Introduction to prismatic cohomology

June 20, 2023

Note: generally everything is sourced from Bhatt–Scholze [2] unless otherwise specified.

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- Example: Fontaine–Winterberger theorem $\mathrm{Gal}_{\mathbb{Q}_p(p^{1/p^\infty})_p^\wedge} \simeq \mathrm{Gal}_{\mathbb{F}_p((T^{1/p^\infty}))_T^\wedge}$

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- Generalize: if perfectoid spaces are analogous to perfect $\mathbb{F}_p$-algebras, how can we drop the "perfect"?

Perfect prisms

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So the data of $A_{\inf}(R)$ (depending only on $R^\flat$) and (ξ) together recover R.

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Further since $R^\flat$ is perfect, ϕ is an isomorphism.

Definition

A **perfect prism** is a pair (A, I) where A is a δ-ring such that the Frobenius is an isomorphism and I is a principal ideal such that $p \in (I, \phi(I))$ and A is derived (p, I)-complete.

Perfect prisms

Theorem

The functors $R \mapsto (A_{\mathrm{inf}}(R), \ker(A_{\mathrm{inf}}(R) \to R))$ *and* $(A, I) \mapsto A/I$ *are inverse functors defining an equivalence between perfectoid rings and perfect prisms.*

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To "deperfect" perfectoid rings, we should look for a version without the "perfect" condition.

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Definition

A **prism** is a pair (A, I) where A is any δ-ring and I is a locally principal ideal such that $p \in (I, \phi(I))$ and A is (p, I)-complete.

A morphism of prisms is a map of δ-rings compatible with the chosen ideals.

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It turns out $(A, (p))$ is a prism iff A is p-torsion-free.

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Write R uniquely as A/I for a perfect prism (A, I). Then maps $R \to R'$ lift uniquely to $(A, I) \to (A', I')$ and so induce $A/p = R^\flat \to A'/p = R'^\flat$ and similarly in reverse, which gives a bijection by rigidity. $\square$

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In fact we have no restriction on (A', I') so actually this is a stronger version!

Prismatic site

For the tilting equivalence we study perfectoid algebras over a perfectoid R, or equivalently perfect prisms (A, I) together with a map $R \to A/I$. Now we want to generalize to non-perfectoid rings:

Definition

The **(absolute) prismatic site** $(R)_{\Delta}$ of a ring R is the category of prisms (A, I) together with a map $R \to A/I$, equipped with the $[BZZZT]$ topology.

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There is also a relative version, which is what we'll mostly use:

Definition

For a fixed prism (A, I) and an A/I-algebra R, the **prismatic site** $(R/A)_{\Delta}$ of R over (A, I) is the category of prisms (B, J) together with compatible maps $(A, I) \to (B, J)$ and $R \to B/J$, equipped with the $[BZZZT]$ topology.

Aside

Note: these are not really sites, because the arrows are the wrong way! Really these should be the opposites of these categories, which works well for replacing R by a scheme (e.g. $\operatorname{Spec} A/I \to \operatorname{Spec} R$).

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- $\Delta_{R} = R\Gamma((R)_{\Delta}, \mathcal{O}_{\Delta})$, the absolute prismatic cohomology;
- $\overline{\Delta}_{R/A} = R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$, relative Hodge–Tate cohomology, and similarly for the absolute version

Prismatic cohomology

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Example

Suppose $R = A/I$. Then

$$\Delta_{R/A} = A, \qquad \overline{\Delta}_{R/A} = A/I = R.$$

Perfectoidization

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Then we can define the perfectoidization in two ways: first, let R be an S-algebra for S perfectoid, so we can write $S = A/I$ for a perfect prism (A, I). Then

$$R_{\text{perfd}} := \Delta_{R/A,\text{perf}} \otimes^{\mathbb{L}}_A A/I.$$

Perfectoidization

This is sort of a "perfected Hodge–Tate cohomology." Note that it is in general a derived ring (in your preferred model, e.g. an animated ring or a commutative algebra object in $D(R)$). However a priori it depends on the choice of a presentation $A/I = S \to R$.

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Fortunately we can define this in another way which makes the independence clear by modifying the site: let $(R)_{\Delta}^{\mathrm{perf}}$ be the **(absolute) perfect prismatic site** of R, consisting of **perfect** prisms (A, I) with maps $R \to A/I$. Then

$$R_{\mathrm{perfd}} \simeq R\Gamma((R)_{\Delta}^{\mathrm{perf}}, \overline{\mathcal{O}}_{\Delta}).$$

Comparison theorems

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- Crystalline cohomology
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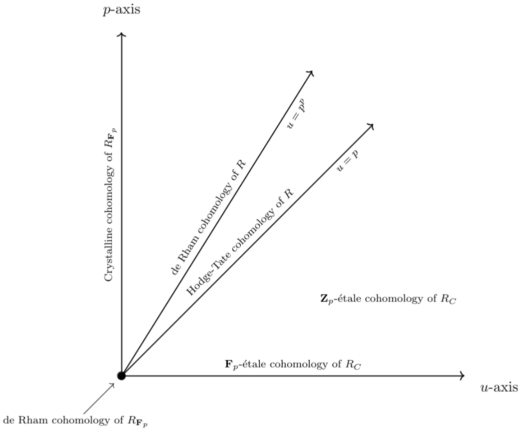
This is the origin of the picture motivating the term "prism" (stolen from [1, Lecture 1]) over $\operatorname{Spec} \mathbb{Z}_p[[u]]$ (should really be Σ):



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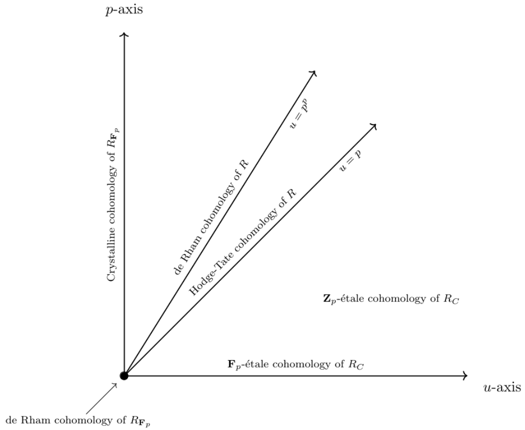

We'll leave details for future talks, but let's spell out the comparison in one example:

Crystalline comparison

Suppose (A, I) is crystalline and R is an A/I-algebra. Then $\Delta_{R/A}$ is **almost** equal to the classical crystalline cohomology $R\Gamma_{\mathrm{crys}}(R/A)$:

$$R\Gamma_{\mathrm{crys}}(R/A) = \phi^* \Delta_{R/A} = \Delta_{R/A} \widehat{\otimes}^{\mathbb{L}}_{A,\phi_A} A.$$

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When (A, I) is any perfect prism, we can then think of $\Delta_{R/A}$ as generalizing crystalline cohomology to mixed-characteristic perfectoids (after correcting by a Frobenius twist).

Qrsp rings and the Nygaard filtration

There is a site of $\mathbb{Z}_p$-algebras called the quasisyntomic site, with a basis of quasiregular semiperfectoid (qrsp) rings (roughly, those which are quotients of perfectoid rings by quasiregular ideals). Prismatic cohomology is well-behaved on these:

Proposition

If R is qrsp, the absolute prismatic site $(R)_{\Delta}$ has an initial object $(\Delta_R^{\mathrm{init}}, (d))$, and for any prism (A, I) with a map $A/I \to R$ we have

$$\Delta_{R/A} = \Delta_R = \Delta_R^{\mathrm{init}}.$$

In particular the prismatic cohomology of R is concentrated in degree 0, and $R\Gamma((R)_{\Delta}, \mathcal{I}_{\Delta}) = d\Delta_R$ is principal.

Qrsp rings and the Nygaard filtration

Thus we can define a filtration $\mathcal{N}^{\geq i}\Delta_R = \phi^{-1}(d^i\Delta_R)$, or (equivalently) $\mathcal{N}^{\geq i}\Delta_{R/A} = \phi^{-1}(d^i\Delta_{R/A})$.

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Then by descent we can define this filtration on the prismatic cohomology (absolute or relative) of any quasisyntomic ring R.

Write $\mathcal{N}^i\Delta_R$, $\mathcal{N}^i\Delta_{R/A}$ for the graded pieces, and $\hat{\Delta}_R$, $\hat{\Delta}_{R/A}$ for the completion with respect to this filtration.

Topological connections

Now let's briefly review topological Hochschild homology: for an E_∞-ring spectrum R (which for us will be a usual discrete ring), $\mathrm{THH}(R)$ can be defined as the universal S^1-equivariant E_∞-ring spectrum over R. The S^1-action means we can form $\mathrm{TC}^-(R) := \mathrm{THH}(R)^{hS^1}$ and $\mathrm{TP}(R) := \mathrm{THH}(R)^{tS^1}$, which come with a natural map $\mathrm{TC}^-(R) \to \mathrm{TP}(R)$. (We take all of these with $\mathbb{Z}_p$-coefficients, i.e. p-completed.)

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Bökstedt's theorem: $\pi_{2*}\,\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p$, and odd homotopy groups vanish. There is a generalization to perfectoid rings R: $\pi_{2*}\,\mathrm{THH}(R) = R$, and odd groups vanish.

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What about qrsp rings?

Topological connections

The evenness is still true, but the description is more complicated: $\pi_{2*}\,\mathrm{THH}(R) = \mathcal{N}^n \hat{\mathbb{\Delta}}_R\{*\}$! (Here $\{*\}$ is a twist we'll skim over.)

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Moreover, $\mathrm{TC}^-(R)$ and $\mathrm{TP}(R)$ also have even homotopy groups, and

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In particular $\pi_0\,\mathrm{TC}^-(R) = \pi_0\,\mathrm{TP}(R) = \hat{\Delta}_R$. Since qrsp rings form a basis for the quasisyntomic site, it follows that the quasisyntomic sheafification of $\pi_0\,\mathrm{TC}^-(-) = \pi_0\,\mathrm{TP}(-)$ is $\hat{\Delta}_-$.

Topological connections

One can use the cyclotomic structure on THH to get Frobenius maps $\mathrm{TC}^-(R) \to \mathrm{TP}(R)$ and take the equalizer to get $\mathrm{TC}(R)$; doing the same sheafification process gives the **syntomic cohomology** of R, which previously was not defined in this generality. One can also expand to even ring spectra other than $\mathrm{THH}(R)$ for R qrsp to get the even and motivic filtrations [4].

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In particular, p-locally the cyclotomic trace $K(R) \to \mathrm{TC}(R)$ is an equivalence by [3], so we can say something about K-theory as a consequence: e.g. quasisyntomic locally its homotopy groups are even, i.e. the quasisyntomic sheafification of $\pi_n K(-, \mathbb{Z}_p)$ vanishes for n odd.

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