THE SMASHING SPECTRUM OF CYCLOTOMIC SPECTRA

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Earlier this year I tried to think about the idempotent algebras in cyclotomic spectra. It turned out to not be too hard, so I decided to write up a little blurb on it for my website in case I don't end up doing anything else with it. We claim that the smashing spectrum of cyclotomic spectra in terms of the smashing spectrum of spectra. (TC does not commute with colimits, so the unit is not compact and we don't quite have the usual tt geometry story). The story starts with genuine $\mathbb{T}$-equivariant spectra modulo finite equivalences, $\mathbb{T}Sp_{\mathcal{F}}$, which is the localization of genuine $\mathbb{T}$-spectra at those maps which are equivalences after applying geometric fixed points for every finite subgroup of $\mathbb{T}$. (If one uses just genuine $\mathbb{T}$-spectra, the same construction creates an older form of cyclotomic spectra that has been mostly replaced with this model). Now, we construct CycSp^{gen} as in Nikolaus-Scholze, the pullback in stably symmetric monoidal ∞-categories (or just symmetric monoidal ∞-categories, it doesn't really matter):

$$\begin{tikzcd}
\mathrm{CycSp}^{gen} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{p\in\mathbb{P}} \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}^{\mathbb{I}} \arrow[d, "{(p_1,p_0)}"] \\
\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \arrow[r, "{(\Phi^{C_p},id)_{p\in\mathbb{P}}}"] & \prod_{p\in\mathbb{P}} \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \times \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}
\end{tikzcd}$$

where here $\mathbb{I}$ denotes the free-living isomorphism, and putting it in the exponent is just taking the cotensor. In any case, we note that taking the smashing spectrum of a category (which we will just call $\mathrm{Idem}(-)$ here and treat it just as a poset, the poset of idempotent algebras that is, Clausen-Scholze have a geometric theory of these building on Balmer-Favi idempotents in lecture 5 of complex analytic geometry) commutes with all limits[1] (Ko Aoki has a paper proving that Idem is a right adjoint, which is one way to see that this holds, but it's also rather easy to see that Idem preserves limits directly from the definitions). In any case, this means that $\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}^{\mathbb{I}})$ is isomorphic to $\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}})$, and the above pullback diagram upon applying $\mathrm{Idem}(-)$ gives us a pullback

$$\begin{tikzcd}
\mathrm{Idem}(\mathrm{CycSp}^{gen}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{p\in\mathbb{P}} \mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}) \arrow[d, "\Delta"] \\
\mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}) \arrow[r, "{(\Phi^{C_p},id)}"] & \prod_{p\in\mathbb{P}} \mathrm{Idem}(\mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}).
\end{tikzcd}$$

The tensor idempotents are exactly those idempotent algebras A such that $\Phi^{C_p}A \simeq A$, as genuine $\mathbb{T}$-spectra with finite equivalences inverted. So, we necessarily have that for any finite subgroup G of $\mathbb{T}$, $\Phi^G A \simeq A$, and in particular this holds upon passing to underlying spectra (writing $\Phi^G A \simeq \Phi^e A$ may make this seem more clear what we are doing). It suffices to show that if A, B are such idempotent cyclotomic spectra with the same underlying spectrum, then they are

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[1]Where here posets are treated as categories for cotensors.

equivalent. But we have maps of algebras (induced by the units) $A \to A \otimes B$, and $B \to A \otimes B$, where $\Phi^e(A \otimes B) \simeq \Phi^e(A) \otimes \Phi^e(B)$ has the same underlying spectrum as A and B since their underlying spectra are idempotent algebras. Hence, we may assume that there is a map of algebras $f : A \to B$ to prove the claim. But then $\Phi^G(f)$ (as a non-equivariant spectrum, perhaps $\Phi^e\Phi^G(f)$ is clearer notation) is an equivalence for all finite $G \subseteq \mathbb{T}$, and since we are working with finite equivalences inverted, f itself is an equivalence.

The only bit left to see that the idempotent algebras in cyclotomic spectra are in bijection with those in spectra is to actually construct such an A associated to any idempotent algebra in spectra. To see that we have these, note that Sp is initial among cocomplete stably symmetric monoidal ∞-categories where the tensor product commutes with colimits in each variable separately (where maps are colimit-preserving symmetric monoidal functors), so there is an induced map $L : \mathrm{Sp} \to \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}}$. Since $\Phi^{C_n} : \mathbb{T}\,\mathrm{Sp}_{\mathcal{F}} \to \mathrm{Sp}$ is symmetric monoidal and preserves colimits, so does $\Phi^{C_n} \circ L$, which is then equivalent to the identity, and we see that given any idempotent algebra A in Sp, $L(A)$ gives the desired tensor idempotent in cyclotomic spectra.

More generally, given a $\mathbb{E}_\infty$-algebra A in CycSp^{gen}, we can look at modules over A in CycSp^{gen}, call this category $A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$. If A has underlying spectrum B, we can examine the symmetric monoidal (conservative) functor $F : A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}} \to B - \mathrm{Mod}$, where $B - \mathrm{Mod}$ denotes the category of B-module spectra. Since this is conservative and symmetric monoidal, again given any two idempotent A-algebras R and S with the same underlying spectrum, we can assume there is an algebra map $R \to S$ up to replacing S by $R \otimes_A S$, which is then an equivalence after applying F, so by conservativity of F, it was an equivalence to start with. Hence, $\mathrm{Idem}(A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}) \hookrightarrow \mathrm{Idem}(B - \mathrm{Mod})$. Now, we note that $\mathrm{End}_{A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}}}(A) \simeq B$ as a spectrum, so that the full stable subcategory of $A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$ generated by A is equivalent to the category of compact B-modules[2]. But then we get by $B - \mathrm{Mod}^\omega \to A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$, an induced functor $H : B - \mathrm{Mod} \to A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}$ taking B to A and commuting with colimits. Since $B - \mathrm{Mod}$ is generated by the unit, and $B \mapsto A$, the unit in the target category, this functor is automatically symmetric monoidal, and $F \circ H \simeq id_{B-\mathrm{Mod}}$, so that the image of any idempotent algebra in $B-\mathrm{Mod}$ under H gives an idempotent algebra in $A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}}$. Hence, $\mathrm{Idem}(A - \mathrm{Mod}_{\mathrm{CycSp}^{gen}}) \simeq \mathrm{Idem}(B - \mathrm{Mod})$.

This says that in general, smashing spectra for module categories of $\mathbb{E}_\infty$-rings in genuine cyclotomic spectra are rather degenerate, reducing to just the underlying spectra.

[2] $G := \mathrm{Hom}_{A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}}}(A, -)$ determines the functor to compact B-modules, which hits B, so is essentially surjective, and is fully faithful since it is when we apply it to A itself, and in general, the property of $\mathrm{Hom}_{A-\mathrm{Mod}_{\mathrm{CycSp}^{gen}}}(N, M) \to \mathrm{Hom}_{B-\mathrm{Mod}}(G(N), G(M))$ is closed under direct sums, shifts and two-out-of-three in both variables, so by definition of this subcategory, we get the claim.