# Math 107 Lecture 20

## Eigenvalues and Eigenvectors

by Dr. Kurianski
on November 13, 2024

# » Announcements

## Announcements

- Skill Check 6 is next Wed (11/20, 110 mins)
- Solutions to Homeworks 1-10 *available* in Canvas Modules
- Skill Check 1-5 solution videos *available*
- Pre-Notes due before start of next lecture
- Assignments Due Friday (11/15):
  - HW11 Handwritten Questions
  - HW11 Coding Problems
  - HW11 MATLAB File Upload
- SOQs

# » Student Opinion Questionnaires

## What are SOQs?

* **Anonymous surveys** that are used by the department and university to evaluate instructor performance.
* Share your experience in this course with the department and university.
* Access SOQs in your **CSUF Student Portal** (**`https://my.fullerton.edu/`**).
* Available from Nov. 9 until **Friday, Nov. 29, 2024**.
* More info on Canvas

# » Objectives

## Objectives

- Find the eigenvalue and eigenvector pairs of a given matrix
- Interpret eigenvectors and eigenvalues geometrically
- Explore properties of eigenvalues and eigenvectors

## » Question

**Question:** What must be true about $\det(A)$ for $A\vec{x} = \vec{0}$ to have infinitely many solutions?

## » Warm-up

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Find all the values of the scalar $\lambda$ for which the matrix $B$ defined by $B = A - \lambda I_3$ is not invertible.

## » Shortcut for $3 \times 3$ matrices

(Reference: Pgs 259-261 of Main Textbook)

**Example:** Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 0 & 4 & -4 \end{bmatrix}$

**Example:** Find the determinant of $B = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

**Question:** (True or False?) The matrix $B$ above is invertible.

## » Question

**Question:** Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Compute $A\mathbf{w}$ and $A\vec{v}$. Sketch the vectors $\mathbf{w}$, $\vec{v}$, $A\mathbf{w}$, and $A\vec{v}$.

# Definitions of Eigenvalues and Eigenvectors

# » Eigenvalues and Eigenvectors

## Definition

**Definition:** Let $A$ be an $n \times n$ matrix, let $\vec{v}$ be a nonzero $n \times 1$ column vector, and let $\lambda$ be a scalar. If

$$A\vec{v} = \lambda\vec{v}$$

then $\vec{v}$ is an **eigenvector** of $A$ and $\lambda$ is an **eigenvalue** of $A$.

**Remark:** There are two important points here:

1. $\vec{v} \neq \vec{0}$
2. $\vec{v}$ and $\lambda$ are linked to each other

# Finding Eigenvalues

# » Finding eigenvalues

## Procedure

**Procedure:** Let $A$ be an $n \times n$ matrix. To find the eigenvalues of $A$, we need to find all values of $\lambda$ such that

$$\det(A - \lambda I) = 0.$$

# » Finding eigenvalues

## Example

**Example:** Find the eigenvalues of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

# » Finding eigenvalues

## Characteristic Polynomial

**Definition:** Let $A$ be an $n \times n$ matrix. The **characteristic polynomial** of $A$ is the $n$th degree polynomial

$$p(\lambda) = \det(A - \lambda I).$$

**Remark:** To find the eigenvalues of $A$, solve $p(\lambda) = 0$ for $\lambda$.

# » Finding eigenvalues

## Question: Chat Blast

**Question:** Find the eigenvalues of

$$A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$$

# Finding Eigenvectors

# » Finding eigenvectors

## Example

**Example:** Recall that one eigenvalue of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is $\lambda = 5$.

We know that $A\vec{v} = \lambda\vec{v}$ or in other words

$$(A - \lambda I)\vec{v} = \vec{0}.$$

We need to solve the above equation for $\vec{v}$.

# » Finding eigenvectors

## Exercise

**Question:** Given $A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$, find one eigenvector for each eigenvalue ($\lambda = 6, -7$).

# » Finding eignevalues and eigenvectors

## Example

**Example:** Find the eigenvalues of $A$ and, for each eigenvalue, give one eigenvector.

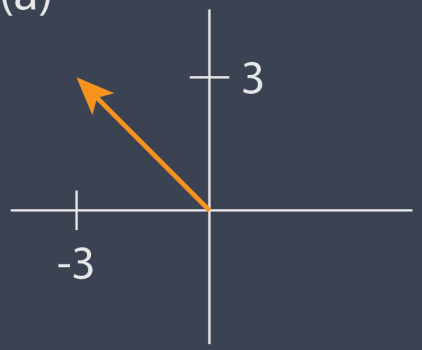
$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 3 & 4 \end{bmatrix}$$

## » Geometry of eigenvalues/eigenvectors

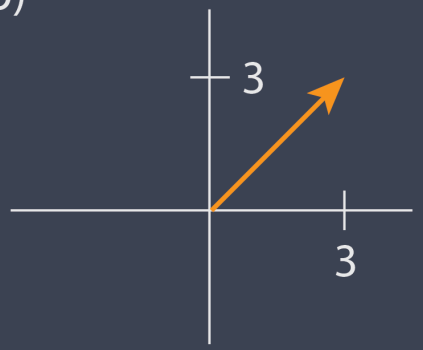
Question: Poll

**Question:** Let $A$ be a $2 \times 2$ matrix. Suppose that $A$ has an eigenvalue of $\lambda = -3$ with corresponding eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Which of the following represents $A\vec{v}$?

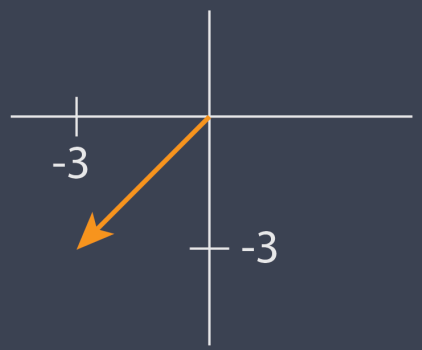
(a)



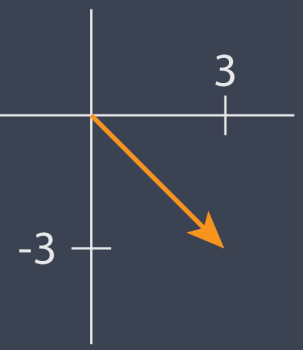

(b)




(c)




(d)



# » Properties of Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ invertible matrix. The following are true:

1. If $A$ is triangular, then the diagonal elements of $A$ are the eigenvalues of $A$.
2. If $\lambda$ is an eigenvalue of $A$ with eigenvector $\vec{v}$ then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$ with eigenvector $\vec{v}$.
3. If $\lambda$ is an eigenvalue of $A$ then $\lambda$ is an eigenvalue of $A^T$.
4. The sum of the eigenvalues of $A$ is equal to the trace of $A$ (sum of the diagonal elements)
5. The product of the eigenvalues of $A$ is equal to $\det(A)$

## » Invertible Matrix Theorem

Let $A$ be an $n \times n$ matrix. The following are equivalent.

1. $A$ is invertible
2. $\det(A) \neq 0$
3. $A$ does not have an eigenvalue of $0$