Approximating the Brachistochrone using Genetic Optimization

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1 Defining the Cost Function

Find the time required for a particle to travel a path f with endpoints $(x_0, f(x_0), (x_n, f(x_n))$. Assume the following:

1. uniform gravitational field,
2. no nonconservative forces,
3. the particle starts at rest.

From these assumptions, the total energy of the system as a function of x is

$$E(x) = K(x) + U_g(x) = \frac{1}{2}m[v(x)]^2 + mgh(x) = \frac{1}{2}m[v(x)]^2 + mgf(x)\,. \quad (1)$$

Because we assume there are no nonconservative forces present, $E(x)$ is constant $\forall x \in \{x_0, \ldots, x_n\}$.

We assume the system starts at rest, so therefore $K(x_0) = 0 \implies E(x_0) = K(x_0) + U_g(x_0) = mgf(x_0)$. Since the total energy of the system is constant,

$$\begin{aligned} \frac{1}{2}m[v(x)]^2 + mgf(x) &= mgf(x_0) \\ \frac{1}{2}m[v(x)]^2 &= mgf(x_0) - mgf(x) \\ [v(x)]^2 &= 2g(f(x_0) - f(x)) \\ v(x) &= \sqrt{2g(f(x_0) - f(x))} \end{aligned} \quad (2)$$

To find the differential displacement along the curve (ds) for a differential change in the x-direction (dx), we can use the arc length formula:

$$ds = dx\sqrt{1 + f'(x)^2}\,. \quad (3)$$

The time required to travel a differential portion of f is given by $dt = \frac{ds}{v(x)}$. Thus, the total time T required is as follows:

$$\begin{aligned} T(f) &= \int dt \\ &= \int_{x_0}^{x_n} \frac{ds}{v(x)} \\ &= \int_{x_0}^{x_n} \sqrt{\frac{1+f'(x)^2}{2g(f(x_0)-f(x))}} dx \\ &\propto \int_{x_0}^{x_n} \sqrt{\frac{1+f'(x)^2}{f(x_0)-f(x)}} dx \,. \end{aligned} \tag{4}$$