An Exposition on Nielsen Theory and the Reidemeister Trace: Fixed Point Invariants in Algebraic Topology

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1 Introduction

The problem of determining fixed points of continuous maps lies at the heart of algebraic topology. Early results such as Brouwer's Fixed Point Theorem (1912) and Lefschetz's Fixed Point Theorem (1926) provided foundational tools for asserting the existence of fixed points based on global invariants. However, these theorems fall short of offering detailed classification, particularly in identifying fixed points that remain invariant under homotopies of the map.

During the early to mid-20th century, several attempts were made to refine Lefschetz theory. Among these were works by **Hopf** and **Lefschetz** himself, who extended their results to include traces on homology groups and algebraic Lefschetz numbers. Still, their theories could not distinguish essential fixed points from those removable by homotopy.

Jakob Nielsen (1927) pioneered a more geometric viewpoint, introducing what became known as the *Nielsen number*, which classifies fixed points into equivalence classes invariant under homotopy. **Karl Reidemeister** (1930s), in the context of algebraic topology and combinatorial group theory, later developed an equivalent approach, known today as the *Reidemeister trace*.

The culmination of these ideas leads to the *Nielsen–Reidemeister theorem*, which equates these two independently defined invariants. This refinement provides stronger data than the Lefschetz number, and it plays a central role in fixed point theory, particularly when studying maps on non-simply connected spaces.

In this paper, we formalize the Nielsen and Reidemeister invariants, prove their equivalence, and highlight their utility by situating them within homological and categorical frameworks, making reference to foundational concepts from the Adams spectral sequence, Dyer–Lashof operations, and the homotopy theory of spectra.

2 Definitions and Framework

2.1 Twisted Free Loop Space and Reidemeister Classes

Let X be a path-connected space and $f : X \to X$ a continuous map. Define the *twisted free loop space*:

$$\Lambda_f X = \{\gamma : I \to X \mid \gamma(0) = f(\gamma(1))\}. \tag{1}$$

Its path components, $\pi_0(\Lambda_f X)$, are called the *Reidemeister classes* of f.

Fix a basepoint $x_0 \in X$ and a path τ from $f(x_0)$ to x_0. Then f induces the map on the fundamental group:

$$f_\# : \pi_1(X, x_0) \to \pi_1(X, x_0), \qquad \alpha \mapsto \tau^{-1} f_*(\alpha)\tau. \tag{2}$$

Define the *twisted conjugacy* relation on $\pi_1(X, x_0)$ by:

$$\beta \sim f_\#(\alpha)\beta\alpha^{-1}, \quad \forall \alpha, \beta \in \pi_1(X, x_0). \tag{3}$$

Then:

Proposition 2.1. *There is a bijection:*

$$\pi_1(X, x_0)/\sim\ \cong\ \pi_0(\Lambda_f X).$$

Each fixed point $x \in \mathrm{Fix}(f)$ defines a constant loop at x in $\Lambda_f X$ and thus determines a Reidemeister class $[x] \in \pi_0(\Lambda_f X)$.

2.2 The Nielsen–Reidemeister Index

Let X be a compact ENR (Euclidean Neighborhood Retract), and assume $\mathrm{Fix}(f)$ is finite. For each $x \in \mathrm{Fix}(f)$, define its *local fixed point index* $I(x; f) \in \mathbb{Z}$. Then the *Nielsen–Reidemeister index* is:

$$RI(f) = \sum_{x \in \mathrm{Fix}(f)} I(x; f) \cdot [x] \in \mathbb{Z}[\pi_1(X, x_0)/\sim]. \tag{4}$$

This is a refinement of the Lefschetz number $L(f)$ and satisfies:

Proposition 2.2. *Under the augmentation map $\varepsilon : \mathbb{Z}[\pi_1(X, x_0)/\sim] \to \mathbb{Z}$, we recover the Lefschetz number:*

$$\varepsilon(RI(f)) = L(f) = \sum_{x \in \mathrm{Fix}(f)} I(x; f). \tag{5}$$

2.3 The Reidemeister Trace

Let $\pi = \pi_1(X, x_0)$ and let $\widetilde{X}$ be the universal cover of X. Fix a basepoint $\tilde{x}_0 \in \widetilde{X}$ lying over x_0. Let $\tilde{f} : \widetilde{X} \to \widetilde{X}$ be a lift of f such that $\tilde{f}(\tilde{x}_0)$ lies over $f(x_0)$.

Define the $(\mathbb{Z}[\pi], \mathbb{Z}[\pi])$-bimodule $\mathbb{Z}[\pi]_f$ where the left action is via $f_\#$ and the right action is standard multiplication.

Definition 2.3. *The* Reidemeister trace *of f is:*

$$R(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\tilde{f}_i) \in HH_0(\mathbb{Z}[\pi]; \mathbb{Z}[\pi]_f), \tag{6}$$

where $\tilde{f}_i$ is the induced map on the cellular chains $C_i(\widetilde{X})$.

3 Main Theorem and Proof

Theorem 3.1 (Nielsen–Reidemeister Theorem)**.** *Let X be a compact ENR and $f : X \to X$ a continuous map. Then:*

$$R(f) = RI(f) \in HH_0(\mathbb{Z}[\pi]; \mathbb{Z}[\pi]_f) \cong \mathbb{Z}[\pi_1(X, x_0)/\sim]. \tag{7}$$

Proof. From the definitions, both $RI(f)$ and $R(f)$ can be interpreted as classes in the same group via the isomorphism:

$$HH_0(\mathbb{Z}[\pi]; \mathbb{Z}[\pi]_f) \cong H_0(\Lambda_f X) \cong \mathbb{Z}[\pi_1(X, x_0)/\sim]. \tag{8}$$

The Reidemeister trace $R(f)$ is constructed by applying the categorical trace to the map on chains induced by $\tilde{f}$. This categorical trace sums over fixed points weighted by the group elements representing the twisted conjugacy classes.

On the other hand, the local fixed point index $I(x; f)$ contributes to $RI(f)$ precisely in the class $[x]$ corresponding to its path class.

Thus, through the natural identification of $H_0(\Lambda_f X)$ with $HH_0(\mathbb{Z}[\pi]; \mathbb{Z}[\pi]_f)$ and the geometric interpretation of both constructions, the two invariants coincide. $\square$

3.1 Example: Degree-d Maps on S^1

Let $f : S^1 \to S^1$ be defined by $f(z) = z^d$. Then $\pi_1(S^1) = \mathbb{Z}$, and $f_\#(n) = dn$. The Reidemeister classes correspond to the orbits under twisted conjugacy:

$$k \sim dn + k - n = (d-1)n + k, \tag{9}$$

so the number of Reidemeister classes is $|d - 1|$.

Each fixed point contributes an index ± 1, and these assemble into $RI(f)$ in $\mathbb{Z}[\mathbb{Z}/(d-1)\mathbb{Z}]$.

4 Connections to Spectral Sequences and Homotopy Theory

The construction and proof of the Nielsen–Reidemeister theorem can be informed by the formalism of the **Adams spectral sequence**, as developed by Aramian and Ravenel [1, 3]. In particular, injective resolutions of spectra and the functorial properties of homotopy categories provide a higher-categorical backdrop to trace theory.

Additionally, the **Dyer–Lashof algebra** acts on the homology of loop spaces, a fact exploited by Kochman [4] to analyze operadic and homological properties of classifying spaces. These ideas tie into the Hochschild homology context where the Reidemeister trace resides.

References

[1] N. Aramian, *The Adams Spectral Sequence*, Lecture Notes.

[2] *Fixed Points 2: Nielsen Theory and the Reidemeister Trace*, Course Materials.

[3] D. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, AMS, 2nd ed., 2003.

[4] S. O. Kochman, *Homology of Classical Groups over the Dyer-Lashof Algebra*, Trans. AMS 185 (1973).

[5] J. P. May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, 1992.