Linear Algebra PILOT Week 1

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1. Gauss-Jordan Elimination

1.1) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \\ 7 & 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{c} x - z = 0 \\ 2y + 4z = 1 \\ z = z \end{array}$, let $z = t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$$

1.2) $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{c} x_1 + x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 + x_3 = 0 \\ x_4 = 0 \end{array}$, let $x_4 = t$

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} t$$

1.3) Let $x = \#$ of \$1, $y = \#$ of \$5, and $z = \#$ of \$10.

Set up a system of equations:

$$\begin{array}{l} x + y + z = 32 \\ x + 5y + 10z = 100 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 1 & 5 & 10 & 100 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 0 & 4 & 9 & 68 \end{array}\right)$$

Note $4y + 9z = 68$ and $y, z \geq 0$ and $y, z \in \mathbb{N}$.

Thus $z \in \{0, 1, 2, \ldots, 7\} \Rightarrow 9z \in \{0, 9, 18, \ldots, 63\}$

$\Rightarrow 4y = 68 - 9z \in \{68, 59, 50, 41, 32, 23, 14, 5\}$.

The only values which follow the above criteria are

$(y, z) = (17, 0)$ and $(y, z) = (8, 4)$.

Now plug in the \$1s:

- $(y,z) = (17,0) \Rightarrow x = 32 - y - z = 15.$

  $x + 5y + 10z = 15 + 5(17) + 0 = 100.$ ✓

  so $(15,17,0)$ is a solution!

- $(y,z) = (8,4) \Rightarrow x = 32 - y - z = 20.$

  $x + 5y + 10z = 20 + 5(8) + 10(4) = 100.$ ✓

  so $(20,8,4)$ is a solution!

$$\boxed{(15,7,0) \text{ and } (20,8,4)}$$

2. Classic constants

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & k & 4 & 6 \\ 1 & 2 & k+2 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & k-2 & 1 & 2 \\ 0 & 0 & k-1 & 2 \end{array}\right)$$

No soln: $k = 1$

Inf soln: $k = 2$

Unique soln: $k \in \mathbb{R} \setminus \{1,2\}$

3. Working backwards

3.1) $(1,1,1)$ & $(3,5,0) \rightarrow \vec{x} = -\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} t + \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$

$$\begin{cases} x = -2t + 3 \\ y = -4t + 5 \\ z = +t \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

3.2) $x = 6 + 5t, \quad y = 4 + 3t, \quad z = 2 + t$

$$\Rightarrow t = z - 2$$

$$\Rightarrow \begin{cases} x = 6 + 5(z-2) = 5z - 4 \\ y = 4 + 3(z-2) = 3z - 2 \end{cases}$$

Note $2y - x = 2 + t \Rightarrow t = -x + 2y - 2$

$$\Rightarrow z = -x + 2y$$

$$\Rightarrow x - 2y + z = 0$$

$$\therefore \begin{cases} x - 5z = -4 \\ y - 3z = -2 \\ x - 2y + z = 0 \end{cases}.$$