7.4 Hermite 插值

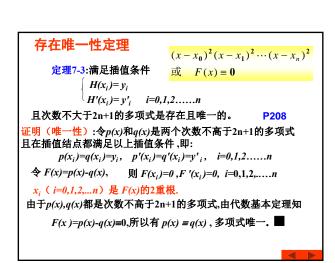
• 一.问题描述

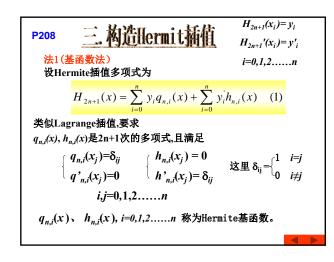
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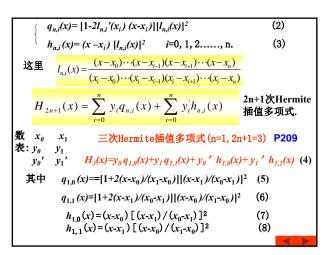
- <u>二.定义</u>
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一. 问题描述

Hermite插值也叫带指定微商值的插值,它要构造一个插值函数,不但在给定节点上取函数值,而且取已知微商值,使插值函数和被插函数的密和程度更好。







$$H_{2n+1}(x) = \sum_{i=0}^{n} y_{i} q_{n,i}(x) + \sum_{i=0}^{n} y_{i} h_{n,i}(x)$$

三次Hermite插值函数的构造(n=1,2n+1=3)

已知数表:

(7)

(8)

构造三次Hermite插值多项式H₃(x).

) 其中

 $H_3(x)=y_0 q_{1,0}(x)+y_1 q_{1,1}(x)+y_0' h_{1,0}(x)+y_1' h_{1,1}(x)$

 $q_{1,0}(x) = [1+2(x-x_0)/(x_1-x_0)][(x-x_1)/(x_0-x_1)]^2$

(6) $q_{1,1}(x)=[1+2(x-x_1)/(x_0-x_1)][(x-x_0)/(x_1-x_0)]^2$

 $h_{1.0}(x) = (x-x_0)[(x-x_1)/(x_0-x_1)]^2$

 $h_{1,1}(x) = (x-x_1)[(x-x_0)/(x_1-x_0)]^2$

例1: 求过0、1两点构造一个三次插值多项式H3(x),满足 条件: H₃(0)=1, H₃ '(0)=1/2, H₃(1)=2, H₃ '(1)=1/2.

•**M**: $x_0=0, x_1=1; y_0=1, y_1=2; y_0'=1/2, y_1'=1/2;$

•由式(5)-(8)得

 $q_{1,0}(x)=(2x+1)(x-1)^2$, $q_{1,1}(x)=(3-2x)x^2$

 $h_{1.0}(x)=x(x-1)^2$, $h_{1.1}(x)=x^2(x-1)$

• $H_3(x)=(1+2x)(x-1)^2+2(3-2x)x^2+0.5(x-1)^2x+0.5(x-1)x^2$ $=-x^3+1.5x^2+0.5x+1$

利用Newton差商插值构造 Hermite 插值多项式

已给数 据表:

x	x ₀	x ₁	 X _n	$H_{2n+I}(x_i) = y_i$
f(x)	f(x0)	f(x1)	 f(xn)	$H_{2n+1}'(x_i)=y'_i$
f '(x)	f'(x0)	f'(x1)	 f'(xn)	
				i=0,1,2n

定义新点列 $z_{2i}=z_{2i+1}=x_i$, (i=0,1,...,n),得: $z_0,z_1,z_2,...,z_{2n+1}$ 定义 $f[z_{2i}, z_{2i+1}] = f'(x_i)$, (i=0,1,...,n),构造差商表:

z f(z) 1阶差商

2阶差商 … 2n阶差商

 $z_0 = x_0$ $f(z_0) - f[z_0, z_1] = f'(x_0) - f[z_0, z_1, z_2]$ ··· $f[z_0, z_1, \dots, z_{2n+1}]$

 $z_1 = x_0$ $f(z_1, \overline{z_2})$ $f[z_1, z_2, z_3]$... $f(z_2) - f[z_2, z_3] = f'(x_2) - f[z_2, z_3, z_4] \cdots$

 $z_3 = x_1 \qquad f(z_3)$

 $f[z_3,z_4]$ $f[z_3,z_4,z_5]$...

 $z_{2n} = x_n$ $f(z_{2n}) f[z_{2n}, z_{2n+1}] = f'(x_n)$

 $z_{2n+1} = x_n \quad f(z_{2n+1})$

Hermite 插值多项式为

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0)(x - z_1) \dots (x - z_{k-1})$$

特别,三次Hermite插值多项式为(n=1,2n+1=3)

 $H_3(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, z_1](x - z_0)(x - z_$ $+ f[z_0, z_1, z_2, z_3](x-z_0)(x-z_1)(x-z_2)$

 $= f(x_0) + f'(x_0)(x - x_0) + f[z_0, z_1, z_2](x - x_0)^2 +$

 $f[z_0, z_1, z_2, z_3](x-x_0)^2(x-x_1)$

例1: 求过0、1两点构造一个三次插值多项式H₃(x),满足 条件: $H_3(0)=1$, $H_3'(0)=1/2$, $H_3(1)=2$, $H_3'(1)=1/2$.

解:构造差商表 z; f(z;) 1阶差商 2阶差商 3阶差商

定义 z₀=z₁=0, $z_2 = z_3 = 1$

 $\underline{1} \longrightarrow \underline{1/2} \longrightarrow \underline{1/2}$ - 11 -1/2 1^{-}

2 1/2 2.

 $H_3(x) = f(x_0) + f'(x_0)(x - x_0) + f[z_0, z_1, z_2](x - x_0)^2 +$ $f[z_0, z_1, z_2, z_3](x-x_0)^2(x-x_1)$

所求 $H_3(x)=1+(1/2)x+(1/2)x^2-x^2(x-1)$

四. 余项定理

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<mark>定理7-4</mark> 若f(x)在区间[a,b]存在2n+2阶导数,则其 Hermite插值余项为:

 $\frac{\gamma(2n+2)}{2}(\xi)[\omega_{n+1}^{2}(x)] \quad \xi \in (a,b) \quad (1)$ $R_{2n+1}(x) = f(x) - H_{2n+1}(x) = \frac{f(x)}{x^2}$ (2n+2)!

 $\omega_{n+1}(x) = (x-x_0)(x-x_1)....(x-x_n)$

特别,三次Hermite插值余项为

 $R_3(x) = f(x) - H_3(x) = \frac{f^{(4)}(\xi)[\omega_2^2(x)]}{4!} \quad \xi \in (a,b) \quad (10)$

 $\omega_2(x) = (x - x_0)(x - x_1)$

定理7-4 若f(x)在区间[a, b]存在2n+2阶导数,则其 Hermite插值余项为:

$$R_{2n+1}(x) = f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)[\omega_{n+1}^2(x)]}{(2n+2)!} \quad \xi \in (a,b) \quad (1)$$

 $\omega_{n+1}(x) = (x-x_0)(x-x_1)....(x-x_n)$

证明

当 $x=x_i$,i=0,1,2....时,左右两端为零,公式成立.

以下考虑
$$x_{\neq x_i}, x \in [a,b]$$
。 因为在结点 x_0, x_1, \dots, x_n 上
$$\begin{cases} f(x_i) = H_{2n+1}(x_i) \\ f'(x_i) = H_{2n+1}'(x_i) \end{cases}$$
 所以
$$\begin{cases} R_{2n+1}(x_i) = f(x_i) - H_{2n+1}(x_i) = 0 \\ R_{2n+1}'(x_i) = f'(x_i) - H_{2n+1}'(x_i) = 0, \end{cases}$$

因此 $x_i(i=0,1....n)$ 为 $R_{2n+1}(x)$ 的二重零点。 $R_{2n+1}(x)$ 可设为: $R_{2n+1}(x) = k(x) \left[\omega_{n+1}(x) \right]^2$

 $R_{2n+1}(x)$ 可设为: $R_{2n+1}(x) = k(x) [\omega_{n+1}(x)]^2$

- k(x)为待定函数。做辅助函数
- $F(z)=f(z)-H_{2n+1}(z)-k(x)[\omega_{n+1}(z)]^2$
- F(x)=0,所以z=x是F(z)的一个零点,此外 x_0 x_n 都是F(z)的二重零点,F(z)在[a,b]上有n+2个零点。
- 由洛尔定理, F'(z)在[a,b]上至少有2n+2个零点,
- F"(z)在[a,b]上至少有2n+1个零点.依此类推,
- ・ $F^{(2n+2)}(x)$ 在插值区间[a,b]中至少存在一个零点,设为 ξ ,即 $F^{(2n+2)}(\xi)=0$ 。
- 故有 $0=F^{(2n+2)}(\xi)=f^{(2n+2)}(\xi)-0-(2n+2)!k(x)$

注意 $|\omega_n \frac{k(x) = f^{(2n+2)}(\xi)/(2n+2)!}{\blacksquare}$ 页式, $H_{2n+1}(z)$ 是 2n+1次多项式,

$$R_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)[\omega_{n+1}^2(x)]}{(2n+2)!} \quad \xi \in (a,b)$$

五.一般插值

实际问题中还会有其它的插值问题,这类问题可用多种方法解决.

例2 已知数据表: x

$$\begin{array}{c|cc} x & 0 & 1 \\ \hline f(x) & y_0 & y_1 \end{array}$$

 $f'(x) y_0'$

P210,例7-5

求过0,1两点的插值多项式 p(x),满足条件 $p(0)=y_0$, $p'(0)=y_0'$,, $p(1)=y_1$,并估计余项。

解:(k-1) (基函数法)它有三个条件,故 p(x)可设为二次多项式

 $p(x) = y_0 L_0(x) + y_1 L_1(x) + y_0' h_0(x)$

这里, $L_0(x)$, $L_1(x)$, $h_0(x)$ 是基函数.

 $p(x) = y_0 L_0(x) + y_1 L_1(x) + y_0 ' h_0(x)$ 要求 $L_0(x)$, $L_1(x)$, $h_0(x)$ 都是二次多项式, 且满足

对 x_0 =0有 $L_0(0)=1$ $L_1(0)=0$ $h_0(0)=0$ $L_0'(0)=0$ $L_1'(0)=0$ $h_0'(0)=1$

对 $x_1=1$ 有 $L_0(1)=0$ $L_1(1)=1$ $h_0(1)=0$

经计算,得 $L_0(x)=(-x-1)(x-1)=1-x^2$

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 $^{\bullet}L_1(x)=x^2$, $h_0(x)=x(1-x)$

•从而 $p(x)=y_0(1-x^2)+y_1x^2+y_0'(1-x)x$

法二) 令 p(x)=f(0)+f[0,1]x+ax(x-1), 这里

$$f(0) = y_0, \quad f[0,1] = \frac{y_1 - y_0}{1 - 0} = y_1 - y_0$$

从而
$$p(x)=y_0+(y_1-y_0)x+ax(x-1)$$

$$p'(x)=y_1-y_0+a(x-1)+ax$$
, $\boxplus p'(0)=y_0'$

得 $a=y_1-y_0-y_0$,

从而 $p(x)=y_0+(y_1-y_0)x+(y_1-y_0-y_0')x(x-1)$

法三) 令 $p(x)=ax^2+bx+c$

由: $p(0)=y_{\theta}$, $p(1)=y_{1}$, $p'(0)=y_{\theta}'$

得: $c = y_{\theta}$, $b = y_{\theta}'$, $a = y_1 - y_{\theta} - y_{\theta}'$

从而 $p(x) = (y_1 - y_\theta - y_\theta')x^2 + y_\theta'x + y_\theta$

法四)构造差商表 $z_i f(z_i)$ I阶差商 2阶差商

0 $\underline{y_0}'$ $\underline{y_0}$ $\underline{y_1 - y_0 - y_0}$ y_0 $y_1 - y_0$

$$p(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1)$$

所求多项式为: $p(x)=y_0 + y_0'x + (y_1-y_0-y_0')x^2$

其余项表达式为 $R(x) = f(x) - p(x) = \frac{f^{(3)}(\xi)}{2}x^2(1-x)$

 $R(\theta) = R'(\theta) = R(1) = \theta$

- 设R(x)=k(x)(1-x) x^2 ,k(x)待求, $x\neq0$,1; 作辅助函数
- $g(t)=f(t)-p(t)-k(x)(1-t)t^2$, g(t)有三个零点: 0、1、x; 利用洛尔定理,g'(t)有三个零点: ξ_{x0} 、 ξ_{x1} 、0
- ..., $g^{(3)}(t)$ 至少有一个零点 ξ ,即 $g^{(3)}(\xi)=0$ 。
- 由此可得 $k(x)=f^{(3)}(\xi)/3!$

作业

习题 7 P232: 14

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