Dyer–Lashof Operations and the Homology of Infinite Loop Spaces

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1 Introduction

In algebraic topology, understanding the homology of iterated loop spaces is central to computing stable homotopy groups and cohomology theories. The **Dyer–Lashof operations** form a family of cohomology and homology operations that act on the homology of E_∞ and H_∞ ring spectra, providing powerful algebraic tools to study infinite loop spaces.

First introduced by E. Dyer and R. Lashof in the 1960s, these operations give rise to the *Dyer–Lashof algebra*, which organizes the structure of operations on the homology of free loop spaces and classifying spaces such as BO, BU, and QS^0. Further foundational contributions by J. P. May and S. Kochman deepened their role in the structure theory of spectra and loop spaces.

We present here an exposition of the definitions, algebraic structure, and applications of Dyer–Lashof operations. We also explore their interaction with spectral sequences such as the Adams and Adams–Novikov sequences, highlighting their utility in stable homotopy theory.

2 The Algebraic Framework

2.1 Definitions

Let $p = 2$. The Dyer–Lashof operations Q^i are stable homology operations defined on the homology of H_∞-spaces (infinite loop spaces). Let X be such a space. Then:

$$Q^i : H_n(X; \mathbb{F}_2) \to H_{n+i}(X; \mathbb{F}_2) \tag{1}$$

is a natural transformation satisfying Cartan-type formulas and Adem relations.

The Dyer–Lashof algebra $\mathcal{R}$ over $\mathbb{F}_2$ is the graded non-commutative algebra generated by the Q^i subject to Adem relations:

$$Q^a Q^b = \sum_j \binom{b-1-j}{2a-b-j} Q^{a+b-j} Q^j, \quad a < 2b. \tag{2}$$

2.2 Properties

- Q^0 acts as the identity.
- Q^i raises degree by i.
- $Q^i(xy) = \sum_{j+k=i} Q^j(x) Q^k(y)$ (Cartan formula).
- $Q^i(x) = 0$ for $i < \deg(x)$ (unstable condition).

These operations behave similarly to the Steenrod operations in cohomology and are dual under certain settings. Moreover, they arise naturally in the study of the homology of loop spaces and iterated classifying spaces.

2.3 Admissible Sequences and the Dyer–Lashof Algebra

An *admissible sequence* $I = (i_1, \ldots, i_k)$ satisfies $i_j \leq 2i_{j+1}$ for all j. The monomial $Q^I = Q^{i_1} \cdots Q^{i_k}$ defines a basis of $\mathcal{R}$. The Dyer–Lashof algebra is thus a graded polynomial algebra with basis given by these admissible monomials modulo the Adem relations.

3 Main Theorems and Computations

Theorem 3.1 (Kochman, 1973)**.** *Let $H_*(BO;\mathbb{F}_2)$ and $H_*(BU;\mathbb{F}_2)$ denote the mod-2 homology of the classifying spaces. Then these are free algebras over the Dyer–Lashof algebra $\mathcal{R}$ generated by appropriate polynomial generators.*

Theorem 3.2. *The homology of QS^0 with $\mathbb{F}_2$ coefficients is a polynomial algebra on classes $\{Q^I[1]\}$, where I ranges over admissible sequences.*

Let us fix $QS^0 = \Omega^\infty \Sigma^\infty S^0$. Then:

$$H_*(QS^0;\mathbb{F}_2) \cong \mathbb{F}_2[Q^I(\iota)] \tag{3}$$

where ι is the unit in degree 0 and Q^I runs over admissible monomials.

Let $x \in H_0(S^0) \cong \mathbb{F}_2$. Then $Q^i(x) \in H_i(QS^0)$. For instance,

$$\begin{aligned} Q^1(\iota) &= \text{nontrivial in } H_1(QS^0), \\ Q^2(Q^1(\iota)) &= Q^2 Q^1(\iota) \in H_3(QS^0). \end{aligned}$$

These form part of the basis elements of $H_*(QS^0)$.

4 Spectral Sequences and Applications

The Dyer–Lashof operations appear prominently in the E_2-term of the Adams spectral sequence, where they serve to define higher structure maps. Their presence provides key insights into the vanishing lines, differentials, and extensions.

In Ravenel's framework [3], the Dyer–Lashof operations underpin the structure of generalized homology theories like MU and BP. Their action can be interpreted in terms of the comodule structure of homology over the Hopf algebroid (BP_*, BP_*BP).

Proposition 4.1. *Let E be a ring spectrum with an H_∞-structure. Then the action of $\mathcal{R}$ on $H_*(E; \mathbb{F}_2)$ is compatible with the structure maps in the Adams spectral sequence.*

References

[1] S. O. Kochman, *Homology of Classical Groups over the Dyer-Lashof Algebra*, Trans. AMS 185 (1973).

[2] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, 1972.

[3] D. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, AMS, 2nd ed., 2003.

[4] E. Dyer and R. Lashof, *Homology of Iterated Loop Spaces*, Amer. J. Math. 84 (1962), 35–88.