THE SMASHING SPECTRUM OF CYCLOTOMIC SPECTRA

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Earlier this year I tried to think about the idempotent algebras in cyclotomic spectra. It turned out to not be too hard, so I decided to write up a little blurb on it for my website, and I will be periodically updating this. We describe the smashing spectrum of cyclotomic spectra in terms of the smashing spectrum of spectra. We will treat both the genuine and naive cases simultaneously.

Our story begins with genuine $\mathbb{T}$-equivariant spectra modulo finite equivalences, $\mathbb{T}Sp_{\mathcal{F}}$, which is the localization of genuine $\mathbb{T}$-spectra at those maps which become equivalences after applying geometric fixed points for every finite subgroup of $\mathbb{T}$. (If one uses just genuine $\mathbb{T}$-spectra, the same construction creates an older form of cyclotomic spectra that has been mostly replaced with this model). Now, we construct CycSp^{gen} as in Nikolaus-Scholze, the pullback in stably symmetric monoidal ∞-categories:

$$\begin{tikzcd} \mathrm{CycSp}^{gen} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\in\mathbb{P}} \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}^{\mathbb{I}} \arrow[d, "{(p_1,p_0)}"] \\ \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \arrow[r, "{(\Phi^{C_p},id)_{p\in\mathbb{P}}}"] & \prod_{p\in\mathbb{P}} \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \times \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \end{tikzcd}$$

where here $\mathbb{I}$ denotes the free-living isomorphism. Nikolaus-Scholze also define a category of naive cyclotomic spectra, CycSp, as the category of Borel $\mathbb{T}$-equivariant spectra together with a lift of Frobenius. More formally, this is the lax equalizer

$$\mathrm{LEq}(\mathrm{Sp}^{\mathrm{B}\,\mathbb{T}} \underset{(-)^{tC_p}}{\overset{id}{\rightrightarrows}} \prod_{p\in\mathbb{P}} \mathrm{Sp}^{\mathrm{B}\,\mathbb{T}})$$

where $(-)^{tC_p}$ denotes the Tate construction (see Nikolaus-Scholze for more details).

Proposition 1. *The functor taking a cyclotomic spectrum to its underlying non-equivariant spectrum induces an equivalence* $\mathrm{Idem}(\mathrm{CycSp}^{gen}) \to \mathrm{Idem}(\mathrm{Sp})$*, and similarly for* CycSp.

Proof. We first give a proof specifically for genuine cyclotomic spectra, which does not work a priori for the naive case since the Tate construction is only lax symmetric monodial. The claim for naive cyclotomic spectra will follow from an easier argument in the next lemma. We note that the functor taking the smashing spectrum of a category (which we will call $\mathrm{Idem}(-)$ here and treat it as a poset-valued functor, with ordering as in lecture 5 of complex analytic geometry) commutes with all limits[1]. Ko Aoki has a paper proving that Idem is a right adjoint, which is one way to see that this holds, but it's also rather easy to see that Idem preserves limits directly from the definitions. In any case, this means that $\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}^{\mathbb{I}})$ is isomorphic to $\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}})$, and the above pullback diagram upon applying $\mathrm{Idem}(-)$ gives us a pullback

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[1]Where here posets are treated as categories for cotensors.

$$\begin{tikzcd}
\mathrm{Idem}(\mathrm{CycSp}^{gen}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_{p\in\mathbb{P}} \mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}) \arrow[d, "\Delta"] \\
\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}) \arrow[r, "(\Phi^{C_p}, id)"] & \prod_{p\in\mathbb{P}} \mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}).
\end{tikzcd}$$

The tensor idempotents in CycSp^{gen} are exactly those idempotent algebras A such that $\Phi^{C_p}A \simeq A$, as genuine $\mathbb{T}$-spectra with finite equivalences inverted. So, we necessarily have that for any finite subgroup G of $\mathbb{T}$, $\Phi^G A \simeq A$, and in particular this holds upon passing to underlying spectra (writing $\Phi^G A \simeq \Phi^e A$ may make this seem more clear what we are doing). It suffices to show that if A, B are such idempotent cyclotomic spectra with the same underlying spectrum, then they are equivalent. But we have maps of algebras (induced by the units) $A \to A \otimes B$, and $B \to A \otimes B$, where $\Phi^e(A\otimes B) \simeq \Phi^e(A) \otimes \Phi^e(B)$ has the same underlying spectrum as A and B since their underlying spectra are idempotent algebras. Hence, we may assume that there is a map of algebras $f : A \to B$ to prove the claim. But then $\Phi^G(f)$ (as a non-equivariant spectrum, perhaps $\Phi^e\Phi^G(f)$ is clearer notation) is an equivalence for all finite $G \subseteq \mathbb{T}$, and since we are working with finite equivalences inverted, f itself is an equivalence.

The only bit left to see that the idempotent algebras in cyclotomic spectra are in bijection with those in spectra is to actually construct such an A associated to any idempotent algebra in spectra. To see that we have these, note that Sp is initial among cocomplete stably symmetric monoidal ∞-categories where the tensor product commutes with colimits in each variable separately (where maps are colimit-preserving symmetric monoidal functors), so there is an induced map $L : \mathrm{Sp} \to \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}$. Since $\Phi^{C_n} : \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \to \mathrm{Sp}$ is symmetric monoidal and preserves colimits, so does $\Phi^{C_n} \circ L$, which is then equivalent to the identity, and we see that given any idempotent algebra A in Sp, $L(A)$ gives the desired tensor idempotent in cyclotomic spectra. $\square$

More generally, given a $\mathbb{E}_\infty$-algebra A in CycSp^{gen} (resp. CycSp), we can look at modules over A in CycSp^{gen} (CycSp), call this category $A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}}$ ($A-\mathrm{Mod}_{\mathrm{CycSp}}$).

Lemma 1. *If A is an $\mathbb{E}_\infty$-algebra in* CycSp^{gen} *(resp.* CycSp*) with underlying nonequivariant $\mathbb{E}_\infty$-algebra B, then the forgetful functor $F : A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}} \to B-\mathrm{Mod}$ ($F : A-\mathrm{Mod}_{\mathrm{CycSp}} \to B-\mathrm{Mod}$) induce injections after applying* Idem*. If A is in the image of the unique colimit-preserving symmetric monoidal functor* $\mathrm{Sp} \to \mathrm{CycSp}^{gen}$ *(resp.* $\mathrm{Sp} \to \mathrm{CycSp}$*), then this map is an isomorphism after applying* Idem*.*

Proof. Note that the functors F above are symmetric monoidal, exact, and conservative (since if a cyclotomic spectrum has underlying spectrum zero, it must itself be zero). The first claim now follows from the general fact that if we have an exact symmetric monoidal conservative functor $G : \mathcal{C} \to \mathcal{D}$ between stably symmetric monoidal ∞-categories, $\mathrm{Idem}(G)$ is injective. To see this, suppose towards a contradiction that we had idempotent algebras $A \not\simeq B$ in $\mathcal{C}$ such that $G(A) \simeq G(B)$, then in particular, since G is symmetric monoidal, $G(A\otimes B) \simeq G(A) \otimes G(B) \simeq G(A)$. Note that in virtue of being an idempotent algebra, we have a map $A \to A \otimes B$, which becomes an equivalence after applying G. But by conservativity of G, $A \simeq A \otimes B$. Similarly, $B \simeq A \otimes B$, so that $A \simeq B$ in $\mathcal{C}$.
For the second claim, if A is in the image of the unique colimit-preserving symmetric monoidal

functor $\mathrm{Sp} \to \mathcal{C}$ for $\mathcal{C} = \mathrm{CycSp}^{gen}$ or CycSp (or any cocomplete stably symmetric monoidal ∞-category with a conservative colimit-preserving symmetric monoidal functor to Sp), then we have an induced functor $B - \mathrm{Mod} \to A - \mathrm{Mod}_{\mathcal{C}}$ which composes with $A - \mathrm{Mod}_{\mathcal{C}} \to B - \mathrm{Mod}$ to an equivalence. Applying Idem shows that $\mathrm{Idem}(F)$ is a split surjection, and by the first claim, we get that $\mathrm{Idem}(F)$ is a bijection. □

Remark. In general, if we denote by C the $\mathbb{E}_\infty$-ring spectrum $\mathrm{Hom}_{\mathrm{CycSp}^{gen}}(\mathbb{S}, A)$, we have an induced symmetric monoidal cocontinuous functor $C - \mathrm{Mod} \to A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$, which induces a comparison $\mathrm{Idem}(C - \mathrm{Mod}) \to \mathrm{Idem}(B - \mathrm{Mod})$. If A is compact in A-modules in cyclotomic spectra, then the canonical functor $C - \mathrm{Mod} \to A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$ is a full embedding, and the comparison $\mathrm{Idem}(C - \mathrm{Mod}) \to \mathrm{Idem}(B - \mathrm{Mod})$ is injective, this is the case for instance, if A is an $\mathbb{E}_\infty$-algebra over $\mathrm{H}\,\mathbb{F}_p^{triv}$, by a result of Clausen-Mathew-Morrow.

It is not so clear whether or not $\mathrm{Idem}(A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}) \hookrightarrow \mathrm{Idem}(B - \mathrm{Mod})$ is an isomorphism without any conditions on A. We can at least prove this in the case $A = \mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$ with CycSp in place of $\mathrm{CycSp}^{\mathrm{gen}}$ at first (note that $\mathrm{CycSp}^{\mathrm{gen}} \to \mathrm{Sp}$ factors over CycSp, so this is a priori slightly weaker). Using a result of Dell'Ambrogio and Stanley, we know that $\mathrm{Spc}(\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod})$ is in bijection with the Zariki spectrum of the graded ring $\mathbb{F}_p[u]$ with $|u| = 2$. Thus, we can simply read off the three idempotent algebras explicitly as the unit $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$ itself, $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)[u^{-1}]$, and 0. We can invert u $\mathbb{T}$-equivariantly, since u (or rather a lift) appears in $\pi_*(\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)^{h\mathbb{T}}) \simeq \mathbb{Z}_p[u,v]/uv - p$ with $|u| = 2$, $|v| = -2$, and doing so $\mathbb{T}$-equivariantly gives an idempotent algebra R over $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$ in $\mathrm{Sp}^{B\mathbb{T}}$ with underlying spectrum $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)[u^{-1}]$, and $\pi_*(R^{h\mathbb{T}}) \simeq \mathbb{Z}_p[u, u^{-1}]$. Furthermore, it is even true that $R^{tC_p} \simeq 0$, and R thus upgrades to an idempotent algebra in $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p - \mathrm{Mod}_{\mathrm{CycSp}})$. To see this, recall from Nikolaus-Scholze that $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) \simeq sh_p(\mathrm{H}\,\mathbb{Z}_p^{triv})$, where for a connective cyclotomic spectrum X, $sh_p(X)$ is the connective cover of X^{tC_p}. In particular, by the Tate orbit lemma, $(\tau_{\geq 0}(\mathrm{H}\,\mathbb{Z}_p)_{hC_p})^{tC_p} \simeq 0$, and we can check explicitly that we have a fiber sequence on connective covers $\tau_{\geq 0}(\mathrm{H}\,\mathbb{Z}_p)_{hC_p} \to \tau_{\geq 0}\,\mathrm{H}\,\mathbb{Z}_p^{hC_p} \to \tau_{\geq 0}\,\mathrm{H}\,\mathbb{Z}_p^{tC_p}$ (which reduces to the fact that this last map is surjective on π_0). Hitting this with the Tate construction and using our observation with the Tate orbit lemma, $(\tau_{\geq 0}\,\mathrm{H}\,\mathbb{Z}_p^{hC_p})^{tC_p} \simeq (\tau_{\geq 0}\,\mathrm{H}\,\mathbb{Z}_p^{tC_p})^{tC_p}$, and $\tau_{\geq 0}\,\mathrm{H}\,\mathbb{Z}_p^{hC_p} \simeq \mathrm{H}\,\mathbb{Z}_p$. We set $R = \mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$, with the Frobenius induced by applying the Tate construction to the cyclotomic Frobenius on $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$. Now we can use the Tate orbit lemma and the Tate fixpoint lemma on $\mathrm{H}\,\mathbb{Z}_p$ to see that $(\mathrm{H}\,\mathbb{Z}_p^{tC_p})^{tC_p} \simeq 0$. Thus, the map from idempotent algebras over $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$ in (naive) cyclotomic spectra to idempotent algebras over $\Phi^e\,\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)$ is a bijection.

Remark. Note here that this idempotent algebra R is not in the full subcategory generated under colimits by the unit in $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$. Indeed, this subcategory identifies with the image of modules over the endomorphisms of the unit, $\mathrm{End}_{\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}}(\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p)) \simeq \mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)$, mapping to $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$. We have that $\pi_*(\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)) = \mathbb{Z}_p$ in degrees $0, -1$ and is zero otherwise. This is a dga over $\mathbb{Z}_p$, and by examining the explicit complex defining it, we see that there is a $\mathbb{E}_\infty$-$\mathrm{H}\,\mathbb{Z}_p$-algebra map $\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p) \to \mathrm{H}\,\mathbb{Z}_p$. The source here has a finite filtration by copies of $\mathrm{H}\,\mathbb{Z}_p$ considered as a module over it by this map, so in particular, $- \otimes_{\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)} \mathrm{H}\,\mathbb{Z}_p : \mathrm{TC}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod} \to \mathcal{D}(\mathbb{Z}_p)$ is conservative. This allows us to identify the idempotent algebras in $\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}$ as $\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)$ itself, 0, and $\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)[p^{-1}]$. Now, we can examine the image of this last idempotent under the canonical functor to $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$ composed with the forgetful functor to $\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) - \mathrm{Mod}$, which takes $\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p)[p^{-1}] = \mathrm{colim}_{\mathbb{N}}(\mathrm{TC}(\mathrm{H}\,\mathbb{F}_p) \xrightarrow{p} \mathrm{TC}(\mathrm{H}\,\mathbb{F}_p) \xrightarrow{p} \ldots)$ to $\mathrm{colim}_{\mathbb{N}}(\mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) \xrightarrow{0} \mathrm{THH}(\mathrm{H}\,\mathbb{F}_p) \xrightarrow{0} \ldots) \simeq 0$.

Questions to think about next update: Can we generalize the bijection of idempotent algebras over THH and the underlying to when R is a quasiregular semiperfectoid $\mathbb{F}_p$-algebra. Does a quasiregular semiperfectoid $\mathbb{Z}_p$ algebra in general work? What about quasisyntomic? Is there an argument to show that the comparison map on posets of idempotent algebras is a bijection whenever we start with a connective cyclotomic spectrum? Are there any examples where the map fails to be surjective? Does the idempotent R above lift to the genuine cyclotomic spectra level?