Math 107 Lecture 17

Matrix Inverse and Transpose

by Dr. Kurianski
on October 28, 2024

» Announcements and Objectives

Announcements

* Skill Check 5 is next Wed (11/6, 110 mins)
* Solutions to Homeworks 1-8 available in Canvas Modules
* Pre-Notes due before start of next lecture
* Assignments Due Friday (11/1):
    * HW9 Handwritten Questions
    * HW9 Coding Problems
    * HW9 MATLAB File Upload

Objectives

* Define and compute the inverse of a given matrix
* Use inverses to solve $A\vec{x} = \vec{b}$
* Discuss properties of the inverse and transpose
* Discuss special types of matrices and their features

What is an inverse?

» Identity matrix

Remark: The identity matrix I acts like "1" for matrix multiplication (when the multiplication makes sense).

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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But

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{(3\times 1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(3\times 3)} =?$$

doesn't make sense!

Remark: If A is a square matrix, then $AI = IA = A$.

» What's an inverse?

What does it mean to solve $ax = b$?

$$x = \frac{b}{a} \text{ or } x = a^{-1}b.$$

Then we can think of a^{-1} as a number such that

$$aa^{-1} = 1.$$

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Important: Just as the number 0 doesn't have an inverse (because $\frac{1}{0}$ is undefined), the matrix A may not have an inverse.

» Matrix inverse

Definition and Computation

Definition: Let A and X be $n \times n$ matrices where $AX = I = XA$. Then

1. A is **invertible** and
2. X is the **inverse** of A, denoted A^{-1}

» Matrix inverse

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How to compute the inverse of A:

1. Write down the augmented matrix $[\,A \mid I\,]$
2. Perform Gaussian elimination
3. If you end up with the identity in the place where A was (i.e., $[\,A \mid I\,] \to [\,I \mid X\,]$), then the matrix on the right is A^{-1}.
4. If you do not end up with the identity in the place where A was, then A is not invertible.

» Computing an inverse

MATLAB Example

Use MATLAB's `rref()` command to find the inverse of

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix}$$

by solving $AX = I$.

» Computing an inverse

MATLAB Activity

Use MATLAB's `rref()` command to find the inverse of

$$A = \begin{bmatrix} -15 & 45 & -3 & 4 \\ 55 & -164 & 15 & -15 \\ -215 & 640 & -62 & 59 \\ -4 & 12 & 0 & 1 \end{bmatrix}$$

by solving $AX = I$.

» Computing an inverse

Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ by hand.

» Computing an inverse

Key Idea

Key Idea: Let A be an $n \times n$ matrix. To find A^{-1}, put the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ into reduced row echelon form. If the result is of the form $\begin{bmatrix} I & X \end{bmatrix}$, then $A^{-1} = X$. If not, then A is not invertible.

Remark: If the matrix A is invertible, then its reduced row echelon form is I.

Inverse of 2×2 matrices

» Formula for 2×2 matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an invertible 2×2 matrix. Then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

» Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ using the formula for 2×2 matrices.

Solving $A\vec{x} = \vec{b}$

» Solving $A\vec{x} = \vec{b}$

Theorem: Let $\vec{b}$ be any $n \times 1$ column vector. If A is an invertible $n \times n$ matrix, then $A\vec{x} = \vec{b}$ has **exactly one solution** given by $\vec{x} = A^{-1}\vec{b}$.

» Solving $A\vec{x} = \vec{b}$

Theorem: Let $\vec{b}$ be any $n \times 1$ column vector. If A is an invertible $n \times n$ matrix, then $A\vec{x} = \vec{b}$ has **exactly one solution** given by $\vec{x} = A^{-1}\vec{b}$.

Example: Let $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Use A^{-1} to solve $A\vec{x} = \vec{b}$.

Concept questions

» Question 1

If A is **not** invertible, how many solutions is it possible for $A\vec{x} = \vec{b}$ to have? (Select all that apply)

(a) exactly one
(b) infinite
(c) none

» Question 2

True or False? If A is invertible, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

» Activity

Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$.
Compute the following using MATLAB:

(a) A^{-1}
(b) B^{-1}
(c) $(AB)^{-1}$
(d) $A^{-1}B^{-1}$
(e) $B^{-1}A^{-1}$

» Inverse of product

Theorem: Let A and B be two invertible $n \times n$ matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$.

» Diagonal entries

Definition: Let A be an $m \times n$ matrix. The **diagonal** of A consists of the entries $a_{11}, a_{22}, \ldots$.

Example: $A = \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{4} & 5 \\ 0 & 0 & \mathbf{6} \end{bmatrix}$

Example: $B = \begin{bmatrix} \mathbf{3} & 1 & 7 \\ 8 & \mathbf{7} & -2 \\ 0 & 5 & \mathbf{-1} \end{bmatrix}$

» Diagonal matrix

Definition

Definition: A **diagonal matrix** is an $n \times n$ matrix in which the only nonzero entries lie on the diagonal.

Example: $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Remark: For a matrix to be a diagonal matrix, it *must* be square (same number of rows as columns).

» Inverse of diagonal matrices

Fact: If A is a diagonal matrix with diagonal entries d_1, d_2, $\ldots$, d_n, i.e.,

$$A = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & d_n \end{bmatrix}$$

where none of the diagonal entries are zero then A^{-1} exists and is equal to

$$A^{-1} = \begin{bmatrix} 1/d_1 & 0 & \ldots & 0 \\ 0 & 1/d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1/d_n \end{bmatrix}$$

Properties of Transpose

» Transpose

Definition

Definition: Let A be an $m \times n$ matrix. The **transpose** of A, denoted A^T, is the $n \times m$ matrix whose columns are the respective rows of A.

» Properties of the Matrix Transpose

Let A and B be matrices where the following operations are defined, and let k be a scalar. Then:

1. $(A+B)^T = A^T + B^T$
2. $(kA)^T = kA^T$
3. $(AB)^T = B^T A^T$
4. $(A^{-1})^T = (A^T)^{-1}$
5. $(A^T)^T = A$

» Transpose of Product

Question: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 6 & -2 \\ 1 & 0 \end{bmatrix}$.

Use MATLAB to compute the following:

(a) A^T

(b) B^T

(c) $(AB)^T$

(d) $A^T B^T$

(e) $B^T A^T$

» Properties of the Matrix Transpose

Example

Question: Let A and B be any $n \times n$ invertible matrices. Simplify $((A^{-1}B^{-1})^T)^{-1}$.

» Symmetric and Skew Symmetric

Definition: A matrix A is **symmetric** if $A^T = A$

Example: $A = \begin{bmatrix} 1 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}$

Definition: A matrix A is **skew symmetric** if $A^T = -A$.

Example: $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$