A Geometric Approach to Solving Inverse Kinematics for 2-Link Planar Robots

Munachimso Henry

Department of Electrical & Electronics Engineering
Covenant University

December 18, 2025

Abstract

This document presents a geometric derivation of the inverse kinematics solution for a planar two-link robotic manipulator with equal link lengths. The objective is to compute joint angles from a desired end-effector position using trigonometric relations. The derivation is intended for implementation in simulation and educational visualization.

1 Introduction

Inverse kinematics (IK) concerns determining the joint parameters of a robotic manipulator required to place its end-effector at a desired position. This problem is the inverse of forward kinematics, where joint angles are mapped to Cartesian space. Inverse Kinematics is generally more challenging due to its nonlinear nature, and the possibility of multiple or nonexistent solutions for a given target. In this work, a geometric approach is used to derive the inverse kinematics for a simple planar two-link robotic arm. The objective is to obtain clear closed-form expressions for the joint angles corresponding to a desired point (x, y). The emphasis is placed on clarity and practicality rather than exhaustive theoretical work.

2 Problem Definition and System Description

Consider a planar robotic arm with two rigid links of equal length L. The arm operates in the Cartesian plane and is anchored at the origin. The objective is to determine joint angles α_1 and α_2 that place the end-effector at a desired position (x, y).

3 Coordinate Frames and Assumptions

The base frame is defined at the origin. All motion is constrained to the xy-plane. θ_1 is measured counterclockwise from the positive x-axis. θ_2 is measured clockwise from the horizontal line passing through joint B where the two links meet.

In this derivation, intermediate geometric angles are introduced to simplify the trigonometric analysis. The final joint angles commanding the robot are denoted by α_1 and α_2, while auxiliary angles such as θ_1 and β_1 appear only in the derivation.

The following assumptions are made:

- Links are rigid and massless
- Joints are ideal revolute joints
- No joint limits are considered

4 Forward Kinematics

From the previous section we established the coordinate system to be used. Let α_1 and α_2 denote the absolute orientations of the first and second links, respectively.
The forward kinematics equations for the end-effector position are given by:

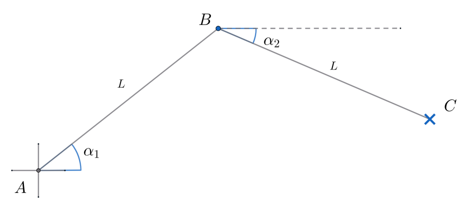

Figure 1: Problem Setup

$$x = L\cos\alpha_1 + L\cos(\alpha_2) \tag{1}$$
$$y = L\sin\alpha_1 + L\sin(\alpha_2) \tag{2}$$

5 Inverse Kinematics Derivation

5.1 Geometric Relations

The desired end-effector position (x, y) defines a triangle formed by the two links of length L, and the distance d from the base to the target point, given by:

$$d = \sqrt{x^2 + y^2} \tag{3}$$

The triangle shown in Figure 2 forms the basis for the inverse kinematics solution. A solution exists only when $0 \leq d \leq 2L$ otherwise the target is unreachable for this arm.

5.2 Joint Angle Solutions

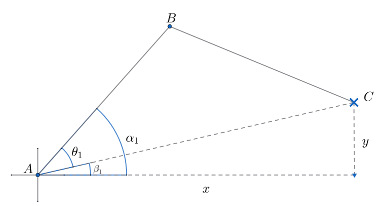

Figure 2: Solving for α_1 with the Law of Cosines

To evaluate the value of α_1, we work with the triangle formed by the two links and the distance from origin to the end-effector. The first link angle satisfies $\alpha_1 = \theta_1 + \beta_1$.
Applying the law of cosines to ΔABC we obtain:

$$L^2 = L^2 + (x^2 + y^2) - 2L\sqrt{(x^2 + y^2)}\cos\theta_1 \tag{4}$$

$$2L(\sqrt{x^2 + y^2})\cos\theta_1 = (x^2 + y^2) \tag{5}$$

$$\cos\theta_1 = \frac{(x^2 + y^2)}{2L\sqrt{x^2 + y^2}} \tag{6}$$

$$\rightarrow \theta_1 = \cos^{-1}\left(\frac{\sqrt{x^2 + y^2}}{2L}\right) \tag{7}$$

Having obtained θ_1, we move to solving for α_1 by summing θ_1 and β_1 From basic trigonometry,

$$\beta_1 = \arctan2(y, x) \tag{8}$$

Having obtained θ_1 and β_1 we can derive α_1 by evaluating the sum of the two.

$$\alpha_1 = \cos^{-1}\left(\frac{\sqrt{x^2+y^2}}{2L}\right) + \arctan2(y, x) \tag{9}$$

We use a similar method to arrive at the angle, α_2 needed for the second joint.

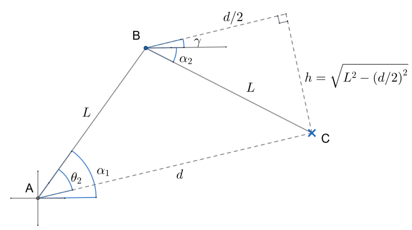

Figure 3: Solving for α_2

Using the cosine rule for ΔABC, we obtain:

$$d^2 = 2L^2 - 2L^2 \cos\theta_2 \tag{10}$$
$$(x^2+y^2) = 2L^2 - 2L^2 \cos\theta_2 \tag{11}$$
$$2L^2 \cos\theta_2 = 2L^2 - (x^2+y^2) \tag{12}$$
$$\cos\theta_2 = 1 - \left(\frac{(x^2+y^2)}{2L^2}\right) \tag{13}$$
$$\rightarrow \theta_2 = \cos^{-1}\left(1 - \frac{(x^2+y^2)}{2L^2}\right) \tag{14}$$

From alternate angles, the obtuse angle obtained at A is equivalent to the obtuse angle at B.

$$\pi - \alpha_1 = \alpha_2 + \theta_2 \tag{15}$$
$$\rightarrow \alpha_2 = \pi - \alpha_1 - \theta_2 \tag{16}$$

Giving us,

$$\alpha_2 = \pi - \left[\cos^{-1}\left(\frac{\sqrt{(x^2+y^2)}}{2L}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right] - \cos^{-1}\left(1 - \frac{(x^2+y^2)}{2L^2}\right) \tag{17}$$

For compactness, the result may be expressed as:

$$\alpha_2 = \pi - (\alpha_1 + \theta_2) \tag{18}$$

6 Implementation Notes

The derived equations were implemented in Python using NumPy for numerical stability. The `arctan2` function was used to ensure correct quadrant selection. Visualization was performed using Matplotlib.

7 Conclusion

This document presented a geometric inverse kinematics derivation for a planar two-link robotic arm. The resulting closed-form expressions are efficient, intuitive, and suitable for real-time simulation and visualization.

Project Resources

The implementation and visualizations associated with this derivation are available at:
`https://github.com/Draycole/iksolver`