Math 107 Lecture 19

Determinants Continued and Intro to Eigenvalues/Eigenvectors

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» Announcements

Announcements

* Skill Check 5 is next Wed (11/6, 110 mins)
* Solutions to Homeworks 1-9 available in Canvas Modules
* Skill Check 4 solution video available
* No class on Monday 11/11 (Veterans Day)
* Pre-Notes due before start of next lecture
* Assignments Due Friday (11/1):
  * HW10 Handwritten Questions
  * HW10 Coding Problems
  * HW10 MATLAB File Upload
* SOQs

» Student Opinion Questionnaires

What are SOQs?

* **Anonymous surveys** that are used by the department and university to evaluate instructor performance.
* Share your experience in this course with the department and university.
* Access SOQs in your **CSUF Student Portal** (`https://my.fullerton.edu/`).
* Available from Nov. 9 until **Friday, Nov. 29, 2024**.
* More info on Canvas

» Objectives

Objectives

* Use the Invertible Matrix Theorem to determine whether given matrices are invertible
* Define and compute the determinant of $n \times n$ matrices using cofactor expansion
* Introduce eigenvalues and eigenvectors

Determinants of $n \times n$ Matrices

» Minor of A

Definition

Definition: Let A be an $n \times n$ matrix. The ***ij*-minor of A**, denoted A_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column of A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 0 & 4 & -4 \end{bmatrix}$. Compute A_{11}, A_{23}, and A_{32}.

» Cofactor

Definition

Definition: Let A be an $n \times n$ matrix. The ***ij*-cofactor of** A, denoted C_{ij} is the number

$$C_{ij} = (-1)^{i+j} A_{ij}.$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 0 & 4 & -4 \end{bmatrix}$. Compute C_{23} and C_{11}.

» Cofactor Expansion

Definition

Definition: Let A be an $n \times n$ matrix. The **cofactor expansion of A along the ith row** is the sum

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The **cofactor expansion of A down the jth column** is the sum

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 0 & 4 & -4 \end{bmatrix}$. Find the cofactor expansion along the 2nd column.

Example: Find the cofactor expansion along the 3rd row.

» Determinant of an $n \times n$ Matrix

Definition

Definition: The determinant of an $n \times n$ matrix A is a number given by the following:

* if A is 1×1, then $A = [a]$ and $\det(A) = a$
* if A is 2×2, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc$
* if A is $n \times n$ with $n > 2$, then

$$\det(A) = \text{cofactor exapansion along any row or column}$$

» Determinant of an $n \times n$ Matrix

Example 1

Example: Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & 0 \\ 7 & 2 & 3 \end{bmatrix}$$

» Determinant of an $n \times n$ Matrix

Example 2

Example: Find the determinant of

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 0 & -2 & 0 \\ 5 & -7 & -1 \end{bmatrix}$$

Properties of Determinants

» Properties of Determinants

Theorems

Theorem: Let A and B be $n \times n$ matrices and let k be a scalar. The following are true:

1. $\det(kA) = k^n \det(A)$
2. $\det(A^T) = \det(A)$
3. $\det(AB) = \det(A)\det(B)$
4. If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Theorem: A matrix A is invertible if and only if $\det(A) \neq 0$.

» Properties of Determinants

Exercise

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ -1 & 5 \end{bmatrix}$.

(a) Compute $\det(A)$ and $\det(B)$.

(b) Use your answers from part (a) to compute $\det(AB)$, $\det(A^T B)$, and $\det(A^{-1}B^{-1})$.

» Invertible Matrix Theorem (revisited)

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent.

* A is invertible.
* The reduced row echelon form of A is I.
* The equation $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
* The equation $A\vec{x} = \vec{0}$ has exactly one solution (namely, $\vec{x} = \vec{0}$).
* $\det(A) \neq 0$

» Shortcut for 3×3 matrices

(Reference: Pgs 259-261 of Main Textbook)

Example: Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 0 & 4 & -4 \end{bmatrix}$

Example: Find the determinant of $B = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 0 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

Question: (True or False?) The matrix B above is invertible.