Math 107 Lecture 13

Roots of Complex Numbers and Fractals

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on October 9, 2024

» Announcements and Objectives

Announcements

- Skill Check 4 is in two weeks (10/23, 110 mins)
- Pre-Notes due before start of next lecture
- Assignments Due Friday (10/11):
  - HW6 Handwritten Questions
  - HW6 Coding Problems
  - HW6 MATLAB File Upload

Objectives

- Compute roots of complex numbers
- Explore fractals

» De Moivre's Theorem

For any complex nuber $z = re^{i\theta}$ and any positive integer k, we have

$$z^k = r^k(\cos(k\theta) + i\sin(k\theta)).$$

Example: Let $z = 2(\cos(\pi/4) + i\sin(\pi/4))$. Compute z^4.

Roots of complex numbers

» Roots of complex numbers

Notice that $z = re^{i\theta}$ can be written equivalently as

$$z = re^{i(\theta + 2\pi k)}$$

for any integer k. This is because the unit circle repeats itself every time we go 2π radians around.

If we want to find the nth roots of the complex number $z = re^{i\theta}$, we might start by writing

$$z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n}e^{i\theta/n}.$$

But because we can add $2\pi k$ to the angle θ for any integer k, this would be the same if we wrote

$$z^{1/n} = r^{1/n}e^{i(\theta+2\pi k)/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right].$$

» Roots of complex numbers

Consider the equation

$$x^4 - 1 = 0.$$

The values

$$x = 1,\ x = -1,\ x = i,\ x = -i$$

are solutions to the equation because when they are each raised to the 4th power, they each equal 1.

In other words, these values solve

$$x = 1^{1/4}.$$

These are called the 4th **roots of unity**.

Cool example

» Cool example

We started our discussion of complex numbers with thinking about the roots of the equation

$$x^2 + 1 = 0.$$

Let's now consider the equation

$$x^4 - 1 = 0$$

and think about the real *and* complex numbers that satisfy it.

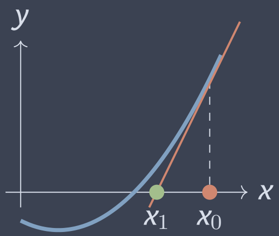
Note: The values $x = 1$, $x = -1$, $x = i$, and $x = -i$ all satisfy $x^4 - 1 = 0$. In other words, they are all **roots** of the equation $f(x) = x^4 - 1$.

» Cool example

One way to find roots of a function numerically is called Newton's method.

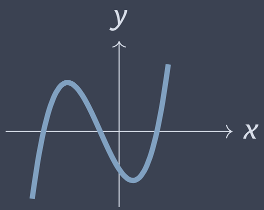
Idea of the method:

1. Make a guess x_0 that you think is close to where $f(x) = 0$.
2. Find the tangent line to the function at your initial guess.
3. Use where the tangent line crosses the x-axis as your next guess for where $f(x) = 0$.
4. Repeat the steps.



» Cool example

What if there are multiple roots (places where $f(x) = 0$)?



Can we use this method to find complex roots of an equation like $f(x) = x^4 - 1$?

» Cool example

Newton fractal

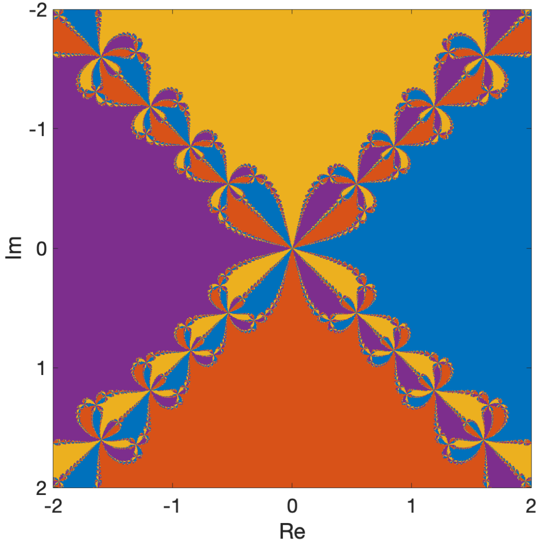
$$x^4 - 1 = 0, \text{ when } x = 1, i, -i, -1$$

Color-code the complex plane using the following rules:

* If the initial guess $a + bi$ converges to the root $x = 1$, color $a + bi$ blue.
* If the initial guess $a + bi$ converges to the root $x = i$, color $a + bi$ orange.
* If the initial guess $a + bi$ converges to the root $x = -i$, color $a + bi$ yellow.
* If the initial guess $a + bi$ converges to the root $x = -1$, color $a + bi$ purple.

» Cool example

Newton fractal



The Sierpinski Triangle

» Creating the Sierpinski Triangle

One way...

1. Start with an equilateral triangle.
2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.
3. Repeat step 2 with each of the remaining smaller triangles infinitely many times.

Another way...

1. Start with an equilateral triangle.
2. Draw a point anywhere on the plane.
3. Choose a corner of the original triangle at random. Draw another point halfway between the previous point you drew and the chosen corner of the original triangle.
4. Repeat step 3 infinitely many times.

» The Sierpinski Triangle

Visualization

» The Sierpinski Triangle

Visualization