Math 107 Lecture 12

Complex Numbers in Polar Coordinates

by Dr. Kurianski
on October 7, 2024

» Announcements and Objectives

Announcements

* Skill Check 3 is this Wednesday (10/9, 60 mins then lecture)
* No Pre-Notes due today. Pre-Notes due before start of next lecture
* Assignments Due Friday (10/11):
  * HW6 Handwritten Questions
  * HW6 Coding Problems
  * HW6 MATLAB File Upload

Objectives

* Find the modulus and conjugate of a given complex number
* Perform algebraic computations with complex numbers
* Convert complex numbers into polar and Cartesian forms
* Use De Moivre's Theorem and Euler's Formula to perform computations

Dividing complex numbers

» Dividing complex numbers

Let $z = a + bi$ and $w = c + di$. To divide z/w, rationalize the fraction by multiplying the top and bottom by $\bar{w}$.

$$\begin{aligned}
\frac{z}{w} &= \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} \\
&= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\
&= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\
&= \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i
\end{aligned}$$

» Dividing complex numbers

Example

Let $v = -3 + 2i$ and $u = 4 - i$. Compute v/u.

$$
\begin{aligned}
\frac{v}{u} &= \frac{-3+2i}{4-i} \cdot \frac{4+i}{4+i} \\
&= \frac{-12 - 3i + 8i + 2(i)^2}{4^2 + 1^2} \\
&= \frac{-12 + 5i - 2}{17} \\
&= \frac{-14 + 5i}{17} \\
&= \frac{-14}{17} + \frac{5}{17}i
\end{aligned}
$$

Modulus

» Modulus of a complex number

Definition: The **modulus** of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

* The word modulus is another word for absolute value or "size."
* The modulus is a real number.
* Do you notice anything familiar about the modulus formula?

Example: The modulus of $z = -1 + 3i$ is

$$|z| = |-1 + 3i| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$$

» Modulus of a complex number

Question

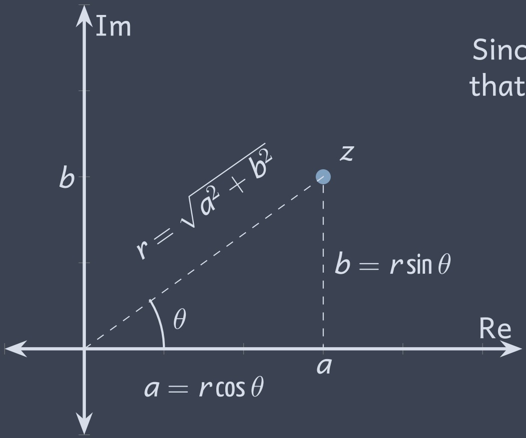
If $z = a + bi$, what is $z\bar{z}$?

$$\begin{aligned} z\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b(i)^2 \\ &= a^2 + b^2 \end{aligned}$$

$$z\bar{z} = |z|^2$$

From Cartesian to Polar Form

» From Cartesian to polar



Since $z = a + bi$ and we saw that $a = r\cos\theta$, $b = r\sin\theta$, we can write

$$z = \underbrace{r\cos\theta}_{a} + (\underbrace{r\sin\theta}_{b})i$$

and hence

$$\boxed{z = r(\cos\theta + i\sin\theta)}.$$

» r and θ

Definitions

Definition: A complex number $z = a + bi$ can be written as $z = r(\cos\theta + i\sin\theta)$ where

$$r = |z| = \sqrt{a^2 + b^2}$$

and θ is the **Principal value of the argument** of z denoted

$$\theta = \text{Arg}(z).$$

Definition: We use the phrase "Principal value" to specify values for θ for which $-\pi < \theta \leq \pi$. This is because $\theta = \theta + 2\pi k$ for any integer k.

» r and θ

Definitions

Given $z = a + bi$, we can find r by computing $|z|$. But how do we find θ?

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

Note that

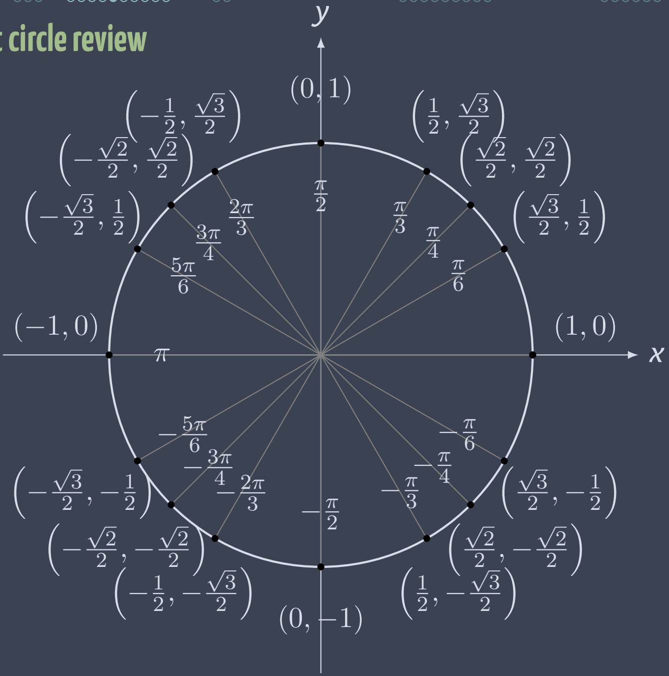
$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta.$$

Solving for θ gives

$$\theta = \arctan\left(\frac{b}{a}\right).$$

Remark: Make sure that θ is
in the **correct quadrant** and is between $-\pi$ and π.

» Unit circle review



» Example

Example: Let $z = 2 - 2i$. What is r?

» r and θ

Definitions

Given $z = a + bi$, we can find r by computing $|z|$. But how do we find θ?

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

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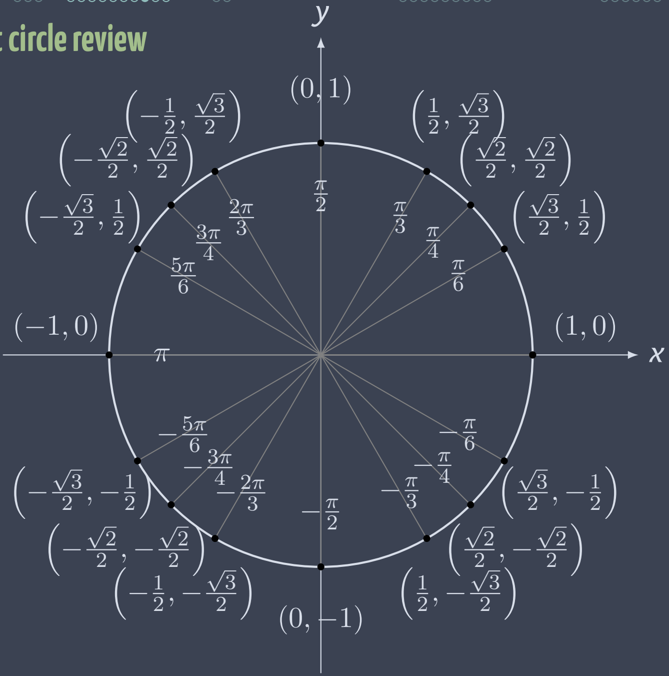


» Example

Example: Let $z = 2 - 2i$. What is θ?

» Question

Example: Let $z = -\sqrt{3} - i$. Write z in polar form as $z = r(\cos\theta + i\sin\theta)$.

Plotting complex numbers in MATLAB

» Plotting complex numbers in MATLAB

MATLAB Syntax:

* `real(z)` – Computes the real part of z
* `imag(z)` – Computes the imaginary part of z

Euler's Formula and De Moivre's Theorem

» Euler's Formula

Euler's formula states that for any real number x,

$$e^{ix} = \cos(x) + i\sin(x).$$

Euler's identity:

$$e^{i\pi} + 1 = 0.$$

Complex numbers:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

» Example

Example: Let $z = -\sqrt{3} - i$. Write z in polar form, i.e., as $z = re^{i\theta}$.

» Question

Question: Let $z = re^{i\theta}$. What is $\bar{z}$ in polar form?

» Multiplying complex numbers

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}.$$

Then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Remark: Notice that the moduli get multiplied and the angles get added.

» Dividing complex numbers

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}.$$

Then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Remark: Notice that the moduli get divided and the angles get subtracted.

» Example

Example: Let $z = 1 - i$ and $w = i$. Convert z and w to polar form. Then compute zw and z/w.

» De Moivre's Theorem

For any complex nuber $z = re^{i\theta}$ and any positive integer k, we have

$$z^k = r^k(\cos(k\theta) + i\sin(k\theta)).$$

» Example

Example: Let $z = 2(\cos(\pi/4) + i\sin(\pi/4))$. Compute z^4.

Roots of complex numbers

» Roots of complex numbers

Notice that $z = re^{i\theta}$ can be written equivalently as

$$z = re^{i(\theta+2\pi k)}$$

for any integer k. This is because the unit circle repeats itself every time we go 2π radians around.

If we want to find the nth roots of the complex number $z = re^{i\theta}$, we might start by writing

$$z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n}e^{i\theta/n}.$$

But because we can add $2\pi k$ to the angle θ for any integer k, this would be the same if we wrote

$$z^{1/n} = r^{1/n}e^{i(\theta+2\pi k)/n} = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right].$$

» Roots of complex numbers

Consider the equation

$$x^4 - 1 = 0.$$

The values

$$x = 1,\ x = -1,\ x = i,\ x = -i$$

are solutions to the equation because when they are each raised to the 4th power, they each equal 1.

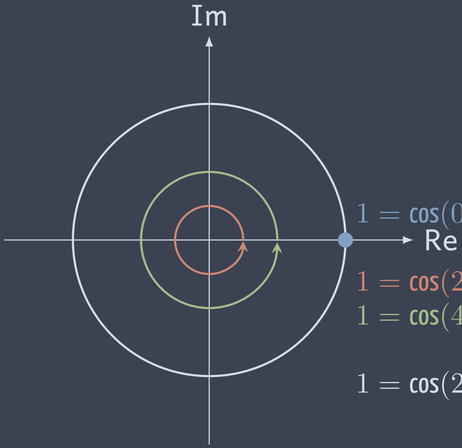
In other words, these values solve

$$x = 1^{1/4}.$$

These are called the 4th **roots of unity**.

» Roots of unity

Let's the polar form of $z = 1$.



$$1 = \cos(0) + i\sin(0)$$

$$1 = \cos(2\pi) + i\sin(2\pi)$$

$$1 = \cos(4\pi) + i\sin(4\pi)$$

$$1 = \cos(2k\pi) + i\sin(2k\pi)$$

» Roots of unity

Definition

As a complex number, 1 can be represented by

$$1 = \cos(2k\pi) + i\sin(2k\pi) = e^{i2k\pi}.$$

Definition: For any positive integer n, the complex number $z = e^{i\theta}$ has exactly n **roots of unity** given by

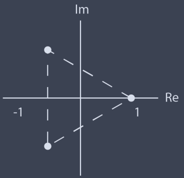
$$1^{1/n} = e^{i2\pi k/n} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

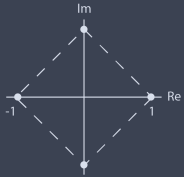
» Roots of unity

Geometric view

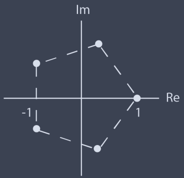
The roots of unity *are* spaced out evenly over the unit circle. This is why they form regular polygons.



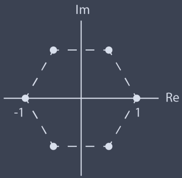
3rd roots of unity



4th roots of unity



5th roots of unity



6th roots of unity

Cool example

» Cool example

We started our discussion of complex numbers with thinking about the roots of the equation

$$x^2 + 1 = 0.$$

Let's now consider the equation

$$x^4 - 1 = 0$$

and think about the real *and* complex numbers that satisfy it.

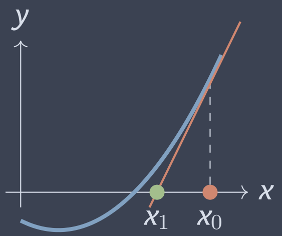
Note: The values $x = 1$, $x = -1$, $x = i$, and $x = -i$ all satisfy $x^4 - 1 = 0$. In other words, they are all **roots** of the equation $f(x) = x^4 - 1$.

» Cool example

One way to find roots of a function numerically is called Newton's method.

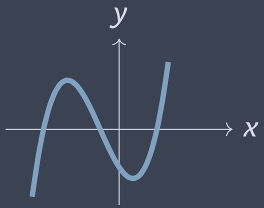
Idea of the method:

1. Make a guess x_0 that you think is close to where $f(x) = 0$.
2. Find the tangent line to the function at your initial guess.
3. Use where the tangent line crosses the x-axis as your next guess for where $f(x) = 0$.
4. Repeat the steps.



» Cool example

What if there are multiple roots (places where $f(x) = 0$)?



Can we use this method to find complex roots of an equation like $f(x) = x^4 - 1$?

» Cool example

Newton fractal

$$x^4 - 1 = 0, \text{ when } x = 1, i, -i, -1$$

Color-code the complex plane using the following rules:

* If the initial guess $a + bi$ converges to the root $x = 1$, color $a + bi$ blue.
* If the initial guess $a + bi$ converges to the root $x = i$, color $a + bi$ orange.
* If the initial guess $a + bi$ converges to the root $x = -i$, color $a + bi$ yellow.
* If the initial guess $a + bi$ converges to the root $x = -1$, color $a + bi$ purple.

» Cool example

Newton fractal