1 Basic Set Theory

Paul R. Halmos, Naive Set Theory.

Definition 1.0.1 (Set)**.** A set is an unordered colleciton of distinct objects, called *elements* or *menbers* of the set. A set is said to contain its elements. We write

- $a \in A$ if a is an element of the set A.

1.1 Set Operation

$$\begin{aligned} U &= \{x \mid x \in X, x \in \mathbb{C}\} \\ &= U_{x \in C} C \end{aligned}$$

1.1.1 Set difference

The set difference of A and B ,denoted by $A - B$, or $A \backslash B$

1.1.2 Symmetric Difference

$$A \Delta B = (A - B) \cup (B - A) \tag{1}$$

1.1.3 Power Set

The power set of A is the set of all subsets of A, denoted by $\mathcal{P}(A)$ or 2^A

1.1.4 Set Algebras

De Morgan's Laws

- $C - (A \cup B) = (C - A) \cap (C - B)$
- $C - (A \cap B) = (C - A) \cup (C - B)$

1.1.5 Cartesian Product

Definition 1.1.1 (Cartesian)**.** The Cartesian product of sets A and B is the set of ordered pairs, such that

$$A \times B = \{(a, b) \mid a \in A, b \in B\} \tag{2}$$

Definition 1.1.2 (Cartesian)**.** By Kuratowski
An ordered pair (a, b) is given by

$$(a, b) := \{\{a\}, \{a, b\}\} \tag{3}$$

Theorem 1 (Cartesian)**.** If $a \in C$ and $b \in C$, then $(a, b) = \mathcal{P}(\mathcal{P}(C))$.
If $a \in A, b \in B$, then take $C = A \cup B$

Theorem 2 (Cartesian)**.** $(x, y) = (a, b)$ *iff* $x = a$ *and* $y = b$

Proof

- $\Leftarrow$: Trival
- $\Rightarrow$: By definition, we need to show that:

$$\underbrace{\{\{x\}, \{x, y\}\}}_{U} = \underbrace{\{\{a\}, \{a, b\}\}}_{V} \Rightarrow x = a \textbf{ and } y = b \tag{4}$$

  - If $x \neq y$, then $|U| = |V| = 2$, by matching size, we have $\{x\} = \{a\}$ $\{y\} = \{b\}$
  - If $x = y$, similarly $x = y = a = b$

Note: a definitio of ordered pairs is rational as long as it can indicate order.

1.1.6 Associative Set Operations

Let $A_1, A_1, \cdots, A_n$ be sets, then

- $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap\limits_{i=1}^{n} A_i$
- for $\cup, \times, \Delta$ are the same.

1.2 Simple Graphs

k-element subsets
Let X be a finit set. For a positive integer k, let $\binom{X}{k}$ denote the set of all k-element subsets. Note that $|\binom{X}{k}| = \binom{|X|}{k}$

Definition 1.2.1 (Graph)**.** A finit simple *graph* G *is a pair* (V, E) *where V is a non-empty finit set and E is a set of 2-element subsets of V, i.e.,* $E \in \binom{V}{2}$ *Elements of V called vertics, also denoted as* $V(G)$*. Elements of E called edges,also denoed as* $E(G)$*.*

2 Logic

2.1 CNF and DNF

- Conjunctive Normal Form (CNF):
  a junction of one or more clauses, where a clause is a disjunction of literals like **roduct of sums** or **AND of ORs**
- Disjunction Normal Forn (DNF):
  like **sum of products** or **OR of ANDs**

Table 1: Conditional truth table

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \to q}$
0	0	1
0	1	1
1	0	0
1	1	1

2.2 Conditional Statements

- $\boldsymbol{p}$: hyposhesis
- $\boldsymbol{q}$: thesis/conclusion

Equivalent forms

- if p, then q
- q is a sufficient condition for q
- q is necessary for p

remark

- either $\boldsymbol{p}$ is false
- or $\boldsymbol{q}$ is true
- i.e. $\neg \boldsymbol{p} \vee \boldsymbol{q}$
- same as $\neg \boldsymbol{q} \to \neg \boldsymbol{p}$

2.3 Tautology and Contradiction

- Tautology: All cases evaluates to 1.
- Contradiction: All cases evaluates to 0.

2.3.1 Tautological Equivalence

- Absorption

$$\boldsymbol{p \wedge (p \vee q) \Leftrightarrow p}$$
$$\boldsymbol{p \vee (p \wedge q) \Leftrightarrow p}$$

- Cases

$$\boldsymbol{(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)}$$
$$\boldsymbol{(p \to q) \vee (p \to r) \Leftrightarrow p \to (q \vee r)}$$
$$\boldsymbol{(p \to r) \wedge (p \to r) \Leftrightarrow (p \vee q) \to r}$$
$$\boldsymbol{(p \to r) \vee (p \to r) \Leftrightarrow (p \wedge q) \to r}$$

- Added premise

$$\begin{aligned}(p \wedge q) \rightarrow r &\Leftrightarrow p \rightarrow (q \rightarrow r)\\ &\Leftrightarrow q \rightarrow (p \rightarrow r)\end{aligned}$$

2.3.2 CNF and DNF

Theorem 3 (Disjunctive Normal Form)**.** For any proposition φ, there is a proposition φ_{dnf} over same Boolean variables and in DNF such that $\varphi \Leftrightarrow \varphi_{dnf}$

Example:

$$\begin{aligned}\varphi &= p \rightarrow q & \varphi_{dnf} &= (\neg p) \vee (q)\\ \varphi &= p \leftrightarrow q & \varphi_{dnf} &= (p \wedge q) \vee (\neg q \wedge \neg p)\\ \varphi &= p \oplus q & \varphi_{dnf} &= (\neg p \wedge q) \vee (p \wedge \neg q)\end{aligned}$$

Just like sum of product.

Theorem 4 (Conjunctive Normal Form)**.** For any proposition φ, there is a proposition φ_{cnf} over same Boolean variables and in DNF such that $\varphi \Leftrightarrow \varphi_{cnf}$

Example:

$$\begin{aligned}\varphi &= p \rightarrow q & \varphi_{dnf} &= (\neg p \vee q)\\ \varphi &= p \leftrightarrow q & \varphi_{dnf} &= (\neg p \vee q) \wedge (\neg q \vee p)\\ \varphi &= p \oplus q & \varphi_{dnf} &= (p \vee q) \wedge (\neg p \vee \neg q)\end{aligned}$$

Just like product of sum.

3 Relations and Functions

3.1 Properties of Relations

Definition 3.1.1 (Binary relation)**.** A binary relation R on A, i.e.$R \subset A$

- reflexive if $aRa \rightarrow true$
- total if $aRb \vee bRa \rightarrow true$
- transitive if $aRb \wedge bRa \rightarrow aRc$
- symmetric if $aRb \leftrightarrow bRa$
- anti-symmetric if $aRb \wedge bRa \rightarrow a = b$
- asymmetric if $aRb \wedge bRa \rightarrow false$

3.2 Important Relations

3.2.1 Partial Order

Definition 3.2.1 (Partial order)**.** A partial order on a set $\boldsymbol{P}$ is a relation that is

- reflexive
- anti-symmetric
- transitive

3.2.2 Irreflexive Relations

Irreflexive and non-reflexive

- Irreflexive: zero self-loops.
- Non-reflexive: missing self-loops

Antisymmetric and non-symmetric

- Antisymmetric:no cycle of length 2
- Non-symmetric: exists directed edge of no return

3.2.3 Terminologies on Partial Orders

- Non-strict Parial Order(e.e.,$\leq$, $\subseteq$)
  A reflexive, weak, or non-strict partial order is a relation $\preceq$ over a set $\boldsymbol{A}$ that is
  - reflexive
  - antisymmetric
  - transitive
- An irreflexive, strong or strict partial order is a relation $\prec$ over a set $\boldsymbol{A}$ that is
  - irreflexive
  - asymmetric
  - transitive

Remark: The term *partial order* typically refers to a non-strict partial order relation.

3.2.4 Equivalence Relation

Definition 3.2.2 (Equivalence relation)**.**

- reflexive
- symmetric
- transitive