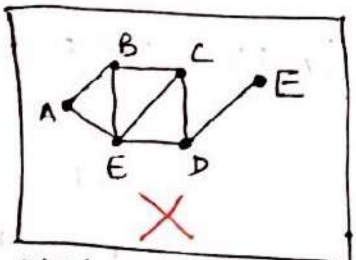


closed trail but not eulerian trail

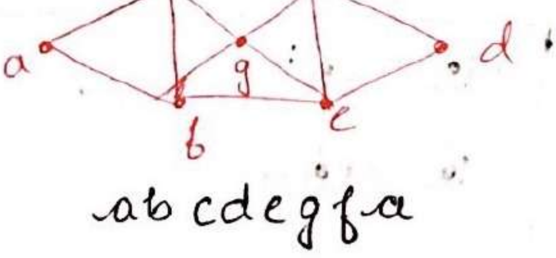
② **Eulerian Graph** : A connected graph is called eulerian if it contains an Eulerian trail.

Example : Above example



Not an eulerian graph (∵ we cannot find even one eulerian trail)

③ **Hamiltonian Cycle** : A cycle that includes every vertex of the graph.



abcdegfa

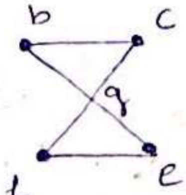
④ **Hamiltonian Graph** : A connected graph is hamiltonian if it contains a hamiltonian cycle.

Example : Above example.

Q-1) Give an example of graph which is Eulerian but not Hamiltonian.

A-1) For a graph to be Eulerian, it must have an eulerian trail in it.

For a graph not to be hamiltonian, it must not contain even a single hamiltonian cycle in it.

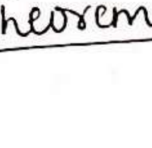


G

This graph G is Eulerian graph because we can find an eulerian trail in it i.e. bcgfegb but the graph G is not hamiltonian.

Some Important Theorems :

Theorem 1 : Let G be a graph in which every vertex has even degree.
Then G can be split into cycles, no two of which have an edge in common.

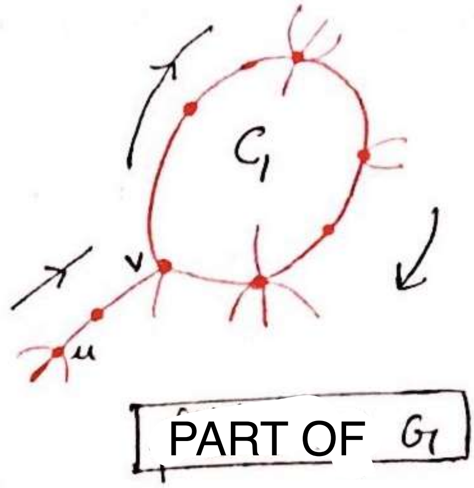


Proof 1 : Let G be a graph in which each vertex has even degree. We obtain our first cycle in G by starting at any vertex 'u' and traversing edges in an arbitrary manner, never repeating any edge.

Because each vertex has even degree, we can always enter and leave any vertex via different edges.

Since there is only a finite number of vertices, we must eventually reach a vertex 'V' that we have met before.

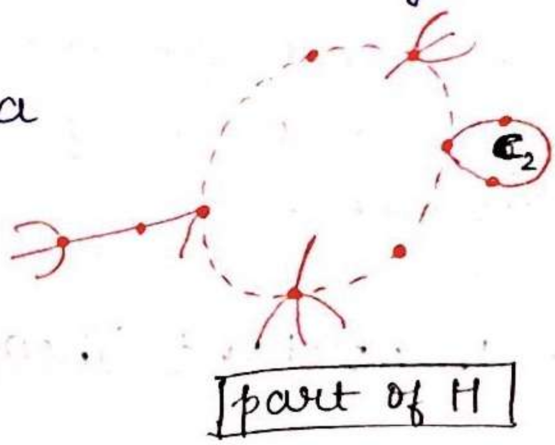
Then, the edges of the trail b/w the 2 occurrences of the vertex 'V' must form a cycle, say C_1



If C_1 includes all edges and vertices of G, then we are done. If not, we now remove from G the edges of C_1.

This leaves a graph H (possibly disconnected) in which each vertex has even degree.

Repeat the above procedure to obtain a cycle C_2 in H with no edges in common with C_1



Removal of the edges of C_2 from H, leaves yet another graph in which each vertex has even degree.

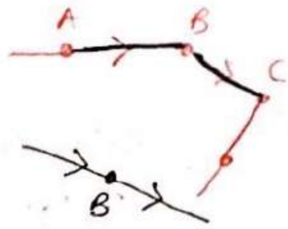
Repeat this process till there are no edges left.

Thus, we obtain a finite no. of cycles $C_1, C_2, \ldots C_K$ that together covers all edges of G, no two of which have any edges in common.

Theorem 2 : A connected graph is Eulerián iff each vertex has even degree.

Proof 2 : Let G be Eulerián Graph.

Then G has an eulerian trail. Whenever this trail passes through a vertex, it contributes 2 to the degree of that vertex. Since each edge is used exactly once, the degree of each vertex is a sum of 2s & hence even.

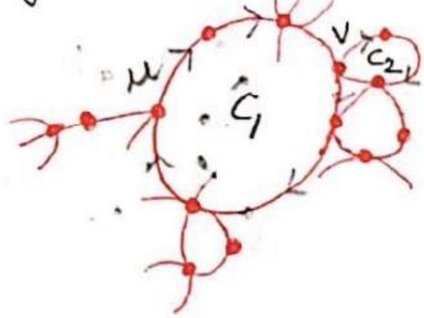


Let each vertex of G have even degree.

Then by thm 1, G can be split into cycles, no 2 of which have an edge in common.

We now fit these cycles together to make an Eulerián trail. We start at any vertex of a cycle C_1 and travel round C_1 until we meet a vertex of another cycle, say C_2.

We traverse the edges of this cycle and then resume travelling round C_1. This gives a closed trail that includes C_1 and C_2.



If this trail includes all the edges in G then we have the required Eulerian trail & hence E.G.

If not, we travel round our new closed trail and add a new cycle, say C_3, when we

(Note : since G is connected there will always be atleast one cycle to add to our trail)

come to it, we continue this process till all cycles have been covered and hence get the required trail.

Theorem 3 : An Eulerian graph can be split into cycles, no two of which have an edge in common.

Lecture 7
(Graph Theory)

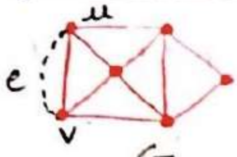
Definition

① Semi Eulerian Trail : An open trail that covers every edge in the graph.

② Semi Eulerian Graph : A connected graph is semi eulerian if there is a semi eulerian trail in the graph.

Theorem 4 : A connected graph is semi-eulerian iff it has exactly 2 vertices of odd degree.

Proof 4 : (a) To prove : If G is semi-Eulerian graph then G has exactly 2 vertices of odd degree.



Let G be semi Eulerian.
Then G has a semi-eulerian trail.
Let 'u' & 'v' be the starting and ending vertices of this semi-eulerian trail.
Now, add an edge joining u & v. This gives us an Eulerian trail. Thus G becomes an eulerian graph and by (thm 2) each vertex has even degree.

If we now recover G by removing the edge e, we see that 'v' & 'u' are the only vertices of odd degree.

(b) $\boxed{\text{To prove}}$: If a connected graph G has exactly 2 vertices of odd degree then G is semi-Eulerian.

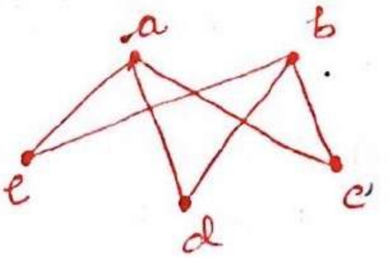
Let G be a connected graph having exactly 2 vertices of odd degree, 'u' & 'v'.
Let us add an edge 'e' joining 'u' & 'v'.
Then we obtain a connected graph in which each vertex has even degree.
Therefore, by $\boxed{\text{thm 2}}$ G is an Eulerian Graph and thus has an eulerian trail.

Removal of edge e from this Eulerian trail gives us an open trail including every edge of the graph G. Thus G is semi-eulerian.

Definitions :

(3) Semi-Hamiltonian path :
A path that includes every vertex of the graph is called as semi-hamiltonian path.

(4) Semi-Hamiltonian Graph : A connected graph is semi-hamiltonian if there is a path which is semi-hamiltonian path.



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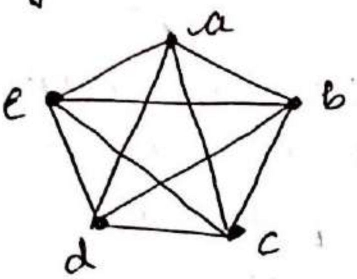
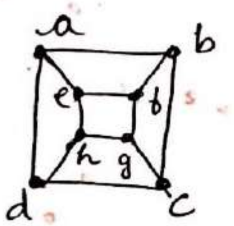
Assignment 4

Q-1) Which of the following graphs are Eulerian and/or Hamiltonian.
Write down Eulerian trail or Hamiltonian cycle where possible

(a)



(b)



(c)