第一章 线性规划

第四节 单纯形法

- 典式
- 迭代原理
- 单纯形法举例
- 两阶段法

基变量个数与方程个数一致

$$(LP)\ \min S = CX$$

$$AX = b \qquad R(A) = m$$

$$X \geq 0$$

$$A = (\underbrace{p_1, p_2, \cdots, p_m}_{B(\text{可行基})}, \underbrace{p_{m+1}, p_{m+2}, \cdots, p_n}_{N}) = (B, N)$$

$$B^{-1}b \geq 0$$

$$X = (\underbrace{x_1, x_2, \cdots, x_m}, \underbrace{x_{m+1}, x_{m+2}, \cdots, x_n})^T = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$C = (\underbrace{c_1, c_2, \cdots, c_m}_{C_B}, \underbrace{c_{m+1}, c_{m+2}, \cdots, c_n}_{C_N}) = (C_B, C_N)$$

复习

$$(LP)\ \min S = CX \qquad C=(C_B, C_N)$$
$$AX = b \qquad A=(B,N) \qquad X=\begin{pmatrix} X_B \\ X_N \end{pmatrix}$$
$$X \geq 0$$

$$AX=b \Rightarrow (B,N)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = b \Rightarrow BX_B + NX_N = b$$

$$\Rightarrow X_B + B^{-1}NX_N = B^{-1}b \Rightarrow X_B = B^{-1}b - B^{-1}NX_N$$

$$S = CX = (C_B, C_N)\begin{pmatrix} X_B \\ X_N \end{pmatrix} = C_B X_B + C_N X_N$$

$$= C_B(B^{-1}b - B^{-1}NX_N) + C_N X_N$$

$$= C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N$$

$$\min S = CX$$
$$AX = b$$
$$X \ge 0$$

复习 $C - C_B B^{-1} A \ge 0 \Leftrightarrow C_N - C_B B^{-1} N \ge 0$

$$C - C_B B^{-1} A = (C_B, C_N) - C_B B^{-1}(B, N)$$
$$= (C_B, C_N) - (C_B, C_B B^{-1} N)$$
$$= (0, C_N - C_B B^{-1} N) \blacksquare$$

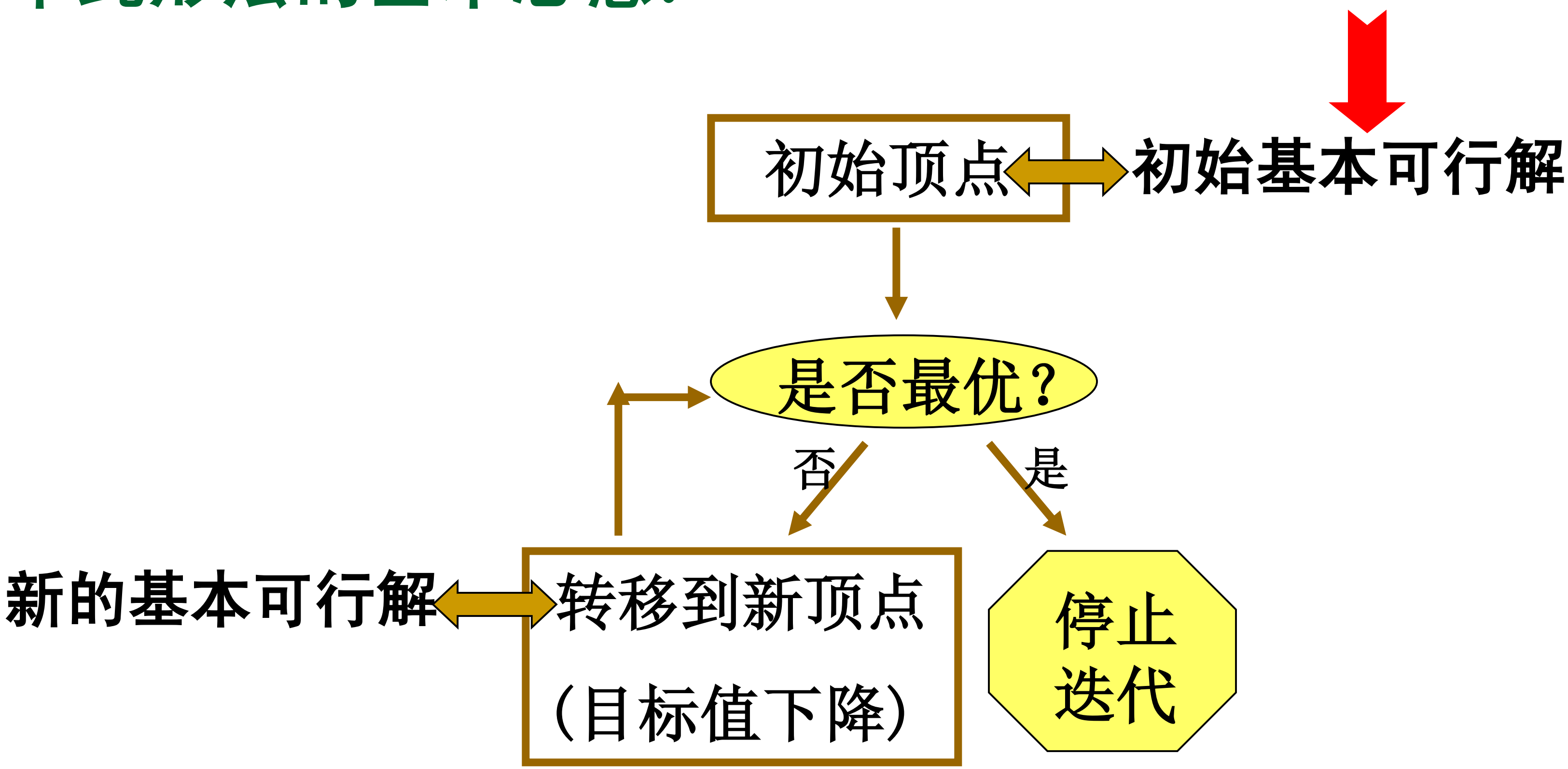
$$X = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

定理1-1（最优性判别定理）

对于 (LP) 的基 B，若有 $X_B^* = B^{-1} b \ge 0$ 且 $C - C_B B^{-1} A \ge 0$ $(C_N - C_B B^{-1} N \ge 0)$ ，则基本可行解 $X^* = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix}$ 是 (LP) 的最优解，称为**最优基本可行解**，B 称为**最优基**。

检验数向量　非基变量检验数向量

单纯形法的基本思想：



第一章 线性规划

一. 典式

- 典式的向量形式
- 典式的分量形式
- 典式的表格形式(单纯形表)

基本可行解

1. 典式的向量形式

$S = CX = C_B B^{-1}$

一个典式唯一地对应一个基本可行解

$$(LP)\ \min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$X_B = B^{-1}b - B^{-1}NX_N$$

可行基 B

$$\min S = C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = X_B$$

$$X_B + B^{-1}NX_N = B^{-1}b$$
$$X_B \geq 0 \quad X_N \geq 0$$

基本可行解

$$X = \begin{pmatrix} B^{-1}b \geq 0 \\ 0 \end{pmatrix}$$

$$S = C_B B^{-1}b$$

称为(LP)的以 $x_1, x_2, \cdots, x_m$ 为基变量的典式

典式的向量形式

$$\min S = C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N$$
$$X_B + B^{-1}NX_N = B^{-1}b$$
$$X_B \ge 0,\quad X_N \ge 0$$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$
$$x_j \ge 0,\quad j = 1,2,3,4$$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$B^{-1}(A,b) = B^{-1}\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$$

(B、N、b → E、$B^{-1}N$、$B^{-1}b$)

初等行变换亦可

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases} \Rightarrow \begin{cases} x_1 \qquad - 2x_3 + x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \end{cases}$$

$$C_B B^{-1}b = (1,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad C_N - C_B B^{-1}N = (2,2) - (1,1)\begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix} = (2,1)$$

$$\min S = C_B B^{-1}b + (C_N - C_B B^{-1}N)X_N = 1 + (2,1)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 1 + 2x_3 + x_4$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \ge 0, \quad X_N \ge 0$$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \ge 0, \quad j = 1,2,3,4$$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array} \quad (B = (x_1, x_2),\ N = (x_3, x_4)) \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad R(A) = 2$$

$$B^{-1}(A,b) = B^{-1}\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad (B,\ N,\ b \to E,\ B^{-1}N,\ B^{-1}b) \quad C_B B^{-1} b = 1$$

典式(基变量为x_1, x_2)：

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \qquad - 2x_3 + x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \ge 0, j = 1,2,3,4 \end{cases}$$

基本可行解：

$$X = (0,1,0,0)^T$$

目标值： $S = 1$

$$X = \begin{pmatrix} B^{-1} b \ge 0 \\ 0 \end{pmatrix}$$

$$S = C_B B^{-1} b$$

基变量：只出现在其中一个方程中，且系数为1。

注： 可行基 $\Longleftrightarrow$ 典式 $\Longleftrightarrow$ 基本可行解

2. 典式的分量形式

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0, \quad X_N \geq 0$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$A = (\underbrace{p_1, p_2, \cdots, p_m}_{B(\text{可行基})}, \underbrace{p_{m+1}, p_{m+2}, \cdots, p_n}_{N}) = (B, N)$$

$$B^{-1} b \geq 0$$

$$B^{-1} N = (B^{-1} p_{m+1}, B^{-1} p_{m+2}, \cdots, B^{-1} p_n)$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \geq 0 \qquad = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$X_B + B^{-1}NX_N = B^{-1}b$$

$$B^{-1}b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \geq 0 \qquad B^{-1}N = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$X = (\underbrace{x_1, x_2, \cdots, x_m}, \underbrace{x_{m+1}, x_{m+2}, \cdots, x_n})^T = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix}$$

$$X_B + B^{-1}NX_N = B^{-1}b$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix} \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix}$$

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + y_{1m+2}x_{m+2} + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + y_{2m+2}x_{m+2} + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + y_{mm+2}x_{m+2} + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$C = (\underbrace{c_1, c_2, \cdots, c_m}_{C_B}, \underbrace{c_{m+1}, c_{m+2}, \cdots, c_n}_{C_N}) = (C_B, C_N)$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \qquad B^{-1} N = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$C_B B^{-1} b = y_{00} \qquad (C_N - C_B B^{-1} N) = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

$\min S = C_B B^{-1}b + (C_N -$

$X_B + B^{-1}NX_N = B$

$X_B \geq 0 \quad X_N \geq 0$

1. 每个方程都有且仅有一个基变量，基变量仅出现在一个方程中且系数为1。

2. 非基变量的系数是其检验数。

$$\min S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$$

典式的分量形式

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + y_{1m+2}x_{m+2} + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + y_{2m+2}x_{m+2} + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + y_{mm+2}x_{m+2} + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

$C_B B^{-1}b = y_{00} \qquad C_N - C_B B^{-1}N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$C_N - C_B B^{-1} N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

$$C_N - C_B B^{-1} N$$
$$= (c_{m+1}, c_{m+2}, \cdots c_n) - C_B B^{-1}(p_{m+1}, p_{m+2}, \cdots, p_n)$$
$$= (c_{m+1}, c_{m+2}, \cdots c_n) - (C_B B^{-1} p_{m+1}, \quad C_B B^{-1} p_{m+2}, \cdots, \quad C_B B^{-1} p_n)$$
$$= (\underline{c_{m+1} - C_B B^{-1} p_{m+1}}, \quad \underline{c_{m+2} - C_B B^{-1} p_{m+2}}, \cdots, \quad \underline{c_n - C_B B^{-1} p_n})$$

非基变量 x_j 的检验数

$$\left.\begin{aligned} y_{0m+1} &= c_{m+1} - C_B B^{-1} p_{m+1} \\ y_{0m+2} &= c_{m+2} - C_B B^{-1} p_{m+2} \\ &\vdots \\ y_{0n} &= c_n - C_B B^{-1} p_n \end{aligned}\right\} \quad y_{0j} = c_j - C_B B^{-1} p_j$$

$$j = m+1, m+2, \cdots, n$$

典式的分量形式:

$y_{00} = C_B B^{-1} b \qquad y_{0j} = c_j - C_B B^{-1} p_j$

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0,\ j = 1,2,\cdots n$$

$$x_1 \quad x_2 \quad \cdots \quad x_m \quad x_{m+1} \quad x_{m+2} \quad \cdots \quad x_n$$

基本可行解: $X^0 = (y_{10}, y_{20}, \cdots, y_{m0}, 0, \quad 0, \quad \cdots \quad ,0)^T = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0$

目标值: $S^0 = y_{00} = C_B B^{-1} b$

3. $-y_{00} = -S + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$

$$\min S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$$

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + y_{1m+2}x_{m+2} + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + y_{2m+2}x_{m+2} + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + y_{mm+2}x_{m+2} + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

单纯形表

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$C = (\underbrace{c_1, c_2, \cdots, c_m}_{C_B}, \underbrace{c_{m+1}, c_{m+2}, \cdots, c_n}_{C_N}) = (C_B, C_N)$$

$$B^{-1} b = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix} \qquad B^{-1} N = \begin{pmatrix} y_{1m+1} & y_{1m+2} & \cdots & y_{1n} \\ y_{2m+1} & y_{2m+2} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{mm+1} & y_{mm+2} & \cdots & y_{mn} \end{pmatrix}$$

$$C_B B^{-1} b = y_{00} \qquad C_N - C_B B^{-1} N = (y_{0m+1}, y_{0m+2}, \cdots, y_{0n})$$

单纯形表:

$y_{0j}=c_j-C_BB^{-1}p_j$ $\quad C-C_BB^{-1}A$

$-C_BB^{-1}b$	x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j} $\quad -y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1 $\quad y_{10}$	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2 $\quad y_{20}$	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$ $\quad \vdots$								
x_m $\quad y_{m0}$	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

$B^{-1}N$

$B^{-1}b$

$$C-C_BB^{-1}A=(C_B,C_N)-C_BB^{-1}(B,N)$$
$$=(C_B,C_N)-(C_B,C_BB^{-1}N)$$
$$=(0,C_N-C_BB^{-1}N)$$
$$=(0,0,\cdots,0,y_{0m+1},y_{0m+2},\cdots,y_{0n})$$

单纯形表：$S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$

$-C_B B^{-1}b$		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_1	y_{10}	1	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

$B^{-1}b \geq 0$

$$\begin{array}{cccccccc} x_1 & x_2 & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_n \\ \Downarrow & \Downarrow & & \Downarrow & \Downarrow & \Downarrow & & \Downarrow \end{array}$$

基本可行解: $X^0 = (y_{10}, y_{20}, \cdots, y_{m0}, 0, \quad 0, \quad \cdots \quad , 0)^T = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0$

目标值: $S^0 = y_{00} = C_B B^{-1}b$

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

$\xrightarrow{\text{典式}}$

$\min S = 1 + 2x_3 + x_4$

$$\begin{cases} x_1 \qquad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1,2,3,4 \end{cases}$$

基本可行解：

$X = (0, 1, 0, 0)^T$

目标值： $S = 1$

单纯形表：

		x_1	x_2	x_3	x_4
y_{0j}	-1	0	0	2	1
x_1	0	1	0	-2	1
x_2	1	0	1	2	0

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

典式 $\longrightarrow$

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \qquad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1, 2, 3, 4 \end{cases}$$

c_j			1	1	2	2
			x_1	x_2	x_3	x_4
C_B	y_{0j}	-1	0	0	2	1
1	x_1	0	1	0	-2	1
1	x_2	1	0	1	2	0
		$B^{-1}b$	$B^{-1}p_1$	$B^{-1}p_2$	$B^{-1}p_3$	$B^{-1}p$

基本可行解：

$$X = (0, 1, 0, 0)^T$$

目标值： $S = 1$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ & & & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$$

$$\boxed{y_{0j} = c_j - C_B B^{-1} p_j}$$

$$y_{00} = C_B B^{-1} b = (1,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

$$y_{03} = c_3 - C_B B^{-1} p_3 = 2 - (1,1)\begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2$$

$$y_{04} = c_4 - C_B B^{-1} p_4 = 2 - (1,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

p_4	b	x_1	x_2	x_3	x_4
y_{0j}	-1	0	0	2	1
x_1	0	1	0	-2	1
x_2	1	0	1	2	0

例： $\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \geq 0, \quad j = 1,2,3,4$$

典式 $\longrightarrow$

$$\min S = 1 + 2x_3 + x_4$$

$$\begin{cases} x_1 \qquad - 2x_3 + \ x_4 = 0 \\ \qquad x_2 + 2x_3 + 0x_4 = 1 \\ x_j \geq 0, j = 1,2,3,4 \end{cases}$$

		x_1	x_2	x_3	x_4
y_{0j}	0	1	1	2	2
x_1	1	1	1	0	1
x_2	2	1	2	2	1
y_{0j}	-1	0	0	2	1
x_1	1	1	1	0	1
x_2	1	0	1	2	0
y_{0j}	-1	0	0	2	1
x_1	0	1	0	-2	1
x_2	1	0	1	2	0

B_1

E

基本可行解：

$$X = (0, 1, 0, 0)^T$$

目标值： $S = 1$

		x_1	x_2	x_3	x_4
y_{0j}	-1	0	0	2	1
x_1	0	1	0	-2	1
x_2	1	0	1	2	0

例：$\min S = x_1 + x_2 + 2x_3 + 2x_4$

$$\begin{cases} x_1 + x_2 + 0x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + x_4 = 2 \end{cases}$$

$$x_j \ge 0, \quad j = 1,2,3,4$$

$\xrightarrow{\text{典式}}$

$\min S = 2 - x_2 + x_4$

$$\begin{cases} x_1 + x_2 + x_4 = 1 \\ 1/2\, x_2 + x_3 + 0x_4 = 1/2 \\ x_j \ge 0, j = 1,2,3,4 \end{cases}$$

		x_1	x_2	x_3	x_4
y_{0j}	0	1	1	2	2
x_1	1	1	1	0	1
x_3	2	1	2	2	1
y_{0j}	-1	0	0	2	1
x_1	1	1	1	0	1
x_3	1	0	1	2	0
y_{0j}	-2	0	-1	0	1
x_1	1	1	1	0	1
x_3	1/2	0	1/2	1	0

基本可行解：

$$X = (1, 0, 1/2, 0)^T$$

目标值：$S = 2$

第一章 线性规划

第四节 单纯形法

- 典式
- 迭代原理
- 单纯形法举例
- 两阶段法

1.典式的向量形式:

复习

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$\min\ S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0 \quad X_N \geq 0$$

$\Downarrow$

2.典式的分量形式:

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

1.典式的向量形式:

基本可行解

复习

$$\min S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$

$$X_B + B^{-1} N X_N = B^{-1} b$$

$$X_B \geq 0 \quad X_N \geq 0$$

$$X = \begin{pmatrix} B^{-1} b \geq 0 \\ 0 \end{pmatrix}$$

$$S = C_B B^{-1} b$$

2.典式的分量形式:

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

基本可行解: $X^0 = (y_{10}, y_{20}, \cdots, y_{m0}, 0, 0, \cdots, 0)^T = \begin{pmatrix} B^{-1} b \geq 0 \\ 0 \end{pmatrix}$

目标值: $S^0 = y_{00}$

3. 典式的表格形式(单纯形表)

$$\min S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + y_{0n}x_n$$

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + y_{1m+2}x_{m+2} + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + y_{2m+2}x_{m+2} + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + y_{mm+2}x_{m+2} + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

		x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

单纯形表

单纯形表: $y_{0j}=c_j-C_BB^{-1}p_j$ $\quad C-C_BB^{-1}A\geq 0$

	$-C_BB^{-1}b$	x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n
	$-y_{00}$	0	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2n}
$\vdots$	$\vdots$								
x_m	y_{m0}	0	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mn}

$B^{-1}N$

最优单纯形表

$B^{-1}b\geq 0$

基本可行解: $X^0=(y_{10},y_{20},\cdots,y_{m0},0,0,\cdots,0)^T$

目标值: $S^0=y_{00}$

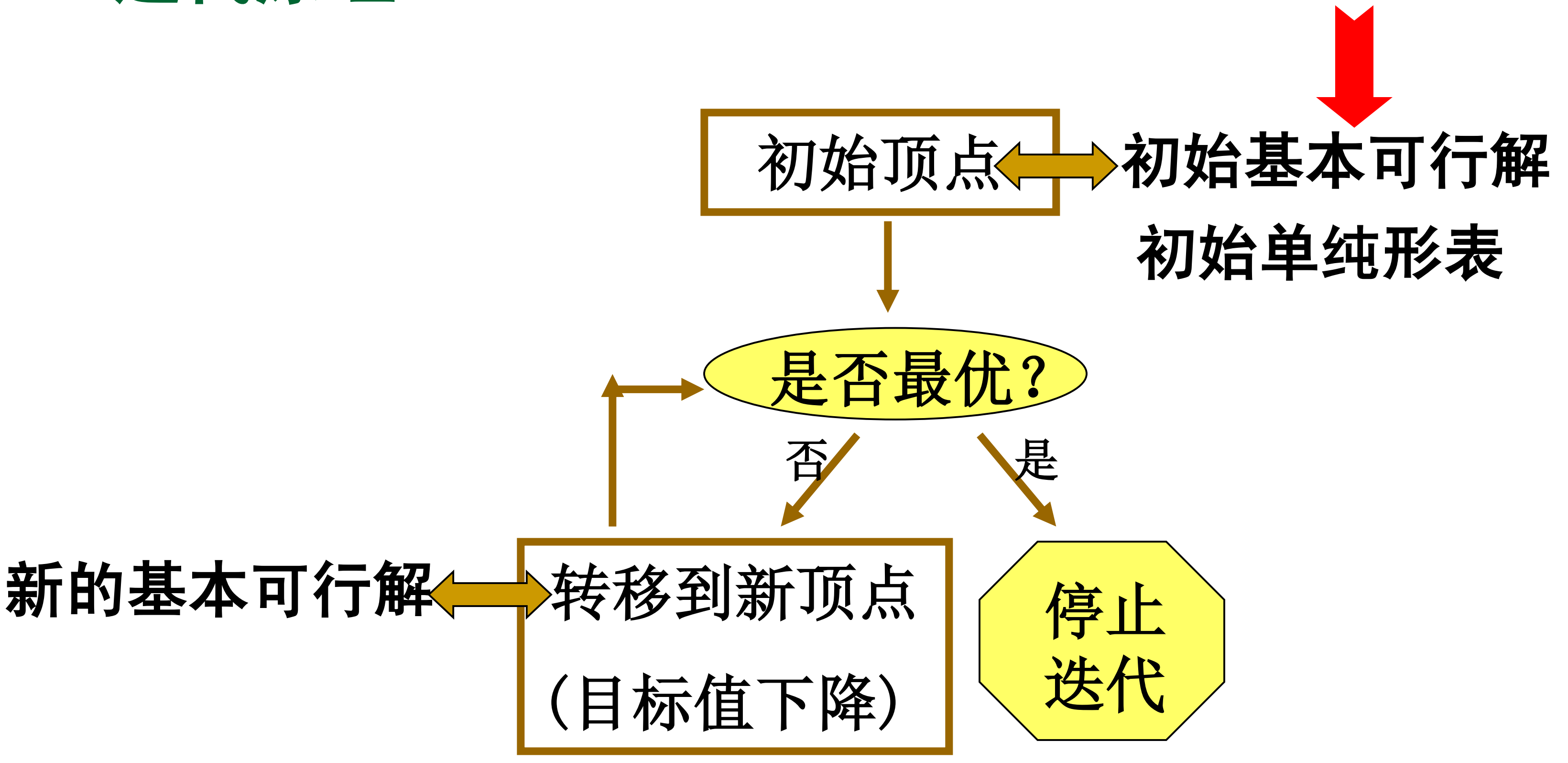
最优解

第一章 线性规划

第四节 单纯形法

- 典式
- 迭代原理
- 单纯形法举例
- 两阶段法

二. 迭代原理：



1. 初始单纯形表

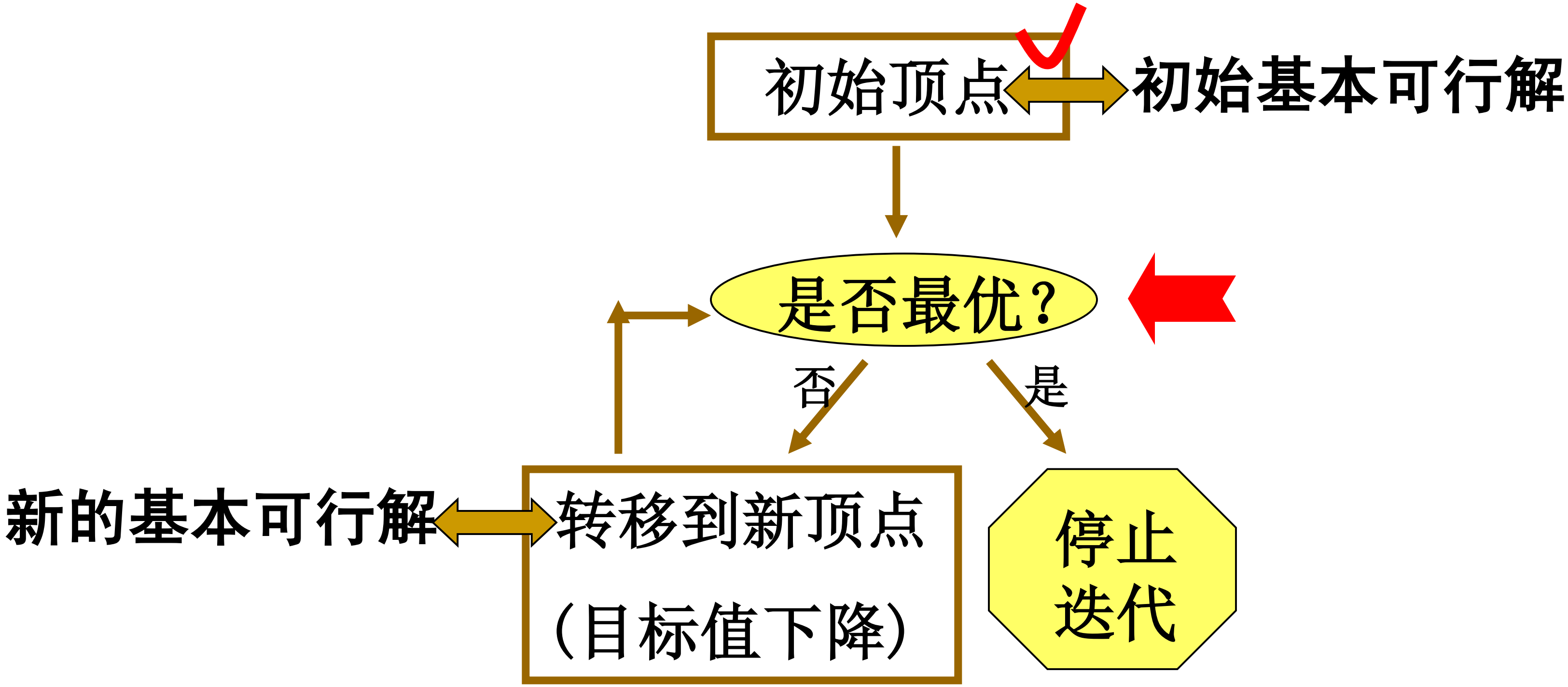
		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0q}	$\cdots$	y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

$B^{-1}b \geq 0$

$$\begin{array}{l} \quad\ x_1 \quad x_2 \ \cdots \ x_p \ \cdots \ x_m \quad x_{m+1} \quad x_{m+2} \ \cdots \ x_q \ \cdots \ x_n \\ X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, 0, \quad 0, \quad \cdots \quad 0, \cdots 0)^T \end{array}$$

初始基本可行解

二. 迭代原理:



2. 判断当前基本可行解是否是最优解：

1）若 $\underline{C-C_BB^{-1}A\geq 0}$ 或 $\underline{C_N-C_BB^{-1}N\geq 0}$,

即非基变量 x_j 的检验数 $y_{0j}=c_j-C_BB^{-1}p_j$ 都 ≥ 0,

则 X^0 是最优解。

2）若有某些检验数 $\underline{y_{0j}<0}$ ，例如：$y_{0q}<0\,(m+1\leq q\leq n)$,

则 X^0 不是最优解(在非退化情况下)。

1）单纯形表 $y_{0j}=c_j-C_BB^{-1}p_j\geq 0$ $C_N-C_BB^{-1}N\geq 0$

		x_1	$x_2\cdots$	$x_p\cdots$	x_m	x_{m+1}	$x_{m+2}\cdots$	$x_q\cdots$	x_n
y_{0j}	$-y_{00}$	0	$0\cdots$	$0\cdots$	0	y_{0m+1}	$y_{0m+2}\cdots$	$y_{0q}\cdots$	y_{0n}
x_1	y_{10}	1	$0\cdots$	$0\cdots$	0	y_{1m+1}	$y_{1m+2}\cdots$	$y_{1q}\cdots$	y_{1n}
x_2	y_{20}	0	$1\cdots$	$0\cdots$	0	y_{2m+1}	$y_{2m+2}\cdots$	$y_{2q}\cdots$	y_{2n}
x_p	y_{p0}	0	$0\cdots$	$1\cdots$	0	y_{pm+1}	$y_{pm+2}\cdots$	$y_{pq}\cdots$	y_{pn}
x_m	y_{m0}	0	$0\cdots$	$0\cdots$	1	y_{mm+1}	$y_{mm+2}\cdots$	$y_{mq}\cdots$	y_{mn}

$B^{-1}b\geq 0$

$$X^0=(y_{10},y_{20},\cdots,y_{p0},\cdots y_{m0},0,\quad 0,\ \cdots\quad 0,\cdots 0)^T$$

$\Downarrow$

是最优解

2）单纯形表

		x_1	x_2	$\cdots x_p$	$\cdots x_m$	x_{m+1}	x_{m+2}	$\cdots x_q$	$\cdots x_n$
y_{0j}	$-y_{00}$	0	0	$\cdots 0$	$\cdots 0$	y_{0m+1}	y_{0m+2}	$\cdots y_{0q}<0$	y_{0n}
x_1	y_{10}	1	0	$\cdots 0$	$\cdots 0$	y_{1m+1}	y_{1m+2}	$\cdots y_{1q}$	$\cdots y_{1n}$
x_2	y_{20}	0	1	$\cdots 0$	$\cdots 0$	y_{2m+1}	y_{2m+2}	$\cdots y_{2q}$	$\cdots y_{2n}$
x_p	y_{p0}	0	0	$\cdots 1$	$\cdots 0$	y_{pm+1}	y_{pm+2}	$\cdots y_{pq}$	$\cdots y_{pn}$
x_m	y_{m0}	0	0	$\cdots 0$	$\cdots 1$	y_{mm+1}	y_{mm+2}	$\cdots y_{mq}$	$\cdots y_{mn}$

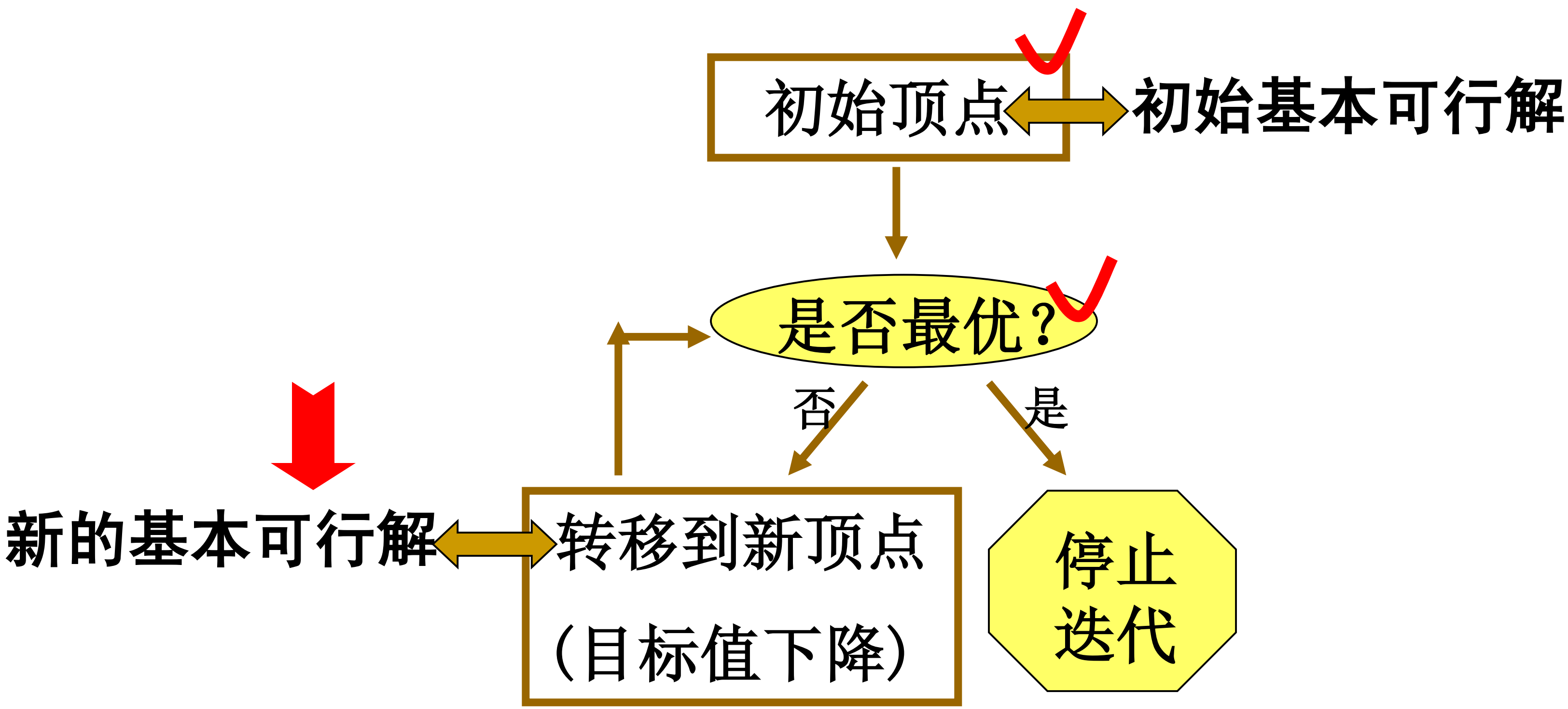
$$B^{-1}b \geq 0$$

$$X^0=(y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, 0, \quad 0, \cdots \quad 0, \cdots 0)^T$$

⇓

不是最优解

二. 迭代原理：



3. 转移到新的基本可行解：

- 确定进基变量
- 确定离基变量
- 进行换基运算

1.典式的向量形式:

复习

$$\min S = CX$$
$$AX = b$$
$$X \geq 0$$

$$\min\ S = C_B B^{-1} b + (C_N - C_B B^{-1} N) X_N$$
$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0 \quad X_N \geq 0$$

2.典式的分量形式:

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

1) 确定进基变量：

$$\min S = y_{00} + y_{0m+1}x_{m+1} + y_{0m+2}x_{m+2} + \cdots + \underset{}{\overset{<0}{y_{0q}}}x_q + \cdots + y_{0n}x_n$$

$$x_1 \quad x_2 \ \ldots \ x_p \ \ldots x_m \quad x_{m+1} x_{m+2} \quad x_q \ldots x_n$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0 \cdots, 0, \cdots 0)^T \qquad S^0 = y_{00}$$

$$\Downarrow$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0}, \ 0, \ 0 \cdots, \theta > 0, 0)^T$$

$$S^1 = y_{00} + y_{0q}\theta \ < S^0 = y_{00} \downarrow$$ x_q称为**进基变量**

确定进基变量的准则：

将检验数 < 0 的非基变量进基做基变量，可使新的基本可行解相应目标函数值下降。

确定进基变量的准则：

如果有不止一个负检验数，则有两种方法确定进基变量 x_q：

1. ***Bland*** **规则：** $$q=\min\{j \mid y_{0j}<0, j=1,2,\cdots,n\}$$

2. **最负检验数法：** $$y_{0q}=\min\{y_{0j} \mid y_{0j}<0, j=1,2,\cdots,n\}$$

单纯形表

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	$y_{0q}<0$		y_{0n}
x_1	y_{10}	1	0	$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	1	$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	0	$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	$\cdots$	y_{mn}

采用***Bland*规则**方法确定进基变量 x_q.

3. 转移到新的基本可行解：

- 确定进基变量
- 确定离基变量
- 进行换基运算

1.典式的向量形式：

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$$X_B + B^{-1} N X_N = B^{-1} b$$
$$X_B \geq 0 \quad X_N \geq 0$$

2.典式的分量形式：

$$\min S = y_{00} + y_{0m+1} x_{m+1} + y_{0m+2} x_{m+2} + \cdots + y_{0n} x_n$$

$$\begin{cases} x_1 + y_{1m+1} x_{m+1} + y_{1m+2} x_{m+2} + \cdots + y_{1n} x_n = y_{10} \\ x_2 + y_{2m+1} x_{m+1} + y_{2m+2} x_{m+2} + \cdots + y_{2n} x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1} x_{m+1} + y_{mm+2} x_{m+2} + \cdots + y_{mn} x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

2) 确定离基变量：

$$y_{0q} < 0$$

$$\quad x_1 \quad x_2 \ \cdots\ x_p \ \cdots x_m \quad x_{m+1} x_{m+2} \quad x_q \cdots x_n$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0 \cdots, 0, \cdots 0)^T \quad S^0 = y_{00}$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0}, \ 0, \ 0 \cdots, \theta > 0, 0)^T$$

$$S^1 = y_{00} + y_{0q}\theta < S^0 = y_{00} \downarrow$$

$$\begin{cases} x_1 + y_{1m+1}x_{m+1} + \cdots + y_{1q}x_q + \cdots + y_{1n}x_n = y_{10} \\ x_2 + y_{2m+1}x_{m+1} + \cdots + y_{2q}x_q + \cdots + y_{2n}x_n = y_{20} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m + y_{mm+1}x_{m+1} + \cdots + y_{mq}x_q + \cdots + y_{mn}x_n = y_{m0} \end{cases}$$

$$x_j \geq 0, j = 1,2,\cdots n$$

$$y'_{10} = y_{10} - y_{1q}\theta \geq 0$$

$$y'_{20} = y_{20} - y_{2q}\theta \geq 0$$

$$\cdots\cdots$$

$$y'_{m0} = y_{m0} - y_{mq}\theta \geq 0$$

2) 确定离基变量:

$$\frac{y_{p0}}{y_{pq}} < \frac{y_{10}}{y_{1q}} \rightarrow y_{1q} \cdot \frac{y_{p0}}{y_{pq}} < y_{10}$$

$$x_1 \quad x_2 \ \cdots \ x_p \ \cdots x_m \quad x_{m+1} x_{m+2} \quad x_q \cdots x_n$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0 \cdots, 0, \cdots 0)^T \qquad S^0 = y_{00}$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0}, \ 0, \ 0 \cdots, \theta, \cdots 0)^T \qquad S^1 = y_{00} + y_{0q}\theta$$

$$y_{0q} < 0$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, 0, \cdots \quad y'_{m0}, 0, \ 0 \cdots, \frac{y_{p0}}{y_{pq}}, 0)^T$$

$$y'_{20} = y_{20} - y_{2q}\theta \geq 0 \Longrightarrow \theta \leq \frac{y_{20}}{y_{2q}} (y_{2q} > 0)$$

$$y'_{p0} = y_{p0} - y_{pq}\theta \geq 0 \Longrightarrow \theta \leq \frac{y_{p0}}{y_{pq}} (y_{pq} > 0)$$

$$y'_{m0} = y_{m0} - y_{mq}\theta \geq 0 \Longrightarrow \theta \leq \frac{y_{m0}}{y_{mq}} (y_{mq} > 0)$$

$$\boxed{\theta = \min\{\frac{y_{i0}}{y_{iq}} \,\Big|\, y_{iq} > 0, 1 \leq i \leq m\} = \frac{y_{p0}}{y_{pq}}}$$

$\Longrightarrow$ x_p称为**离基变量**

最小非负比值准则

单纯形表

		x_1	$x_2 \cdots$	$x_p \cdots$	x_m	x_{m+1}	$x_{m+2} \cdots$	x_q	$\cdots$	x_n
y_{0j}	$-y_{00}$	0	$0 \cdots$	$0 \cdots$	0	y_{0m+1}	$y_{0m+2} \cdots$	$y_{0q} < 0$		y_{0n}
x_1	y_{10}	1	$0 \cdots$	$0 \cdots$	0	y_{1m+1}	$y_{1m+2} \cdots$	y_{1q}	$\cdots$	y_{1n}
x_2	y_{20}	0	$1 \cdots$	$0 \cdots$	0	y_{2m+1}	$y_{2m+2} \cdots$	y_{2q}	$\cdots$	y_{2n}
x_p	y_{p0}	0	$0 \cdots$	$1 \cdots$	0	y_{pm+1}	$y_{pm+2} \cdots$	y_{pq}	$\cdots$	y_{pn}
x_m	y_{m0}	0	$0 \cdots$	$0 \cdots$	1	y_{mm+1}	$y_{mm+2} \cdots$	y_{mq}	$\cdots$	y_{mn}

第一种情况：假设 $y_{1q}, y_{2q}, \cdots y_{pq}, \cdots y_{mq} > 0$

$$\theta = \min\left\{\frac{y_{10}}{y_{1q}}, \frac{y_{20}}{y_{2q}}, \cdots, \frac{y_{p0}}{y_{pq}}, \cdots, \frac{y_{m0}}{y_{mq}}\right\} = \frac{y_{p0}}{y_{pq}} \Rightarrow x_p$$ 称为**离基变量**

单纯形表

		x_1	$x_2 \cdots x_p \cdots x_m$	x_{m+1}	$x_{m+2} \cdots$	x_q	$\cdots$ x_n
y_{0j}	$-y_{00}$	0	$0 \cdots 0 \cdots 0$	y_{0m+1}	$y_{0m+2} \cdots$	$y_{0q}<0$	y_{0n}
x_1	y_{10}	1	$0 \cdots 0 \cdots 0$	y_{1m+1}	$y_{1m+2} \cdots$	y_{1q}	$\cdots y_{1n}$
x_2	y_{20}	0	$1 \cdots 0 \cdots 0$	y_{2m+1}	$y_{2m+2} \cdots$	y_{2q}	$\cdots y_{2n}$
x_p	y_{p0}	0	$0 \cdots 1 \cdots 0$	y_{pm+1}	$y_{pm+2} \cdots$	y_{pq}	$\cdots y_{pn}$
x_m	y_{m0}	0	$0 \cdots 0 \cdots 1$	y_{mm+1}	$y_{mm+2} \cdots$	y_{mq}	$\cdots y_{mn}$

第二种情况： 若 $y_{iq} \le 0, i = 1,2,\cdots,m,$ 则对 θ 没限制，

$y'_{10} = y_{10} - y_{1q}\theta \ge 0$

$y'_{20} = y_{20} - y_{2q}\theta \ge 0$

$y'_{m0} = y_{m0} - y_{mq}\theta \ge 0$

而 $S^1 = y_{00} + y_{0q}\theta \xrightarrow[\theta\to+\infty]{} -\infty,$

即(*LP*)没有有限的最优解。

第三种情况：无穷多最优解判别条件：

$$\begin{array}{ccccccccccccc} x_1 & x_2 & \ldots & x_p & \ldots x_m & x_{m+1} & x_{m+2} & x_q & \ldots x_n \end{array}$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0 \cdots, 0, \cdots 0)^T \quad S^0 = y_{00}$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, y'_{p0}, \cdots y'_{m0}, \ 0, \ 0 \cdots, \theta > 0, 0)^T \quad S^1 = y_{00} + y_{0q}\theta$$

$$\begin{cases} y'_{10} = y_{10} - y_{1q}\theta \geq 0 \\ y'_{20} = y_{20} - y_{2q}\theta \geq 0 \\ y'_{p0} = y_{p0} - y_{pq}\theta \geq 0 \\ y'_{m0} = y_{m0} - y_{mq}\theta \geq 0 \end{cases}$$

非基变量 x_q 的检验数 $y_{0q} = 0$,
则 $S^1 = y_{00} + y_{0q}\theta = y_{00} \Rightarrow$ 最优值

例：

若对于某个基本可行解，所有检验数都非负，且存在一个非基变量的检验数=0，则(LP) 有无穷多个最优解。

3. 转移到新的基本可行解：

- 确定进基变量
- 确定离基变量
- 进行换基运算

3) 换基运算：

$$y'_{i0} = y_{i0} - y_{iq}\frac{y_{p0}}{y_{pq}}$$

		x_1	x_2	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q		$\cdots$	x_n
y_{0j}	$-y_{00}$	0	0	$\cdots$	0	$\cdots$	0	y_{0m+1}	y_{0m+2}	$\cdots$	y_{0q}	0	$\cdots$	y_{0n}
x_1	y_{10} $-y_{1q}\cdot\frac{y_{p0}}{y_{pq}}$			$\cdots$	0	$\cdots$	0	y_{1m+1}	y_{1m+2}	$\cdots$	y_{1q}	0	$\cdots$	y_{1n}
x_2	y_{20} $-y_{2q}\cdot\frac{y_{p0}}{y_{pq}}$			$\cdots$	0	$\cdots$	0	y_{2m+1}	y_{2m+2}	$\cdots$	y_{2q}	0	$\cdots$	y_{2n}
$\leftarrow x_q$	$\frac{y_{p0}}{y_{pq}}$	0	0	$\cdots$	1	$\cdots$	0	y_{pm+1}	y_{pm+2}	$\cdots$	y_{pq}	1	$\cdots$	y_{pn}
x_m	y_{m0} $-y_{mq}\cdot\frac{y_{p0}}{y_{pq}}$			$\cdots$	0	$\cdots$	1	y_{mm+1}	y_{mm+2}	$\cdots$	y_{mq}	0	$\cdots$	y_{mn}

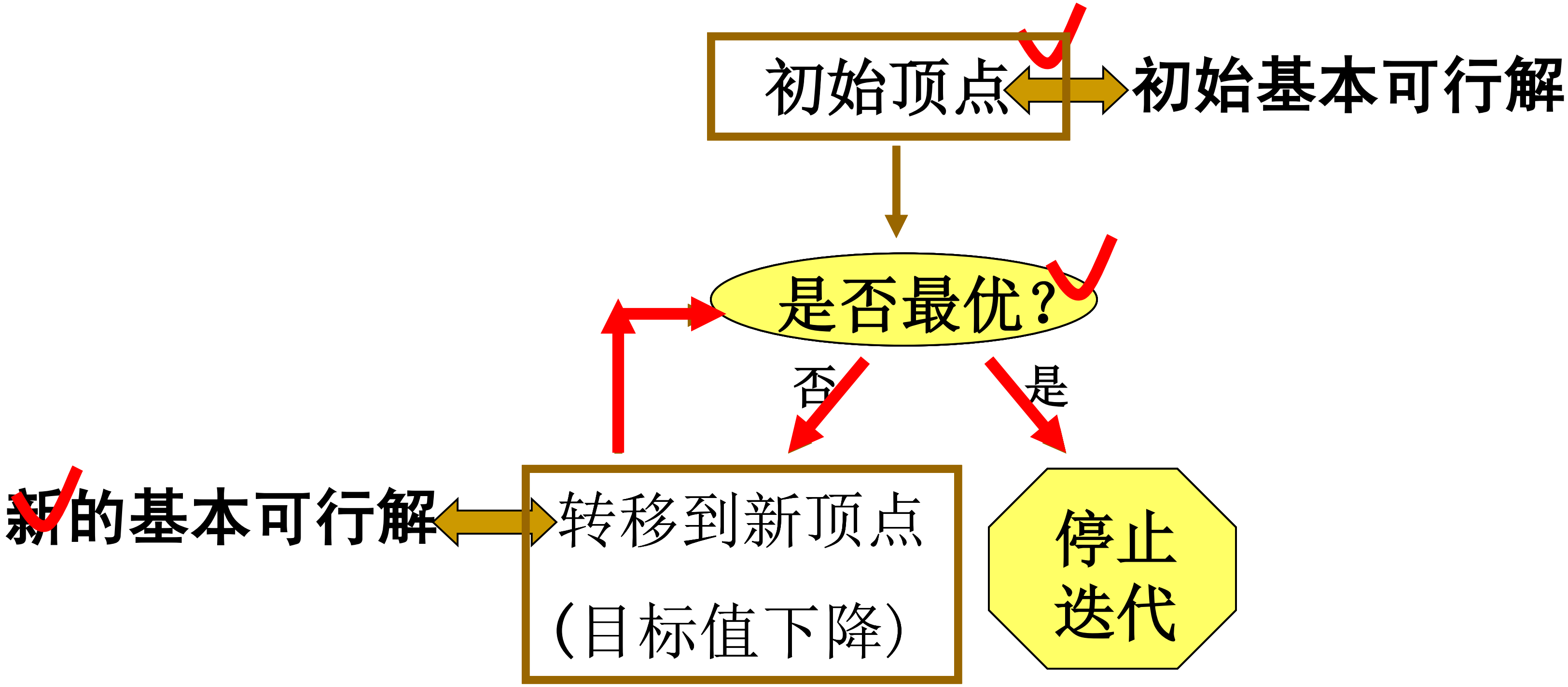
因为 x_q 代替 x_p 成为第p 个方程的基变量，所以它只能出现在第p个方程中，且系数为1，不能出现在其他方程中，检验数也为0。

$$\begin{array}{cccccccccccccc} & & x_1 & x_2 & \cdots & x_p & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_q & \cdots & x_n \\ X^1 & = (& y'_{10}, & y'_{20}, & \cdots, & 0, & \cdots & y'_{m0}, & 0, & 0 & \cdots, & \frac{y_{p0}}{y_{pq}}, & \cdots & 0)^T \end{array}$$

3. 转移到新的基本可行解：

- 确定进基变量
- 确定离基变量
- 进行换基运算

二. 迭代原理：



迭代原理：

$$\begin{array}{cccccccc} x_1 & x_2 & x_p & x_m & x_{m+1} & x_{m+2} & x_q & x_n \end{array}$$

$$X^0 = (y_{10}, y_{20}, \cdots, y_{p0}, \cdots y_{m0}, \ 0, \ 0\cdots, 0, \cdots 0)^T$$

$$X^1 = (y'_{10}, y'_{20}, \cdots, 0, \cdots \ y'_{m0}, 0, \ 0\cdots, \frac{y_{p0}}{y_{pq}}, 0)^T$$

$$S^1 = y_{00} + y_{0q}\theta < S^0 = y_{00} \downarrow$$

初始基本可行解X^0

$$B_0 = (p_1, p_2, \cdots p_p \cdots p_m)$$

$y_{0j} \geq 0?$ $j = 1, 2, \cdots, n$

Y：X^0是最优解

N

$$y_{0j} = c_j - C_B B^{-1} p_j \geq 0$$

$$y_{0q} < 0$$

确定进基变量x_q

$$q = \min\{j \mid y_{0j} < 0, j = 1, 2, \cdots, n\}$$

确定离基变量x_p

$$\theta = \min\{\frac{y_{i0}}{y_{iq}} \mid y_{iq} > 0, 1 \leq i \leq m\} = \frac{y_{p0}}{y_{pq}}$$

换基运算

若 $y_{iq} \leq 0$, (LP)没有有限的最优解。

新的基本可行解X^1

$$B_1 = (p_1, p_2, \cdots p_q \cdots p_m)$$

第一章 线性规划

第四节 单纯形法

- 典式
- 迭代原理
- 单纯形法举例
- 两阶段法

例1-10 求解线性规划问题：

典式 $\sigma_j = c_j - C_B B^{-1} p_j$　$X = (6, 0, 0, 0, 3, 4)^T$　$\sigma_{00} = C_B B^{-1} b$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$x_1 + x_2 + 3x_3 + x_4 = 6$$

3

4

,4

$$B_0 = (p_1, p_5, p_6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

标准形 $\longrightarrow$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$\begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 + x_5 = 3 \\ -x_2 + 6x_3 - x_4 + x_6 = 4 \\ x_j \geq 0, j = 1,2,3,4,5,6 \end{cases}$$

单纯形表

	C_j		1	-2	1	-3	0	0
			x_1	x_2	x_3	x_4	x_5	x_6
C_B	X_B	-6	0	-3	-2	-4	0	0
1	x_1	6	1	1	3	1	0	0
0	x_5	3	0	-2	1	1	1	0
0	x_6	4	0	-1	6	-1	0	1
		b	p_1	p_2	p_3	p_4	p_5	p_6

例1-10

$y_{0j} = c_j - C_B B^{-1} p_j \geq 0$　$C - C_B B^{-1} A \geq 0$

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	4	0	-1	6	-1	0	1

×2

x_1　x_2　x_3　x_4　x_5　x_6

$X^0 = (6,\ 0,\ 0,\ 0,\ 3,\ 4)^T$　$S^0 = 6$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$

$$X^0 = (6,\ 0,\ 0,\ 0,\ 3,\ 4\ )^T \qquad S^0 = 6$$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	-6	0	-3	-2	-4	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$

$$X^0 = (6,\ 0,\ 0,\ 0,\ 3,\ 4\ )^T \qquad S^0 = 6$$

例1-10

表一		x_1	x_2	x_3	x_4	x_5	x_6
X_B	12	3	0	7	-1	0	0
x_2	6	1	1	3	1	0	0
x_5	15	2	0	7	3	1	0
x_6	10	1	0	9	0	0	1

×3

$$\begin{array}{cccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ X^0 & = & (6, & 0, & 0, & 0, & 3, & 4)^T \end{array} \qquad S^0 = 6$$

例1-10

$y_{0j} = c_j - C_B B^{-1} p_j \ge 0$ $\quad C - C_B B^{-1} A \ge 0$

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 6 & -1 & 0 & 1 \end{pmatrix}, \quad B_0 = (p_1, p_5, p_6)$$

$$B_1 = (p_2, p_5, p_6) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

		x_1	x_2	x_3	x_4	x_5	x_6	
x_2	6	1	1	3	1	0	0	
x_4	5	2/3	0	7/3	1	1/3	0	$\times \frac{1}{3}$
x_6	10	1	0	9	0	0	1	

$X^0 = (6, 0, 0, 0, 3, 4)^T \quad S^0 = 6$

$X^1 = (0, 6, 0, 0, 15, 10)^T \quad S^1 = -12$

例1-10

表二		x_1	x_2	x_3	x_4	x_5	x_6
X_B	**12**	**3**	**0**	**7**	**-1**	**0**	**0**
x_2	**1**	**1/3**	**1**	**2/3**	0	**-1/3**	**0**
x_4	**5**	**2/3**	**0**	**7/3**	**1**	**1/3**	**0**
x_6	**10**	**1**	**0**	**9**	**0**	**0**	**1**

$\times(-1)$

$$\begin{array}{cccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ X^0 & = (& 6, & 0, & 0, & 0, & 3, & 4 \ )^T \\ X^1 & = (& 0, & 6, & 0, & 0, & 15, & 10 \ )^T \end{array}$$

$S^0 = 6$

$S^1 = -12$

例1-10

表二		x_1	x_2	x_3	x_4	x_5	x_6
X_B	17	11/3	0	28/3	0	1/3	0
x_2	1	1/3	1	2/3	0	-1/3	0
x_4	5	2/3	0	7/3	1	1/3	0
x_6	10	1	0	9	0	0	1

$$\begin{array}{lccccccl} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ X^0 = (& 6, & 0, & 0, & 0, & 3, & 4 &)^T \quad S^0 = 6 \\ X^1 = (& 0, & 6, & 0, & 0, & 15, & 10 &)^T \quad S^1 = -12 \end{array}$$

$$\min S = x_1 - 2x_2 + x_3 - 3x_4 \qquad\qquad \min S = x_1 - 2x_2 + x_3 - 3x_4$$

$$\begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 \le 3 \\ -x_2 + 6x_3 - x_4 \le 4 \\ x_j \ge 0, j = 1,2,3,4 \end{cases} \xrightarrow{\text{标准形}} \begin{cases} x_1 + x_2 + 3x_3 + x_4 = 6 \\ -2x_2 + x_3 + x_4 + x_5 = 3 \\ -x_2 + 6x_3 - x_4 + x_6 = 4 \\ x_j \ge 0, j = 1,2,3,4,5,6 \end{cases}$$

x_2	1	1/3	1	2/3	0	-1/3	0
x_4	5	2/3	0	7/3	1	1/3	0
x_6	10	1	0	9	0	0	1

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 6 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

$B_1 = (p_2, p_5, p_6)$

$$, B^* = (p_2, p_4, p_6) = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

例1-11 求解线性规划问题：

$y_{0j} = c_j - C_B B^{-1} p_j$ $y_{00} = C_B B^{-1} b$

$$\max S = x_1 + 3x_2 + 4x_3 \qquad \min(-S) = -x_1 - 3x_2 - 4x_3$$

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 \le 10 \\ 2x_1 + 3x_2 + x_3 \le 5 \\ x_j \ge 0,\ j = 1,2,3 \end{cases} \xrightarrow{\text{标准形}} \begin{cases} 3x_1 + 5x_2 - 4x_3 + x_4 = 10 \\ 2x_1 + 3x_2 + x_3 + x_5 = 5 \\ x_j \ge 0,\ j = 1,2,3,4,5 \end{cases}$$

单纯形表

		C_j	-1	-3	-4	0	0
			x_1	x_2	x_3	x_4	x_5
C_B	X_B	0	-1	-3	-4	0	0
0	x_4	10	3	5	-4	1	0
0	x_5	5	-2	3	1	0	1

例1-11

$y_{0j}=c_j-C_B B^{-1}p_j \geq 0$ $C-C_B B^{-1}A \geq 0$

		x_1	x_2	x_3	x_4	x_5
X_B	**0**	**-1**	**-3**	**-4**	**0**	**0**
x_1	**10/3**	**1**	**5/3**	**-4/3**	**1/3**	**0**
x_5	**5**	**-2**	**3**	**1**	**0**	**1**

$\times(1/3)$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$X^0=(0,\ 0,\ 0,\ 10,\ 5\ )^T \quad S^0=0$$

例1−11

		x_1	x_2	x_3	x_4	x_5
X_B	10/3	0	-4/3	-16/3	1/3	0
x_1	10/3	1	5/3	-4/3	1/3	0
x_5	5	-2	3	1	0	1

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ X^0 = (& 0, & 0, & 0, & 10, & 5 &)^T \end{array} \quad S^0 = 0$$

例1-11

		x_1	x_2	x_3	x_4	x_5
X_B	10/3	**0**	**-4/3**	**-16/3**	**1/3**	**0**
x_1	**10/3**	**1**	**5/3**	**-4/3**	**1/3**	**0**
x_5	**35/3**	**0**	**19/3**	**-5/3**	**2/3**	**1**

×2

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ X^0 = (& 0, & 0, & 0, & 10, & 5 \ )^T \end{array} \quad S^0 = 0$$

例1-11

$y_{0j} = c_j - C_B B^{-1} p_j \geq 0$ $C - C_B B^{-1} A \geq 0$?

		x_1	x_2	x_3	x_4	x_5
X_B	10/3	**0**	**-4/3**	**-16/3**	**1/3**	**0**
x_1	**10/3**	**1**	**5/3**	**-4/3**	**1/3**	**0**
x_5	**35/3**	**0**	**19/3**	**-5/3**	**2/3**	**1**

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$X^0 = (0,\ 0,\ 0,\ 10,\ 5\ )^T$ $S^0 = 0$

$X^1 = (10/3, 0,\ 0,\ 0,\ 35/3)^T$ $S^1 = -10/3$

例1-11

$y_{0j} = c_j - C_B B^{-1} p_j \geq 0$ $C - C_B B^{-1} A \geq 0$?

		x_1	x_2	x_3	x_4	x_5
X_B	10/3	**0**	**-4/3**	**-16/3**	**1/3**	**0**
x_1	**10/3**	**1**	**5/3**	**-4/3**	**1/3**	**0**
x_5	**35/3**	**0**	**19/3**	**-5/3**	**2/3**	**1**

X^2 $S^2 = y_{00} + y_{0q}\theta = -\frac{10}{3} - \frac{16}{3}\theta \xrightarrow[\theta \to +\infty]{} -\infty$

该问题没有有限的最优解

$X^0 = (0, 0, 0, 10, 5)^T$ $S^0 = 0$

$X^1 = (10/3, 0, 0, 0, 35/3)^T$ $S^1 = -10/3$ ↓

第一章 线性规划

第四节 单纯形法

- 典式
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- 两阶段法

作业：P95 6 (1) (2) (3)

作业：P41 6 (1) (2) (3)