Math 107 Lecture 11

Transformations of the Cartesian Plane and Intro to Complex Numbers

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on October 2, 2024

» Announcements and Objectives

Announcements

- Skill Check 3 is NEXT Wednesday (10/9, 60 mins then lecture)
- Pre-Notes due before start of next lecture
- Assignments Due Friday (10/4):
  - HW5 Handwritten Questions
  - HW5 Coding Problems
  - HW5 MATLAB File Upload

Objectives

- Interpret matrix multiplication as a transformation of the Cartesian plane
- Given a transformation of the plane, find the matrix that produces it
- Write a combination of transformations as multiplication by specific matrices

» Warm-up

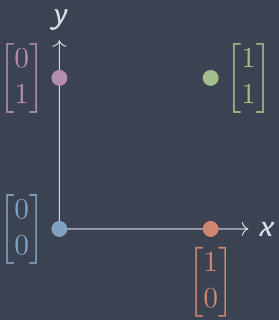
Let $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Compute $A\vec{x}$, $A\vec{y}$, and $A(\vec{x} + \vec{y})$.

Visualizing matrix multiplication

» Considering the unit square

What happens when we multiply *every* vector by some matrix A?

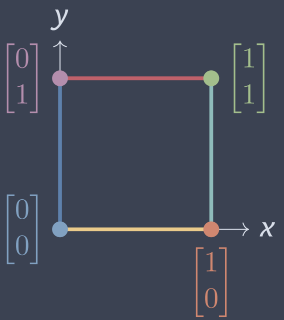
We can start answer this by first considering the unit square.

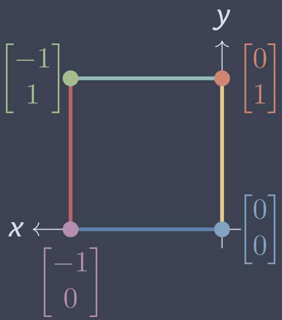


» Considering the unit square

Example 1

For example, let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and multiply each corner of the square by A.



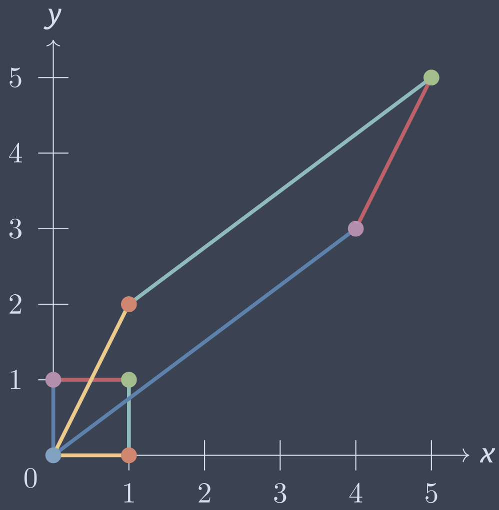


Multiplication by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates vectors by an angle of $\frac{\pi}{2}$ about the origin.

» Considering the unit square

Example 2

Multiplication by $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$



» Relationship to A

Question: How do the entries of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

relate to where the corners of the square ended up?

$$\text{mult by } A \qquad \begin{array}{cccc} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 5 \end{bmatrix} & \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ \text{never changes} & \text{1st col of } A & \text{sum of each row} & \text{2nd col of } A \end{array}$$

» Relationship to A

Why? Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ never changes}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a+0 \\ c+0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ 1st col}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} \text{ sum of rows}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+b \\ 0+d \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \text{ 2nd col}$$

» Elementary basis

Notice that

$$\begin{bmatrix}0\\0\end{bmatrix} = (0)\begin{bmatrix}1\\0\end{bmatrix} + (0)\begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\1\end{bmatrix} = (1)\begin{bmatrix}1\\0\end{bmatrix} + (1)\begin{bmatrix}0\\1\end{bmatrix}$$

We wrote $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ as linear combinations of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.
Since matrix multiplication is distributive, we have

$$A\begin{bmatrix}0\\0\end{bmatrix} = A\left((0)\begin{bmatrix}1\\0\end{bmatrix} + (0)\begin{bmatrix}0\\1\end{bmatrix}\right) = (0)A\begin{bmatrix}1\\0\end{bmatrix} + (0)A\begin{bmatrix}0\\1\end{bmatrix}$$

$$A\begin{bmatrix}1\\1\end{bmatrix} = A\left((1)\begin{bmatrix}1\\0\end{bmatrix} + (1)\begin{bmatrix}0\\1\end{bmatrix}\right) = A\begin{bmatrix}1\\0\end{bmatrix} + A\begin{bmatrix}0\\1\end{bmatrix}$$

So the vectors $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ are somehow redundant.

» Elementary basis

Key take-away

It turns out that we can write *every* 2×1 vector as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Key Take-away: This means we can predict how multiplication by any 2×2 matrix A will transform the Cartesian plane by considering what it does to the unit vectors

$$\vec{e}_1 = \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

» Elementary basis

Example

Write $\begin{bmatrix} \pi \\ -11 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In other words, find a and b such that

$$\begin{bmatrix} \pi \\ -11 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$a = \pi, \; b = -11$$

» Finding the matrix

Now suppose we know how we want to transform the plane. How do we construct a matrix A that will perform the given transformation?

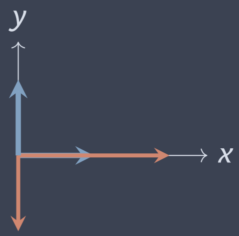
$$A = \begin{bmatrix} - & \quad - \\ - & \quad - \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{where} & \text{where} \\ \vec{e}_1 & \vec{e}_2 \\ \text{ends} & \text{ends} \\ \text{up} & \text{up} \end{array}$$

» Finding the matrix

Example

Find the matrix A that flips the Cartesian plane about the x-axis and then stretches the plane horizontally by a factor of two.



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

» Finding the matrix

Question

Question: Find the matrix A that flips the Cartesian plane about the y-axis and then stretches the plane vertically by a factor of four.

Rotating vectors

» How do we rotate vectors by any angle θ?

We saw that multiplying

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

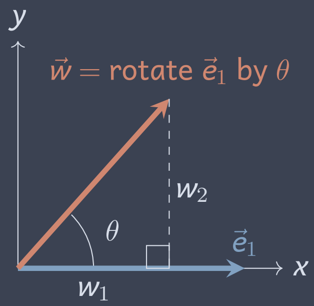
rotates the vector $\vec{v}$ by an angle of $\frac{\pi}{2}$ about the origin.

How do we rotate vectors by any angle θ?

» How do we rotate vectors by any angle θ?

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Start with what happens to $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{w_1}{\|\vec{w}\|_2}$$

$$\implies w_1 = \|\vec{w}\|_2 \cos\theta$$

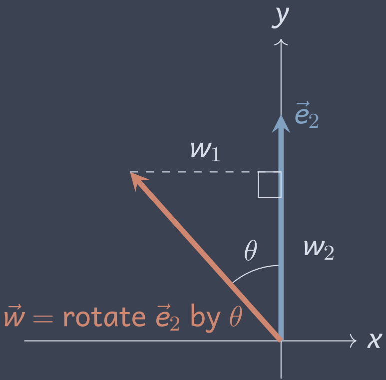
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{w_2}{\|\vec{w}\|_2}$$

$$\implies w_2 = \|\vec{w}\|_2 \sin\theta$$

$$\|\vec{w}\|_2 = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 = \sqrt{1^2 + 0} = 1 \implies \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}$$

» How do we rotate vectors by any angle θ?

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{w_2}{\|\vec{w}\|_2} = w_2$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{w_1}{\|\vec{w}\|_2} = w_1$$

But w_1 is negative so
$w_1 = -\sin\theta$

$$\boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}}$$

» How do we rotate vectors by any angle θ?

Rotation matrix

Multiply by the rotation matrix

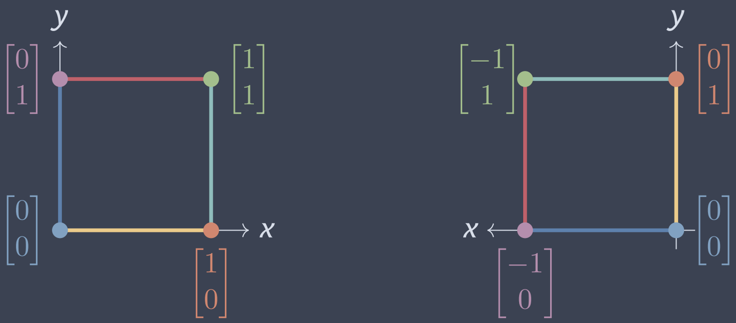
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Question: What is the matrix that will rotate vectors by an angle of $\frac{\pi}{2}$?

» Considering the unit square

Example 1

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Multiplication by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates vectors by an angle of $\frac{\pi}{2}$ about the origin.

» Activity 1: `rotMat`

Write a function called `rotMat` that rotates a vector $\vec{v}$ by a desired angle θ. The inputs should be the 2×1 vector v to be rotated and and angle `theta`. The output is the vector `w` which is the vector v after it has been rotated by `theta`. Use the rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and regular matrix multiplication in MATLAB (*). Before rotating the vector, write a conditional statement that checks if v has size 2×1.

To see if your function works, test it by rotating $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $\pi/2$. You can also test it by rotating $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $\pi/2$.

Multiple transformations

» Two transformations

Multiplying a vector by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

flips it across the x axis.

Multiplying a vector by

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

rotates it by $\pi/2$ about the origin.

What would we multiply $\vec{v}$ by if we wanted to flip a vector $\vec{v}$ across the x-axis and *then* rotate by $\pi/2$?

» More transformations

Main Text pg.s 121-123

* Horizontal stretch by k: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
* Vertical stretch by k: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
* Horizontal shear by k: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
* Vertical shear by k: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
* Reflection across y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
* Reflection across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Real and imaginary numbers

» Real numbers

We are used to thinking about real numbers, that is, numbers that we can write on a number line.

Example: $1,\ 0,\ -2.3,\ \pi,\ \frac{13}{27}$

» Real numbers

Solutions to equations

We've also seen that we can use real numbers to solve equations.

Example: The equation

$$x^2 - 2 = 0$$

has two solutions: $x = \sqrt{2}$ and $x = -\sqrt{2}$.

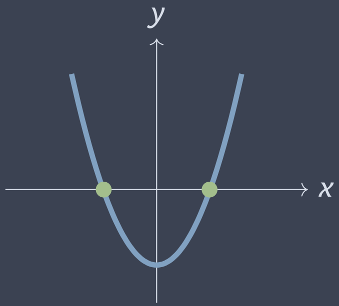
» Real numbers

Solutions to equations

On a graph, the solutions to

$$x^2 - 2 = 0$$

are the points where $y = x^2 - 2$ crosses the x-axis.

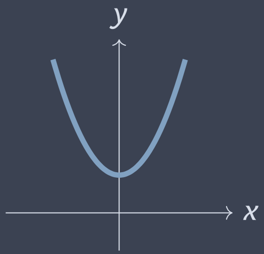


» Real numbers

Solutions to equations

But what *are* the solutions to the equation

$$x^2 + 1 = 0?$$



In calculus, we said that this equation has no **real** solutions. But what if we expand our thinking?

» Some history

Rafael Bombelli

In the 1500s, an Italian mathematician named Rafael Bombelli started using the notation

$$\sqrt{-1}$$

to solve equations like

$$x^2 + 1 = 0.$$

René Descartes thought this was so ridiculous, that he called the numbers "imaginary."

And the name stuck.

» Imaginary numbers

Definition: The **imaginary unit** i is defined to be the number such that $i^2 = -1$.

MATLAB Syntax: In MATLAB, i is recognized by the letter `i` or the command `1i`.

Definition: An **imaginary number** is any multiple of i. It is a number of the form ci where c is a real number. Moreover, $(ci)^2 = -c^2$.

Note: If you're using the imaginary number i in a loop, do not make your loop index the variable `i`.

» Imaginary numbers

Multiplying

Since $i^2 = -1$, multiplication by i goes in a kind of cycle.

Example:

$$i^1 = i$$

$$i^2 = -1 \text{ (by definition)}$$

$$i^3 = (i^2)i = (-1)i = -i$$

$$i^4 = (i^2)(i^2) = (-1)(-1) = 1$$

$$i^5 = (i^4)i = (1)i = i$$

$$\vdots$$

Complex numbers

» Complex numbers

Definitions

Definition: A **complex number** z is a number of the form

$$z = a + bi$$

where a and b are real numbers.
Example: $-1+2i$, $5-4i$, $3+i$

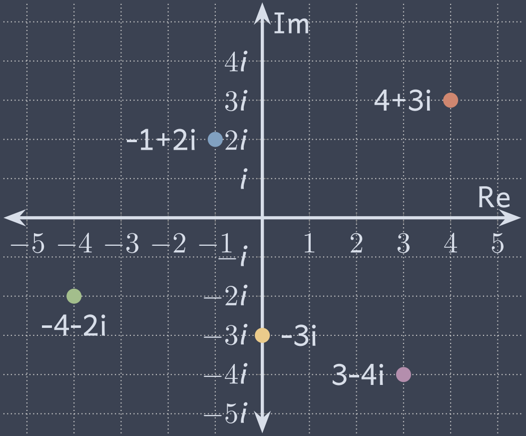
Definition: The **real part** of z is a and is denoted by $\mathrm{Re}(z)$.
Example: $\mathrm{Re}(-1+2i) = -1$

Definition: The **imaginary part** of z is b and is denoted by $\mathrm{Im}(z)$.
Example: $\mathrm{Im}(-1+2i) = 2$

Important: The imaginary part of a complex number $a + bi$ is *just* b by itself. It is *not* bi.

Complex plane

» Complex plane $\mathbb{C}$



Complex conjugate

» Complex conjugate

Definition: The complex conjugate $\bar{z}$ of the complex number $z = a + bi$ is

$$\bar{z} = a - bi.$$

Example: If $z = 3 + 9i$, then $\bar{z} = 3 - 9i$.

» Complex conjugate

Question

Question: Find the complex conjugate of $z = -1 + i$.

Adding and multiplying complex numbers

» Adding complex numbers

To add two complex numbers $z = a + bi$ and $w = c + di$, add the real parts together and the imaginary partys together:

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Example: If $z = 3 - i$ and $w = -6 + 8i$, then

$$z + w = (3 - i) + (-6 + 8i) = (3 - 6) + (-1 + 8)i = -3 + 7i$$

» Adding complex numbers

Question 1

Question: If $w = -1 + i$, what is $w + \bar{w}$?

» Adding complex numbers

Question 2

Question: If $z = a + bi$, what is $z + \bar{z}$?

Question: If $z = a + bi$, what is $z - \bar{z}$?

» Multiplying complex numbers

To multiply two complex numbers $z = a + bi$ and $w = c + di$, use FOIL:

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= ac + adi + bci + bd(i)^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Example: Let $z = 1 + 2i$ and $w = 3 - 4i$. Then

$$zw = (1+2i)(3-4i) = 3-4i+6i+(-8)(i^2) = (3+8)+(6-4)i = 11+2i$$

» Multiplying complex numbers

Question: Let $z = 2 - i$ and $w = -1 - i$. Find zw.