Math 107 Lecture 16

Existence and Uniqueness and Writing Solutions in Vector Form

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» Announcements and Objectives

Announcements

* Skill Check 4 is this Wed (10/23, 110 mins)
* Solutions to Homeworks 1-7 *available* in Canvas Modules
* Pre-Notes due before start of next lecture
* Assignments Due Friday (10/25):
    * HW8 Handwritten Questions
    * HW8 Coding Problems
    * HW8 MATLAB File Upload

Objectives

* Explore *applications* to cryptography
* Find *all* solutions to *a* given consistent system, including identifying free variables
* Give examples of particular solutions for systems with infinitely many solutions

Cryptography

» Cryptography

What is it and how is it used?

What is it?

* Encrypted (coded) messages transmitted to a receiver which can decrypt (decode) and read the message.

How is it used?

* Credit card transactions
* Passwords
* Secure web browsing
* ATMs

» Caesar cipher

Activity

1. Enumerate the letters in alphabetical order

a	b	c	d	e	f	g	h	i	j	k	l	m	n
0	1	2	3	4	5	6	7	8	9	10	11	12	13

o	p	q	r	s	t	u	v	w	x	y	z
14	15	16	17	18	19	20	21	22	23	24	25

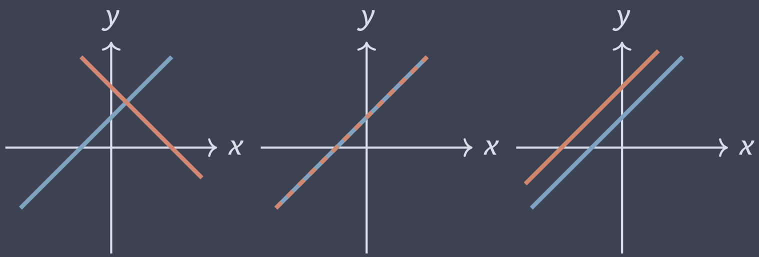
2. Turn a word into a vector of associated numbers. For example, ‘cat’ becomes $\begin{bmatrix} 2 \\ 0 \\ 19 \end{bmatrix}$.
3. Add 3 to each element to create an encrypted word. For example,

$$\begin{bmatrix} 2 \\ 0 \\ 19 \end{bmatrix} \to \begin{bmatrix} 5 \\ 3 \\ 22 \end{bmatrix}$$

Existence and Uniqueness

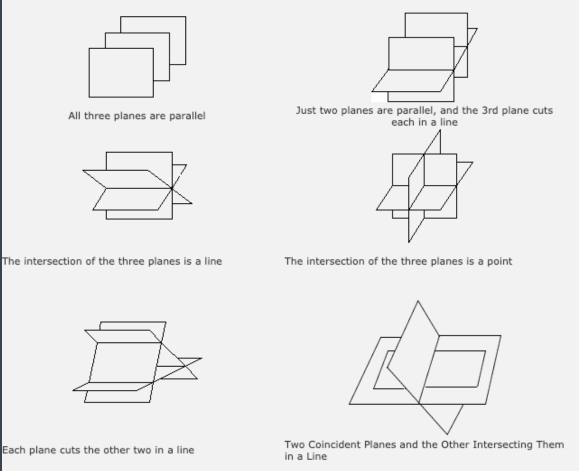
» How many solutions?

Two equations, two unknowns



» How many solutions?

Three equations, three unknowns



» How many solutions?

Theorem

Theorem: Every system of linear equations has exactly one solution, infinitely many solutions, or no solutions.

» Consistent or Inconsistent

Definition

Definition: A system of linear equations is **consistent** if it has *a* solution (perhaps more than one). A linear system is **inconsistent** if it does not have *a* solution.

» Dependent and Independent (free) Variables

Definition

Definition: Consider the reduced row echelon form of an augmented matrix of a linear system of equations. Then a variable that corresponds to a leading 1 is a **dependent variable**. A variable that does not correspond to a leading 1 is a **free** (or **independent**) variable.

» Particular Solution

Definition

Definition: Consider a linear system of equations with infinite solutions. A **particular solution** is one solution out of the infinite set of possible solutions.

» Some Key Ideas

* A consistent linear system of equations will have exactly one solution if and only if there is a leading 1 for each variable in the system.
* If a consistent linear system of equations has a free variable, it has infinite solutions.
* If a consistent linear system has more variables than leading 1s, then the system will have infinite solutions.
* A consistent linear system with more variables than equations will always have infinite solutions.

Matrix-vector equations

» What do we mean by solving an equation?

Analogy to algebra

When we solve an algebra equation like $ax = b$, we want to find a value for x that satisfies the equation.

We can solve similar equations for matrices. The notation looks like

$$A\vec{x} = \vec{b}$$

where $\vec{x}$ is a vector of unknowns and $\vec{b}$ is a vector of constants.

» Matrix-vector equation

The equation $A\vec{x} = \vec{b}$ is another way of writing a system of linear equations.

In other words, solving $A\vec{x} = \vec{b}$ for $\vec{x}$ is the same as solving a linear system of equations. And any system of linear equations can be written in the form $A\vec{x} = \vec{b}$.

» Why is $A\vec{x} = \vec{b}$ important?

* Cryptography
* Modeling physical systems (solving differential equations on a computer)
* Analyzing trends in data
* And more!

Homogeneous and inhomogeneous equations

» Homogeneous linear system

Definition

Definition: A system of linear equations is **homogeneous** if the constants in each equation are zero.

Remark: The constants are the numbers we put in the last column of the augmented matrix (e.g., the right-hand side).

Notation: The mathematical notation for a homogeneous system is

$$A\vec{x} = \vec{0}$$

where $\vec{0}$ is the zero vector. Its size depends on the context of the problem.

Remark: $\vec{x} = \vec{0}$ is *always* a solution to a homogeneous system, but there might also be infinite solutions.

» Question

Poll

Question: Which of the following systems are homogeneous? (Select all that apply.)

(a) $$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(d) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

» Example

Example: Solve $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Write your answer in vector form.

» Inhomogeneous linear system

Definition

Definition: A system of linear equations is **inhomogeneous** if the constants in each equation are not all zero.

Remark: The constants are the numbers we put in the last column of the augmented matrix (e.g., the right-hand side). Note that some of the constants in an inhomogeneous system can be zero, just not all of them.

Notation: The mathematical notation for a homogeneous system is

$$A\vec{x} = \vec{b}$$

where $\vec{b}$ is a vector that contains at least one nonzero element.

Remark: An inhomoegeneous linear system can have one, zero, or infinite solutions.

» Question

Poll

Question: Which of the following systems are inhomogeneous? (Select all that apply.)

(a) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(d) $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

» Example

Example: Solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Write your answer in vector form.

» Example

Example: Solve $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 5 & -4 & -1 \\ 1 & 0 & -2 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Write your answers in vector form.

» Inhomogeneous linear system

Key idea

Let $A\vec{x} = \vec{b}$ be a **consistent** system of linear equations.

* If $A\vec{x} = \vec{0}$ has exactly one solution (i.e., $\vec{x} = \vec{0}$), then $A\vec{x} = \vec{b}$ also has exactly one solution.
* If $A\vec{x} = \vec{0}$ has exactly infinite solutions, then $A\vec{x} = \vec{b}$ also has infinite solutions.