Math 107 Lecture 14

Fractals and Intro to Linear Systems

by Dr. Kurianski
on October 14, 2024

» Announcements and Objectives

Announcements

* Skill Check 4 is next Wed (10/23, 110 mins)
* Pre-Notes due before start of next lecture
* Assignments Due Friday (10/18):
  * HW7 Handwritten Questions
  * HW7 Coding Problems
  * HW7 MATLAB File Upload

Objectives

* Explore fractals
* Convert word problems into systems of linear equations
* Write systems of linear equations as augmented matrices
* Solve systems of linear equations using row reduction

Cool example

» Cool example

We started our discussion of complex numbers with thinking about the roots of the equation

$$x^2 + 1 = 0.$$

Let's now consider the equation

$$x^4 - 1 = 0$$

and think about the real *and* complex numbers that satisfy it.

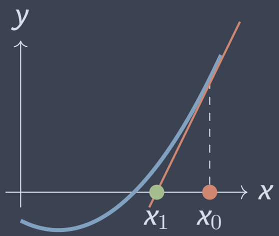
Note: The values $x = 1$, $x = -1$, $x = i$, and $x = -i$ all satisfy $x^4 - 1 = 0$. In other words, they are all **roots** of the equation $f(x) = x^4 - 1$.

» Cool example

One way to find roots of a function numerically is called Newton's method.

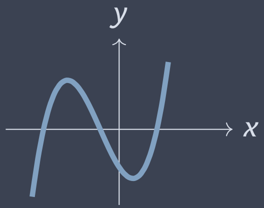
Idea of the method:

1. Make a guess x_0 that you think is close to where $f(x) = 0$.
2. Find the tangent line to the function at your initial guess.
3. Use where the tangent line crosses the x-axis as your next guess for where $f(x) = 0$.
4. Repeat the steps.



» Cool example

What if there are multiple roots (places where $f(x) = 0$)?



Can we use this method to find complex roots of an equation like $f(x) = x^4 - 1$?

» Cool example

Newton fractal

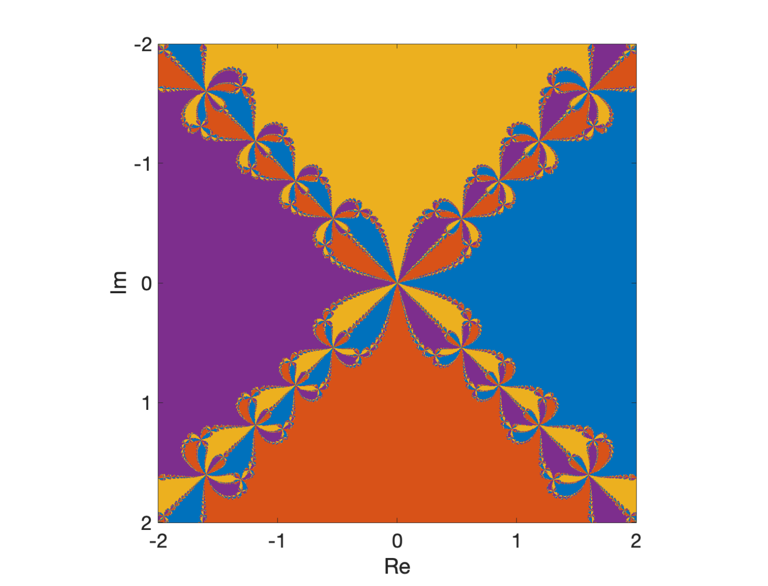
$$x^4 - 1 = 0, \text{ when } x = 1, i, -i, -1$$

Color-code the complex plane using the following rules:

* If the initial guess $a + bi$ converges to the root $x = 1$, color $a + bi$ blue.
* If the initial guess $a + bi$ converges to the root $x = i$, color $a + bi$ orange.
* If the initial guess $a + bi$ converges to the root $x = -i$, color $a + bi$ yellow.
* If the initial guess $a + bi$ converges to the root $x = -1$, color $a + bi$ purple.

» Cool example

Newton fractal



The Sierpinski Triangle

» Creating the Sierpinski Triangle

One way...

1. Start with an equilateral triangle.
2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.
3. Repeat step 2 with each of the remaining smaller triangles infinitely many times.

Another way...

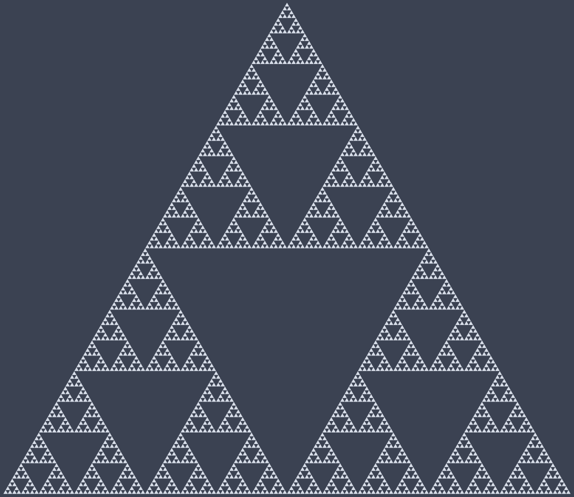
1. Start with an equilateral triangle.
2. Draw a point anywhere on the plane.
3. Choose a corner of the original triangle at random. Draw another point halfway between the previous point you drew and the chosen corner of the original triangle.
4. Repeat step 3 infinitely many times.

» The Sierpinski Triangle

Visualization

» The Sierpinski Triangle

Visualization



Linear Equations

» Warm-up

Find scalars x and y such that

$$x\begin{bmatrix}1\\2\end{bmatrix}+y\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}-1\\8\end{bmatrix}$$

We can write this as a system of linear equations:

$$\begin{aligned} x+y&=-1\\ 2x-3y&=8 \end{aligned}$$

We can then solve using elimination and back substitution.

» Linear equations

Why do we need them?

Systems of linear equations have applications in:

* business
* engineering
* computer graphics
* economics
* operations research
* and more...

But instead of two or three variables, these real-world systems can rely on thousands or millions of variables. So we need a more efficient way to solve them than elimination.

» Linear equations

Definitions

Definition: A **linear equation** is an equation that can be written in the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = k$$

where the x_i are variables (unknowns), the c_i are coefficients, and k is a constant.

Example:

* $3x + y - z = 7$
* $2x_1 + 7x_2 - 3x_3 + x_4 = -1$

Nonlinear equations are equations that have terms involving the variables that are not linear (coefficient times variable).

Example:

* $3x^2 + y - z = 7$
* $\sin(x) + 2y = 3$
* $xy = 1$

» Linear equations

Poll 1

Which of the following are examples of linear equations? (Select all that apply.)

(a) $2x + 3y - 7z = 29$

(b) $y_1 + 14^2 y_2 + 4 = y_2 + 13 - y_1$

(c) $\ln(y) = 2x$

(d) $3x + \pi \sin(3\pi/4) y = z - 4x$

(e) $x^2 + 7x = 4y$

» Linear equations

Poll 2

Which of the following are examples of linear equations? (Select all that apply.)

(a) $3xy = 0$

(b) $x_1 + \frac{7}{2}x_2 + x_3 - x_4 + 17x_5 = \sqrt[3]{-10}$

(c) $3^x + 4 = y$

(d) $\sqrt{7}\,r + \pi s + \frac{3t}{4} = \cos(\pi/4)$

(e) $6y + 3z = \sin(3x)$

Systems of Linear Equations

» Systems of linear equations

Definitions

Definition: A **system of linear equations** is a set of linear equations that involve the same variables.

Example:

$$\begin{aligned} 3x + y - z &= 7 \\ x + z &= 4 \\ z - y &= 6 \end{aligned}$$

Example:

$$\begin{aligned} a - 3b &= -7 \\ 2a &= -2 \end{aligned}$$

» Systems of linear equations

Definitions

Definition: A **solution** to *a* system of linear equations is *a* set of values for the variables x_i such that each equation in the system is satisfied simultaneously.

Example:

$$\begin{aligned} a - 3b &= -7 \\ 2a &= -2 \end{aligned}$$

$$a = -1, \; b = 2$$

» System of linear equations from word problems

Example: A jar contains red, blue, and green marbles. There are a total of 30 marbles in the jar. There are twice as many red marbles as green ones. The number of blue marbles is the same as the sum of the red and green marbles. How many marbles of each color are there?

We can start to think about solving this problem by writing down a system of equations:

$$\begin{aligned} r + b + g &= 30 \\ r &= 2g \\ b &= r + g \end{aligned}$$

Word problems

» How to write word problem as a system

1. Determine what is being asked.
2. Identify the unknowns.
3. Give the unknowns names (variables).
4. Write down system of equations from relationships.

Example: The LA Zoo sells tickets for \$17 for children, \$22 for adults, and \$19 for seniors. Attendance on a certain day is 4,000 and the total gate revenue is \$60,000. There were twice as many children's tickets sold as adults. How many of each type of ticket were sold?

1. # of tickets sold of each type
2. # adult, # children, # seniors
3. a, c, s
4. Write down the equations

Solving systems

» Solving systems

Motivation

In the warm-up, we solved

$$\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$$

using elimination and back substitution.

When we have a system of 3 or more variables, solving by elimination can get really time consuming. Instead, we'll convert the system to an **augmented matrix** and keep track of our computations using **row reduction**.

» Coefficient matrix

Consider the system

$$\begin{aligned} r + b + g &= 30 \\ r &= 2g \\ b &= r + g \end{aligned}$$

1. Put all variables on the left and constants on the right:

$$\begin{aligned} r + b + g &= 30 \\ r - 2g &= 0 \\ r + g - b &= 0 \end{aligned}$$

2. Line up the variables
3. Write coefficients as a matrix

» Coefficient matrix

Consider the system

$$\begin{aligned} r+b+g &= 30 \\ r &= 2g \\ b &= r+g \end{aligned}$$

1. Put all variables on the left and constants on the right
2. Line up the variables

$$\begin{aligned} r+b+g &= 30 \\ r - 2g &= 0 \\ r-b+g &= 0 \end{aligned}$$

3. Write coefficients as a matrix

» Coefficient matrix

Consider the system

$$\begin{aligned} r + b + g &= 30 \\ r &= 2g \\ b &= r + g \end{aligned}$$

1. Put all variables on the left and constants on the right
2. Line up the variables
3. Write coefficients as a matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

The above matrix is called the **coefficient matrix**.

» Coefficient matrix

Notice that multiplying the **coefficient matrix** by the column vector $[r; b; g]$ produces the system of equations without the right-hand sides.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ b \\ g \end{bmatrix} = \begin{bmatrix} r+b+g \\ r-2g \\ r-b+g \end{bmatrix}$$

» Augmented matrix

To form the **augmented matrix**, write the right-hand side of the system as a column vector and concatenate on the right of the coefficient matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & 0 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

» Augmented matrix

Example 1

Example: Write the following system of equations as an augmented matrix.

$$\begin{aligned} 4x - 7 + y &= 0 \\ 2 - 5z + 3y &= x \\ z &= 4x \end{aligned}$$

» Augmented matrix

Example 2

Example: Write the following system of equations as an augmented matrix.

$$\begin{aligned} 3x + 4y &= -x \\ z - 3 &= y \end{aligned}$$

» Augmented matrix

Example 3

Example: Convert the following augmented matrix into *a* system of linear equations using variables x_1 and x_2.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right]$$

» Augmented matrix

Example 4

Example: Convert the following augmented matrix into a system of linear equations using variables x_1, x_2. and x_3.

$$\left[\begin{array}{ccc|c} -3 & 4 & 7 & 9 \\ 0 & 1 & -2 & -1 \\ 3 & 0 & 0 & 4 \end{array}\right]$$

Row reduction

» Row reduction

Goal

The goal of reduced row reduction is to perform row operations on an augmented matrix until there are ones on the diagonal of the coefficient matrix (if possible) and zeros above and below them.

Example:

$$\left[\begin{array}{ccc|c} a & b & c & d \\ f & g & h & j \\ k & l & m & n \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right]$$

Remark: It might not be possible to get a “leading one” in every row. We’ll explore this idea in the next lecture!

» Row reduction

Example 1

Solve the following system of equations using reduced row reduction on the augmented matrix. Check your work by performing each row operation in MATLAB.

$$\begin{aligned} 2x_1 - 2x_2 &= -10 \\ 4x_1 + x_2 &= -10 \end{aligned}$$