TOWARDS THE NERVES OF STEEL CONJECTURE

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Abstract. Given a local $\otimes$-triangulated category, and a fiber sequence $y \xrightarrow{g} \mathbb{1} \xrightarrow{f} x$, one may ask if there is always a nonzero object z such that either $z \otimes f$ or $z \otimes g$ is $\otimes$-nilpotent. The claim that this property holds for all local $\otimes$-triangulated categories is equivalent to Balmer's "nerves of steel conjecture" [Bal, Remark 5.15]. In the present paper, we will see how this property can fail if the category we start with is not rigid, discuss a large class of categories where the property holds, and ultimately prove that the nerves of steel conjecture is equivalent to a stronger form of this property.

1. Introduction

In [Bal, Remark 5.15], Balmer had nerves of steel so as not to conjecture that the comparison map between the homological spectrum and the Balmer spectrum of a $\otimes$-triangulated category (tt-category) $\mathcal{T}$ is always an isomorphism. Despite his best efforts, this conjecture has come to be known as Balmer's "nerves of steel conjecture." The problem we study takes a slightly different form, but is in much the same spirit. To begin, let us make the following definition.

Definition 1.1. We say that the *exact-nilpotence condition* holds for a local tt-category $\mathcal{T}$ if whenever we have a fiber sequence

$$y \xrightarrow{g} \mathbb{1} \xrightarrow{f} x,$$

there exists a nonzero object $z \in \mathcal{T}$ such that either $z \otimes g$ or $z \otimes f$ is $\otimes$-nilpotent.

The connection between this property and the nerves of steel conjecture follows from further work of Balmer summarized in the following theorem.

Theorem 1.2 ([Bal20])**.** *The nerves of steel conjecture holds if and only if the exact-nilpotence condition holds for every tt-category $\mathcal{T}$.*

Proof. It follows from [Bal20, Theorem A.1] that the nerves of steel conjecture is equivalent to the statement that in any local tt-category $\mathcal{T}$, given any morphisms f, g with $f \otimes g \simeq 0$, there is some nonzero object z with $z \otimes f$ or $z \otimes g$ is $\otimes$-nilpotent. If we have a fiber sequence with f and g as in Definition 1.1, then $f \otimes g \simeq 0$, proving one direction. Conversely, assume the exact-nilpotence condition always holds, and take any $g, f \in \mathcal{T}$ with $f \otimes g \simeq 0$. We may assume without loss of generality that $g : \mathbb{1} \to y$ and $f : \mathbb{1} \to x$. Then, $(f \otimes id_y) \circ g \simeq 0$, so that g factors over $\mathrm{fib}(f) \otimes y \to y$, and the exact-nilpotence condition applied to the fiber sequence $\mathrm{fib}(f) \to \mathbb{1} \to x$ proves the claim. $\square$

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In §2, we will show that if one drops the requirement that $\mathcal{T}$ be rigid, the exact-nilpotence condition can fail. This is proved by looking at free constructions. In particular, we prove,

Theorem 1.3. *If $\mathcal{T}$ is the subcategory of compact objects in the free stably symmetric monoidal stable ∞-category on an object with a map from the unit over any local tt-category, then $\mathcal{T}$ is a local category for which the exact-nilpotence condition is false.*

On the positive side, we prove in §3 that the exact-nilpotence condition holds for a large class of local tt-categories which are generated by their unit. The class of such tt-categories is closed under filtered colimits with local transition maps. We will discuss many categories where the condition is known to hold, and add onto this by showing, for instance:

Proposition 1.4. *If R is a connective $\mathbb{E}_\infty$-ring such that $\pi_*(R)$ is a local ring, then $\mathcal{C} = \mathrm{Mod}_R^{perf}$ is a local tt category. If $\pi_*(R)$ is a Noetherian ring, then the exact-nilpotence condition holds for $\mathcal{C}$.*

This extends to showing that the exact-nilpotence condition holds for the category Mod_R^{perf} over any connective rational $\mathbb{E}_\infty$-ring R with $\pi_0(R)$ a local ring, the proof of which takes the majority of the section.

In the final section, we slightly strengthen Theorem 1.2, to the following claim

Theorem 1.5. *The following are equivalent,*

- *The nerves of steel conjecture holds.*
- *For every local tt-category $\mathcal{T}$, the exact-nilpotence condition holds.*
- *There exists an integer n such that for every local tt-category $\mathcal{T}$, and any fiber sequence as in Definition 1.1, there exists a nonzero object $z \in \mathcal{T}$ such that either $z \otimes g^{\otimes n} \simeq 0$ or $z \otimes f^{\otimes n} \simeq 0$.*

1.A. **Conventions.** For the purposes of this paper, we will often use ∞-categorical language, following Lurie [Lur08][Lur17]. We list off some definitions that will be used throughout the paper.

Definition 1.6. We say that a symmetric monoidal stable ∞-category $\mathcal{C}$ is *local* if it is nonzero, and given $x, y \in \mathcal{C}$ with $x \otimes y \simeq 0$, then either $x \simeq 0$ or $y \simeq 0$.

Definition 1.7. The term *tt-category* will refer to an idempotent complete rigid symmetric monoidal stable ∞-category, with a *local tt category* being a tt-category which is also local.

Definition 1.8. A symmetric monoidal stable functor $F : \mathcal{C} \to \mathcal{D}$ between local tt-categories will itself be called *local* if given $c \in \mathcal{C}$, $F(c) \simeq 0$ iff $c \simeq 0$.

When working with $\mathbb{E}_\infty$-rings over a characteristic zero field k, we will denote by $\Lambda_k[var_{2j+1}]$ the free $\mathbb{E}_\infty$-k-algebra on a class in degree $2j+1$, where var will be a variable name. We similarly denote by $k[var_{2n}]$ the free $\mathbb{E}_\infty$-k-algebra on a class in degree $2j$. If R is an arbitrary $\mathbb{E}_\infty$-k-algebra with a class $y_{2n} \in \pi_{2n}(R)$, we denote by R/y_{2n} the ring $R \otimes_{k[y_{2n}]} k$, which identifies with the cofiber $\mathrm{cofib}(y_{2n} : \Sigma^{2n} R \to R)$.

Definition 1.9. When working with an $\mathbb{E}_\infty$-ring R, we write Mod_R^{perf} for the category of compact objects in the category Mod_R of R-module spectra. We often will abbreviate and write $\mathrm{Spc}(R)$ for the Balmer spectrum $\mathrm{Spc}(\mathrm{Mod}_R^{perf})$ of this category.

1.B. **Acknowledgments.** Do Later

2. The Non-Rigid Case

We begin this section with a review of universal constructions to motivate what is to come. The uninterested reader may skip straight to §2.B for the counterexample to the exact-nilpotence condition if rigidity is dropped.

In the following, when working in the category Fin_* of finite pointed sets, we write $\langle n\rangle$ for the pointed set $\{*\}\coprod\{1,\dots,n\}$, pointed at $*$.

2.A. Universal constructions.

Given a symmetric monoidal ∞-category $\mathcal{C}$, pointed objects $\mathbb{1}\to x$ in $\mathcal{C}$ may be viewed as algebra objects over the ∞-operad $\mathrm{Poi}^{\otimes}\simeq\mathbb{E}_0$. This operad is defined as the subcategory of finite pointed sets Fin_* containing all objects, and morphisms are maps $f:*\coprod S\to *\coprod T$ such that $|f^{-1}(t)|\leq 1$ for all $t\in T$ [Lur17, Example 2.1.1.19]. As in [Lur17, Construction 2.2.4.1], we can form the monoidal envelope

$$\mathrm{Env}(\mathbb{E}_0):=\mathbb{E}_0\times_{\mathrm{Fun}(\{0\},\mathrm{Fin}_*)}\mathrm{Act}(\mathrm{Fin}_*)_{/\langle 1\rangle},$$

where $\mathrm{Act}(\mathrm{Fin}_*)_{/\langle 1\rangle}$ is the category of active morphisms in Fin_* with target $\{*\}\coprod\{1\}$.[1] There is a unique active morphism from any object to $\langle 1\rangle$, which provides us with an identification of $\mathrm{Env}(\mathbb{E}_0)$ with the category Fin^{inj} of finite sets and injective maps between them. Summarizing, we have:

Lemma 2.1. *The free symmetric monoidal ∞-category on a pointed object is given by the category Fin^{inj} of finite sets with injective maps.*

□

Now, to arrive at the free situation in tt geometry, we proceed as follows. Let $\mathcal{T}$ be an idempotent complete symmetric monoidal stable ∞-category. By [Lur17, Lemma 5.3.2.11], there is an equivalence between the category of such categories (with symmetric monoidal exact functors between them) and the category of compactly generated presentably symmetric monoidal stable ∞-categories. This equivalence leads us to study the following:

Lemma 2.2 ([Lur17])**.** *The free presentably symmetric monoidal stable ∞-category over $\mathrm{Ind}(\mathcal{T})$ on a pointed object is the category $\mathrm{Fun}((\mathrm{Fin}^{inj})^{op},\mathrm{Ind}(\mathcal{T}))$ of $\mathrm{Ind}(\mathcal{T})$-valued presheaves on finite sets with injective maps. This category is equipped with the Day convolution product.*

Proof. Given a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, a map $\mathrm{Ind}(\mathcal{T})\to\mathcal{C}$, and a pointed object classified by a symmetric monoidal functor $x:\mathrm{Fin}^{inj}\to\mathcal{C}$, left Kan extension of x along the Yoneda embedding $\mathrm{Fin}^{inj}\to\mathrm{Fun}((\mathrm{Fin}^{inj})^{op},\mathrm{Ind}(\mathcal{T}))$ the map displaying the desired universal property. □

For the purposes of this paper, we work with small categories, so what we care about is the companion statement:

Corollary 2.3. *The free idempotent complete symmetric monoidal stable ∞-category over such a category $\mathcal{T}$ is the category*

$$\mathrm{Fun}((\mathrm{Fin}^{inj})^{op},\mathrm{Ind}(\mathcal{T}))^{\omega}$$

of compact objects in $\mathrm{Fun}((\mathrm{Fin}^{inj})^{op},\mathrm{Ind}(\mathcal{T}))$.

[1] Recall that a morphism f is active if $f^{-1}(*)=*$.

□

It will be convenient to introduce the following notation.

Notation 2.1. Given a small symmetric monoidal ∞-category $\mathcal{C}$, denote the category $\mathrm{Fun}((\mathrm{Fin}^{inj})^{op}, \mathrm{Ind}(\mathcal{C}))^{\omega}$ considered above by $\mathcal{C}[\mathrm{Poi}]$. Additionally, we denote by $\mathcal{C}[X]$ the category $\mathrm{Fun}((\mathrm{Fin}^{\simeq})^{op}, \mathrm{Ind}(\mathcal{C}))^{\omega}$ of compact objects in $\mathrm{Ind}(\mathcal{C})$-valued presheaves on the category of finite sets and bijective maps.

Remark 2.4. The second category appearing above, $\mathcal{C}[X]$, can be shown to be the free idempotent complete symmetric monoidal stable ∞-category over $\mathcal{C}$ on an object. This serves as a sort of "non-rigid" tt-affine line, which the notation was chosen to reflect.

2.B. **The Counterexample.** We come to the main theorem of this section.

Theorem 2.5. *Let $\mathcal{C}$ be a local idempotent complete symmetric monoidal stable ∞-category. Then the category $\mathcal{C}[\mathrm{Poi}]$ described in the previous section is a local symmetric monoidal stable ∞-category for which the exact-nilpotence condition fails.*

Before proving this theorem, we must first embark on a journey to study the "affine line" $\mathcal{C}[X]$ over $\mathcal{C}$.

Proposition 2.6. *If $\mathcal{C}$ is a local idempotent complete symmetric monoidal stable ∞-category, then so too is $\mathcal{C}[X]$.*

Proof. Note that the category $\mathcal{C}[X]$ decomposes as a sum

$$\mathcal{C}[X] \simeq \coprod_{n\geq 0} \mathrm{Fun}(B\Sigma_n, \mathrm{Ind}(\mathcal{C}))^{\omega},$$

such that every object $f \in \mathcal{C}[X]$ factors as a finite direct sum $f \simeq \oplus_{n\geq 0} f_n$, where $f_n : (\mathrm{Fin}^{\simeq})^{op} \to \mathcal{C}$ are presheaves with the property that $f_n(S) = 0$ if $|S| \neq n$. To show that $\mathcal{C}[X]$ is local, it suffices to show that any two nonzero objects of the form $f = f_n : B\Sigma_n \to \mathcal{C}$ and $g = g_m : B\Sigma_m \to \mathcal{C}$ have nonzero tensor product. Given two such nonzero objects f and g, the object

$$f \otimes g : B\Sigma_n \times B\Sigma_m \to \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

is nonzero, since $\mathcal{C}$ is local. The tensor product $f \otimes_{\mathcal{C}[X]} g$, given by Day convolution, is described through inducing this representation $\mathrm{Ind}_{B\Sigma_n \times B\Sigma_m}^{B\Sigma_{m+n}}(f \otimes g)$.

Thus, our claim reduces to showing that the functor

$$\mathrm{Ind}_{B\Sigma_n \times B\Sigma_m}^{B\Sigma_{m+n}} : \mathcal{C}^{B\Sigma_n \times B\Sigma_m} \to \mathcal{C}^{B\Sigma_{n+m}}$$

is faithful on objects. This follows from the description

$$\mathrm{Ind}_{B\Sigma_n \times B\Sigma_m}^{B\Sigma_{m+n}}(-) \simeq \mathbb{1}[\Sigma_{n+m}] \otimes_{\mathbb{1}[\Sigma_n \times \Sigma_m]} -,$$

and the fact that $\mathbb{1}[\Sigma_{n+m}]$ is a direct sum of copies of $\mathbb{1}[\Sigma_n \times \Sigma_m]$. □

Remark 2.7. For general $\mathcal{C}$, not every Σ_n-equivariant object of $\mathcal{C}$ is compact as an object of $\mathrm{Fun}(B\Sigma_n, \mathrm{Ind}(\mathcal{C}))$, the former category being possibly larger. However, the above proof still goes through if we work with the larger category $\coprod_{n\geq 0} \mathcal{C}^{B\Sigma_n}$, which could be used equally well in place of $\mathcal{C}[X]$ for what follows.

Proof of Theorem 2.5. First, note that the inclusion $\mathrm{Fin}^{\simeq} \to \mathrm{Fin}^{inj}$ gives rise to a functor $\mathrm{Res} : \mathcal{C}[\mathrm{Poi}] \to \mathcal{C}[X]$. It is easy to see that this functor exists at the level of presentable categories. In order to see that it actually takes compact objects to compact objects, select a choice of compact generators $\{c_i\}_{i \in I}$ for $\mathcal{C}$. Then, the collection $\{よ([n]) \otimes_{\mathcal{C}} c_i\}_{i \in I, n \in \mathbb{N}}$ for naturals n (where $[n]$ is the n-element set $\{1, ..., n\}$, and $よ : \mathrm{Fin}^{inj} \to \mathcal{C}[\mathrm{Poi}]$ is the Yoneda embedding) forms a collection of compact generators for the category $\mathcal{C}[\mathrm{Poi}]$. In particular, $\mathcal{C}[\mathrm{Poi}]$ identifies with the thick stable closure of this collection in the category $\mathrm{Fun}((\mathrm{Fin}^{inj})^{op}, \mathrm{Ind}\,\mathcal{C})$, so to see that any object of $\mathcal{C}[\mathrm{Poi}]$ lands in $\mathcal{C}[X]$, it suffices to test on these test objects, where it can be seen directly.

A priori, Res need not be symmetric monoidal. However, in our case, it actually is. Since this functor arises as the right adjoint of a symmetric monoidal functor (which classifies the free object of $\mathcal{C}[\mathrm{Poi}]$), Res is lax symmetric monoidal, so our goal is to show that this lax structure can be promoted to a genuine symmetric monoidal structure. Consider the full subcategory $\mathcal{T}$ of $\mathcal{C}[\mathrm{Poi}] \times \mathcal{C}[\mathrm{Poi}]$ consisting of all pairs (f, g) for which the natural map $Res(f) \otimes Res(g) \to Res(f \otimes g)$ is an equivalence. The category $\mathcal{T}$ is a thick stable subcategory of $\mathcal{C}[\mathrm{Poi}]$, so we can use our test objects $\{よ([n]) \otimes_{\mathcal{C}} c_i\}$ from before. Note that $\mathrm{Res}(よ([0])) \simeq よ([0])$, such that Res takes the unit to the unit. Using that $よ([1])$ generates the Yoneda image under tensor powers, we can reduce to checking that tensor powers of $よ([1])$ map to tensor powers of $\mathrm{Res}(よ([1]))$. The object $\mathrm{Res}(よ[1])$ is given as $よ[0] \oplus よ[1] \in \mathcal{C}[X]$. Thus,

$$\mathrm{Res}(1)^{\otimes m} \simeq \oplus_{n \geq 0} よ([n])^{\oplus \binom{m}{n}},$$

whereas,

$$\mathrm{Res}((よ([1]))^{\otimes m}) \simeq \mathrm{Res}(よ([m])) \simeq \oplus_{n \geq 0} \mathrm{Fin}^{inj}([n], [m]) \simeq \oplus_{n \geq 0} よ([n])^{\oplus \binom{m}{n}},$$

and the map provided by the lax symmetric monoidal structure is an equivalence.

Now, since Res is faithful on objects, we find that $\mathcal{C}[\mathrm{Poi}]$ is local by Proposition 2.6. Note that $\mathrm{Res}(よ([0]) \to よ([1]))$ is split, so the free map cannot become $\otimes$-nilpotent after tensoring with a nonzero object of $\mathcal{C}[\mathrm{Poi}]$. Now, note that there is a canonical equivalence $\mathcal{C}^{op}[\mathrm{Poi}] \simeq \mathcal{C}[\mathrm{Poi}]^{op}$ taking the free map in $\mathcal{C}^{op}[\mathrm{Poi}]$ to the fiber of the free map $\mathbb{1} \to よ([1])$ in $\mathcal{C}[\mathrm{Poi}]$. In particular, the above proof run for $\mathcal{C}^{op}$ in place of $\mathcal{C}$ shows that there cannot be any nonzero object in $\mathcal{C}$ such that $\mathrm{fib}(\mathbb{1} \to よ[1])$ becomes $\otimes$-nilpotent after tensoring with this object. Thus, the exact-nilpotence condition fails for $\mathcal{C}[\mathrm{Poi}]$, as claimed. $\square$

Remark 2.8. It is tempting to attempt to perform a similar proof in the rigid setup. The first obstruction to this idea is the fact that the free tt-category on a pointed object over a given tt-category $\mathcal{C}$ is inherently rather complicated. Instead of a presheaf category on finite sets and injective maps, one has a presheaf category on a variant of the category of oriented 1-dimensional cobordisms where one allows half-open intervals. Further complicating matters, this category is not going to be local, and its spectrum is rather complicated, similar to the spectrum of the rigid tt affine line, which will be studied in forthcoming work of cite Tobias, Greg, Tomer, Ko and Anish. Nevertheless, if we understood all of the local categories in the free case over the category of spectra, this would lead either to a proof of the nerves of steel conjecture or a counterexample.

3. Some Positive Results

Although the exact-nilpotence condition does fail without rigidity, it may yet be the case that it always holds when working in a tt-category. As a first step towards the goal of a general proof, we present a number of cases where the condition holds, summarized by the following theorem.

Theorem 3.1. *The class of local tt-categories for which the exact-nilpotence condition holds is closed under filtered colimits along local transition maps, and contains the following classes of tt-categories*

- *Localizations of the categories* $(\mathrm{Sp}_G^{gen})^\omega$ *of compact objects in the category spectra of genuine equivariant* G*-spectra for a compact Lie group* G*. [Bal, Corollary 5.10]*
- *The category* Mod_R^{perf} *of perfect complexes over an ordinary local ring* R*. [Bal, Corollary 5.11]*
- *Localizations of the stable category of finite-dimensional Lie superalgebra representations* $\mathrm{stab}(\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\overline{0}})})$ *over a complex Lie superalgebra. [HN24]*
- *The category of perfect modules over a connective* $\mathbb{E}_\infty$*-ring* R *such that* $\pi_*(R)$ *is a Noetherian ring and* $\pi_0(R)$ *is local. (Proposition 3.2).*
- *The category of perfect modules over a rational connective* $\mathbb{E}_\infty$*-ring* R *with* $\pi_0(R)$ *local.*

Proof. To prove the first claim, let $\mathcal{C} := \varinjlim_{i\in I}\mathcal{C}_i$ be a filtered colimit along local transition maps of local tt-categories for which the exact-nilpotence condition holds. Consider a fiber sequence

$$y \xrightarrow{g} \mathbb{1} \xrightarrow{f} x$$

in $\mathcal{C}$. Then, there exists some $i \in I$ and some $x' \in \mathcal{C}_i$ such that x is a summand of the image of x'. Up to adding the complementary summand to x and y (which won't affect the nilpotence of f, g tensored with a nonzero object), we may assume that $x = x'$. There is some $j \in I$, $j > i$, and a map $\mathbb{1} \xrightarrow{f'} x'$ in $\mathcal{C}_j$ having image $\mathbb{1} \xrightarrow{f} x$ in $\mathcal{C}$. Taking $0 \neq z \in \mathcal{C}_j$ such that $z \otimes f'$ or $z \otimes \mathrm{fib}(f')$ is $\otimes$-nilpotent, the image of z has this same property in $\mathcal{C}$. As the transition maps between the $\mathcal{C}_i$ were local, the image of z in $\mathcal{C}$ is nonzero, and the exact-nilpotence condition holds for $\mathcal{C}$.

For the final bullet-point, let R be any connective rational $\mathbb{E}_\infty$-ring with $\pi_0(R)$ local. Then, R can be written as a filtered colimit of localizations of finitely presented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebras, by first writing R as an arbitrary filtered colimit of finitely presented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebras, and localizing every term S in the colimit at the pullback to $\pi_0(S)$ of the maximal ideal in $\pi_0(R)$. The proof of Proposition 3.2 reveals that a map $R \to S$ of connective local $\mathbb{E}_\infty$-rings induces a local map on their categories of perfect modules if $\pi_0(R) \to \pi_0(S)$ is a local ring homomorphism. By [Lur17, Corollary 4.8.5.13], this gives a presentation of Mod_R^{perf} as a filtered colimit along local transition maps of categories Mod_S^{perf} with S the localization of a finitely presented connective rational $\mathbb{E}_\infty$-ring, so we may assume that R has these same properties by the first claim.

Note that $\pi_0(R)$ is a Noetherian ring, hence we can write the maximal ideal as $m = (x_1, \ldots, x_n)$. The $\mathbb{E}_\infty$-ring $R/(x_1, \ldots, x_n)$ is a connective, finitely presented

$\mathbb{E}_\infty$-$\pi_0(R)/m$-algebra, with $\pi_0(R)/m$ a field. $R/(x_1, \ldots, x_n)$ is also a perfect R-module, so the exact-nilpotence condition holds for Mod_R^{perf} if and only if it holds for $\mathrm{Mod}_{R/(x_1,\ldots,x_n)}^{perf}$, where the claim follows by Theorem 3.12. $\square$

Proposition 3.2. *If R is a connective $\mathbb{E}_\infty$-ring such that $\pi_0(R)$ is a local ring, then $\mathcal{C} = \mathrm{Mod}_R^{perf}$ is a local tt category. If $\pi_*(R)$ is a Noetherian ring, then the exact-nilpotence condition holds for $\mathcal{C}$.*

Proof. Since R is a connective $\mathbb{E}_\infty$-ring, [Lur17, Proposition 7.1.3.15] shows that $\pi_0(R) = \tau_{\leq 0}R$ is canonically an $\mathbb{E}_\infty$-R-algebra. Writing k for the residue field of $\pi_0(R)$, this supplies us with a symmetric monoidal functor $- \otimes_R k : \mathrm{Mod}_R^{perf} \to \mathcal{D}^b(k)$. Since the target is local, in order to see that Mod_R^{perf} is local, it suffices to show that if $M \otimes_R k \simeq 0$, then $M \simeq 0$. Since $M \otimes_R \pi_0(R)$ is a perfect R-module, $M \otimes_R k \simeq 0$ is equivalent to $M \otimes_R \pi_0(R) \simeq 0$. From the fiber sequence

$$\tau_{\geq 1}R \to R \to \pi_0(R),$$

we obtain a fiber sequence, for any R-module M,

$$M \otimes_R \tau_{\geq 1}R \to M \to M \otimes_R \pi_0(R).$$

Now, if M is a perfect R-module, there is some maximal $i \in \mathbb{Z} \cup \{+\infty\}$ such that $\pi_j(M) = 0$ for all $j < i$, with $i = \infty$ iff $M \simeq 0$. If M is nonzero, then $\pi_i(M \otimes_R \tau_{\geq 1}R) = 0$, so that $M \otimes_R \tau_{\geq 1}R \to M$ cannot be an equivalence, and $M \otimes_R \pi_0(R)$ is nonzero as well, from which the claim follows.

Now, assume that $\pi_*(R)$ is Noetherian, write $\mathfrak{m}$ for the maximal ideal in $\pi_*(R)$. We can take a presentation $\mathfrak{m} = (a_1, ..., a_n)$ for some a finite set of generators a_i. Define the perfect R-module

$$z := \bigotimes_{R,i=1}^{n} \mathrm{cofib}(\Sigma^{|a_i|}R \xrightarrow{a_i} R).$$

Although a_i may not act as zero on $\pi_*(z)$, we at least know that a_i^2 acts as zero on the homotopy groups of z. We learn that $\pi_*(z)$ is a finitely generated graded module over the graded ring $\pi_*(R)/(a_1^2, ..., a_n^2)$. In particular, z only has finitely many nonzero homotopy groups, and $\pi_i(z)$ has a finite filtration whose associated graded pieces are all k-vector spaces, for all i.

Now, consider a fiber sequence

$$y \xrightarrow{g} \mathbb{1} \xrightarrow{f} x$$

between perfect R-modules. Since k is a field, up to replacing this sequence by its dual, we may assume that $f \otimes_R k$ is split or equivalently that $g \otimes_R k \simeq 0$. We claim that $z \otimes_R g$ is $\otimes$-nilpotent. Since z has only finitely many nonzero homotopy groups, the Whitehead filtration on z is a finite filtration, so it suffices to show that $\pi_i(z) \otimes_R g$ is $\otimes$-nilpotent for each i. Using that $\pi_i(z)$ has a finite filtration with associated graded pieces given by k-vector spaces, we deduce that $\pi_i(z) \otimes_R g$ is $\otimes$-nilpotent, since $k \otimes_R g \simeq 0$ $\square$

Remark 3.3. In the above proof, when $\pi_*(R)$ is Noetherian, we constructed a a single object such that the exact-nilpotence condition for any fiber sequence always holds with this taking the role of z in the statement of the condition. This can only happen when the closed point is the support locus of a single object.

In [Mat16], Mathew computes the Balmer spectrum of the category of modules over rational $\mathbb{E}_\infty$-rings with Noetherian π_*. As a corollary of Mathew's result, we find that

Corollary 3.4. *If R is a rational Noetherian $\mathbb{E}_\infty$-ring, then Mod_R^{perf} is local if and only if $\pi_*(R)$ is a graded local ring.*

□

It is tempting to try to use Theorem 3.1 to prove that the exact-nilpotence condition holds for modules over any finitely presented local rational $\mathbb{E}_\infty$-ring. Unfortunately, we run into the same issue as Mathew

Example 3.5. ([Mat16, Proposition 8.8]) The $\mathbb{E}_\infty$-ring $R\Gamma(\mathbb{A}^2_{\mathbb{Q}}\backslash\{0\})$ is finitely presented, but not Noetherian.

One may hope the connective case is better. The situation is not so serendipitous, and in fact, non-Noetherian compact $\mathbb{E}_\infty$-$\mathbb{Q}$-algebras are quite plentiful, as the following example shows.

Example 3.6. Start with the free $\mathbb{E}_\infty$-ring $\mathbb{Q}[x_2]\otimes\Lambda_{\mathbb{Q}}[y_1]$ on generators y_1 in degree 1 and x_2 in degree 2. Let R denote the $\mathbb{E}_\infty$ quotient by x_2y_1, that is,

$$\mathbb{Q}\otimes_{\Lambda_{\mathbb{Q}}[z_3]}(\mathbb{Q}[x_2]\otimes\Lambda_{\mathbb{Q}}[y_1]),$$

where z_3 maps to x_2y_1. Considering $\mathbb{Q}[x_2]\otimes\Lambda_{\mathbb{Q}}[y_1]$ as a $\Lambda_{\mathbb{Q}}[z_3]$-module, the generator y_1 generates a $\Sigma\mathbb{Q}$-summand, and when we take the tensor product, we will get generators in π_{4n+1} which multiply with x, y and each other to zero. In particular, $\pi_*(R)$ is far from Noetherian.

In line with Balmer's vision that there should be a good notion of "Noetherian" in tt-geometry, one may expect that any finitely presented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebra is "Noetherian," whatever this should mean. One expected property is that the Balmer spectrum of a Noetherian tt-category should be a Noetherian topological space. We only prove this in the main case of interest, though remark that a careful examination of the proof will show how to adapt the following to prove such a claim in the connective case. Before we begin, let's recall the notion of a collection of ring maps detecting nilpotence, the form we take is from [Mat16, Definition 4.3]

Definition 3.7. Given an $\mathbb{E}_\infty$-ring R, and a collection $\{R\to S_i\}_{i\in I}$ of $\mathbb{E}_\infty$-ring maps, we say that this collection *detects nilpotence* (over R) if given any associative algebra object T in $\mathrm{Ho}(\mathrm{Mod}_R)$, then for any $x\in\pi_*(T)$, x is nilpotent if and only if the image of x under $\pi_*(T)\to\pi_*(S_i\otimes_R T)$ is nilpotent in $\pi_*(S_i\otimes_R T)$ for every $i\in I$.

For a detailed account about detecting nilpotence, we refer to loccite. We state the main properties that will be useful to us in the following.

Proposition 3.8. *Let R be an $\mathbb{E}_\infty$-ring, and $\{R\to S_i\}_{i\in I}$ a collection of $\mathbb{E}_\infty$-ring maps which detects nilpotence. The following hold:*

- *If $f: R\to A$ is an $\mathbb{E}_\infty$-ring map, then $\{A\to A\otimes_R S_i\}_{i\in I}$ detects nilpotence over A.*
- *If $h: x\to y$ is a morphism in Mod_R^{perf}, then h is $\otimes$-nilpotent if and only if $h\otimes_R S_i$ is $\otimes$-nilpotent in $\mathrm{Mod}_{S_i}^{perf}$ for every $i\in I$.*

- *The induced map on Balmer spectra* $\coprod_{i\in I} \mathrm{Spc}(S_i) \to \mathrm{Spc}(R)$ *is surjective.*[2]
- *If* $\{S_i \to T_{ij}\}_{j\in J_i}$ *detects nilpotence over each* S_i*, then* $\{R \to T_{ij}\}_{i\in I, j\in J_i}$ *detects nilpotence over* R*.*
- *If* k *is a field of characteristic zero, and* $n \in \mathbb{Z}$*, the collections* $\{\Lambda_k[z_{2n+1}] \to k\}$ *and* $\{k[z_{2n}] \to k, k[z_{2n}] \to k[z_{2n}^{\pm 1}]\}$ *detect nilpotence over* $\Lambda_k[z_{2n+1}]$ *and* $k[z_{2n}]$*, respectively.*
- *Given an* $\mathbb{E}_\infty$*-ring map* $A \to B$*, if the thick* $\otimes$*-ideal of* Mod_A *generated by* B *is all of* Mod_A*, then* $\{A \to B\}$ *detects nilpotence over* A*.*

Proof. For the first condition, if T is any A-algebra, it is also an R-algebra, and there is an equivalence $T \otimes_R S_j \simeq T \otimes_A (A \otimes_R S_j)$, from which the result follows. The second, fourth, fifth, and sixth bulletpoints are Proposition 4.4, Proposition 4.6, Example 4.7 and Example 4.8 of [Mat16], respectively. The third claim follows from [Bar+23, Theorem 1.3]. ☐

With these preliminaries in hand, we come to the main theorem of the section.

Theorem 3.9. *Let* R *be a finitely presented connective* $\mathbb{E}_\infty$*-*$\mathbb{Q}$*-algebra with* $\pi_0(R) = k$ *a field. Then there exists a finite collection* $\{S_1, \dots, S_n\}$ *of* $\mathbb{E}_\infty$*-*R*-algebras under* R *such that the collection* $\{S_1, \dots, S_n, k\}$ *detects nilpotence for* R*, where each* S_i *has a unit in* $\pi_2(S_i)$*,* $\pi_1(S_i) = 0$*, and* $\pi_0(S_i)$ *is a* k*-smooth integral domain.*

Proof. Let R be a rational connective $\mathbb{E}_\infty$-algebra, with $\pi_0(R)$ a field k. Mathew proves that R is a k-algebra, and since π_0 commutes with colimits of connective $\mathbb{E}_\infty$-rings, k is a finite extension of $\mathbb{Q}$, so is finitely presented.

Our goal is to build the collection $\{S_1, \dots S_n\}$ by induction on the number of cells in R if R has finitely many cells. The general claim will then follow since any finitely presented R is a retract of some algebra with finitely many cells. To explicitly build this, use that $\pi_i(R)$ is a finite dimensional k-vector space for any $i \geq 0$, and then build $T^{(0)} = k \to T^{(1)} \to \dots \to R$ such that $T^{(i)} \to R$ is i-connected, and $T^{(i+1)}$ is obtained from $T^{(i)}$ by first adjoining new generators in degree $i+1$ and then quotienting out relations in this degree. Then $R = \varinjlim T^{(i)}$, and compactness of R implies that $T^{(n)} \to R$ splits for some $n \gg 0$.

In the case $R = k$, we may take the empty collection, since k detects nilpotence over k. Suppose by induction we have constructed such a collection $\{S_1, \dots, S_n\}$ for T, and we attach to T a new generator in degree i or quotient out a relation in degree i to get to R. We split into cases, the first three being the easiest:

Case 1. i **is odd and we are adjoining a new generator in degree** i**.** Our new algebra is $R \simeq T \otimes_k \Lambda_k[t_i]$, with t_i in degree i. Proposition 3.8 implies that $\Lambda_k[t] \to k$ satisfies descent, and hence detects nilpotence. Hence,

$$T \otimes_k \Lambda_k[t_i] \otimes_{\Lambda_k[t_i]} k \simeq T$$

detects nilpotence over R, and we may take the same collection $\{S_1, \dots, S_n\}$ used for T for R.

Case 2. i **is odd and we are adjoining a relation in degree** i**.** In this case, there is a pushout diagram

[2] Here and later on in the paper, a coproduct of Balmer spectra denotes the coproduct in the category of topological spaces.

$$\begin{tikzcd} \Lambda_k[t_i] \arrow[r] \arrow[d] & T \arrow[d] \\ k \arrow[r] & R. \end{tikzcd}$$

By basechange compatibility, the collection $S_j \otimes_{\Lambda_k[t_i]} k$ together with $k \otimes_{\Lambda_k[t_i]} k$ detect nilpotence over R. Since the S_j are even, any map $\Lambda_k[t_i] \to S_j$ must factor over k, and $S_j \otimes_{\Lambda_k[t_i]} k$ is also even, with

$$\pi_0(S_j \otimes_{\Lambda_k[t_i]} k) \simeq \pi_0(S_j)[x]$$

a polynomial ring over $\pi_0(S_j)$. Now, $k \otimes_{\Lambda_k[t_i]} k \simeq k[\sigma t_i]$ is a free algebra on a class σt_i in degree $i+1$. By Proposition 3.8, the collection $\{k, k[(\sigma t_i)^{\pm 1}]\}$ detects nilpotence over $k[\sigma t_i]$. Now, we can use that there is a map $k[(\sigma t_i)^{\pm 1}] \to k[(x_2)^{\pm 1}]$ to the free algebra on an invertible class in degree 2 (taking σt_i to $x_2^{(i+1)/2}$) which admits descent, hence detects nilpotence. Whence, we may take for R the collection

$$\{S_1 \otimes_{\Lambda_k[t_i]} k, \ldots, S_n \otimes_{\Lambda_k[t_i]} k, k[x_2^{\pm 1}]\},$$

satisfying the desired properties.

Case 3. i is even and we are adjoining a new generator in degree i. This is similar to Case 2. In this case, $R = T \otimes_k k[t_i]$ with t_i a polynomial generator in degree i, and we may use for R the collection

$$\{S_1 \otimes_k k[t_i], \ldots, S_n \otimes_k k[t_i], k[x_2^{\pm 1}]\}.$$

Case 4. i is even and we are adjoining a relation in degree i. In this case, there is an algebra map $k[t_i] \to T$, with $R \simeq T \otimes_{k[t_i]} k$. By basechange compatibility, the collection of $S_j \otimes_{k[t_i]} k$ and $k \otimes_{k[t_i]} k$ jointly detects nilpotence over R. $k \otimes_{k[t_i]} k \simeq \Lambda_k[\sigma t_i]$ is a free algebra on a generator in odd degree, so k detects nilpotence over this algebra.

We reduce to showing that if S is an even $\mathbb{E}_\infty$-k-algebra with a unit in $\pi_2(S)$ and with $\pi_0(S)$ a smooth integral domain over k, then for any $x \in \pi_0(S)$, there exists algebras $S'_1, \ldots, S'_m$ with the same property, and maps $S/x \to S'_j$ which jointly detect nilpotence. By [Mat16, Theorem 1.3/4], we find that a collection of maps $\{S/x \to S'_j\}$ between even Noetherian $\mathbb{E}_\infty$-k-algebras with a unit in degree 2 detects nilpotence if and only if the induced map

$$\coprod_{j=1}^{m} \mathrm{Spec}(\pi_0(S'_j)) \to \mathrm{Spec}(\pi_0(S/x))$$

is surjective. Given an even 2-periodic $\mathbb{E}_\infty$-k-algebra S, there are a number of operations we can apply to S make other algebras with these same properties. Namely, we may adjoint a polynomial variable to $\pi_0(S)$, we may localize $\pi_0(S)$ at any multiplicatively closed set, and we may quotient out $\pi_0(S)$ by a regular sequence. In particular, starting from our S, we can get maps to étale covers, Zariski covers, and (affine covers of) blowups along smooth centers.

Consider the divisor cut out by $x \in \pi_0(S)$ on $\mathrm{Spec}(S)$. By Hironaka's theorem on embedded resolution of singularities [Hir64, Corollary 3], there is a sequence of blowups along smooth centers $X_r \to \ldots \to X_0 = \mathrm{Spec}(\pi_0(S))$ such that the (reduced subscheme structure on the) pullback of (x) to X_r is a normal crossing divisor. By what we have said, X_r has an affine open cover by schemes $\mathrm{Spec}(\pi_0(A_j))$

for even 2-periodic smooth $\mathbb{E}_\infty$-S-algebras A_j. Refining this by an étale cover to assume x pulls back to a strict normal crossing divisor, and then further if need be to assume that every irreducible component of the pullback of x is given by a principal divisor, we may assume that we have a collection of maps $S \to A_j$ of even 2-periodic smooth $\mathbb{E}_\infty$-rings detecting nilpotence (since the induced map of Zariski spectra of their π_0 is jointly surjective) such that the image of x in each A_j is either a unit, or can be written as $x = y_1 \dots y_{r_j}$ with $\pi_0(A_j)/y_n$ a k-smooth integral domain for each n. Finally, we use that we have maps $S/x \to A_j/y_k$ which jointly detect nilpotence, again using the result of Mathew and the fact that the induced map on Zariski spectra is surjective. Therefore, taking the collection $\{A_j/y_k\}_{j,k}$ as our $S'_1, \dots, S'_m$, the claim is shown. $\square$

Corollary 3.10. *If R is a connective finitely presented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebra with $\pi_0(R)$ a field, then $\mathrm{Spc}(R)$ is a Noetherian topological space, and there is a collection of "residue fields" for R. These come in the form of $\mathbb{E}_\infty$-algebra maps $R \to L_j$ for $j \in J$ some index set, with each L_j an even 2-periodic $\mathbb{E}_\infty$-k-algebra with $\pi_0(L_j)$ a field, such that the collection $\{R \to L_j\} \cup \{R \to k\}$ detects nilpotence over R.*

Proof. By Theorem 3.9 and Proposition 3.8, there is a surjection from the Noetherian topological space $\coprod_{i=1}^n \mathrm{Spc}(S_i) \coprod \mathrm{Spc}(k) \to \mathrm{Spc}(R)$ is surjective, so that $\mathrm{Spc}(R)$ must be Noetherian as well, proving the first claim. The second claim follows by Theorem 3.9 and [Mat16, Theorem 1.3/4]. $\square$

Remark 3.11. We only prove existence of residue fields in the above, not "uniqueness." Below, we will prove "uniqueness" of the residue field at the closed point. We do not say anything about whether or not there can be residue fields $R \to L_1$, $R \to L_2$, both having the same image (different from the closed point) under the induced map of Balmer spectra, but with $L_1 \otimes_R L_2 \simeq 0$. One would not expect this to happen, but we do not rule out the possibility in the present work.

Theorem 3.12. *If R is a connective finitely presented $\mathbb{E}_\infty$-$\mathbb{Q}$-algebra with $\pi_0(R) = k$ a field, then the exact-nilpotence condition holds for Mod_R^{perf}.*

Proof. Since $\mathrm{Spc}(R)$ is Noetherian, there is some object $Z \in \mathrm{Mod}_R^{perf}$ with $\mathrm{supp}(Z)$ equal to the unique closed point, we fix a choice of such an object. Take a fiber sequence

$$y \xrightarrow{g} \mathbb{1} \xrightarrow{f} x$$

of perfect R-modules, and suppose without loss of generality (up to dualizing this sequence) that $g \otimes_R k \simeq 0$. We claim that $Z \otimes g$ is $\otimes$-nilpotent. By Corollary 3.10, there is a set of residue fields $\{L_j\}_{j \in J} \cup \{k\}$ for R, and so it suffices to check that $L_j \otimes_R (Z \otimes g)$ is $\otimes$-nilpotent for each of our constructed residue fields L_j. If the map $\mathrm{Spc}(L_j) \to \mathrm{Spc}(R)$ has image different from the closed point, then $L_j \otimes_R Z \simeq 0$, so we are reduced to the case when $L_j \otimes_R -$ has trivial kernel on perfect R-modules. Over L_j, either $g \otimes_R L_j$ or $f \otimes_R L_j$ is zero, and we claim that $g \otimes_R L_j$ is zero, which would follow if we knew that $k \otimes_R L_j$ was nonzero.

Write $L_j =: L[x_2^{\pm 1}]$, with $L = \pi_0(L_j)$ a field and x_2 a chosen unit in $\pi_2(L_j)$, and write $L[x_2]$ for the connective cover of $L[x_2^{\pm 1}]$ (noting $R \to L[x_2^{\pm 1}]$ factors over $R \to L[x_2]$ by connectivity of R). Suppose that $L[x_2^{\pm 1}] \otimes_R k \simeq 0$. This will lead to a contradiction following a number of steps. First, note that $\pi_*(R) \to \pi_*(L[x_2^{\pm 1}])$ factors over k, or else there is some $y \in \pi_{2n}(R)$ mapping to a unit multiple of

x_2^n for some $n > 0$, and then $\mathrm{cofib}(y) \otimes_R L[x_2^{\pm 1}] \simeq 0$, contradicting the choice of map $R \to L[x_2^{\pm 1}]$. Suppose that we have inductively constructed an $\mathbb{E}_\infty$-R-algebra R_i which is perfect as an R-module, with $R_0 = R$, such that $R \to L[x_2]$ factors uniquely as $R \to R_i \to L[x_2]$. We extend this sequence in the following ways:
Case 1. If $\pi_*(R_i) \to \pi_*(L[x_2])$ has image larger than k in π_0, set $R_{i+1} := R_i$.
Case 2. If $\pi_*(R_i) \to \pi_*(L[x_2^{\pm 1}])$ factors over k, and if there is some $n > 0$ with $\pi_{2n}(R_i) \neq 0$, take $n > 0$ minimal with this property, choose a nonzero $z_{2n} \in \pi_{2n}(R_i)$, and define $R_{i+1} := R \otimes_{k[z_{2n}]} k$. Upon applying $\mathrm{Hom}_{\mathrm{CAlg}}(-, L[x_2^{\pm 1}])$ to the pushout diagram

$$\begin{array}{ccc} k[z_{2n}] & \longrightarrow & R_i \\ \downarrow & & \downarrow \\ k & \longrightarrow & R_{i+1}, \end{array}$$

we obtain a fiber sequence of spaces

$$\mathrm{Hom}_{\mathrm{CAlg}}(R_{i+1}, L[x_2^{\pm 1}]) \to \mathrm{Hom}_{\mathrm{CAlg}}(R_i, L[x_2^{\pm 1}]) \to \mathrm{Hom}_{\mathrm{CAlg}}(k[z_{2n}], L[x_2^{\pm 1}]).$$

Since $\pi_1(\mathrm{Hom}_{\mathrm{CAlg}}(k[z_{2n}], L[x_2^{\pm 1}])) \simeq 0$, the extension of $R_i \to L[x_2^{\pm 1}]$ to $R_i \to R_{i+1} \to L[x_2^{\pm 1}]$ is unique.
Case 3. If $\pi_{2n}(R_i) = 0$ for $n > 0$, define $R_{i+1} := R_i$.

Lemma 3.13. *In the above situation, we never end up in Case 1.*

Proof of lemma. Indeed, suppose that the map $\pi_*(R_i) \to \pi_*(L[x_2^{\pm 1}])$ has image larger than k, so that there is some element $y \in \pi_{2n}(R_i)$ mapping to a unit multiple of x_2^n for some $n > 0$. Since R_i is a perfect R-module by construction, it suffices to show that, should such a y exist, then

$$R_i/y \otimes_R L[x_2^{\pm 1}] \simeq 0,$$

or equivalently that x_2 is nilpotent in the algebra

$$R_i/y \otimes_R L[x_2].$$

The splitting $R_i \otimes_R L[x_2] \to L[x_2]$ shows that the image of $y \otimes 1$ in $\pi_*(R_i \otimes_R L[x_2])$ is not nilpotent. Again, by construction of R_i, $R_i \otimes_R L[x_2]$ is a perfect $L[x_2]$-module, so that the (graded) ring

$$A := \pi_{2n*}(R_i \otimes_R L[x_2])$$

is finitely generated over the polynomial subring $L[y \otimes 1]$, and in particular, $1 \otimes x_2^n$ is integral over this subring. Whence, there is some monic polynomial $f(z)$ over $L[y \otimes 1]$, which must be homogeneous for $|y \otimes 1| = 2n$, $|z| = 2n$, and in particular, takes the form

$$f(z) = z^j + a_{j-1}(y \otimes 1)z^{j-1} + \ldots + a_1(y \otimes 1)^{j-1}z + a_0(y \otimes 1)^j,$$

for some $j > 0$ and some $a_i \in L$. From this, it follows that $1 \otimes x^{nj} \simeq 0$ in $R_i/y \otimes_R L[x_2]$, and hence $R_i/y \otimes_R L[x_2^{\pm 1}] \simeq 0$, contradicting the fact that $M \otimes_R L[x_2^{\pm 1}]$ is nonzero for every nonzero perfect R-module M. $\square$

Thus, our sequence $R_i \to R_{i+1} \to \dots$ terminates only if R_i has $\pi_{2n}(R_i) = 0$ for all $n > 0$ at some finite stage of the construction. In any case, since $\pi_n(R)$ is finite dimensional for all $n > 0$, the maps $\dots \to R_i \to R_{i+1} \to \dots$ become increasingly connective, and upon taking the colimit $A := \varinjlim_i R_i$, one finds that $\pi_{2n}(A) = 0$ for $n > 0$, and $R \to L[x_2^{\pm 1}]$ factors as $R \to A \to L[x_2^{\pm 1}]$. Now, there is a map $A \otimes_k k[x_2^{\pm 1}] \to L[x_2^{\pm 1}]$, and the ring $A \otimes_k k[x_2^{\pm 1}]$ satisfies the hypothesis of [Mat16, Proposition 4.9] by construction. Now, by Mathew's Proposition 4.9, maps

$$A \otimes_k k[x_2^{\pm 1}] \to L[x_2^{\pm 1}]$$

are in bijection with maps

$$\pi_*(A \otimes_k k[x_2^{\pm 1}] \to L[x_2^{\pm 1}]) \to \pi_*(L[x_2^{\pm 1}]).$$

In this way, we see that the map $A \otimes_k k[x_2^{\pm 1}] \to L[x_2^{\pm 1}]$ is homotopic to the composite

$$A \otimes_k k[x_2^{\pm 1}] \to k \otimes_k k[x_2^{\pm 1}] \to L[x_2^{\pm 1}].$$

Finally, the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \otimes_k k[x_2^{\pm 1}] \arrow[d] \\ k \arrow[r] & k \otimes_k k[x_2^{\pm 1}], \end{tikzcd}$$

shows that $A \to L[x_2^{\pm 1}]$ factors as $A \to k \to L[x_2^{\pm 1}]$, so the same holds of $R \to A \to k \to L[x_2^{\pm 1}]$. But then

$$k \otimes_R L[x_2^{\pm 1}] \simeq (k \otimes_R k) \otimes_k L[x_2^{\pm 1}],$$

which cannot be zero, as $k \otimes_R k$ is nonzero. This yields the desired contradiction. $\square$

4. Strengthening Theorem 1.2

Finally, we will explain how the statement of Theorem 1.2 can be improved to include a bound on the order of nilpotence. The idea for this comes from recent ultraproduct constructions in higher algebra [BSS20][Lev23]. We recall the definitions here.

Definition 4.1. Fix a $\mathbb{N}$-indexed collection $\{\mathcal{C}_i\}$ of tt-categories, and some non-principal ultrafilter $\mathcal{U}$ on the natural numbers $\mathbb{N}$. Then the ultraproduct of this collection with respect to $\mathcal{C}_i$ is defined as:

$$\prod_{\mathcal{U}} \mathcal{C}_i = \varinjlim_{I \in \mathcal{U}} \prod_{i \in I} \mathcal{C}_i.$$

Remark 4.2. Since products and filtered colimits of symmetric monoidal stable ∞-categories commute with passing to the homotopy category, and both operations make sense for triangulated categories, the above construction could be done equally well at the triangulated level, for the more finitely-inclined.

The formalism of ultraproducts makes the following proof rather simple:

Theorem 4.3. *The following are equivalent,*

- *The nerves of steel conjecture holds.*
- *For every local tt-category $\mathcal{T}$, the exact-nilpotence condition holds.*
- *There exists an integer n such that for every local tt-category $\mathcal{T}$, and any fiber sequence as in Definition 1.1, there exists a nonzero object $z \in \mathcal{T}$ such that either $z \otimes g^{\otimes n} \simeq 0$ or $z \otimes f^{\otimes n} \simeq 0$.*

Proof. Theorem 1.2 is the equivalence of the first two statements, so we need only see that the second condition implies the third. Suppose otherwise, that we had a collection $\mathcal{C}_n$ of local tt-categories, such that for each n, there is a fiber sequence

$$y_n \xrightarrow{g_n} \mathbb{1} \xrightarrow{f_n} x_n$$

such that there exists $0 \neq z_n \in \mathcal{C}_n$ with $z_n \otimes g_n^{\otimes n} \simeq 0$, but there is no $0 \neq z'_n \in \mathcal{C}_n$ with $z'_n \otimes g^{\otimes n-1} \simeq 0$. Fix some non-principal ultrafilter $\mathcal{U}$ on the naturals, and let $\mathcal{T} := \prod_{\mathcal{U}} \mathcal{C}_n$ be the ultraproduct of these categories. Then, $\mathcal{T}$ remains a local tt-category, since given nonzero objects $(x_n), (y_n) \in \mathcal{T}$, the set of n such that x_n (resp. y_n) is nonzero is contained in $\mathcal{U}$, so too is their intersection, and since $\mathcal{C}_n$ were all local, $x_n \otimes y_n$ is nonzero when both terms are, hence $(x_n) \otimes (y_n)$ is a nonzero object of $\mathcal{T}$. Rigidity, and the stable/triangulated structure both happen pointwise. Now, there is a fiber sequence

$$(y_n)_n \xrightarrow{(g_n)_n} \mathbb{1} \xrightarrow{(f_n)_n} (x_n)_n$$

in $\mathcal{T}$, where $(f_n)_n$ is not $\otimes$-nilpotent on any nonzero object, and neither is $(g_n)_n$. Indeed, if $(g_n)_n$ were $\otimes$-nilpotent on a nonzero object z, we could take k such that $((g_n)_n)^{\otimes k} \otimes z \simeq 0$, and then choosing some $I \in \mathcal{U}$ with z nonzero on I, and any $r \in I$ with $r > k$, we would be supplied with a nonzero object $z_r \in \mathcal{C}_r$ such that $z_r \otimes g_r^{\otimes k} \simeq 0$, contradicting the choice of $\mathcal{C}_r$. $\square$

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