Classical Theorems in Stable Homotopy Theory: New and Old

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1 Introduction

Chapter 1.1 of [1] presents a rich survey of foundational theorems in algebraic topology that have proven essential in the development of stable homotopy theory. This document focuses on five landmark results: the Hurewicz theorem, the Freudenthal suspension theorem, the Serre finiteness theorem, the Nishida nilpotence theorem, and the Cohen–Moore–Neisendorfer exponent theorem.

Each of these theorems marks a crucial step in our understanding of the relationship between homotopy and homology, the behavior of loop spaces, the structure of the stable homotopy groups of spheres, and the role of torsion and nilpotence. We will explore the historical context of each result, provide detailed statements and full proofs where possible, and explain their ongoing influence in modern topology.

2 The Hurewicz Theorem (Theorem 1.1.2)

2.1 Statement

Let X be an $(n-1)$-connected space, $n \geq 2$. Then the Hurewicz homomorphism

$$h_k : \pi_k(X, x_0) \to H_k(X; \mathbb{Z})$$

is an isomorphism, and $H_i(X) = 0$ for $i < n$.

2.2 Historical Context

Proven in 1909 by Witold Hurewicz, this was among the earliest results relating homotopy and homology. It offered one of the first bridges from the more intuitive world of singular homology to the intricate structure of homotopy groups. The basic idea to prove this theorem is to consider the Postnikov decomposition of X and use the relative homotopy groups and the long exact sequences. We use the following lemma. Below, $F_j I^k$ denotes the set of j-dimensional faces of the k-cube.

2.3 Lemma

Let (X, x_0) be a pointed space.

1. For any $k \geq 1$, if $f : (S^k, p) \to (X, x_0)$ is homotopic, without fixing the base points, to a constant map, then $[f] = 0 \in \pi_k(X, x_0)$.
2. For any $k \geq 2$, let $f : \partial I^{k+1} \to X$ be a map sending every $(k-1)$-dimensional face to x_0. Then
$$[f] = \sum_{\sigma \in F_k(I^{k+1})} [f|_\sigma] \in \pi_k(X, x_0)$$

Proof. We will sketch the argument and leave the details as an exercise. (a) Given a homotopy f_t with $f_0 = f$ and f_1 constant, one can use the trajectory of p, i.e., the path $\gamma : I \to X$ sending $t \mapsto f_t(p)$, to modify this to a homotopy sending p to x_0 at all times. The endpoint of the homotopy will then be the composition of γ with a map $(S^k, p) \to (I, 0)$. Since I is contractible, we can further "homotope" this base point to a constant map. (b) We use a homotopy to shrink the restrictions of f to the k-dimensional faces of I^{k+1}, so that most of ∂I^{k+1} is mapped to x_0. Identifying $\partial I^{k+1} \cong S^k$, and using the fact that $k \geq 3$, we can then move around these k-cubes until they are lined up as int he definition of composition π_k.

Proof of the Hurewicz theorem. Using singular homology with cubes, we define a map

$$\Psi : C_k(X) \longrightarrow \pi_k(X, x_0)$$

as follows. The idea is that a generator of $C_k(X)$ is a cube whose boundary may map anywhere in X, and we have to modify it, via a chain homotopy, to obtain a cube whose boundary maps to x_0. To do so, we define a map

$$K : C_i(X) \longrightarrow C_{i+1}(X)$$

for $0 \leq i \leq k$ as follows.

Since X is path connected, for each 0-cube $p \in X$ we can choose a path $K(p)$ from x_0 to p. For each 1-cube $\sigma : I \to X$, there is a map $\partial I^2 \to X$ sending the four faces to $x_0, \sigma, K(\sigma(0))$, and $K(\sigma(1))$. Since X is simply connected, this can be extended to a map $K(\sigma) : I^2 \to X$ such that

$$\partial K(\sigma) = \sigma - K(\sigma(1)) + K(\sigma(0)).$$

Continuing by induction on i, if $1 \leq i < k$, then for each i-cube $\sigma : I^i \to X$, we can choose[4] an $(i+1)$-cube $K(\sigma) : I^{i+1} \to X$ which sends the faces to x_0, σ, and the faces of $K(\partial\sigma)$, and therefore satisfies equation (2.3). Finally, if $\sigma : I^k \to X$ is a k-cube, then there is a map $\partial I^{k+1} \to X$ sending the faces to x_0, σ, and the faces of $K(\partial\sigma)$. Let F denote the face sent to σ. Identifying $(\partial I^{k+1}/F, F) \cong (I^k, \partial I^k)$, this gives an element $[\sigma] \in \pi_k(X, x_0)$. Moreover, it is easy to see that the sum of the faces other than F is homologous to $\Psi(\sigma)$ regarded as a cube, i.e. there is a cube $K(\sigma)$ with

$$\partial K(\sigma) = \sigma - K\partial\sigma - \Phi(\Psi(\sigma)).$$

While Ψ may depend on the above choices, we claim that Ψ induces a map on homology which is inverse to Φ. To start, we claim that if $\sigma : I^{k+1} \to X$ is a $(k+1)$-cube, then

$$\Psi(\partial\sigma) = 0.$$

To see this, first note that by Lemma 2.2(b), we have $\Psi(\partial\sigma) = [f]$, where $f : \partial I^{k+1} \to X$ sends each k-dimensional face of I^{k+1} to Ψ of the corresponding face of σ. By canceling stuff along adjacent faces, f is homotopic, without fixing base points, to $\sigma|_{\partial I^{k+1}}$. This is homotopic to a constant map since it extends over I^{k+1}. So by Lemma 2.2(a) we conclude that $[f] = 0$. It follows that Ψ induces a map $\Psi_* : H_k(X) \to \pi_k(X, x_0)$.

It follows immediately that $\Phi \circ \Psi = \mathrm{id}_{H_k(X)}$. Also, we can make the choices in the definition of K so that:

> (*) If $i < k$ and if $\sigma : I^i \to X$ is a constant map to x_0, then $K(\sigma)$ is also a constant map to x_0.

It is then easy to see that $\Psi_* \circ \Phi = \mathrm{id}_{\pi_k(X,x_0)}$. □

3 The Freudenthal Suspension Theorem (Theorem 1.1.4)

3.1 Statement

Theorem 3.1 (Freudenthal). *Let X be an n-connected space. Then the suspension map*

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \leq 2n$ and a surjection for $k = 2n + 1$.

3.2 Historical Context

This result, proved by Hans Freudenthal in the 1930s, laid the groundwork for the stable homotopy category. It shows that repeated suspension "stabilizes" homotopy groups, thus introducing the modern concept of stable homotopy.

Proof. Consider the natural map $\pi_k(X) \to \pi_{k+1}(\sum X)$. If a space X is n-connected, then by the relative homotopy long exact sequence, the pair of spaces (CX, X) is $(n+1)$-connected, where CX is the reduced cone over X. Thus, we can decompose $\sum X$ as two copies of CX, say $(CX)_+, (CX)_-$, whose intersection is X. Then, homotopy excision says that the inclusion map:

$$((CX)_+, X) \subset (\sum X, (CX)_-)$$

induces isomorphisms on π_i, $i < 2n+2$ and a surjection on π_{2n+2}. From the same relative long exact sequence, $\pi_i(X) = \pi_{i+1}(CX, X)$. Recall that cones are contractible, so

$$\pi_i(\sum X, (CX)_-) = \pi_i(\sum X).$$

So, we get

$$\pi_i(X) = \pi_{i+1}((CX)_+, X) = \pi_{i+1}((\sum X, (CX)_-) = \pi_{i+1}(\sum X)$$

for $i+1 < 2n+2$. So, for $i = 2n+1$ the left and right maps are isomorphisms, regardless of how connected X is, and the middle one is a surjection by excision. Thus, the composition is a surjection as claimed. □

4 The Serre Finiteness Theorem (Theorem 1.1.8)

4.1 Statement

The homotopy group of spheres $\pi_{n+k}(S^k)$, for $k \geq 1$, are finite abelian groups, hence pure torsion, except

1. for $n = 0$ in which case $\pi_k(S^k) = \mathbf{Z}$;
2. $k = 2m$ and $n = 2m-1$ in which case

   $$\pi_{4m-1}(S^{2m}) \cong \mathbf{Z} \oplus F_m$$

   for F_m some finite group.

4.2 Historical Context

Jean-Pierre Serre proved this in the 1950s using spectral sequences, particularly the Serre spectral sequence. This theorem was a milestone in the understanding of the stable homotopy groups of spheres, establishing their torsion nature beyond degree zero. However, I don't know a freely accessible English version of this proof, so here goes.

Proof. We apply the Serre spectral sequence to the path-loop fibration:

$$\Omega S^{n+1} \to PS^{n+1} \to S^{n+1}.$$

This yields a spectral sequence in homology:

$$E^2_{p,q} = H_p(S^{n+1}; H_q(\Omega S^{n+1})) \Rightarrow H_{p+q}(PS^{n+1}) = 0.$$

The E^2 page has nonzero terms only when $p = 0$ or $p = n+1$, since $H_*(S^{n+1}) = \mathbb{Z}$ in degrees 0 and $n+1$, and 0 elsewhere. This yields:

$$E^2_{0,q} = H_q(\Omega S^{n+1}), \quad E^2_{n+1,q} = H_q(\Omega S^{n+1}).$$

Since PS^{n+1} is contractible, the spectral sequence converges to 0. Thus, all the E^2 terms must cancel in the spectral sequence. This forces relations among the $H_q(\Omega S^{n+1})$.

Now use the fact that $\pi_{k+n}(S^n) \cong \pi_k(\Omega S^n)$. One shows that $H_*(\Omega S^{n+1})$ is a finitely generated abelian group in each degree using the spectral sequence and the finite generation of homology of spheres. Then apply the Hurewicz theorem to obtain that $\pi_k(\Omega S^n)$ is finitely generated.

Inductively, one obtains that $\pi_{n+k}(S^n)$ is a finitely generated abelian group. Furthermore, for $k \geq 1$, it is a finite group except:

- When $k = 0$, $\pi_n(S^n) = \mathbb{Z}$.
- When $n = 2m$ and $k = 2m-1$, $\pi_{4m-1}(S^{2m}) = \mathbb{Z} \oplus F_m$ for finite F_m.

All other $\pi_{n+k}(S^n)$ are finite abelian groups. □

5 The Nishida Nilpotence Theorem (Theorem 1.1.9)

5.1 Statement

Every positive-degree element in the stable homotopy groups of spheres is nilpotent under composition.

5.2 Historical Context

Goro Nishida proved this in 1973, providing an astonishing structural result. This implies that despite the complicated and rich nature of the stable homotopy groups, their multiplicative structure is highly constrained.

Note that there is a stronger generalization due to Devinatz–Hopkins–Smith that states the following:

> (*) For any ring spectrum R the kernel of the canonical morphism $\pi.R \to MU.(R)$ to the MU-homology of R consists of nilpotent elements.

The Nishida theorem is the special case when the spectrum is the sphere. The MU-homologoy of the sphere spectrum is the Lazard ring and hence is torsion free, whereas all positive-degree elements of the stable homotopy ring are torsion by the Serre finiteness theorem and therefore belong to the aforementioned kernel. The full proof is long and technical—see [4], Theorem 3.

6 The Cohen–Moore–Neisendorfer Exponent Theorem (Theorem 1.1.10)

6.1 Statement

For odd primes p, there exists an exponent e such that for any finite complex X, the p-torsion in $\pi_*(\Omega^2\Sigma^2 X)$ is annihilated by p^e.

6.2 Historical Context

Proved in the late 1970s and early 1980s, this result bounds the order of p-torsion in double loop spaces. It is a deep theorem in unstable homotopy theory with significant implications for exponents and power operations. Again, the proof is long and technical—see [5].

Proof Sketch.

1. Begin by constructing a fibration:

$$\Pi_{r+1} \times C(n) \to T^{2n+1}\{p^r\} \to \Omega S^{2n+1}\{p^r\},$$

   where $C(n)$ is the homotopy fibre of the double suspension, and Π_{r+1} is a product of spaces with known geometric exponent p^{r+1}.

2. This fibration, once looped, becomes multiplicative. Since $\Omega S^{2n+1}\{p^r\}$ has geometric exponent p^r, and $C(n)$ has geometric exponent p, the total space $T^{2n+1}\{p^r\}$ has geometric exponent bounded above by p^{2r+1}.

3. However, a sharper bound is achieved using a semi-splitting argument. A map $f : \Pi_{r+1} \times S^{2n-1} \to T^{2n+1}\{p^r\}$ can be extended to a map $h : T^{2n+1}\{p^r\} \to \Pi_r$ such that $h \circ f$ factors through known maps of geometric exponent p^{r+1}.

4. The existence of such a retraction implies that the identity map on $\Omega T^{2n+1}\{p^r\}$ has order p^{r+1}. Since $\Omega^2 P^m(p^r)$ is a retract of $\Omega T^{2n+1}\{p^r\}$, it also has geometric exponent p^{r+1}.

5. Therefore, the homotopy groups $\pi_*(P^m(p^r))$ are annihilated by p^{r+1}. This exponent is sharp, as there exist infinitely many elements in the homotopy groups of $P^m(p^r)$ of exact order p^{r+1}.

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