NOTE ON PRO-ÉTALE STACKS

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1. Formalism

We discuss pro-étale stacks and their uses in classifying algebraic groups over an arbitrary field F. First, we recall that a pro-étale morphism is an inverse limit of étale morphisms of schemes, and the pro-étale site of a field is defined as

Definition 1. For a field F, the pro-étale site $F_{pro\acute{e}t}$ is the category whose objects are pro-étale morphisms over $\operatorname{Spec} F$, and where open covers are given by finite jointly surjective families of (pro-étale) maps.

Étale maps over $\operatorname{Spec} F$ are finite disjoint unions of the spectra of finite separable extensions F, so that pro-étale morphisms include, in particular, all separable extensions of F, such as a separable closure F^{sep} (which we fix once and for all), as well as profinite sets as the inverse limit of finite sets considered as disjoint unions of copies of $\operatorname{Spec} F$. In some sense, combinations of these two classes of objects are all of the objects of $F_{pro\acute{e}t}$. We now define the main object of interest

Definition 2. A stack on $F_{pro\acute{e}t}$ is a sheaf of groupoids in the pro-étale topology.[1]

Let us expand a bit on what this means. As opposed to the classical sheaf condition for $\mathcal{G}$ to be a sheaf, which is, given an open cover $\{U_i\}$ of U,

$$\mathcal{G}(U) \xrightarrow{\sim} \lim\Big(\prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j)\Big)$$

the existence of 1-morphisms in a groupoid means we have to remember one more step in the intersections (the so-called Cech nerve), i.e., the sheaf condition now becomes that we have an equivalence

$$\mathcal{G}(U) \xrightarrow{\sim} \lim\Big(\prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j) \Rrightarrow \prod_{i,j,k} \mathcal{G}(U_i \cap U_j \cap U_k)\Big). \tag{1.1}$$

Let us determine what this means explicitly in the case of the cover $\operatorname{Spec} F^{sep} \to \operatorname{Spec} F$. We note that $\operatorname{Spec} F^{sep} \times_{\operatorname{Spec} F} \operatorname{Spec} F^{sep} = \operatorname{proj\,lim}_{L/F \text{ finite Galois}} \operatorname{Spec} L \times_{\operatorname{Spec} F} \operatorname{Spec} F^{sep}$
$= \operatorname{proj\,lim}_{L/F \text{ finite Galois}} \operatorname{Gal}(L/F) \times \operatorname{Spec} F^{sep} = \Gamma \times \operatorname{Spec} F^{sep}$, where $\Gamma = \operatorname{Gal}(F^{sep}/F)$ is the absolute Galois group of F, considered here as a profinite group (the topology matters!). In this case, our sheaf condition diagram comes from the restrictions

$$\begin{tikzcd} \Gamma \times \Gamma \times \operatorname{Spec} F^{sep} \arrow[r, shift left=2, "f_{01}"] \arrow[r, "f_{02}"] \arrow[r, shift right=2, "f_{12}"'] & \Gamma \times \operatorname{Spec} F^{sep} \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] & \operatorname{Spec} F^{sep} \end{tikzcd}$$

Date: mm/dd/yyyy.

[1]We will work with this naively instead of using the straightening/unstraightening definition.

where here, $f_{01}(\gamma,\rho,x) = (\gamma,\rho(x))$, $f_{02}(\gamma,\rho,x) = (\gamma\rho,x)$, $f_{12}(\gamma,\rho,x) = (\rho,x)$, $g_0(\gamma,x) = \gamma(x)$, and $g_1(\gamma,x) = x$. For concreteness, we will deal with a specific stack of interest (although everything we say goes through in general with minimal modifications) $\mathcal{G} = Tw_G$[2], for G/F^{sep} an algebraic group, where Tw_G is the so-called stack of twisted G-forms, i.e., the groupoid of points $Tw_G(L)$ for L/F a separable extension is the groupoid of algebraic groups G'/L which become isomorphic to G after base extending to $G' \times_{\operatorname{Spec} L} \operatorname{Spec} F^{sep}$, with isomorphisms being those of algebraic groups over L.[3]

Now, in this case, the part of the diagram (1.1) inside the limit translates to

$$\mathcal{G}(F^{sep}) \rightrightarrows \prod_\Gamma \mathcal{G}(F^{sep}) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \prod_{\Gamma\times\Gamma} \mathcal{G}(F^{sep}),$$

(though, not quite, since we are lying a little bit by writing these as products, since Γ has its profinite structure, so really $\mathcal{G}(\Gamma \times \operatorname{Spec} F^{sep})$ is some filtered union over its finite quotients of products of $\mathcal{G}(F^{sep})$, which will be encoded in the preceeding discussion by requiring out assignments from Γ to be continuous). To specify an element of $\mathcal{G}(F)$, we need to take an element of the limit, meaning we need first some $H \in \mathcal{G}(F^{sep})$, and up to isomorphism, we may assume $H = G$. Now, examining the image in the (γ, x) component of the first product, $g_0^*(G) = \gamma^*(G)$, (pullback of $G \to \operatorname{Spec} F^{sep}$ along $\gamma : \operatorname{Spec} F^{sep} \to \operatorname{Spec} F^{sep}$), and $g_1^*(G) = G$, so we need some isomorphism $h_\gamma : \gamma^*G \to G$. Precomposing with the natural isomorphism $\gamma^{-1} : G \to \gamma^*G$, defined by

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\gamma^{-1}"'] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} F^{sep} \arrow[d, "\gamma^{-1}"] \\
\gamma^*(G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} F^{sep} \arrow[d, "\gamma"] \\
G \arrow[r] & \operatorname{Spec} F^{sep}
\end{tikzcd}$$

we may replace h_γ by the data of an automorphism $f_\gamma : G \to G$. Now, the last three arrows encode compatibility that these f_γ must satisfy, namely that going from vertex 0 to 1 to 2 is the same as going directly from 0 to 2. From 0 to 2 is $(\gamma\rho, x)$, which translates to the automorphism $f_{\gamma\rho} : G \to G$. From 0 to 1 we start with (γ, x), and since the map lands in $(\gamma, \rho(x))$, we pull back along ρ to the automorphism $\rho^*(\gamma^*(G)) \xrightarrow{\rho^* h_\gamma} \rho^*(G)$, and from 1 to 2 is just h_ρ, so precomposing with our $\rho^*\gamma^{-1} \circ \rho^{-1}$ at the start, and adding in $\rho^{-1} \circ \rho$ before h_ρ, going from 0 to 1 to 2 is the morphism $h_\rho \circ \rho^{-1} \circ \rho \circ \rho^*(f_\gamma) \circ \rho^{-1} = f_\rho \circ \rho \circ \rho^*(f_\gamma) \circ \rho^{-1}$. To any automorphism h of G, we can define $\rho(h) = \rho \circ \rho^*(h) \circ \rho^{-1}$, this determining a continuous action of Γ on $\operatorname{Aut}(G)$ in the discrete topology, so that the condition here, claled the cocycle condition, is $f_\rho \circ \rho(f_\gamma) = f_{\gamma\rho}$. This may look different than the ordinary cocyle condition, since really the action of Γ defined and all of the f_γ should really be how γ^{-1} act and $f_{\gamma^{-1}}$, but we did it this way for notational convenience. These give precisely the cocycles coming up in Galois cohomology, and one can show using the above

[2]Twisted G-forms are usually defined for G/F an algebraic group, and that case corresponds to taking $G_{F^{sep}}$ in our discussion.

[3]One can see that this is a stack by noting we get descent for algebraic objects defined in terms of vector spaces, tensor products, and maps between them where morphisms are those preserving these structures, this including Hopf algebras.

that the isomorphism classes of objects in $\mathcal{G}(F)$ are in bijection with the first Galois cohomology $H^1(\Gamma, \mathrm{Aut}(G))$ (see here for a more complete discussion).

2. Examples

Now we will show how to use this in practice. The general strategy is to take the algebraic group G we are looking at, and try to find a stack $\mathcal{F}$ which we understand better, with am isomorphism $\mathcal{F} \to Tw_G$. To check any morphism of this form is an isomorphism, it suffices to check this in F^{sep}-valued points, (and that it is compatible with the Γ-action), since any finite separable L/F admits a cover from F^{sep}, and the values on these points determine the value everywhere else (see section 5 of Bhatt-Scholze for more information on this topic). We will now demonstrate a couple of examples to illustrate the strategy

Examples 1) Let $G = \mathbb{G}_m$. Then, $\mathrm{Aut}(G) = \mathbb{Z}/2\mathbb{Z}$, with automorphisms being $t \mapsto t^{-1}$ and the identity. These are defined over $\mathrm{Spec}\,\mathbb{Z}$ in the isomorphism $\mathbb{G}_m \simeq \mathrm{Spec}\,F^{sep} \times \mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$, so that Γ acts trivially on the automorphism group. If we think about a scheme over $\mathrm{Spec}\,F^{sep}$ with automorphism group $\mathbb{Z}/2\mathbb{Z}$, a naive guess is the spectrum of the algebra $F^{sep} \times F^{sep}$. We define a map from the stack of quadratic étale algebras to Tw_G taking, on L-valued points, $K/L \mapsto K^\times/L^\times$ (the group taking an L-algebra R to $R_K^\times/R^\times$). On F^{sep}-valued points, every quadratic étale extension is $F^{sep} \times F^{sep}$, this algebra having automorphism group $\mathbb{Z}/2\mathbb{Z}$, and since the functor described here induced an isomorphism $\mathrm{Aut}_{q\acute{E}t(F^{sep})}(F^{sep} \times F^{sep}) \xrightarrow{\sim} \mathrm{Aut}_{Tw_G(F^{sep})}(\mathbb{G}_m)$, it is an equivalence on F^{sep}-valued points, and since it is compatible with the Γ-action, it is an isomorphism. Thus, 1-dimensional tori over a field F are in bijection with quadratic étale extensions of F. For instance, in the case $F = \mathbb{R}$, we have two up to isomorphism, $\mathbb{R} \times \mathbb{R}$, corresponding to the split torus, and $\mathbb{C}$, corresponding to the algebraic group $\mathbb{C}^\times/\mathbb{R}^\times$, the circle (also described as $U(1)$ or $SO(2)$, with coordinate ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$).

2) In the case of $\mathrm{SL}_2 = G$, one can show that $\mathrm{Aut}(G) = \mathrm{PGL}_2(F^{sep}) = \mathrm{Aut}(M_2(F))$. Taking the stack of degree 2 (so dimension 4) central simple F-algebras (equivalently quaternion F-algebras), we can map this to Tw_G taking A to the algebraic group $\mathrm{SL}_1(A)$, the kernel of the reduced norm map $Nrd : \mathrm{GL}_1(A) \to \mathbb{G}_m$, where $\mathrm{GL}_1(A)(R) = (A_R)^\times$ for an F-algebra R. This can be checked to similarly give an isomorphism of stacks, classifying all twisted forms of SL_2. In the case $F = \mathbb{R}$, there are two (up to isomorphism) degree 2 central simple $\mathbb{R}$-algebras, $M_2(\mathbb{R})$, corresponding to $\mathrm{SL}_2(\mathbb{R})$, and the Hamilton quaternions $\mathbb{H}$, corresponding to $\mathrm{SL}_1(\mathbb{H})$, which has $\mathbb{R}$-valued points just the 3-sphere S^3, the unit sphere of $\mathbb{H}$.

For more results like this, see section 26 of the Book of Involutions, where a similar strategy (although the language is mildly different, it amounts to the same thing) is used to classify semi-simple groups over an arbitrary field F.