Math 107 Lecture 18

Matrix Invertibility and Properties

by Dr. Kurianski
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» Announcements and Objectives

Announcements

- Skill Check 5 is next Wed (11/6, 110 mins)
- Solutions to Homeworks 1-8 available in Canvas Modules
- Pre-Notes due before start of next lecture
- Assignments Due Friday (11/1):
  - HW9 Handwritten Questions
  - HW9 Coding Problems
  - HW9 MATLAB File Upload

Objectives

- Discuss properties of the inverse and transpose
- Discuss special types of matrices and their features
- Use the Invertible Matrix Theorem to determine whether given matrices are invertible
- Define and compute the determinant of $n \times n$ matrices

» Diagonal entries

Definition: Let A be an $m \times n$ matrix. The **diagonal** of A consists of the entries $a_{11}, a_{22}, \ldots$.

Example: $A = \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{4} & 5 \\ 0 & 0 & \mathbf{6} \end{bmatrix}$

Example: $B = \begin{bmatrix} \mathbf{3} & 1 & 7 \\ 8 & \mathbf{7} & -2 \\ 0 & 5 & \mathbf{-1} \end{bmatrix}$

» Triangular Matrices

Definition

Definition: An $n \times n$ matrix for which all elements above the main diagonal are zero is called a **lower triangular matrix**.

Example: $C = \begin{bmatrix} 0.5 & 0 & 0 \\ 6 & 21 & 0 \\ 7 & 5\pi & -1 \end{bmatrix}$

Definition: An $n \times n$ matrix for which all elements below the main diagonal are zero is called an **upper triangular matrix**.

Example: $C = \begin{bmatrix} -10 & 34 & 9 \\ 0 & \sqrt{2} & 57 \\ 0 & 0 & -4 \end{bmatrix}$

» Diagonal matrix

Definition

Definition: A **diagonal matrix** is an $n \times n$ matrix in which the only nonzero entries lie on the diagonal.

Example: $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Remark: For a matrix to be a diagonal matrix, it *must* be square (same number of rows as columns).

» Inverse of diagonal matrices

Fact: If A is a diagonal matrix with diagonal entries d_1, d_2, $\ldots$, d_n, i.e.,

$$A = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & d_n \end{bmatrix}$$

where none of the diagonal entries are zero then A^{-1} exists and is equal to

$$A^{-1} = \begin{bmatrix} 1/d_1 & 0 & \ldots & 0 \\ 0 & 1/d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1/d_n \end{bmatrix}$$

Matrix Transpose

» Transpose

Definition

Definition: Let A be an $m \times n$ matrix. The **transpose** of A, denoted A^T, is the $n \times m$ matrix whose columns are the respective rows of A.

» Properties of the Matrix Transpose

Let A and B be matrices where the following operations are defined, and let k be a scalar. Then:

1. $(A+B)^T = A^T + B^T$
2. $(kA)^T = kA^T$
3. $(AB)^T = B^T A^T$
4. $(A^{-1})^T = (A^T)^{-1}$
5. $(A^T)^T = A$

» Transpose of Product

Question: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 6 & -2 \\ 1 & 0 \end{bmatrix}$.

Use MATLAB to compute the following:

(a) A^T

(b) B^T

(c) $(AB)^T$

(d) $A^T B^T$

(e) $B^T A^T$

» Properties of the Matrix Transpose

Example

Question: Let A and B be any $n \times n$ invertible matrices. Simplify $((A^{-1}B^{-1})^T)^{-1}$.

» Symmetric and Skew Symmetric

Definition: A matrix A is **symmetric** if $A^T = A$

Example: $A = \begin{bmatrix} 1 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}$

Definition: A matrix A is **skew symmetric** if $A^T = -A$.

Example: $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

Invertible Matrix Theorem

» Invertible Matrix Theorem

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent.

* A is invertible.
* The reduced row echelon form of A is I.
* The equation $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
* The equation $A\vec{x} = \vec{0}$ has exactly one solution (namely, $\vec{x} = \vec{0}$).

Remark: We will add more statements to this theorem!

Determinant of 2×2 Matrices

» Determinants of 2×2 matrices

Definition

Definition: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A, denoted $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is given by

$$\det(A) = ad - bc.$$

Connection to inverse:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

» Determinants of 2×2 matrices

Examples

Example: Compute the determinants of the following matrices:

1. $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$
2. $B = \begin{bmatrix} -2 & -1 \\ 5 & 7 \end{bmatrix}$
3. $C = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

» Question

Question: (True or False?) Let B be a 2×2 matrix with $\det(B) = 0$. Then B is not invertible.

Determinants of $n \times n$ Matrices

» Minor of A

Definition

Definition: Let A be an $n \times n$ matrix. The ***ij*-minor of** A, denoted A_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column of A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compute A_{11}, A_{23}, and A_{32}.

» Cofactor

Definition

Definition: Let A be an $n \times n$ matrix. The ***ij*-cofactor of** A, denoted C_{ij} is the number

$$C_{ij} = (-1)^{i+j} A_{ij}.$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compute C_{23} and C_{11}.

» Cofactor Expansion

Definition

Definition: Let A be an $n \times n$ matrix. The **cofactor expansion of A along the ith row** is the sum

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The **cofactor expansion of A down the jth column** is the sum

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find the cofactor expansion along the 2nd row.

Example: Find the cofactor expansion down the 3rd column.

» Determinant of an $n \times n$ Matrix

Definition

Definition: The determinant of an $n \times n$ matrix A is a number given by the following:

* if A is 1×1, then $A = [a]$ and $\det(A) = a$
* if A is 2×2, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc$
* if A is $n \times n$ with $n > 2$, then

$$\det(A) = \text{cofactor exapansion along any row or column}$$

» Determinant of an $n \times n$ Matrix

Example 1

Example: Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & 0 \\ 7 & 2 & 3 \end{bmatrix}$$

» Determinant of an $n \times n$ Matrix

Example 2

Example: Find the determinant of

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 0 & -2 & 0 \\ 5 & -7 & -1 \end{bmatrix}$$

Properties of Determinants

» Properties of Determinants

Theorems

Theorem: Let A and B be $n \times n$ matrices and let k be a scalar. The following are true:

1. $\det(kA) = k^n \det(A)$
2. $\det(A^T) = \det(A)$
3. $\det(AB) = \det(A)\det(B)$
4. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Theorem: A matrix A is invertible if and only if $\det(A) \neq 0$.

» Properties of Determinants

Exercise

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ -1 & 5 \end{bmatrix}$. Compute $\det(A)$, $\det(B)$, and $\det(AB)$.

» Invertible Matrix Theorem (revisited)

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent.

* A is invertible.
* The reduced row echelon form of A is I.
* The equation $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
* The equation $A\vec{x} = \vec{0}$ has exactly one solution (namely, $\vec{x} = \vec{0}$).
* $\det(A) \neq 0$

» Shortcut for 3×3 matrices

(Reference: Pgs 259-261 of Main Textbook)

Example: Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Example: Find the determinant of $B = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

Question: (True or False?) The matrix B above is invertible.