PHYS 225
Fundamentals of Physics: Mechanics

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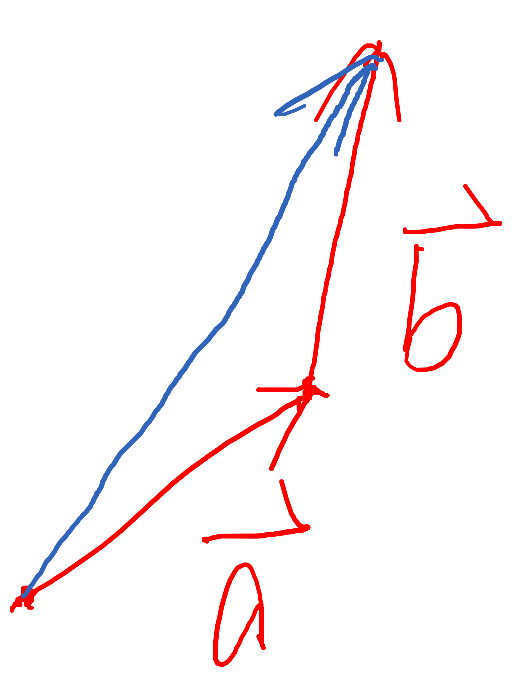
Fall 2024

Lecture 7: Vector multiplication

CALIFORNIA STATE UNIVERSITY
FULLERTON

Learning goals for today

$\vec{c} = \vec{a} + \vec{b}$



- Summarize vector addition
- Vector multiplications

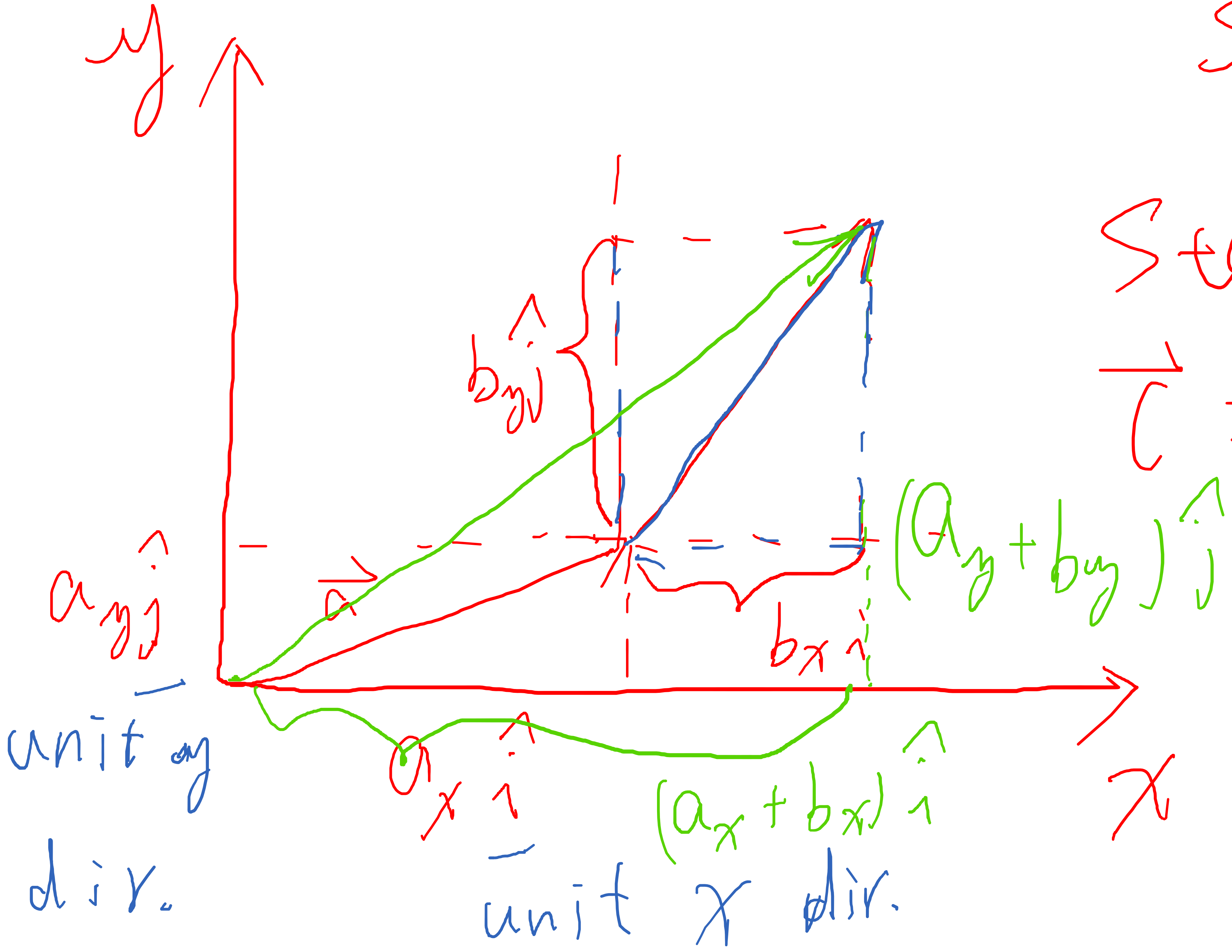
Head tail rule

Add by components

Step1: Decomposition:

Step2:

$$\vec{c} = \vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j}$$



Example 1

by components

Given: $\vec{a}$, $\vec{b}$

Goal: $\vec{c} = \vec{a} + \vec{b}$

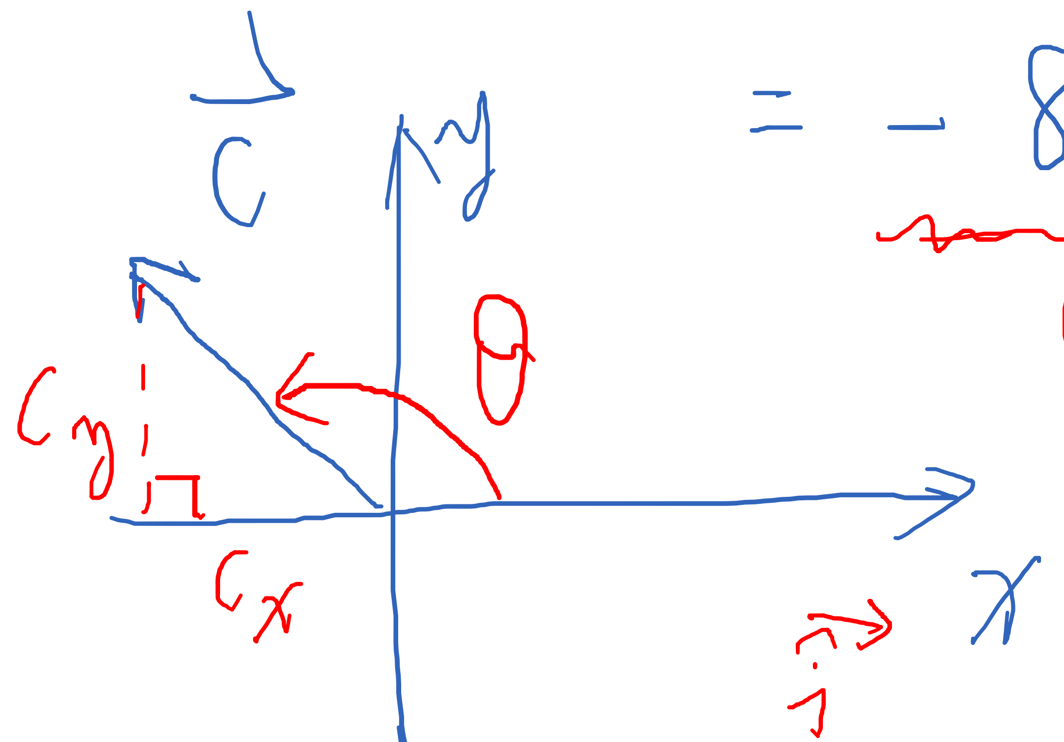
(a) In unit-vector notation, what is the sum of $\vec{a} = (\underbrace{3.7\ \text{m}}_{}\overset{a_x}{})\hat{i} + (\overset{a_y}{1.7\ \text{m}})\hat{j}$ and $\vec{b} = (\overset{b_x}{-12.0\ \text{m}})\hat{i} + (\overset{b_y}{6.8\ \text{m}})\hat{j}$. What are **(b)** the magnitude and **(c)** the direction of $\vec{a} + \vec{b}$ (relative to $\hat{i}$)?

a) Step 1: $\vec{c} = \vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j}$

$$= (3.7\text{m} - 12.0\text{m})\hat{i} + (1.7\text{m} + 6.8\text{m})\hat{j}$$

$$= \underbrace{-8.3\text{m}}_{c_x}\hat{i} + \underbrace{8.5\text{m}}_{c_y}\hat{j}$$

b) Sketch



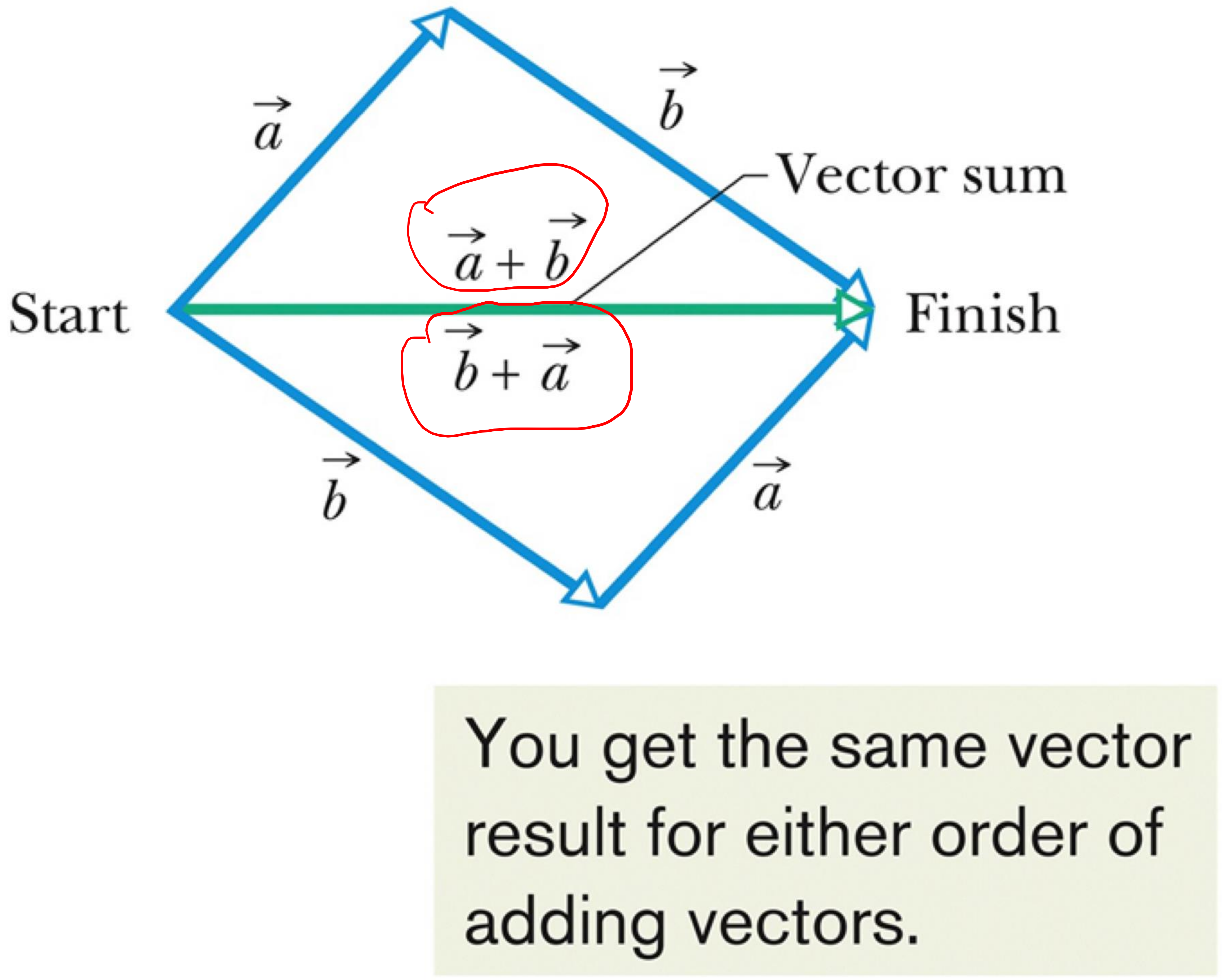
$$|\vec{c}| = \sqrt{c_x^2 + c_y^2} = \sqrt{(-8.3\text{m})^2 + (8.5\text{m})^2} = 11.9\text{m}$$

c) $90^\circ < \theta < 180^\circ$, range of atan is $(-90^\circ, 90^\circ)$: $\theta = 180^\circ + \text{atan}\frac{c_y}{c_x} = 134^\circ$

Properties of vector addition: I

- Vector addition is commutative:

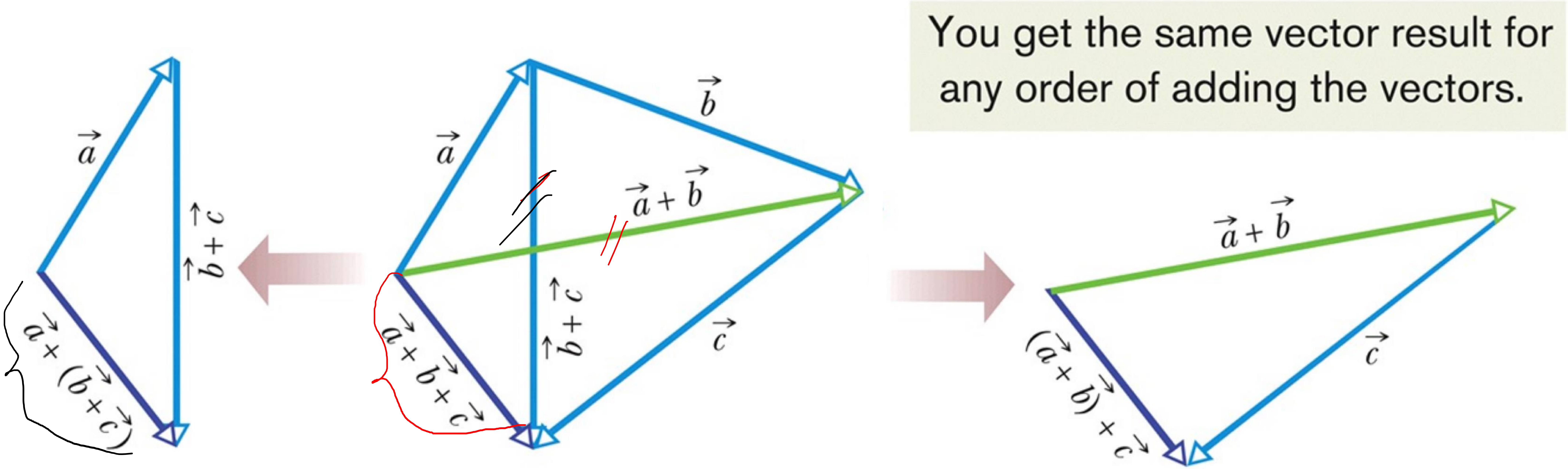
  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$



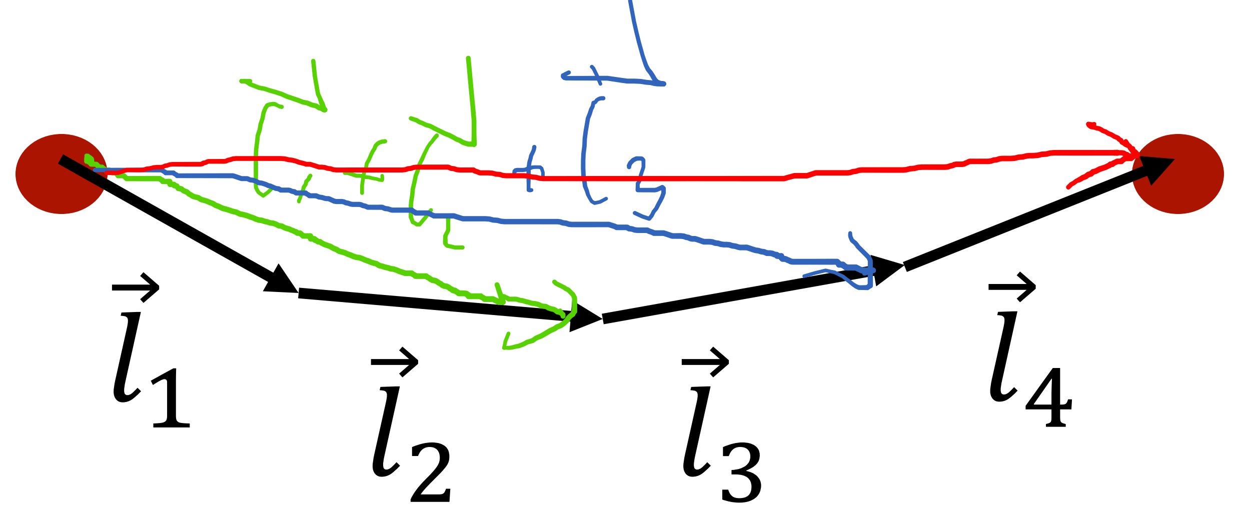
Properties of vector addition: II

- Vector addition is associative

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$



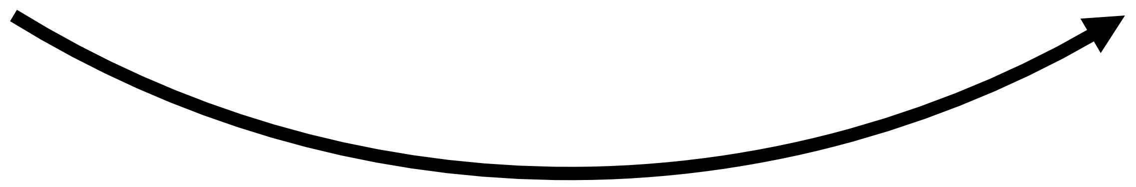
Clicker question



- What is the vector sum of vectors $\vec{l}_1$,$\vec{l}_2$,⋯, $\vec{l}_4$ above: $\sum_{i=1}^{i=4} \vec{l}_i$?

A

B



Vector addition summary

- Vector addition by head-tail convention
- Vector addition by components
- Properties of vector addition
  - Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
  - Associative: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

Chapter 3.2: Vector multiplication

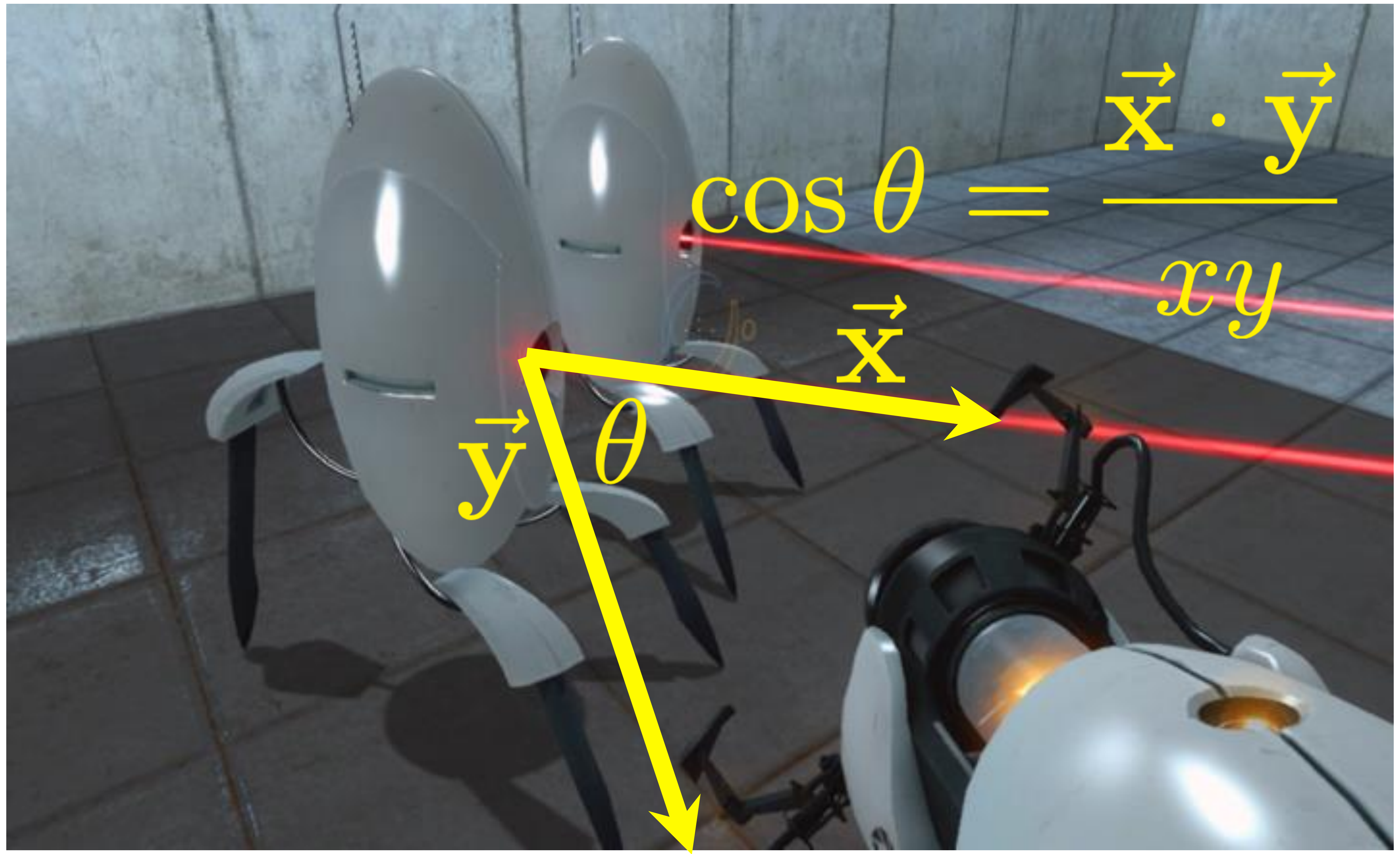
- Multiply a vector by a scalar
- Dot product $\cdot$
- Cross product $\times$

Motivation

Re: Dot Product Applications

by **duckshirt** » Sun Apr 18, 2010 12:43 pm UTC

Yesterday I used dot products when programming a 3D-ish game. As far as I know, it's the easiest way to find the angle between two vectors; since M•N = |M||N|cos(theta), theta = arccos(M•N / [|M||N|]). And the cross product came up even more often. Just another example in case you weren't convinced already...



How to determine the aiming angle, θ, given $\vec{x}$ and $\vec{y}$?

Dot product

Input : Vectors

Output: A scalar

- **Dot Product** (or scalar product): creates a new **scalar.**

For example: $\vec{a} \cdot \vec{b}$

  - In terms of **vector components**:

$$\underbrace{\boxed{\vec{a} \cdot \vec{b}}}_{\text{Scalar}} = a_x b_x + a_y b_y + a_z b_z$$

Scalar

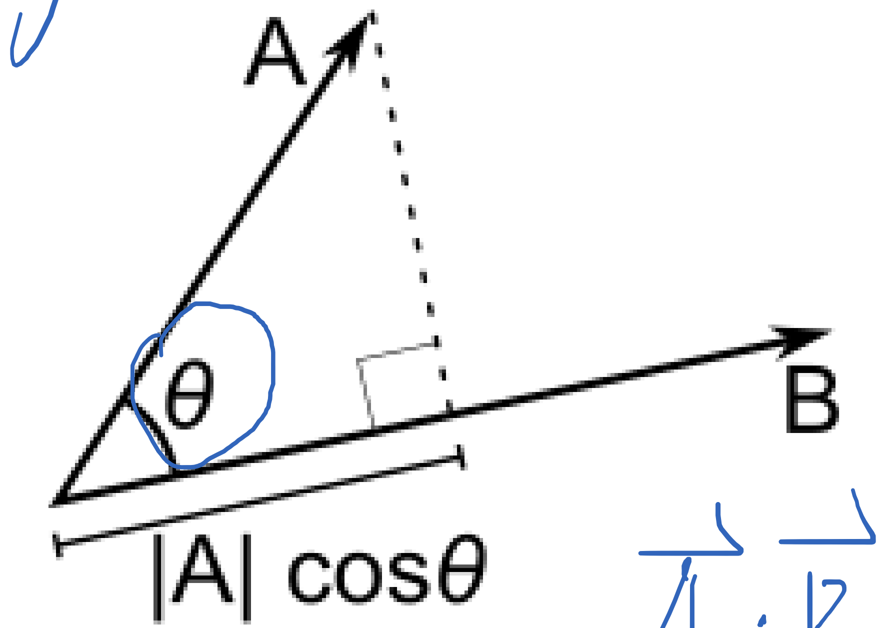
vector vector

(In 2D, $a_z = b_z = 0$)

  - In terms of **geometry**:

$$\vec{a} \cdot \vec{b} = \underbrace{|\vec{a}|}_{\text{magnitude of } \vec{a}}\underbrace{|\vec{b}|}_{\text{magnitude of } \vec{b}} \cos\theta$$

angle between $\vec{a}$ & $\vec{b}$



$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

Clicker question 1

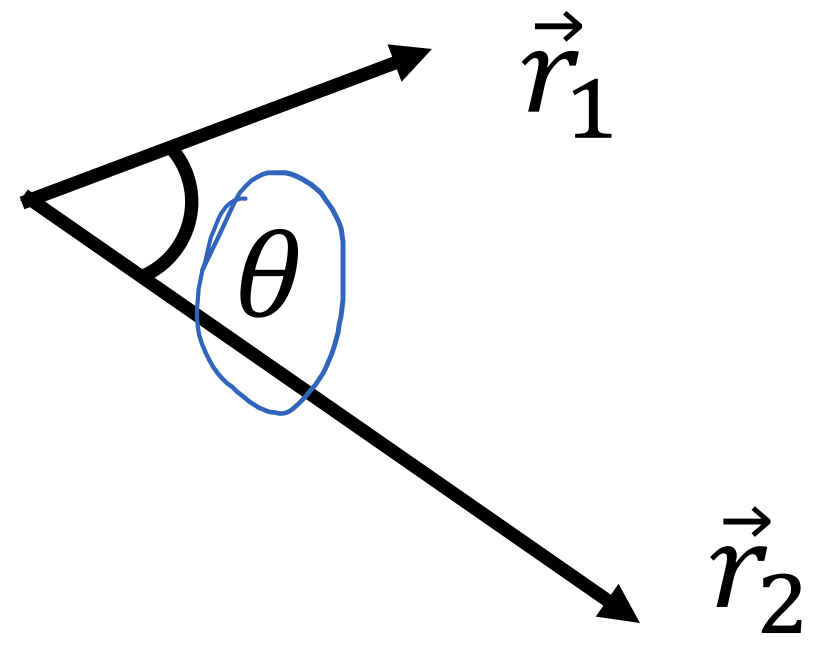
$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

- Which of the following is true regarding the dot product between vectors $\vec{r}_1$ and $\vec{r}_2$ below? Here $\theta < 90°$.

A $\quad \vec{r}_1 \cdot \vec{r}_2 > 0$

B $\quad \vec{r}_1 \cdot \vec{r}_2 = 0$

C $\quad \vec{r}_1 \cdot \vec{r}_2 < 0$



If $0 < \theta < 90°$,
then $\cos\theta > 0$

If $90° < \theta < 180°$
then $\cos\theta < 0$

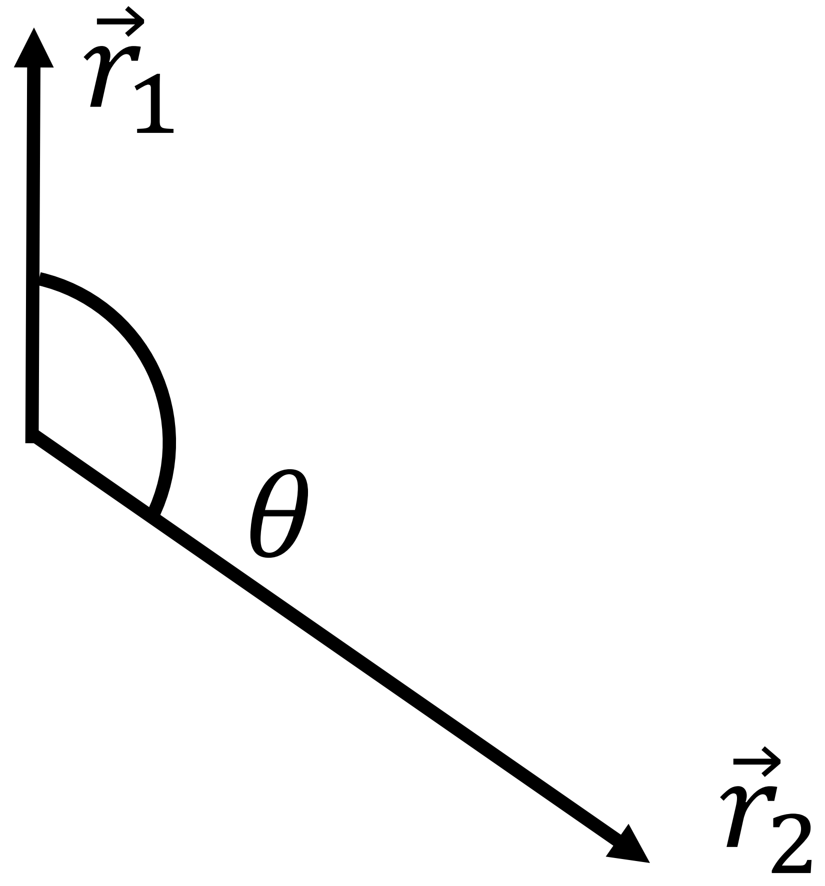
Clicker question 2

- Which of the following is true regarding the dot product between vectors $\vec{r}_1$ and $\vec{r}_2$ below? Here $\theta > 90°$.

A $\quad \vec{r}_1 \cdot \vec{r}_2 > 0$

B $\quad \vec{r}_1 \cdot \vec{r}_2 = 0$

C $\quad \vec{r}_{1_1} \cdot \vec{r}_2 < 0$



Example 2

Given: $\vec{a}$, $\vec{b}$

Goal: θ between $\vec{a}$ & $\vec{b}$

- What's the angle between the two vectors $\vec{a}$ and $\vec{b}$,where $\vec{a} = 5.00\hat{\imath} + 9.00\hat{\jmath} + 1.00\hat{k}$,$\vec{b} = 2.00\hat{\imath} + 8.00\hat{\jmath} + 3.00\hat{k}$? (Assume the angle is between 0° and 180°)

Step 1: $\vec{a} \cdot \vec{b}$ by components: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$= 5.00 \times 2.00 + 9.00 \times 8.00 + 1.00 \times 3.00 = 85$

Step 2: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \rightarrow \cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{85}{\sqrt{107}\sqrt{77}} \approx 0.936$

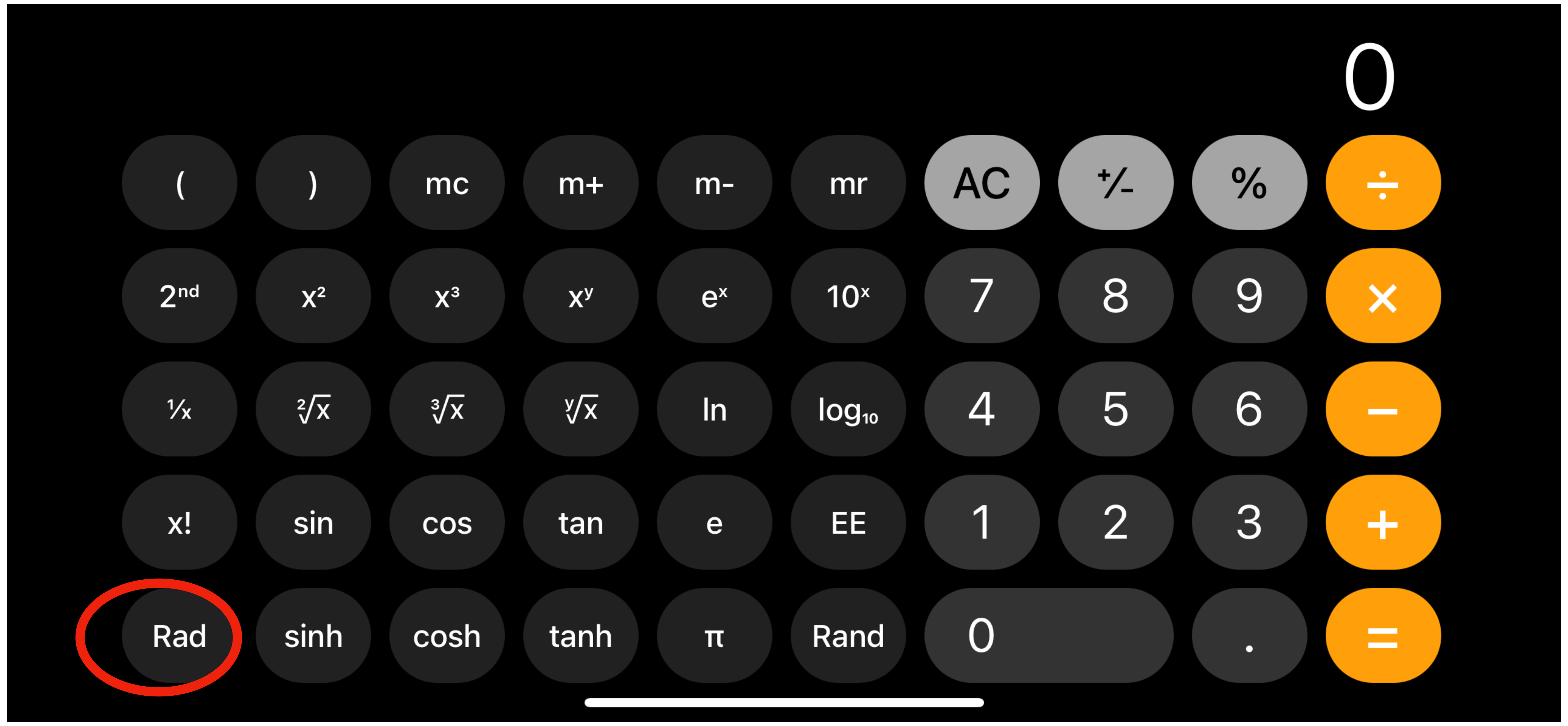
$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{5.00^2 + 9.00^2 + 1.00^2} = \sqrt{107}$

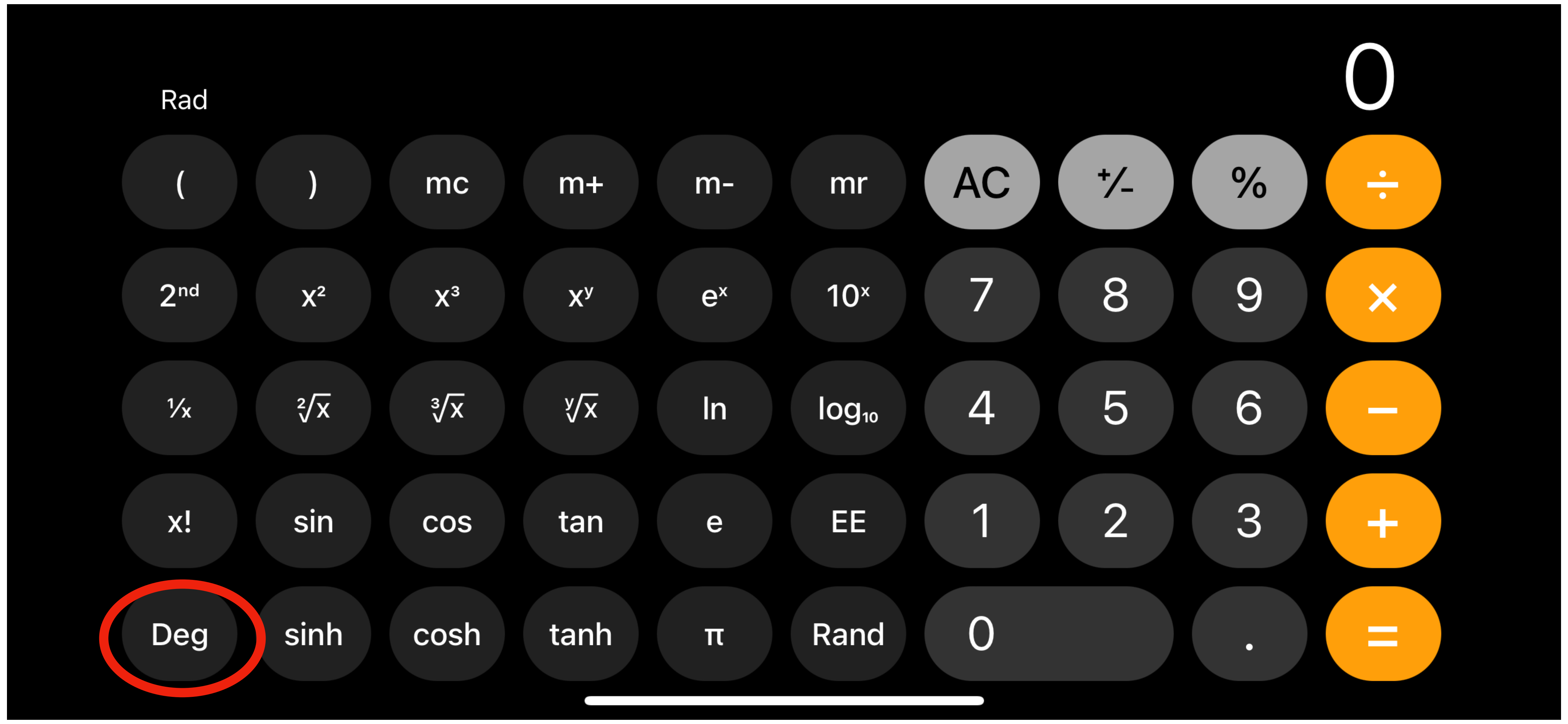
$|\vec{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2} = \sqrt{2.00^2 + 8.00^2 + 3.00^2} = \sqrt{77}$

$\theta = a\cos 0.936 \approx 20.5°$

Calculator set up: Rad or degree?

- Depends on what the question is asking for

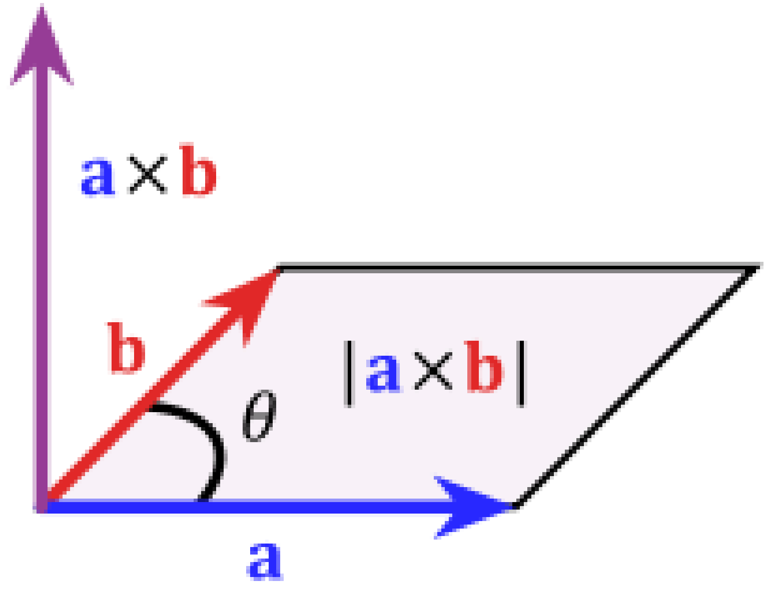




If you are unsure if your calculator is in rad or degree mode, do simple calculations to validate!

Cross product

Why is the cross-product useful?



it help us define surfaces

https://static.dezeen.com/

Vector multiplication: Cross product

Input : Vectors
Output : Vector

- Cross Product: Creates a new **vector.**

Method 1

  - In terms of vector components:

For example: $\vec{a} \times \vec{b}$

Vector

$$\boxed{\vec{a} \times \vec{b}} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

vector vector

- You don't have to memorize everything, remember the mnemonics instead!

Step 1: List the x, y, z components of vec. 1 & vec. 2 in order

Step 2: Determinant

$$\vec{a} \times \vec{b} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ a_x & a_y & a_z & a_x & a_y \\ b_x & b_y & b_z & b_x & b_y \end{array} \right.$$

Red terms are negative

Step 2: Copy 1st 2 columns to the right

Blue terms are positive

Step 3: ↘ are +, ↗ are −

Vector multiplication: Cross product

- Cross Product: Creates a new **vector.**
  - In terms of vector components:

Vector $$\boxed{\vec{a} \times \vec{b}} = |\vec{a}||\vec{b}||\sin\theta|\,\hat{u}$$

$\hat{u} \perp \vec{a}$ and $\hat{u} \perp \vec{b}$

  - In terms of geometry: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}||\sin\theta|\,\hat{u}$ $\hat{u} \perp \vec{a}$ and $\hat{u} \perp \vec{b}$

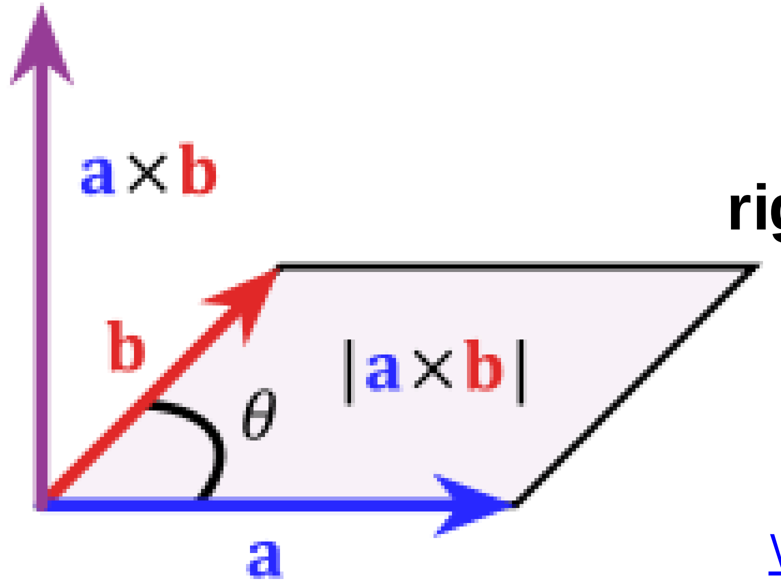
angle between $\vec{a}$ & $\vec{b}$

magnitudes of $\vec{a}$ & $\vec{b}$

unit vector

by right hand rule (RHR)

The vector product is a new vector that's **perpendicular** to each of the two vectors!



right hand rule

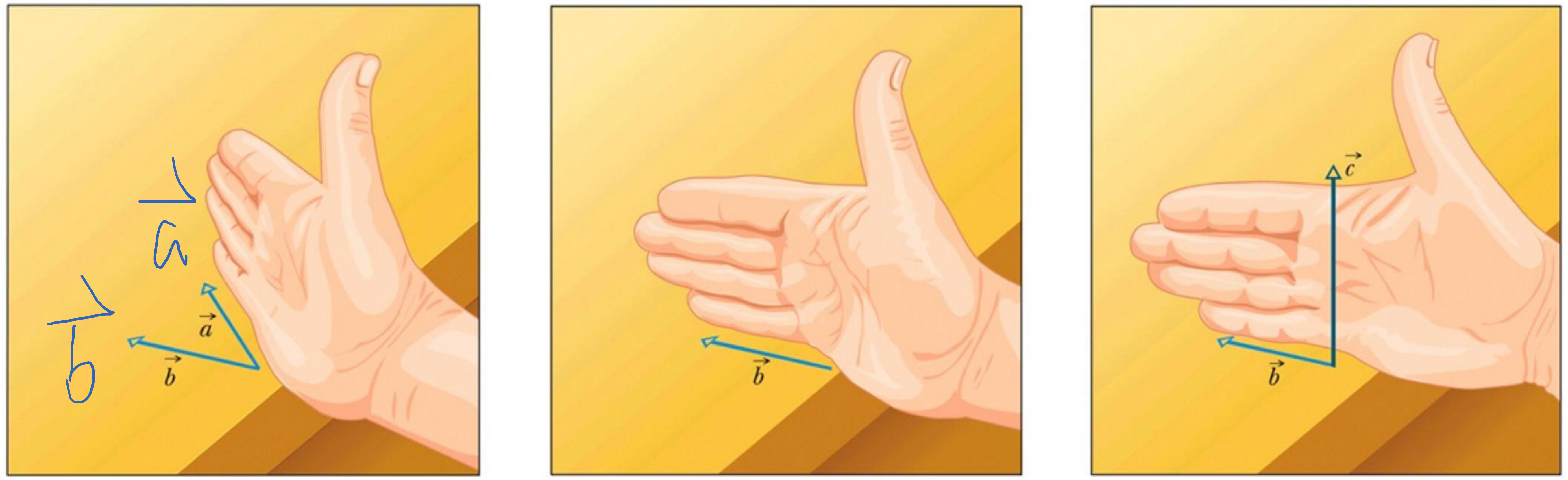
wikipedia.org

Cross product: Right-hand rule

RHR

- The direction of cross product can be determined by **right-hand rule**

$\vec{a} \times \vec{b}$



1. Point your 4 fingers to the 1st vector;
2. Curl the 4 fingers towards the 2nd vector;
3. The thumb points to the cross product.

$\vec{b} \times \vec{a}$

$$b \times a$$

<u>The order matters!</u>

Clicker question 3

- Given that the cross product $\vec{r}_1 \times \vec{r}_2 = \vec{a}$,what is the cross product $\vec{r}_2 \times \vec{r}_1$?

  Hint: Think about right hand rule.

A $\vec{r}_2 \times \vec{r}_1 = \vec{a}$

B $\vec{r}_2 \times \vec{r}_1 = -\vec{a}$

C $\vec{r}_2 \times \vec{r}_1 = \vec{r}_2$

D $\vec{r}_2 \times \vec{r}_1 = 0$

Clicker question 4

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta \ \hat{u}$$

angle between $\vec{a}$ & $\vec{b}$

RHR

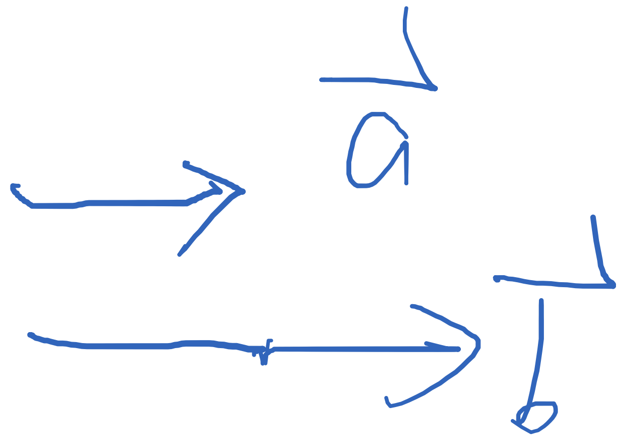
- Vectors $\vec{a} = 5.00\hat{\imath}$,$\vec{b} = 8.00\hat{\imath}$,what is $\vec{a} \times \vec{b}$?

A $40.0\ \hat{k}$

B $-40.0\ \hat{k}$

C 0

D $40.0\ \hat{\imath}$

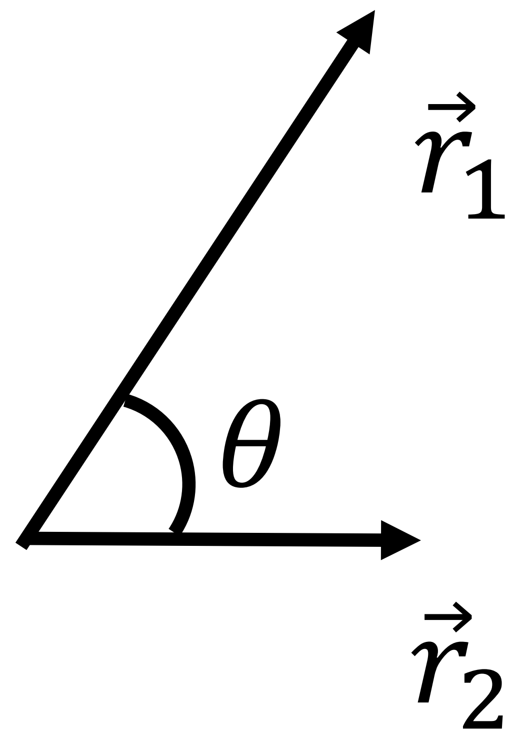


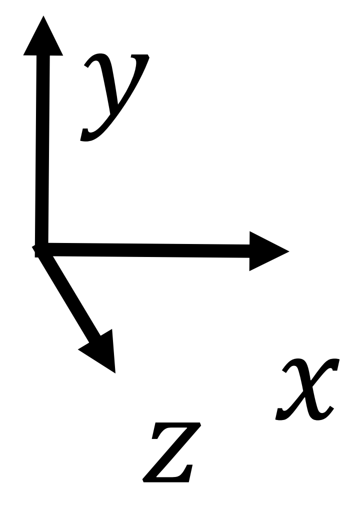
$\theta = 0°$

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{u}$$

Group activity

- The magnitude of $\vec{r}_1$ is $|\vec{r}_1| = 2.0$,and $\vec{r}_1$ is in the xy plane and is $\theta = 60°$ counterclockwise from the x-axis; the magnitude of $\vec{r}_2$ is $|\vec{r}_2| = 1.0$,and $\vec{r}_2$ is along the +x direction.
    - What is the magnitude and direction of $\vec{r}_1 \times \vec{r}_2$?
    - Please express $\vec{r}_1 \times \vec{r}_2$ in unit vector notation.





Properties of vector multiplication

- Vector scaling, dot product and cross product are **distributive** over addition:

$$c(\vec{a}+\vec{b}) = c\vec{a} + c\vec{b}$$

$$\vec{c}\cdot(\vec{a}+\vec{b}) = \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b}$$

$$\vec{c}\times(\vec{a}+\vec{b}) = \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$$

- Dot product is **commutative**

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

- Cross product is neither commutative or associative

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a},\ but \quad \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$$

Practice questions

Clicker question 5

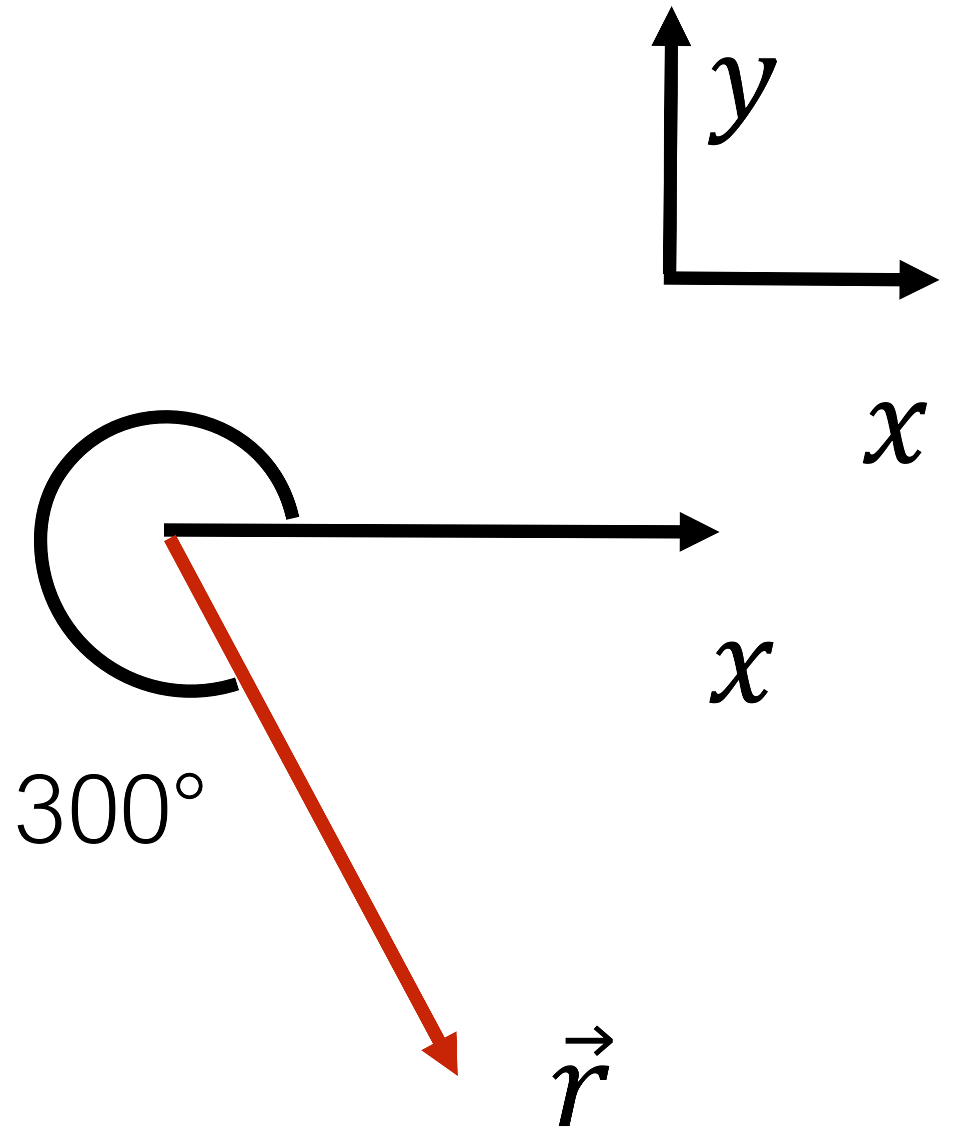
- A vector, $\vec{r}$,has a magnitude of 3.50 units, and is in the direction of $300°$ as measured counterclockwise from the positive x axis. Please find the x and y components of $\vec{r}$,r_x and r_y.

A $\vec{r} = 3.03\hat{\imath} + 1.75\hat{\jmath}$

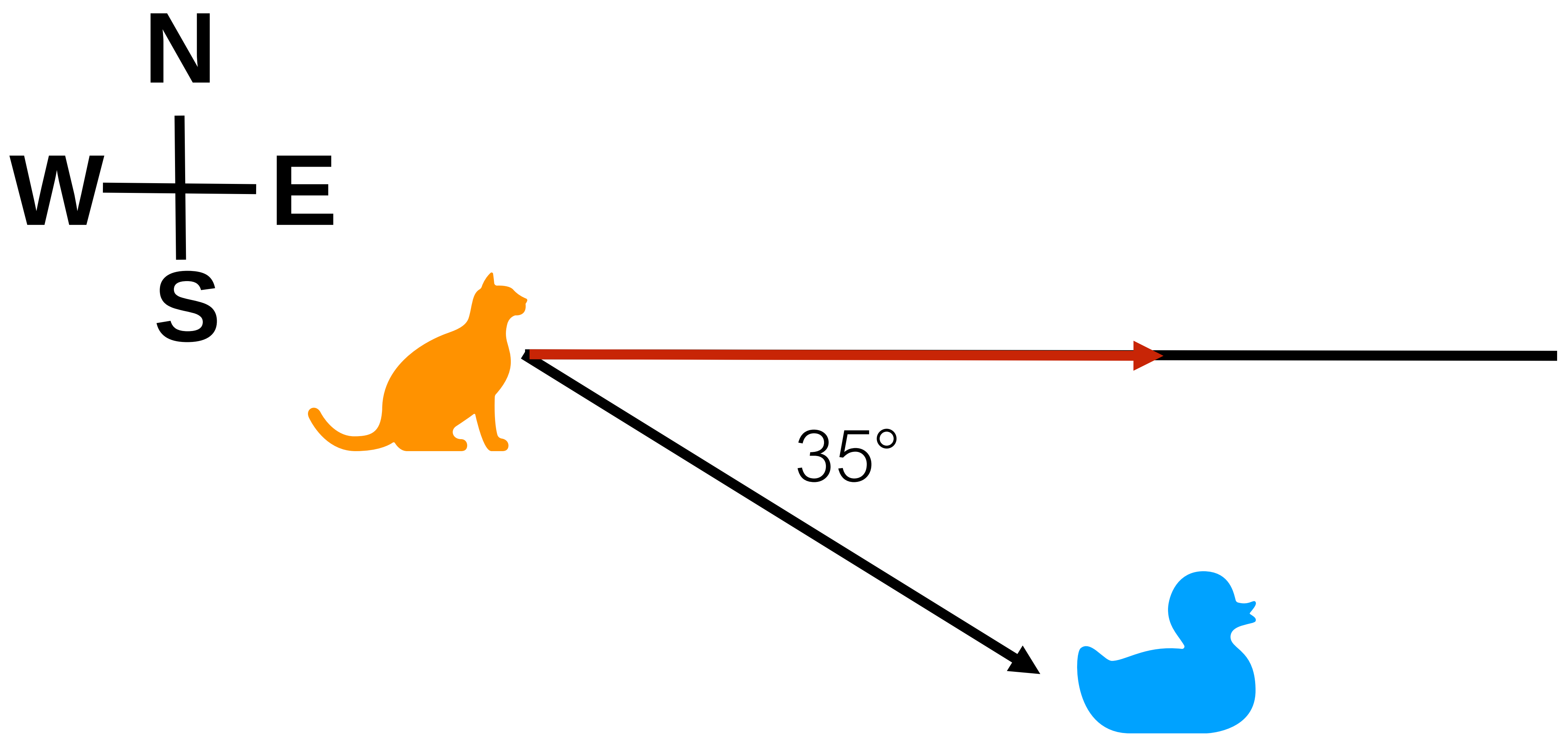
B $\vec{r} = 1.75\hat{\imath} + 3.03\hat{\jmath}$

C $\vec{r} = 1.75\hat{\imath} + (-3.03)\hat{\jmath}$

D $\vec{r} = 3.03\hat{\imath} + (-1.75)\hat{\jmath}$



Clicker question 7



- Which of the following is correct?

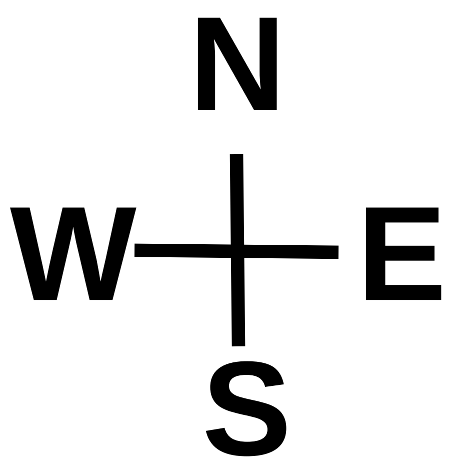
A The duck is 35° to the east of north from the cat.

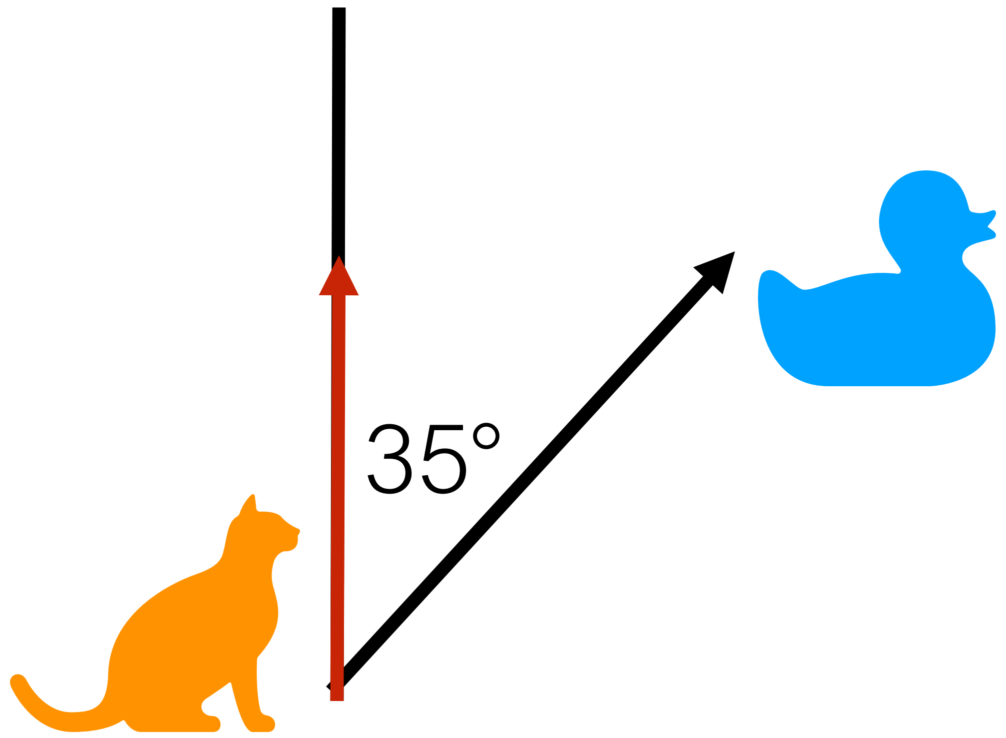
B The duck is 35° to the north of east from the cat.

C The duck is 35° to the south of east from the cat.

D The duck is 35° to the west of north from the cat.

Clicker question 8





- Which of the following is correct?

A The duck is 35° to the south of east from the cat.

B The duck is 35° to the east of north from the cat.

C The duck is 35° to the north of east from the cat.

D The duck is 35° to the west of north from the cat.

Summary of chapter 3

- Learning objectives
  - Vectors: Magnitude (size) and direction
  - Vector decomposition
  - Vector addition, vector scaling
  - Properties of vector addition: Commutative and associative
  - Vector multiplication:
    - ❖Vector scaling, vector multiplied by a scalar;
    - ❖dot product, $\vec{vector_1} \cdot \vec{vector_2}$;
    - ❖cross product, $\vec{vector_1} \times \vec{vector_2}$
  - Properties of dot product: Commutative
  - Properties of cross product: **Anti-commutative, and not associative**

Homework

- Homework assignment in Module 3.4: assignment, due in a week

Pre-lecture survey for Chapter 4, Section 1

- Pre-lecture survey: Module 4.1.1 (before the next lecture)