OPTIMAL CONTROLLER SYNTHESIS FOR DECENTRALIZED SYSTEMS

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John Swigart

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Sanjay Lall, Primary Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Stephen Boyd

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Stephen Rock

Approved for the Stanford University Committee on Graduate Studies.

Patricia J. Gumport, Vice Provost Graduate Education

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Abstract

Optimal control theory can be classified, broadly speaking, as either centralized or decentralized. Centralized systems are generally thought of as a single decision maker, taking all sensor measurements and deciding all actuator inputs. By contrast, decentralized systems are typically characterized by multiple subsystems which each make decisions based on different sets of information.

Optimal control, as the name suggests, is aimed at finding control policies which minimize a particular norm of the closed-loop map. For centralized problems, the only constraint on the set of admissible control policies is that the resulting system be internally stable. These types of problems have been solved for quite some time, and a variety of methods now exist to explicitly construct the optimal policies. Decentralization imposes additional constraints on admissible policies. These constraints can, in many cases, make the optimization problem intractable. Thus, for decentralized problems, our goal is two-fold. First, we try to identify classes of systems for which tractable representations of the optimization problem exist. Second, we need to synthesize the optimal decentralized controllers.

This research is focused at decentralized problems which are represented by subsystems connected over graphs. The decentralization property is given as sparsity constraints on the admissible control policies. For such systems, there exists a simple condition which is sufficient for tractability of the optimization problem. For these systems, we provide a method for constructing the optimal decentralized controllers. These results provide significant insight into the structure of the optimal controllers. Moreover, they establish the state dimension of the optimal policies.

Preface

Autonomous systems are ubiquitous. Consider the average household. Most homes have refrigerators, ovens, and heating/air-conditioning units. For each of these systems, the user simply sets a temperature and lets the system perform the necessary functions to achieve and maintain that temperature. The user does not need to stand by the furnace and adjust the heat input to the room manually every minute. Similar systems exist to regulate the electricity, light intensity, and water temperature, although some might question the efficacy of the latter when they get in the shower.

Taking a step out of the house, people's cars contain a large number of autonomous systems. Such systems can control engine temperature and pressure, anti-lock brakes and cruise control. At intersections, some traffic lights respond to traffic patterns to minimize congestion. The list of autonomous systems continues; those listed here are a fraction of those encountered before ever reaching the first traffic light.

In all of these examples, each system consists of sensors, which measure the current state of the system, and actuators, which can affect a change to the system state. Sensors can measure temperature, voltage, etc., and may be subject to noise or errors. Actuators could be valves, motors, or many other things. Much research and development has gone into improving the efficiency, sensitivity, and accuracy of sensors and actuators; this is beyond the scope of this work.

The question that arises is how to map the sensor measurements to the actuator input, in order to achieve the desired system behavior. Control theory is aimed at answering this question for all types of systems. The work presented here represents a small piece of the answer to this general question.

Acknowledgments

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It is important that I also thank the members of my research group. I probably learned as much about control theory in group meetings and one-on-one discussions with them as I did in reading textbooks.

Last, but certainly not least, I must thank my fiancée Cate Crowley. Cate, you are, without a doubt, the reason that I am here today. I could not have done this without your support. I love you!

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Chapter 1

Introduction

1.1 Motivation

Control theory is typically classified as either centralized or decentralized. Centralized systems can be represented as a single dynamical system connected to an individual local controller. On the other hand, decentralized systems usually consist of multiple subsystems whose dynamics may be coupled. Each subsystem has a local controller, and these controllers may have different sets of information about the global system with limited communication between each other.

Centralized control is considered the classical version of these problems, since solutions to these problems have been known for quite some time, for a large variety of problems. In particular, for the classic LQG problem (Linear dynamics, Quadratic cost, Gaussian noise), the optimal control policy is known to be linear and satisfy a constant gain times a Kalman filter estimator [32]. Thus, for control of a single system, many results exist for finding an optimal policy.

However, for decentralized systems, the story is not so straightforward. Given the success of centralized controller synthesis, it was generally believed that optimal decentralized policies would have similarly clean solutions. However, the Witsenhausen counterexample demonstrated that even simple decentralized problems can be intractable [30]. This example consists of a two-player linear system. The objective function is quadratic, and the system signals are scalar. Nevertheless, it was shown that all linear policies are strictly suboptimal for this problem. In fact, for this problem the optimal policy remains unknown. This notion that decentralized problems are, in general, intractable was later generalized in [2].

Nevertheless, decentralized control problems are becoming increasingly prevalent. One approach to finding optimal control policies is to simply add a central computer or decision maker, which takes in all observations and decides each subsystem's action; thus, turning the problem into a centralized one. However, for many systems such a method is impossible or impractical, given the structure of the problem.

Consider the following examples. The internet has become an incredibly powerful tool for transmitting information. However, in order for information on one computer to get transmitted to another, this information must get sent through a number of routers. These routers receive many messages and must decide which messages to pass and where to pass them. Of course, the best routing policy could be implemented if a single computer had access to all routing requests and could make decisions for all the routers. However, it is clear that such a strategy would require a significant amount of bandwidth for transmitting all coordination information. Moreover, the amount of computation required for such a strategy would be prohibitive; not to mention the practical difficulties of such a policy. Thus, it is required that each router have its own local controller to decide how to pass messages, such that the internet as a whole still functions efficiently.

As another example, one development in aerospace engineering relates to trying to observe extra-solar planets. Unfortunately, such planets are incredibly difficult to see against the brightness of the star that they orbit. As a result, very large telescopes are required to make these observations. As an alternative, it is possible to obtain the same results by arranging smaller satellite telescopes in a particular orientation. However, this approach requires very accurate maintenance of the constellation over very large distances. While communication may be possible in some cases, this would add complexity and significantly increase the power requirements for the satellites. Thus, a decentralized policy is required for this problem. Other examples include formation flight, systems consisting of teams of vehicles, or other large, spatially distributed systems such as the power grid.

1.2 Previous Work

The general optimal control problem can be written as the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\| \\ \text{subject to} \quad & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \mathcal{S} \end{aligned}$$

The matrices P represent the dynamics of the system, and we seek stabilizing controllers $\mathcal{K}$ belonging to some set $\mathcal{S}$.

Since the general version of this problem remains intractable, much of the research in decentralized control theory has been aimed at characterizing which problems are tractable [7, 11, 13, 1]. Recently, these results were unified and generalized under the concept of *quadratic invariance* [17]. Quadratic invariance is an algebraic property:

$$\mathcal{K}P_{22}\mathcal{K} \in \mathcal{S} \qquad \text{for all } \mathcal{K} \in \mathcal{S}$$

In its simplest form, if P_{22} and the set $\mathcal{S}$ satisfy this property, then the optimization problem admits a convex parametrization. This is similar to the classic Youla parametrization in the centralized case.

For distributed systems over networks, conditions for quadratic invariance, and thus tractability, of such systems were provided in [22, 16]. Similar results were obtained in [20] using a poset-based framework.

Tractability, as it is used here, means that the underlying optimization problem has a convex parametrization [4]. Though convex, the formulations remain infinite-dimensional. In particular, the approach in [15] requires a change of variables via the Youla parametrization, and optimization over this parameter. Since the parameter itself is a linear stable system, a standard approximation would be via a finite basis for the impulse response function [3]. This is in contrast to the centralized case, for which explicit state-space formulae can be constructed.

For controller synthesis, one of three classical approaches are typically used.

Semi-definite programming attempts to reformulate the optimization as a set of linear matrix inequalities. For the types of decentralized problems considered here, a number of SDP approaches have been proposed in [19, 14, 31, 8]. Unfortunately, solutions via this method are numerical. In other words, because the solutions are not analytic, there is limited intuition that can be obtained from these approaches. In particular, for the classic centralized problem, the Riccati methods are apparent in the SDP formulations; this is not currently the case for these decentralized problems. Moreover, the estimation structure of the optimal solutions, presented in this work, cannot be obtained from those results.

Another approach to synthesis is via dynamic programming. For the continuous time, two-player version of our work, a dynamic programming approach was used in [28]. Though dynamic programming is not presented in this work, many of the results presented herein were also obtained via this method [21, 23, 24]. Also, in [6], an approximate dynamic programming scheme for solving decentralized control problems was suggested.

Lastly, factorization methods can be applied to these problems. This is the technique used in this work. A factorization method was also suggested, though not implemented, in [29]. For the quadratic case, vectorization [15] reduces the problem to a centralized one, but loses the intrinsic structure and results in high-order controllers.

However, in none of these approaches have explicit state-space formulae been derived for these problems. In the work presented here, a spectral factorization approach is taken to construct explicit state-space formulae for these problems. Such formulae offer the practical advantages of computational reliability and simplicity. As a result, we can efficiently and analytically compute the optimal controllers for the decentralized problem. Moreover, we gain significant insight into the form of the solutions which previous approaches do not provide. In addition, we establish the order of the optimal controllers for these systems, which is an open problem for general decentralized systems, even in the simplest cases.

This work is organized as follows. As suggested above, the first step in decentralized control is the convex parametrization of the optimization problem. Chapter 2 introduces the class of problems that will be considered here. The interconnection structure of these decentralized systems are represented by graphs. The set of tractable problems can then be represented in terms of these graphs.

Since spectral factorization is the approach used in this work, Chapter 3 develops this approach by considering both the classical, centralized problem, as well as a couple non-classical information structures. The remaining chapters apply this methodology to three different problems. Since the simplest non-trivial decentralized problem is the two-player system, Chapter 4 addresses the state-feedback version of this problem. These results are then extended to arbitrary graphs in Chapter 5. Lastly, the state-feedback assumption is weakened for the two-player problem. To this end, a partial output feedback structure is considered in Chapter 6. Some concluding remarks are made in Chapter 7.

Chapter 2

Systems Over Graphs

This chapter introduces the notation that will be used throughout this work. This includes both the functional spaces that will considered as well as our mathematical representation of graphs. For the decentralized systems considered here, the set of allowable controllers will be defined by sparsity constraints. Lastly, since decentralized control problems are intractable in general, we establish a simple condition for which tractable formulations of these problems exist.

2.1 Hardy Spaces

We use the following notation in this work. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The complex open unit disc is $\mathbb{D}$; its boundary, the unit circle, is $\mathbb{T}$; and the closed unit disc is $\bar{\mathbb{D}}$. We denote $\mathbb{R}^{m\times n}$ as the set of $m \times n$ matrices in $\mathbb{R}$. This notation will also be used to denote $m \times n$ block matrices, where the dimensions of the blocks are implied by the context.

The set $\mathcal{L}_2(\mathbb{T})$ is the Hilbert space of Lebesgue measurable functions on $\mathbb{T}$, which are square integrable, with inner product

$$\langle F, G\rangle = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}(F^*(e^{j\theta})G(e^{j\theta}))\, d\theta$$

As is standard, $\mathcal{H}_2$ denotes the Hardy space of functions analytic outside the closed unit disc, and at infinity, with square-summable power series.

$$\mathcal{H}_2 = \left\{ f : \{\infty\} \cup \mathbb{C} \setminus \bar{\mathbb{D}} \to \mathbb{C} \;\middle|\; \exists x \in \ell_2(\mathbb{Z}_+) \text{ s.t. } f(z) = \sum_{k=0}^{\infty} x_k z^{-k} \right\}$$

The set $\mathcal{H}_2^\perp$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$. The prefix $\mathcal{R}$ indicates the subsets of proper real rational functions. That is, $\mathcal{RL}_2$ is the set of transfer functions with no poles on $\mathbb{T}$, and $\mathcal{RH}_2$ is the set of transfer functions with no poles on or outside $\mathbb{T}$.

Also, we denote the subspace $\mathcal{L}_\infty(\mathbb{T})$ as the set of Lebesgue measurable functions which are bounded on $\mathbb{T}$. Similarly, $\mathcal{H}_\infty$ is the subspace of $\mathcal{L}_\infty$ with functions analytic outside of $\mathbb{T}$, and $\mathcal{H}_\infty^-$ is the subspace of $\mathcal{L}_\infty$ with functions analytic inside $\mathbb{T}$. Consequently, $\mathcal{RH}_\infty$ is the set of transfer functions with no poles outside of $\mathbb{T}$. Note that, in this case, $\mathcal{RH}_2 = \mathcal{RH}_\infty$; we will use these spaces interchangeably.

For any Hilbert spaces S, T and bounded operator $G : S \to T$, we let $G^* : T \to S$ denote its adjoint operator. The special case is when G is a real matrix; in which case, G^T denotes its transpose. Also, the following notation denotes the image of G.

$$GS = \{G(F) \in T \mid F \in S\}$$

Some useful facts about Hardy spaces which we will make use of in this work are [32]:

- if $G \in \mathcal{L}_\infty$, then $G\mathcal{L}_2 \subset \mathcal{L}_2$
- if $G \in \mathcal{H}_\infty$, then $G\mathcal{H}_2 \subset \mathcal{H}_2$
- if $G \in \mathcal{H}_\infty^-$, then $G\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$

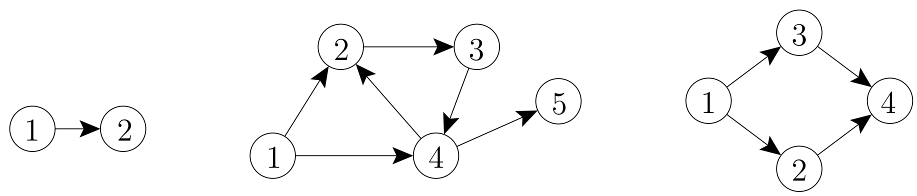
For transfer functions $F \in \mathcal{RL}_2$, we use the notation

$$F(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$$

Lastly, we define $\mathrm{P}_{\mathcal{H}_2} : \mathcal{L}_2 \to \mathcal{H}_2$ as the orthogonal projection onto $\mathcal{H}_2$.

2.2 Graph Notation

We represent a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ by a set of N vertices $\mathcal{V} = \{v_1, \ldots, v_N\}$ and a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. There is a directed edge from vertex v_i to vertex v_j if $(v_i, v_j) \in \mathcal{E}$. We assume that $\mathcal{G}$ has no self-loops; that is, $(v_i, v_i) \notin \mathcal{E}$ for all $v_i \in \mathcal{V}$. Visually, a directed graph can be represented like the examples in Figure 2.1.



(a) (b) (c)

Figure 2.1: Directed Graph Examples

For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we define the *vertex adjacency matrix* $A^{\mathcal{V}} \in \mathbb{R}^{N \times N}$ as

$$A^{\mathcal{V}}_{ij} = \begin{cases} 1 & (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

The *transitive closure* of the graph, the set of all paths, is defined by the matrix $\mathcal{M}^{\mathcal{G}} \in \mathbb{R}^{N \times N}$ as

$$\mathcal{M}^{\mathcal{G}} = (I + A^{\mathcal{V}})^{N-1}$$

That is, $\mathcal{M}^{\mathcal{G}}_{ij} \neq 0$ if and only if there exists a directed path from vertex j to vertex i, or $i = j$.

For each vertex $i \in \mathcal{V}$, we define the set of *ancestors* $\mathcal{A}_i$, and the set of *descendants* $\mathcal{D}_i$ as

$$\mathcal{A}_i = \{j \in \mathcal{V} \mid \mathcal{M}^{\mathcal{G}}_{ij} \neq 0\}$$
$$\mathcal{D}_i = \{j \in \mathcal{V} \mid \mathcal{M}^{\mathcal{G}}_{ji} \neq 0\}$$

In other words, $j \in \mathcal{A}_i$ if and only if there exists a directed path from j to i, or $i = j$. Similarly, $j \in \mathcal{D}_i$ if and only if there exists a directed path from i to j, or $i = j$. Note that $i \in \mathcal{A}_i$ and $i \in \mathcal{D}_i$. Removing i from these sets, we define the sets

$$\begin{aligned} \mathcal{A}_i' &= \mathcal{A}_i \setminus \{i\} \\ \mathcal{D}_i' &= \mathcal{D}_i \setminus \{i\} \end{aligned}$$

2.3 Sparsity Structures

Since we will be dealing with systems represented by directed graphs, we will need to deal with matrices that satisfy certain sparsity constraints. For any $m \times n$ matrix A and ring R, we define the set of matrices in R with similar sparsity structure by

$$\mathrm{Sparse}(A; R) = \{B \in R^{m \times n} \mid B_{ij} = 0 \text{ if } A_{ij} = 0\}$$

Typically, $R = \mathbb{R}$ or $R = \mathcal{RL}_\infty$. We will be particularly concerned with the space $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$. Note that $\mathcal{S}$ has an orthogonal complement, such that

$$G \in \mathcal{S}^\perp \qquad \Leftrightarrow \qquad \begin{array}{ll} G_{ij} \in \mathcal{H}_2^\perp & \text{if } \mathcal{M}_{ij}^{\mathcal{G}} \neq 0 \\ G_{ij} \in \mathcal{L}_2 & \text{if } \mathcal{M}_{ij}^{\mathcal{G}} = 0 \end{array}$$

For $N \times N$ block matrices, whose blocks are indexed by $\mathcal{V} \times \mathcal{V}$, it will be convenient to construct submatrices as follows. For sets $S, T \subset \mathcal{V}$, we define M_{ST} as the $|S| \times |T|$ matrix which is constructed by eliminating the rows of M not in S and the columns of M not in T. In cases where $S = T$, we denote $M_S = M_{SS}$. In addition, when $S = T = \mathcal{V}$, we drop subscripts altogether, since $M_{\mathcal{V}} = M$. This notational convention also applies to constructing subvectors from vectors which are indexed by $\mathcal{V}$.

This subscripting notation will be frequently used with the identity matrix. As an example, suppose $\mathcal{D}_1 = \{v_1, v_3, v_4\}$. Then, $I_{\mathcal{D}_1 v_2} = 0$ since $v_2 \notin \mathcal{D}_1$, and

$$I_{\mathcal{D}_1 v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad I_{\mathcal{D}_1 \mathcal{D}_1'} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{\mathcal{D}_1 \mathcal{V}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \end{bmatrix}$$

2.4 Networked System

Each vertex in $\mathcal{V}$ represents a separate linear time-invariant plant with corresponding controller, as shown in Figure 2.2.

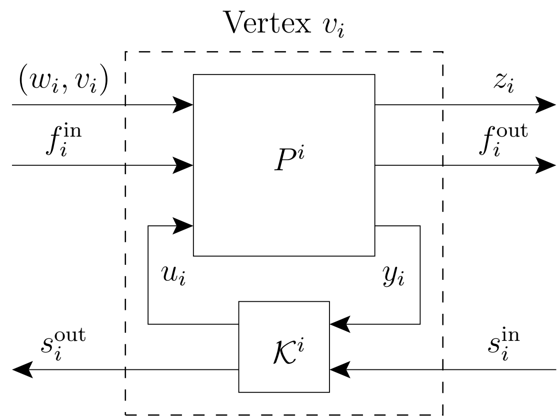


Figure 2.2: Plant and Controller for Each Vertex

The plant dynamics can be represented as

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t) + H_i w_i(t) + f_i^{\text{in}}(t) \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state of system i, and $u_i(t) \in \mathbb{R}^{m_i}$ is the input applied to system i. The individual subsystems are affected by exogenous, independent noise $w_i(t) \in \mathbb{R}^{n_i}$, and by the dynamics of the subsystems which are connected to it, represented by

$$f_i^{\text{in}}(t) = \sum_{j \in \mathcal{A}_i'} A_{ij} x_j(t) + B_{ij} u_j(t)$$

In turn, subsystem i can affect other subsystems, via $f_i^{\text{out}}(t)$. Also, the measured output of each system is $y_i(t) \in \mathbb{R}^{p_i}$, and is given by

$$y_i(t) = C_i x_i(t) + D_i v_i(t) + \sum_{j \in \mathcal{A}_i'} C_{ij} x_j(t)$$

where $v_i(t) \in \mathbb{R}^{p_i}$ is independent sensor noise.

Each controller $\mathcal{K}^i$ makes decision u_i based on the measured output of its own system y_i, and any signals received from neighboring subsystems, via $s_i^{\text{in}}(t)$. It also decides what information to transmit to other controllers, via $s_i^{\text{out}}(t)$.

In actuality, every networked system consists of two underlying graph structures: a graph $\mathcal{G}^P = (\mathcal{V}, \mathcal{E}^P)$ determining the dynamic coupling of the plants, and a graph $\mathcal{G}^K = (\mathcal{V}, \mathcal{E}^K)$ for the allowable communication channels of the controllers. In other words, $\mathcal{G}^P$ determines which subsystems can influence $f_i^{\text{in}}(t)$, and $\mathcal{G}^K$ determines which controllers can send information to $\mathcal{K}^i$, via $s_i^{\text{in}}(t)$.

Connecting the plants according to $\mathcal{G}^P$, we obtain the following discrete time state-space system for the overall system dynamics.

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Hw(t) \\ y(t) &= Cx(t) + Dv(t) \end{aligned} \tag{2.2}$$

where $x(t) = \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T$, and $u(t) = \begin{bmatrix} u_1(t)^T & \cdots & u_N(t)^T \end{bmatrix}^T$, and similarly for $w(t)$ and $v(t)$. It is straightforward to show that the block matrices $A, B, C \in \mathbb{R}^{N\times N}$ satisfy

$$A \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^P}; \mathbb{R}) \qquad B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^P}; \mathbb{R}) \qquad C \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^P}; \mathbb{R})$$

and H and D are block diagonal. We will assume that H is also invertible. Note that H being invertible simply implies that no component of the state evolves noise-free. This assumption merely simplifies our presentation, while not fundamentally affecting our results.

Similarly, we can construct the overall controllers according to $\mathcal{G}^K$. It will be assumed throughout this work that there are no communication delays or bandwidth constraints between the controllers. In other words, if $(i, j) \in \mathcal{E}^K$, then controller j has access to any information that is available to controller i. Consequently, we will search for controllers of the form

$$\begin{aligned} q_i(t+1) &= A_{K_i} q_i(t) + B_{K_i} y_{\mathcal{A}_i^K}(t) \\ u_i(t) &= C_{K_i} q_i(t) + D_{K_i} y_{\mathcal{A}_i^K}(t) \end{aligned} \tag{2.3}$$

for each $i \in \mathcal{V}$. In other words, player i makes decision u_i based on the history of y_j's, for each j which is an ancestor of vertex i in $\mathcal{G}^K$. Equivalently, we want a transfer function $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$, such that

$$u = \mathcal{K}y$$

For the classical finite-horizon LQG problem (Linear dynamics, Quadratic cost, Gaussian noise), we aim to find controllers which minimize the following quadratic cost function.

$$\min_{\mathcal{K}} \quad \mathrm{E} \sum_{t=0}^{T} x(t)^T Q x(t) + u(t)^T R u(t)$$

where $Q \geq 0$, and $R > 0$, and $\mathrm{E}(\cdot)$ denotes expected value. This quadratic cost can be generalized by defining the cost vector

$$z(t) = \hat{C}x(t) + \hat{D}u(t)$$

so that the objective function becomes

$$\min_{\mathcal{K}} \quad \mathrm{E} \sum_{t=0}^{T} \|z\|_2^2$$

Analogously, the infinite-horizon cost is given by

$$\lim_{T \to \infty} \frac{1}{T+1} \mathrm{E} \sum_{t=0}^{T} \|z(t)\|_2^2$$

It is well-known that this cost is equivalent to the $\mathcal{H}_2$ norm of the closed-loop transfer function from $\xi = (w, v)$ to z, discussed below. For simplicity, we will assume that $\hat{D}^T\hat{D} > 0$. Note that no sparsity constraints are assumed for $\hat{C}$ or $\hat{D}$, nor is $\hat{C}^T\hat{D}$ assumed to be zero. This formulation allows for any coupling of the states and actions in the cost.

Consequently, our plant can be expressed as the matrix $P \in \mathcal{RL}_\infty$, where

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix}$$

and

$$P = \left[\begin{array}{c|cc} A & \begin{bmatrix} H & 0 \end{bmatrix} & B \\ \hline \hat{C} & 0 & \hat{D} \\ C & \begin{bmatrix} 0 & D \end{bmatrix} & 0 \end{array}\right] \tag{2.4}$$

Figure 2.3 illustrates the overall feedback system, with plant P given by (2.4) and controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$.

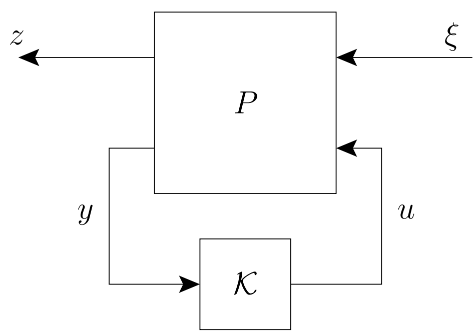


Figure 2.3: Overall Feedback System

In addition to satisfying the constraint that $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$, it is a basic requirement that the resulting closed-loop system be *internally stable*. We will define internal stability in terms of the plant states $x(t)$ and the controller states $x_K(t)$. That is, the closed-loop system is defined to be internally stable if

$$\lim_{t\to\infty} x(t), x_K(t) \to 0$$

for any initial conditions $x(0)$ and $x_K(0)$.

Equivalently, looking at Figure 2.4, the system will be internally stable if and only if the mapping between $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is in $\mathcal{RH}_\infty$.

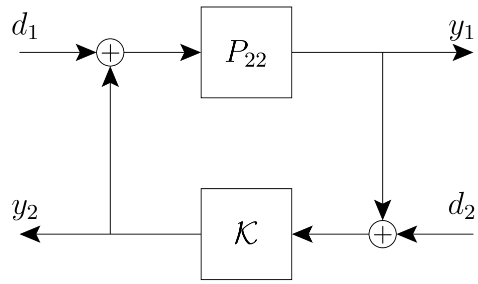


Figure 2.4: Feedback System for Internal Stability

When $\mathcal{K}$ satisfies this property, we say that $\mathcal{K}$ *is stabilizing*, or that $\mathcal{K}$ *stabilizes* P. Using this definition, it is straightforward to show that $\mathcal{K}$ is stabilizing if and only if

$$\begin{bmatrix} QP_{22} & Q \\ P_{22} + P_{22}QP_{22} & P_{22}Q \end{bmatrix} \in \mathcal{RH}_\infty$$

where $Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1}$. Thus, whenever P is stable, it is clear that $\mathcal{K}$ is stabilizing if and only if $\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \in \mathcal{RH}_\infty$.

Lastly, we define $\mathcal{F}(P, \mathcal{K})$ as the linear fractional transformation

$$\mathcal{F}(P, \mathcal{K}) = P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}$$

As noted above, our objective function is the $\mathcal{H}_2$ norm of the closed-loop transfer function from ξ to z, which is given by $\mathcal{F}(P, \mathcal{K})$. In other words, with ξ normally distributed $\mathcal{N}(0, I)$, we have the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|\mathcal{F}(P, \mathcal{K})\|_2 \\ \text{subject to} \quad & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty) \end{aligned} \tag{2.5}$$

This work is aimed at solving this optimal decentralized control problem.

2.5 Tractable Graph Structures

In general, the optimization problem (2.5) is intractable. Thus, before trying to solve this problem, we must restrict our attention to those problems for which we can find convex representations. To this end, the remainder of this chapter is aimed at establishing a class of systems for which convex parametrizations exist.

Since we are minimizing a linear fractional transformation under sparsity constraints, it is necessary to determine the sparsity pattern of the image of this transformation. In particular, we are interested in finding the sparsity pattern of the set

$$\{\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)\}$$

To do this, we must establish an algebra for sparsity sets. It is straightforward to show that for any matrices $A, B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$, both $A + B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$ and $AB \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$. Thus, $\text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$ is closed under both addition and multiplication. However, showing that it is also closed under inversion requires the following lemma.

Lemma 1. *For any matrix $A \in \mathbb{C}^{n\times n}$, define $A^{bin} \in \mathbb{R}^{n\times n}$ as*

$$A^{bin} = \begin{cases} 1 & \text{if } A_{ij} \neq 0 \\ 0 & \text{if } A_{ij} = 0 \end{cases}$$

If $\det(I - A) \neq 0$, *then* $\text{Sparse}((I - A)^{-1}; \mathbb{C}) \subseteq \text{Sparse}((I + A^{bin})^{n-1}; \mathbb{C})$.

Proof. Suppose $Q \in \text{Sparse}((I - A)^{-1}; \mathbb{C})$. We need to show that $Q_{ij} = 0$ whenever $((I + A^{\text{bin}})^{n-1})_{ij} = 0$. To this end, suppose $((I + A^{\text{bin}})^{n-1})_{ij} = 0$. From the construction of A^{bin}, this implies that $((A^{\text{bin}})^m)_{ij} = 0$, and thus $(A^m)_{ij} = 0$, for all $m = 0, \ldots, n-1$. In addition, the Cayley-Hamilton theorem tells us that, for all $p \geq n$,

$$A^p = \sum_{m=0}^{n-1} \beta_m A^m$$

for some $\beta_0, \ldots, \beta_{n-1}$. Consequently, it is clear that $(A^m)_{ij} = 0$, for all $m \geq 0$.

Now, consider the spectral radius $\rho(A)$, and suppose $0 < k < \frac{1}{\rho(A)}$ (if $\rho(A) = 0$, suppose $k > 0$). Then, we know

$$(I - kA)^{-1} = \sum_{m=0}^{\infty} (kA)^m$$

More importantly, using the above result, it is clear that

$$((I - kA)^{-1})_{ij} = \sum_{m=0}^{\infty} ((kA)^m)_{ij} = 0$$

since every element in the summation is zero. However, we also know that $((I - kA)^{-1})_{ij} = (\det(I - kA))^{-1}(\operatorname{adj}(I - kA))_{ij}$, which is a rational expression in k. We know from complex analysis that a rational expression which is not identically zero can be equal to zero at only a finite number of points. Since $((I - kA)^{-1})_{ij} = 0$ for all $0 < k < \frac{1}{\rho(A)}$, then $((I - kA)^{-1})_{ij}$ must be identically zero for all k, including when $k = 1$. As a result, since $(I - A)^{-1}$ exists, then $((I - A)^{-1})_{ij} = 0$. Thus, for any $Q \in \text{Sparse}((I - A)^{-1}; \mathbb{C})$, we must have $Q_{ij} = 0$, as desired. ■

Note that every $\mathcal{M}^{\mathcal{G}}$ is expressed as $(I + A^{\mathcal{V}})^{N-1}$. Thus, for any invertible $A \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$, we have $A^{-1} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$, so $\text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{C})$ is closed under inversion. In addition, this result shows that $\text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ is closed under inversion by applying the lemma point-wise in frequency.

Given this result, it is clear that sparsity constraints are invariant under linear fractional transformations. However, two sparsity sets exist in the expression

$$\mathcal{K}(I - P_{22}\mathcal{K})^{-1}$$

By design, $\mathcal{K}$ must have the sparsity pattern of $\mathcal{M}^{\mathcal{G}^K}$. In addition, by the construction of P_{22} in (2.4), it is clear that P_{22} has the sparsity pattern of $\mathcal{M}^{\mathcal{G}^P}$. Thus, the resulting sparsity pattern of this expression must depend on the coupling of these two sparsity sets. The following theorem formalizes this.

Theorem 2. *Suppose $\mathcal{G}^P$ and $\mathcal{G}^K$ are directed graphs. Let P_{22} be defined as in* (2.4).

Then,

$$\{\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)\} = \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$$

if and only if

$$P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$$

Proof. ($\Leftarrow$) Suppose $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$. It is clear that $\text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$ is closed under multiplication. As a result, we see that $P_{22}\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$ for all $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$. It follows from Lemma 1 that

$$(I - P_{22}\mathcal{K})^{-1} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$$

and so $\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$. Thus,

$$\{\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)\} \subset \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$$

Conversely, let $Q \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$. Then,

$$\mathcal{K} = Q(I + P_{22}Q)^{-1} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$$

by the same argument as above. Moreover, $Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1}$. Consequently,

$$\text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty) \subset \{\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)\}$$

Thus, the two sets are equivalent.

($\Rightarrow$) For notational convenience, let $G = P_{22}$. Suppose $G \notin \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$. Then, there exists i, j such $G_{ij} \neq 0$ and $\mathcal{M}^{\mathcal{G}^K}_{ij} = 0$. Let K_i and K_j be real matrices, of appropriate dimensions, such that $K_i G_{ij} K_j \neq 0$, and let $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathbb{R})$ be defined as

$$\mathcal{K}_{km} = \begin{cases} K_i & \text{if } k = m = i \\ K_j & \text{if } k = m = j \\ 0 & \text{otherwise} \end{cases}$$

With this $\mathcal{K}$, it is straightforward to show that

$$\left(\mathcal{K}(I-G\mathcal{K})^{-1}\right)_{ij} = \left(I-K_iG_{ii}\right)^{-1}K_iG_{ij}K_j\left(I-G_{jj}K_j-G_{ji}(I-K_iG_{ii})^{-1}K_iG_{ij}K_j\right)^{-1}$$

which is clearly non-zero since $K_iG_{ij}K_j \neq 0$. Thus, it follows that

$$\mathcal{K}(I-P_{22}\mathcal{K})^{-1} \notin \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$$

and so $\text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty) \neq \{\mathcal{K}(I-P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)\}$. ■

Thus, Theorem 2 provides a characterization for a class of tractable problems. In other words, if $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$, then by defining $Q = \mathcal{K}(I-P_{22}\mathcal{K})^{-1}$, we see that $Q \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$ if and only if $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$. When P_{22} is stable, the optimization problem in (2.5) is equivalent to the following convex problem.

$$\begin{array}{ll} \text{minimize} & \|P_{11}+P_{12}QP_{21}\|_2 \\ \text{subject to} & Q \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RH}_\infty) \end{array}$$

An analogous result can be obtained in the case where P_{22} is unstable. This will be formalized in later chapters.

It is worth noting that this result can also be obtained using the quadratic invariance property of these systems [16].

For the remainder of this work, we will restrict our attention to those systems for which $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$. In fact, without loss of generality, we will assume that $\mathcal{M}^{\mathcal{G}^K} = \mathcal{M}^{\mathcal{G}^P}$, and drop the superscripts on $\mathcal{M}^{\mathcal{G}}$. Note that this assumption does not require $\mathcal{G}^K = \mathcal{G}^P$, but that the transitive closures of the two graphs are equal. Figure 2.5 demonstrates that the plant graph structure need not equal the controller graph structure, yet such a system satisfies $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K};\mathcal{RL}_\infty)$.

Contrast this example with the fully decentralized problem, where the set of allowable controllers is $\text{Sparse}(I;\mathcal{RL}_\infty)$. From Theorem 2, we see that

$$\{\mathcal{K}(I-P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(I;\mathcal{RL}_\infty)\} = \text{Sparse}(I;\mathcal{RL}_\infty)$$

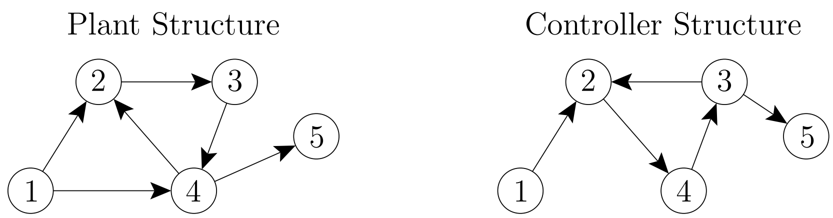


Figure 2.5: Tractable System with $\mathcal{G}^P \neq \mathcal{G}^K$

if and only if $P_{22} \in \text{Sparse}(I; \mathcal{RL}_\infty)$. In other words, this system will only be tractable if the dynamics of the subsystems are fully decoupled. This result supports the well-known conclusion that such decentralized problems under more general plant structures are intractable.

Our last result of this chapter concerns graph structures that contain cycles. To this end, for each $i \in \mathcal{V}$, let us define the set

$$\mathcal{C}_i = \mathcal{A}_i \cap \mathcal{D}_i$$

From our notation in Section 2.2, it clear that there exists a cycle containing vertex $i \in \mathcal{V}$ if and only if $\mathcal{C}_i \neq \{i\}$. In other words, a cycle contains some $j \in \mathcal{V}$ which is both an ancestor and a descendent of $i \neq j$. It is clear that $i \in \mathcal{C}_i$, for all $i \in \mathcal{V}$. Consequently, we say that a graph is *acyclic* if $\mathcal{C}_i = \{i\}$ for all $i \in \mathcal{V}$. In addition, the following lemma provides some additional properties of these sets.

Lemma 3. *For all $i, j \in \mathcal{V}$, the following properties hold:*

i) $i \in \mathcal{C}_j$ if and only if $j \in \mathcal{C}_i$

ii) $i \in \mathcal{A}_j$ if and only if $i \in \mathcal{A}_k$ for all $k \in \mathcal{C}_j$

iii) $i \in \mathcal{D}_j$ if and only if $i \in \mathcal{D}_k$ for all $k \in \mathcal{C}_j$

Proof. i) Suppose $i \in \mathcal{C}_j$. By definition, $i \in \mathcal{A}_j$ and $i \in \mathcal{D}_j$. This is equivalent to $\mathcal{M}^{\mathcal{G}}_{ij} \neq 0$ and $\mathcal{M}^{\mathcal{G}}_{ji} \neq 0$. These two conditions hold if and only if $j \in \mathcal{D}_i$ and $j \in \mathcal{A}_i$, so that $j \in \mathcal{C}_i$.

ii) Clearly, if $i \in \mathcal{A}_k$ for all $k \in \mathcal{C}_j$, it holds when $k = j$. To show the other direction, suppose $i \in \mathcal{A}_j$. Then, for any $k \in \mathcal{C}_j$, we have $k \in \mathcal{D}_j$, or equivalently,

$j \in \mathcal{A}_k$. Since i is an ancestor of j, and j is an ancestor of k, then i is an ancestor of k, so that $i \in \mathcal{A}_k$.

iii) The proof follows the same arguments as ii), considering descendants instead of ancestors. ■

With these results, we see that $\mathcal{C}_i$ represents an equivalence class for vertices. In other words, there is no loss of generality in grouping cycles together as single entities; that is, considering an equivalent graph whose vertices are the directed cycles of the original graph.

$$\mathcal{V}_{\text{new}} = \{\mathcal{C}_i \mid i \in \mathcal{V}\}$$

Intuitively, given our assumptions, every subsystem within a cycle has the same information as every other subsystem within that cycle. Thus, it is equivalent to think of this collection of subsystems as a single centralized system.

Note that this equivalent graph is acyclic. Thus, we can assume, without loss of generality, that our graph is acyclic, and we will assume this for the remainder of the paper.

Lastly, for directed, acyclic graphs, it is intuitive to order the vertices such that all vertices are indexed lower than their descendent vertices. The following lemma formalizes this notion.

Lemma 4. *Suppose $\mathcal{G}$ is a directed, acyclic graph. Then, there exists a partition $\mathcal{V}_1, \ldots, \mathcal{V}_k$ of the vertex set $\mathcal{V}$ which satisfies*

i) $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_k$

ii) $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *for all* $i \neq j$

iii) $\mathcal{A}'_i \subset \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{j-1}$ *for all* $i \in \mathcal{V}_j$

iv) $\mathcal{D}'_i \subset \mathcal{V}_{j+1} \cup \ldots \cup \mathcal{V}_k$ *for all* $i \in \mathcal{V}_j$

Proof. In simpler terms, we must show that there exists a partition of $\mathcal{V}$, such that every ancestor of a vertex $i \in \mathcal{V}_j$ is an element of one of the sets $\mathcal{V}_1, \ldots, \mathcal{V}_{j-1}$.

To this end, we must first show that every directed, acyclic graph contains at least one $v \in \mathcal{V}$ which has no ancestors, other than itself. In other words, $\mathcal{A}'_v = \emptyset$. Suppose that the graph does not contain such a vertex. Then, let us choose the longest path $v_1 \cdots v_n$ in the graph , which has length $n-1$. Since $v_1 \in \mathcal{V}$, there must exist $v_0 \in \mathcal{A}'_{v_1}$. Since our graph is acyclic, this implies that $v_0 \neq v_i$ for all $i = 1, \ldots, n$. However, this implies that $v_0 v_1 \cdots v_n$ is a path of length n in the graph, which contradicts the fact that $v_1 \ldots v_n$ is the longest path in the graph. Hence, there must exist at least one vertex with $\mathcal{A}'_v = \emptyset$.

Thus, we can now construct our partition recursively. Let $\mathcal{V}_1$ be the non-empty set of vertices

$$\mathcal{V}_1 = \{i \in \mathcal{V} \mid \mathcal{A}'_i = \emptyset\}$$

Now, consider the graph $\mathcal{G} - \mathcal{V}_1$, consisting of the original graph without the vertices in $\mathcal{V}_1$. Since $\mathcal{G}$ is acyclic, removing the vertices in $\mathcal{V}_1$ does not change the acyclic structure. Since $\mathcal{G} - \mathcal{V}_1$ is acyclic, it must have at least one vertex $v \in \mathcal{V} - \mathcal{V}_1$ such that $\mathcal{A}'_v = \emptyset$, so we let $\mathcal{V}_2$ be the set of vertices satisfying this condition, and we continue in this manner. Since we remove at least one vertex from the graph at every step, we will eventually partition $\mathcal{V}$, satisfying conditions i) and ii).

Lastly, since every $v \in \mathcal{V}_j$ has no ancestors in the graph $\mathcal{G} - \bigcup_{i=1}^{j-1} \mathcal{V}_i$, then its ancestors in the graph $\mathcal{G}$ must be in the set $\bigcup_{i=1}^{j-1} \mathcal{V}_i$, satisfying condition iii). Condition iv) follows analogously. ∎

From Lemma 4, we can renumber the vertices according to this ordering, so that $i_1 < i_2$ if $i_1 \in \mathcal{V}_{j_1}$, $i_2 \in \mathcal{V}_{j_2}$, and $j_1 < j_2$. As a result, for any $i \in \mathcal{V}$, the ancestors of i satisfy

$$j \in \mathcal{A}_i \Rightarrow j \leq i$$

Consequently, $\mathcal{M}^{\mathcal{G}}_{ij} = 0$ for all $i < j$, so $\mathcal{M}^{\mathcal{G}}$ is lower triangular. This ordering will prove useful in our results in later chapters.

At this point, one might reasonably ask what types of sparsity patterns are allowed within this framework. As an illustrative example, Table 2.1 lists 16 different sparsity patterns for $\mathcal{M}^{\mathcal{G}}$. These represent every possible sparsity pattern for $\mathcal{M}^{\mathcal{G}}$ when $\mathcal{G}$ is a directed acyclic graph with 4 vertices, up to a renumbering of the vertices. Notice that every sparsity pattern is lower triangular. Additionally, Table 2.2 provides a particular graph for each transitive closure of Table 2.1.

2.6 Summary

We began this chapter by introducing the notation that will be used throughout this work. Most of this terminology is standard in control theory. Since this work is aimed at decentralized systems defined by graphs, some additional graph notation and the corresponding sparsity sets were established.

In contrast to classical, centralized control problems, the decentralized optimization problems considered here are subject to these sparsity constraints. Moreover, the plant dynamics may be subject to a different sparsity pattern than the set of allowable controllers. To resolve this issue, a simple condition relating these sparsity patterns was established, under which the closed-loop mapping was invariant with respect to the sparsity constraints. As a result, the optimization can be reformulated as a convex problem. The remaining chapters focus on solving this problem.

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Table 2.1: Sparsity Patterns for the Transitive Closures of Every Directed Acyclic Graph with Four Vertices

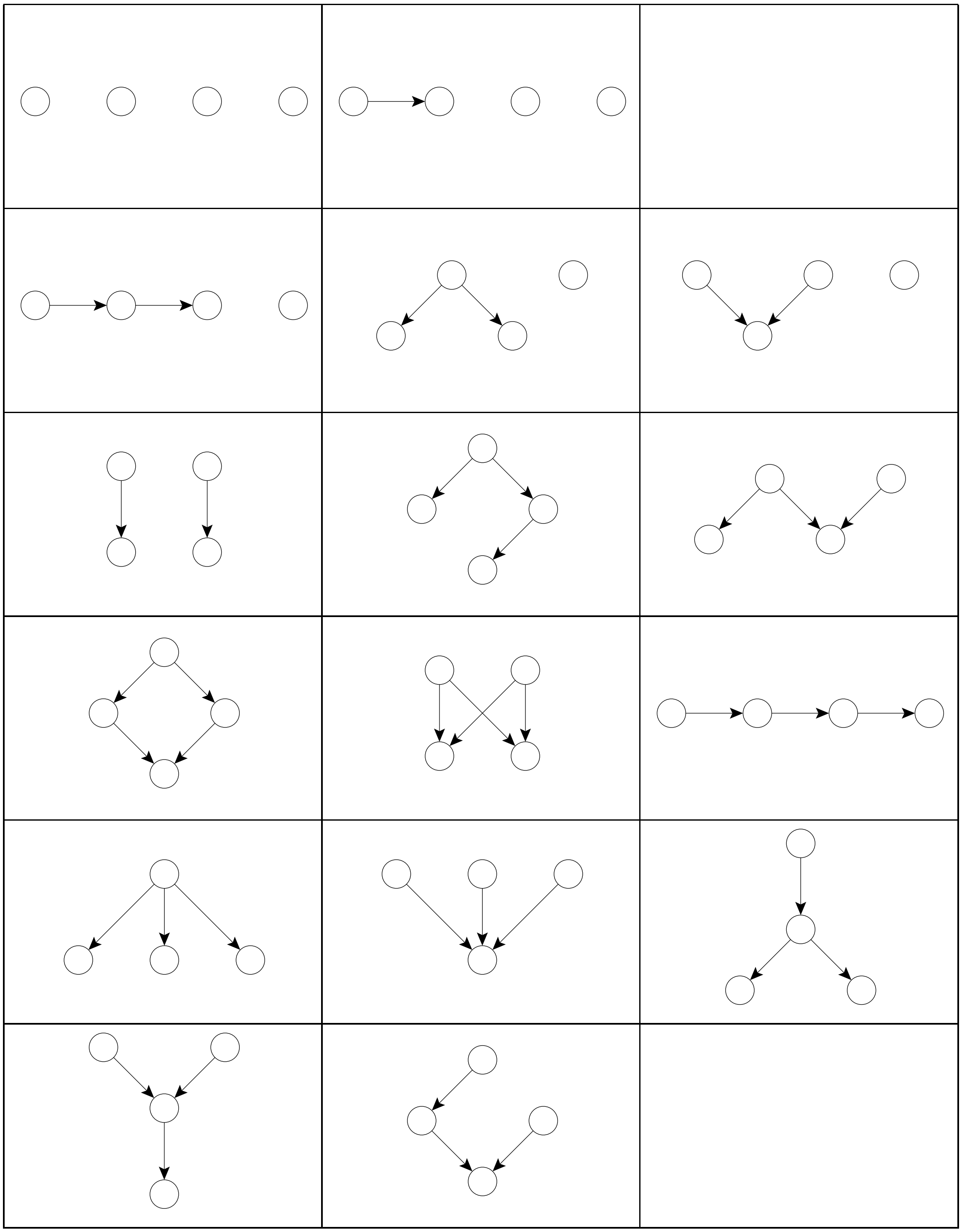


Table 2.2: An Example Graph for Each Transitive Closure of Table 2.1

Chapter 3

Spectral Factorization

We turn now to the controller synthesis problem; that is, finding an optimal controller for (2.5). The method used in this work is called *spectral factorization*. In this chapter, we develop this approach by first considering the classical, centralized, finite-horizon problem. We then extend this approach to a non-classical, finite-horizon problem. Lastly, we discuss the application of this method to the infinite-horizon problem.

3.1 Finite Horizon Case

We begin our discussion with the finite horizon version of the classical, centralized problem. This is a convenient starting point since the finite horizon case simplifies the problem by removing the issue of stability and replacing transfer functions with real matrices.

For the time horizon $t = 0, \dots, N$, we can concatenate all the states as $x = \begin{bmatrix} x(0)^T & \cdots & x(N)^T \end{bmatrix}^T$, and similarly for u, y, ξ. As a result, it is clear that the dynamics of the plant can be written as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix}$$

where the P_{ij} are now real, lower triangular matrices. The lower triangular structure arises because the dynamics are causal. Similarly, a causal controller is required;

equivalently, we want a real lower triangular matrix $\mathcal{K}$, such that $u = \mathcal{K}y$, which is optimal for

$$\begin{array}{ll} \text{minimize} & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\|_F \\ \text{subject to} & \mathcal{K} \text{ lower triangular} \end{array} \tag{3.1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Now, let

$$Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1}$$

Since the set of lower triangular matrices is closed under addition, multiplication, and inversion, and P_{22} is lower triangular, then it is clear that Q is lower triangular for all lower triangular $\mathcal{K}$. Conversely, for any lower triangular Q, we can find the corresponding lower triangular $\mathcal{K}$ as

$$\mathcal{K} = Q(I + P_{22}Q)^{-1}$$

As a result, it is clear that (3.1) is equivalent to the following convex problem.

$$\begin{array}{ll} \text{minimize} & \|P_{11} + P_{12}QP_{21}\|_F \\ \text{subject to} & Q \text{ lower triangular} \end{array} \tag{3.2}$$

This optimization problem is a finite-dimensional convex problem, so there exist many methods for finding an optimal lower triangular matrix Q. However, when we turn to infinite-dimensional problems, in which we consider transfer functions, greater care is required to find solutions. In either case, it is possible to show that Q is optimal if and only if it satisfies a particular optimality condition. The following lemma is applicable to both the finite and infinite-dimensional cases.

Lemma 5. *Suppose H_1 and H_2 are Hilbert spaces, and $F : H_1 \to H_1$, $G_1 : H_2 \to H_1$, $G_2 : H_2 \to H_1$ are bounded operators. Let $\mathcal{S} \subset H_2$ be a subspace. Then, $Q \in \mathcal{S}$*

minimizes

$$\begin{array}{ll} \textit{minimize} & \|F + G_1 Q G_2\| \\ \textit{subject to} & Q \in \mathcal{S} \end{array}$$

if and only if

$$G_1^* F G_2^* + G_1^* G_1 Q G_2 G_2^* \in \mathcal{S}^\perp \tag{3.3}$$

Proof. This result is a version of the classical projection theorem; see for example [9]. We first show that if $Q_0 \in \mathcal{S}$ is a minimizer of $\|F + G_1 Q G_2\|$, then Q_0 satisfies (3.3). Suppose, to the contrary, that there exists $\Gamma \in \mathcal{S}$ such that

$$\langle \Gamma, G_1^* F G_2^* + G_1^* G_1 Q_0 G_2 G_2^* \rangle = \delta \neq 0$$

We can assume that $\|\Gamma\| \leq \|G_1\|^{-1} \|G_2\|^{-1}$. Then, letting $Q_1 = Q_0 - \delta\Gamma$, we have

$$\begin{aligned} \|F + G_1 Q_1 G_2\|^2 &= \|F + G_1 Q_0 G_2 - \delta G_1 \Gamma G_2\|^2 \\ &= \|F + G_1 Q_0 G_2\|^2 - \langle F + G_1 Q_0 G_2 \ , \delta G_1 \Gamma G_2 \rangle \\ &\qquad - \langle \delta G_1 \Gamma G_2 \ , F + G_1 Q_0 G_2 \rangle + |\delta|^2 \|G_1 \Gamma G_2\|^2 \\ &= \|F + G_1 Q_0 G_2\|^2 + |\delta|^2 \|G_1 \Gamma G_2\|^2 \\ &\qquad - 2\delta \operatorname{Re} \langle G_1^* F G_2^* + G_1^* G_1 Q_0 G_2 G_2^* \ , \Gamma \rangle \\ &= \|F + G_1 Q_0 G_2\|^2 + |\delta|^2 \|G_1 \Gamma G_2\|^2 - 2|\delta|^2 \\ &\leq \|F + G_1 Q_0 G_2\|^2 - |\delta|^2 \\ &< \|F + G_1 Q_0 G_2\|^2 \end{aligned}$$

Thus, if $Q_0 \in \mathcal{S}$ does not satisfy (3.3), then Q_0 is not a minimizer.

Conversely, suppose that $Q_0 \in \mathcal{S}$ satisfies (3.3). Then, for any $Q \in \mathcal{S}$, we have

$$\begin{aligned}
\|F + G_1 Q G_2\|^2 &= \|F + G_1 Q_0 G_2 + G_1(Q - Q_0)G_2\|^2 \\
&= \|F + G_1 Q_0 G_2\|^2 + \langle F + G_1 Q_0 G_2 \, , G_1(Q - Q_0)G_2 \rangle \\
&\qquad + \langle G_1(Q - Q_0)G_2 \, , F + G_1 Q_0 G_2 \rangle + \|G_1(Q - Q_0)G_2\|^2 \\
&= \|F + G_1 Q_0 G_2\|^2 + \|G_1(Q - Q_0)G_2\|^2 \\
&\qquad + 2\,\mathrm{Re}\langle Q - Q_0 \, , G_1^* F G_2^* + G_1^* G_1 Q_0 G_2 G_2^* \rangle
\end{aligned}$$

Since $\mathcal{S}$ is a subspace, then $Q - Q_0 \in \mathcal{S}$. As a result, the above inner product term is zero, so we have

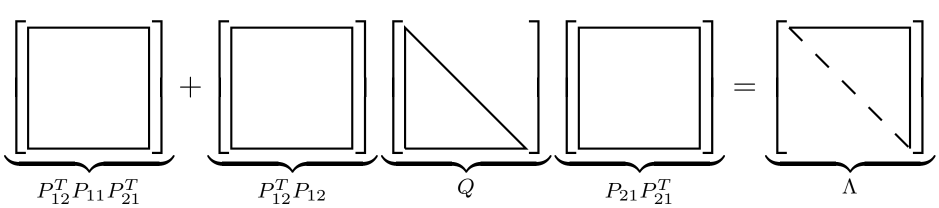
$$\|F + G_1 Q G_2\|^2 = \|F + G_1 Q_0 G_2\|^2 + \|G_1(Q - Q_0)G_2\|^2$$

Thus, $Q = Q_0$ is a minimizer. ∎

For the finite horizon problem in (3.2), the set of lower triangular matrices $\mathcal{S}$ has an orthogonal complement such that $\Lambda \in \mathcal{S}^\perp$ if and only if Λ is strictly upper triangular. From Lemma 5, it is clear that $Q \in \mathcal{S}$ is optimal for (3.2) if and only if

$$P_{12}^T P_{11} P_{21}^T + P_{12}^T P_{12} Q P_{21} P_{21}^T = \Lambda \in \mathcal{S}^\perp \tag{3.4}$$

In other words, we must find a lower triangular Q such that this expression for Λ is strictly upper triangular. It is important to note here that Q and Λ are orthogonal, that they possess complimentary structures. However, these structures are coupled by the matrices $P_{12}^T P_{12}$ and $P_{21} P_{21}^T$. Visually, (3.4) looks like



(3.5)

The difficulty is that $P_{12}^T P_{12}$ and $P_{21} P_{21}^T$ are full matrices which ruin the triangular structure of Q. However, since $P_{12}^T P_{12} > 0$ and $P_{21} P_{21}^T > 0$, it is well-known that

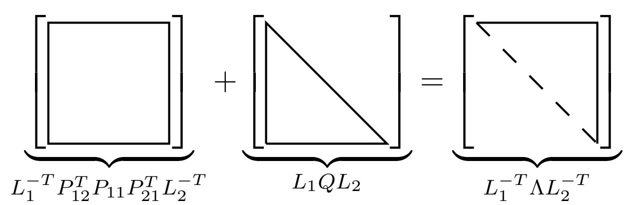
Cholesky factorizations exist which decompose these matrices into the product of an invertible upper triangular matrix and an invertible lower triangular matrix. In particular,

$$P_{12}^T P_{12} = L_1^T L_1 \qquad P_{21} P_{21}^T = L_2 L_2^T$$

where L_1 and L_2 are lower triangular and invertible. With these factorizations, the optimality condition is equivalent to

$$L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T} + L_1 Q L_2 = L_1^{-T} \Lambda L_2^{-T}$$

Since the set of lower triangular matrices is invariant under addition and multiplication (similarly for upper triangular matrices), notice now that the term $L_1 Q L_2$ remains lower triangular, while the $L_1^{-T} \Lambda L_2^{-T}$ term is still strictly upper triangular, as shown below.



Thus, our factorization has succeeded in decoupling our variables, so that they may be solved independently. In other words, if we consider only the lower triangular elements of the above optimality condition, we obtain

$$\mathrm{P}_{\mathrm{Lower}}(L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T}) + L_1 Q L_2 = 0$$

where $\mathrm{P}_{\mathrm{Lower}}(\cdot)$ is the orthogonal projection onto the set of lower triangular matrices. Solving for Q is now straightforward, and is given by

$$Q = -L_1^{-1} \mathrm{P}_{\mathrm{Lower}}(L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T}) L_2^{-1}$$

Once the optimal Q for (3.2) is obtained, the optimal $\mathcal{K}$ for (3.1) can be found by substitution into

$$\mathcal{K} = Q(I + P_{22} Q)^{-1} \tag{3.6}$$

This is the solution to the centralized problem, which has been known for some time. However, the method used here is the basic idea behind the more complicated approaches to follow.

3.2 Temporal/Vertical Skyline Case

Before we finish our discussion on finite-horizon problems, we turn to a couple non-classical problems, which will further illustrate the solution method discussed in the previous section.

In the centralized problem above, the only restriction on the plant and controller was causality. As we showed, this simply amounted to all our matrices being block lower triangular. However, suppose we have the same system, but now each signal $y(t)$ is delayed in reaching the controller. Moreover, these delays can be different for each t, so that some measurements might be delayed more than others. The result is a sparsity pattern on $\mathcal{K}$ which might look something like Figure 3.1, where '$\times$' represents a potentially non-zero entry.

$$\mathcal{K} \sim \begin{bmatrix} \times & & & & & \\ \times & & & & & \\ \times & & \times & & & \\ \times & & \times & \times & & \\ \times & \times & \times & \times & & \\ \times & \times & \times & \times & & \times \end{bmatrix}$$

Figure 3.1: Temporal Skyline Structure

Mathematically, we can define these sets, which we call the *temporal*, or *vertical*, *skyline* information structures, as follows.

Definition 6. *Consider the sequence of N integers $\mathcal{I} = (i_0, \ldots, i_{N-1})$, where i_j satisfies $j \le i_j \le N$, for each $j = 0, \ldots, N-1$. We say that the controller $\mathcal{K}$ has a* Temporal Skyline *(TS) information structure if at time $t \in [0, N-1]$ it has as its information variables the observation set $Y_t = \{y(j) \mid i_j \le t\}$. We define the set of* *matrices with this TS structure by*

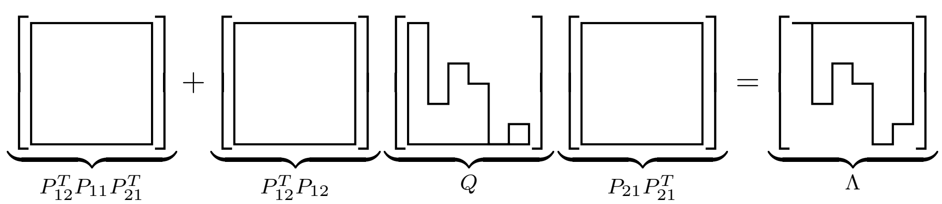
$$\mathcal{TS}_{\mathcal{I}} = \{K \in \mathbb{R}^{N\times N} \mid K_{jk} = 0 \quad \textit{if} \quad j-1 < i_{k-1}\}$$

We also define the complementary structure

$$\mathcal{TS}_{\mathcal{I}}^{\perp} = \{K \in \mathbb{R}^{N\times N} \mid K_{jk} = 0 \quad \textit{if} \quad j-1 \geq i_{k-1}\}$$

In simpler terms, the controller can receive any observation at any time, subject to causality, and remembers the observations in the future. As a result, we see that this implies that $\mathcal{K} \in \mathcal{TS}_{\mathcal{I}}$ has the TS structure defined by $\mathcal{I}$. As seen in Figure 3.1, the sparsity pattern of TS structures looks similar to a city skyline, from which the name is derived. As an example, the sequence $\mathcal{I}$ corresponding to Figure 3.1 would be $\mathcal{I} = (0, 4, 2, 3, 6, 5)$. In comparison, the sequence corresponding to the centralized case would be $\mathcal{I} = (0, 1, 2, \ldots, N-1)$.

It has been previously shown that in a causal system (in which P_{22} is lower triangular) the TS structure produces a quadratically invariant controller [18]. This fact allows the direct use of the previously obtained optimality condition (3.4), where $Q \in \mathcal{TS}_{\mathcal{I}}$ and $\Lambda \in \mathcal{TS}_{\mathcal{I}}^{\perp}$. Analogous to (3.5), the optimality condition might look something like the following visualization.



(3.7)

As in the centralized case of Section 3.1, the goal is to find factorizations for $P_{12}^T P_{12}$ and $P_{21} P_{21}^T$ which preserve the TS structure in (3.7). For the centralized problem, triangular factorizations sufficed. However, this no longer works for general TS structures. As a result, we need a more general set of matrices.

Definition 7. *Given* $\mathcal{I} = (i_0, \ldots, i_{N-1})$*, we define the set of* sparse antisymmetric *matrices $\mathcal{SA}_\mathcal{I}$ by*

$$\mathcal{SA}_\mathcal{I} = \{A \in \mathbb{R}^{N\times N} \mid A_{jk} = 0 \textit{ if } i_{j-1} < i_{k-1}, \textit{ or if } i_{j-1} = i_{k-1} \textit{ and } j < k\}$$

The term *sparse antisymmetric* comes from the fact that, for any matrix $A \in \mathcal{SA}_\mathcal{I}$, $A_{jk} = 0$ whenever $A_{kj} \neq 0$.

In addition, since $\mathcal{TS}$ is a closed subspace, it is clear that every matrix $A \in \mathbb{R}^{N\times N}$ can be written as the sum of a unique $B \in \mathcal{TS}_\mathcal{I}$ and $C \in \mathcal{TS}_\mathcal{I}^\perp$. Consequently, we let $\mathrm{P}_{\mathcal{TS}_\mathcal{I}} : \mathbb{R}^{N\times N} \to \mathcal{TS}_\mathcal{I}$ be the orthogonal projection onto $\mathcal{TS}_\mathcal{I}$, so that $\mathrm{P}_{\mathcal{TS}_\mathcal{I}}(A) = B$. With this notation, the solution to (3.7) can be found with the following theorem.

Theorem 8. *Suppose $\mathcal{I} = (i_0, \ldots, i_{N-1})$. There exist invertible matrices $L_1, L_2 \in \mathbb{R}^{N\times N}$, such that L_1 is lower triangular, $L_2 \in \mathcal{SA}_\mathcal{I}$, and*

$$P_{12}^T P_{12} = L_1^T L_1 \qquad P_{21} P_{21}^T = L_2 L_2^T$$

Moreover, the unique $Q \in \mathcal{TS}_\mathcal{I}$ satisfying

$$P_{12}^T P_{11} P_{21}^T + P_{12}^T P_{12} Q P_{21} P_{21}^T \in \mathcal{TS}_\mathcal{I}^\perp \tag{3.8}$$

is given by

$$Q = -L_1^{-1} \mathrm{P}_{\mathcal{TS}_\mathcal{I}}(L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T}) L_2^{-1} \tag{3.9}$$

We will develop the proof of this theorem in the following lemmas.

In the centralized case, we used the fact that, for any lower triangular matrices L_1, L_2, and Q, their product $L_1 Q L_2$ was also lower triangular. The following lemma provides the analogous result for the TS case.

Lemma 9. *If $Q \in \mathcal{TS}_\mathcal{I}$, $L_2 \in \mathcal{SA}_\mathcal{I}$, and L_1 is a lower triangular matrix, then $L_1 Q L_2 \in \mathcal{TS}_\mathcal{I}$.*

Proof. Looking at the product of L_1QL_2, we have,

$$(L_1QL_2)_{jk} = \sum_{l,m}(L_1)_{jl}Q_{lm}(L_2)_{mk} = \sum_{\substack{j\geq l \\ l\geq i_{m-1}+1 \\ i_{m-1}\geq i_{k-1}}} (L_1)_{jl}Q_{lm}(L_2)_{mk}$$

Looking at the summation, we see that there will be no terms to sum over whenever $j < i_{k-1} + 1$, meaning that $(L_1QL_2)_{jk} = 0$ whenever that condition holds. Hence, $L_1QL_2 \in \mathcal{TS}_{\mathcal{I}}$. ∎

A similar result shows how the sparsity pattern of $\Lambda \in \mathcal{TS}_{\mathcal{I}}^{\perp}$ is preserved.

Lemma 10. *If $\Lambda \in \mathcal{TS}_{\mathcal{I}}^{\perp}$, $L_2 \in \mathcal{SA}_{\mathcal{I}}$ is invertible, and L_1 is an invertible lower triangular matrix, then $L_1^{-T}\Lambda L_2^{-T} \in \mathcal{TS}_{\mathcal{I}}^{\perp}$.*

Proof. This fact can be proven by explicit computation of the entries of $L_1^{-T}\Lambda L_2^{-T}$, in the same fashion as Lemma 9. ∎

Thus, from Lemmas 9 and 10, it is clear that left multiplication by triangular matrices preserves the TS structure. However, instead of triangular matrices for right multiplication, the appropriate set of matrices for this purpose is the set $\mathcal{SA}_{\mathcal{I}}$. Consequently, the last thing that needs to be shown is that there exists such a factorization for $P_{21}P_{21}^T$.

Lemma 11. *For any symmetric, positive definite matrix $A \in \mathbb{R}^{N\times N}$, there exists an invertible matrix $L \in \mathcal{SA}_{\mathcal{I}}$ such that $A = LL^T$.*

Proof. For the set $\mathcal{SA}_{\mathcal{I}}$, we can find a permutation matrix J such that J^TLJ is lower triangular for any $L \in \mathcal{SA}_{\mathcal{I}}$. This follows directly from the fact that L is sparse antisymmetric. Since we already know that a Cholesky factorization exists for any positive definite matrix, then we have

$$A = J(\hat{L}\hat{L}^T)J^T = (J\hat{L}J^T)(J\hat{L}^TJ^T) = LL^T$$

where $\hat{L}\hat{L}^T$ is the Cholesky factorization for J^TAJ. Since $\hat{L}$ is an invertible lower triangular matrix, then $L = J\hat{L}J^T \in \mathcal{SA}_{\mathcal{I}}$ is invertible. ∎

The proof of Theorem 8 follows directly from these results.

Proof of Theorem 8. Since $P_{12}^T P_{12}$ is positive definite, it is clear that the factorization L_1 exists for $P_{12}^T P_{12}$; this is the Cholesky factorization. Similarly, Lemma 11 shows that the factorization $L_2 \in \mathcal{SA}_{\mathcal{I}}$ exists for $P_{21} P_{21}^T$.

Now, in a manner similar to the centralized case in Section 3.1, we see that

$$L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T} + L_1 Q L_2 = L_1^{-T} \Lambda L_2^{-T}$$

where $L_1 Q L_2 \in \mathcal{TS}_{\mathcal{I}}$ from Lemma 9, and $L_1^{-T} \Lambda L_2^{-T} \in \mathcal{TS}_{\mathcal{I}}^{\perp}$ from Lemma 10. Consequently,

$$\mathrm{P}_{\mathcal{TS}_{\mathcal{I}}}(L_1^{-T} P_{12}^T P_{11} P_{21}^T L_2^{-T}) + L_1 Q L_2 = 0$$

and the result follows. ∎

Thus, we obtained the optimal $Q \in \mathcal{TS}_{\mathcal{I}}$ by finding factorizations L_1 and L_2 such that the optimality condition (3.8) decoupled into separate equations for $Q \in \mathcal{TS}_{\mathcal{I}}$ and $\Lambda \in \mathcal{TS}_{\mathcal{I}}^{\perp}$. Lastly, the optimal controller $\mathcal{K} \in \mathcal{TS}_{\mathcal{I}}$ can be found by substitution into (3.6).

3.3 Spatial/Horizontal Skyline Case

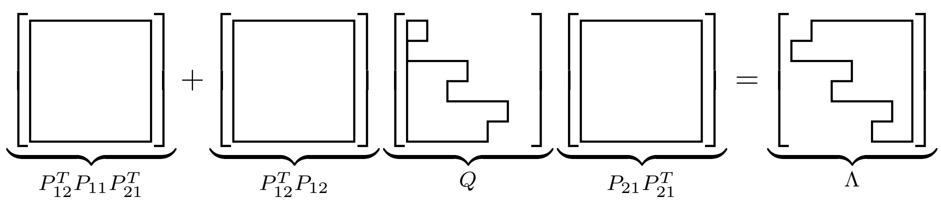
In the previous section we extended the results from the classic information structure to solve the control problem which had a temporal skyline information structure. Fortunately, the above analysis can also be utilized to solve (3.4) for other information structures. One such information structure is the *spatial*, or *horizontal*, *skyline* structure (SS). An example spatial skyline structure can be seen in Figure 3.2.

$$\begin{bmatrix} \times & & & & & \\ & & & & & \\ \times & \times & \times & & & \\ \times & \times & & & & \\ \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & & \end{bmatrix}$$

Figure 3.2: Example Spatial Skyline Structure

While the TS and SS structures are functionally similar, it is important to note that these two structures are physically very different. In the TS case, we can think of the system as a single decision maker who receives information at arbitrary times but remembers everything he receives. On the other hand, in the SS case, we are forced to think of the problem as a multiple player system where each player makes decisions in turn based on different sets of information. In other words, the decision maker at time t may have $y(0), \dots, y(t)$ available to make decision $u(t)$, while decision maker $t+1$ may only have $y(0)$ and $y(1)$ available for deciding $u(t+1)$.

We omit the complete solution here; the details can be found in [23]. However, for this problem the optimality condition (3.4) can be visualized as follows.



(3.10)

In the TS case, the optimality condition could be found by finding a Cholesky factorization for $P_{12}^T P_{12}$ and a sparse antisymmetric factorization for $P_{21} P_{21}^T$. However, in the SS case in (3.10), the opposite factorizations are used. In other words, we must find a sparse antisymmetric factorization for $P_{12}^T P_{12}$ and a triangular factorization for $P_{21} P_{21}^T$. Thus, in this case, the SS structure is preserved by left multiplication by a particular sparse antisymmetric matrix and right multiplication by a lower triangular matrix. Once these factorizations are obtained, the solution for the optimal Q is found in the same fashion as the TS and centralized cases.

3.4 Scalar Transfer Functions

We turn now to discussing the infinite horizon/transfer function case. Once again, we will briefly describe the methodology as it applies to the centralized problem; the decentralized problems are considered in later chapters. While the infinite horizon version of this problem is more complicated, the basic idea remains the same. To begin, when P is stable, $P \in \mathcal{RH}_2$, the classic Youla parametrization reformulates the optimization problem as

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12} Q P_{21}\| && (3.11) \\ \text{subject to} \quad & Q \in \mathcal{RH}_2 && (3.12) \end{aligned}$$

Instead of Q being lower triangular, we search for Q in the set of stable transfer functions. From Lemma 5, the corresponding optimality condition is clearly

$$P_{12}^* P_{11} P_{21}^* + P_{12}^* P_{12} Q P_{21} P_{21}^* = \Lambda \in \mathcal{H}_2^{\perp} \tag{3.13}$$

Since Q is a stable transfer function and $\Lambda \in \mathcal{H}_2^{\perp}$ is an anti-stable transfer function, our goal is again to factor the $P_{12}^* P_{12}$ and $P_{21} P_{21}^*$ terms into the products of a stable transfer function, with stable inverse, and an anti-stable transfer function, with anti-stable inverse. This is known as *spectral factorization*, since the terms stable/anti-stable are references to the spectrum of the operators.

In the finite horizon problem, invertible, lower triangular matrices always have lower triangular inverses. However, when moving to the infinite horizon case, the inverse of a stable transfer function need not be stable. Thus, it is important that we find factorizations whose factors are not only stable, but also have stable inverses.

We begin our discussion of spectral factorization with the case of scalar transfer functions. To this end, we first consider the case of trigonometric polynomials. A *trigonometric polynomial* is a rational function of the form

$$f(z) = \sum_{k=-n}^{n} c_k z^k$$

We also define

$$f^{\sim}(z) = \sum_{k=-n}^{n} \overline{c_k}\, z^{-k}$$

It is straightforward to show that $f(z)$ is real for all $z \in \mathbb{T}$ if and only if

$$c_k = \overline{c_{-k}}$$

for all k. Consequently, if $f(z)$ is real, then

$$f\left(\frac{1}{\bar{z}}\right) = \overline{f(z)}$$

for all $z \in \mathbb{C}$, so that r is a root of f if and only if $\frac{1}{\bar{r}}$ is a root. Notice that if $|r| < 1$, then $|\frac{1}{\bar{r}}| > 1$, so that f has an equal number of stable and unstable roots. Spectral factorization of f takes advantage of this fact, as seen in the following theorem.

Theorem 12. *Suppose f is the trigonometric polynomial*

$$f(z) = \sum_{k=-n}^{n} c_k z^k$$

and $f(z)$ is real for all $z \in \mathbb{T}$. Then,

$$f(z) \geq 0 \qquad \textit{for all } z \in \mathbb{T}$$

if and only if there exists a polynomial

$$q(z) = a(z - r_1)\ldots(z - r_n)$$

with all $|r_i| \leq 1$ such that

$$f(z) = q^{\sim}(z)q(z)$$

Proof. Clearly, if there exists such a polynomial q, then for all $z \in \mathbb{T}$,

$$f(z) = q^{\sim}(z)q(z) = |a|^2|z - r_1|^2 \ldots |z - r_n|^2 \geq 0$$

Conversely, suppose that $f(z)$ is non-negative for all $z \in \mathbb{T}$. Without loss of generality, we assume that $c_{-n} \neq 0$ and let p be the polynomial

$$p(z) = z^n f(z)$$

Since p is a polynomial of degree $2n$, it has $2n$ non-zero roots, and we can factorize it as

$$p(z) = c\prod_{i=1}^{m}(z - \overline{r}_i^{\,-1})(z - r_i)\prod_{j=1}^{s}(z - w_j)^2$$

where $|r_i| < 1$ and $|w_j| = 1$. Note that r_i and $\overline{r}_i^{\,-1}$ are both roots since f is non-negative on $\mathbb{T}$. Since f is also continuous on $\mathbb{T}$, then each root $w_j \in \mathbb{T}$ must have even multiplicity. Consequently, f may be written as

$$f(z) = d\prod_{i=1}^{m}(z^{-1} - \overline{r}_i)(z - r_i)\prod_{j=1}^{s}(z^{-1} - \overline{w}_j)(z - w_j)$$

where $d > 0$ since each pair of terms above is non-negative on $\mathbb{T}$. Thus, letting

$$q(z) = \sqrt{d}\prod_{i=1}^{m}(z - r_i)\prod_{j=1}^{s}(z - w_j)$$

we obtain our desired factorization. ■

While the above theorem, also known as *Wiener-Hopf factorization*, applies to trigonometric polynomials, the extension to scalar rational transfer functions follows directly, as this next theorem demonstrates.

Theorem 13. *Suppose $g \in \mathcal{RH}_\infty$, with no poles or zeros on $\mathbb{T}$. Then, there exists $l \in \mathcal{RH}_\infty$, such that*

$$g^*g = l^*l$$

and $l^{-1} \in \mathcal{RH}_\infty$.

Proof. We can write

$$g(z) = \frac{a(z)}{b(z)}$$

where a and b are polynomials. Consequently, we have

$$g^{\sim}(z)g(z) = \frac{a^{\sim}(z)a(z)}{b^{\sim}(z)b(z)}$$

Using Theorem 12, we can find spectral factors for both the numerator and the denominator, so that

$$a^{\sim}(z)a(z) = \alpha^{\sim}(z)\alpha(z)$$
$$b^{\sim}(z)b(z) = \beta^{\sim}(z)\beta(z)$$

where α and β are the same order and have all their roots in $\mathbb{D}$. Consequently, we let

$$l(z) = \frac{\alpha(z)}{\beta(z)}$$

Then, $g^*g = l^*l$, and l has all its poles and zeros in $\mathbb{D}$, so that $l, l^{-1} \in \mathcal{RH}_\infty$. ■

With this result, we can extend our previous discussion on solving the optimality condition (3.13) to the case of scalar transfer functions. Specifically, suppose that

$$P_{11}(z) = \frac{a(z)}{b(z)} \qquad P_{12}(z) = \frac{c(z)}{d(z)} \qquad P_{21}(z) = \frac{e(z)}{f(z)}$$

Then, to solve (3.13), we can find spectral factorizations

$$L_1^* L_1 = P_{12}^* P_{12} \qquad L_2 L_2^* = P_{21} P_{21}^*$$

such that $L_1, L_1^{-1} \in \mathcal{RH}_\infty$ and $L_2, L_2^{-1} \in \mathcal{RH}_\infty$. Suppose that

$$L_1(z) = \frac{\alpha(z)}{\beta(z)} \qquad L_2(z) = \frac{\gamma(z)}{\delta(z)}$$

Then, since L_1 and L_2 are invertible, the optimality condition is equivalent to

$$L_1^{-*} P_{12}^* P_{11} P_{21}^* L_2^{-*} + L_1 Q L_2 = L_1^{-*} \Lambda L_2^{-*}$$

Once again, since $L_1 Q L_2 \in \mathcal{RH}_2$ and $L_1^{-*} \Lambda L_2^{-*} \in \mathcal{H}_2^\perp$, we can decouple these terms and solve directly for $L_1 Q L_2$. To this end, we have

$$\mathrm{P}_{\mathcal{H}_2}(L_1^{-*} P_{12}^* P_{11} P_{21}^* L_2^{-*}) + L_1 Q L_2 = 0$$

To determine $\mathrm{P}_{\mathcal{H}_2}(\cdot)$, the projection of the first term onto $\mathcal{H}_2$, we can write this term as

$$L_1^{-*}P_{12}^*P_{11}P_{21}^*L_2^{-*} = \frac{\beta^\sim(z)c^\sim(z)a(z)e^\sim(z)\delta^\sim(z)}{\alpha^\sim(z)d^\sim(z)b(z)f^\sim(z)\gamma^\sim(z)}$$

Using a partial fraction decomposition of this term will allow us to write it as the sum of a stable transfer function and an anti-stable transfer function.

$$L_1^{-*}P_{12}^*P_{11}P_{21}^*L_2^{-*} = \frac{n_s(z)}{d_s(z)} + \frac{n_a(z)}{d_a(z)} \qquad\qquad \frac{n_s(z)}{d_s(z)} \in \mathcal{H}_2, \quad \frac{n_a(z)}{d_a(z)} \in \mathcal{H}_2^\perp$$

Thus, the projection onto $\mathcal{H}_2$ is simply the stable term. Lastly, since $L_1^{-1}, L_2^{-1} \in \mathcal{RH}_\infty$, then the solution for $Q \in \mathcal{RH}_\infty$ is

$$\begin{aligned} Q &= -L_1^{-1}\mathrm{P}_{\mathcal{H}_2}(L_1^{-*}P_{12}^*P_{11}P_{21}^*L_2^{-*})L_2^{-1} \\ &= -\frac{\beta(z)n_s(z)\delta(z)}{\alpha(z)d_s(z)\gamma(z)} \end{aligned}$$

3.5 Summary

In this chapter, we provided an introduction to spectral factorization, the principle technique that will be used in the remaining chapters of this work. This technique was applied to the finite horizon, centralized problem, as well as to non-classical Temporal Skyline and Spatial Skyline structures. We concluded the chapter by extending this method to the scalar transfer function case.

The goal in all of these problems was to solve a particular optimality condition. This equation consisted of two variables with complementary structures, which were coupled by full matrices. Thus, spectral factorization is a method for decoupling these variables. In the finite horizon case, spectral factorization reduces to the Cholesky factorization. In the centralized problem, the optimality condition could be solved by Cholesky factorizing both left and right multiplication operators. By contrast, in the Temporal Skyline case, a special factorization is required for the right multiplication operator, and in the Spatial Skyline case, the special factorization is required for the left multiplication operator. In the transfer function case, the added difficulty is that stable factorizations must also have stable inverses.

The one remaining case that has not been considered is the matrix transfer function case. This will be covered in the next chapters where we consider our decentralized problems.

Chapter 4

The 2-Player Problem

With the spectral factorization results of the previous chapter, we can now begin to apply these methods to solve our decentralized control problems. We begin by considering the simplest of these graph structures, a two player system with communication allowed in only one direction. The results of this chapter follow from the work in [25, 26]. Explicit state-space formulae are provided, and the order of the optimal controllers is established.

4.1 Problem Formulation

For decentralized problems represented by graphs, the simplest non-trivial system consists of two vertices (or players) with a single directed edge from vertex 1 to vertex 2. This graph is shown in Figure 4.1.

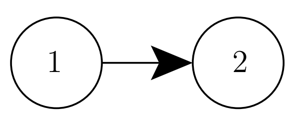


Figure 4.1: Two-Player System

It is clear that this graph can be described by the following expressions.

$$\mathcal{V} = \{1, 2\} \qquad \mathcal{E} = \{(1, 2)\} \qquad \mathcal{M}^{\mathcal{G}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{4.1}$$

As demonstrated in Chapter 2, this problem is only tractable if the plant dynamics satisfy the corresponding sparsity structure for this graph. Thus, without loss of generality, we will assume for the remainder of the chapter that $\mathcal{G}^P = \mathcal{G}^K = (\mathcal{V}, \mathcal{E})$, given by (4.1). Consequently, we are interested in the following state-space system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

This corresponds to a two-player system, in which player 1's state can influence player 2's state. We are interested in finding controllers of the form

$$\begin{aligned} q_1(t+1) &= A_{K1} q_1(t) + B_{K1} x_1(t) \\ u_1(t) &= C_{K1} q_1(t) + D_{K1} x_1(t) \end{aligned}$$

and

$$\begin{aligned} q_2(t+1) &= A_{K2} q_2(t) + B_{K2} x(t) \\ u_2(t) &= C_{K2} q_2(t) + D_{K2} x(t) \end{aligned}$$

That is, player 1 makes decision u_1 based only on the history of his own state x_1, while player 2 makes decision u_2 based on the history of both states x_1 and x_2. Implicit in this construction is that state feedback is assumed for both players. This assumption will be discussed further in later sections. This controller can be represented by the transfer functions $\mathcal{K}_{11}, \mathcal{K}_{21}, \mathcal{K}_{22} \in \mathcal{RL}_\infty$, such that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$. We also define the space

$$\mathcal{S} = \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$$

Note that the space $\mathcal{S} \subset \mathcal{L}_2$ has an orthogonal complement, such that $G \in \mathcal{S}^\perp$ if and only if $G_{11}, G_{21}, G_{22} \in \mathcal{H}_2^{\perp}$ and $G_{12} \in \mathcal{L}_2$

Using our notation from Chapter 2, our cost is now the vector

$$z(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

and $D^T D > 0$. Thus, the overall plant dynamics are given by

$$P = \left[\begin{array}{c|cc} A & H & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array}\right] \tag{4.2}$$

where $A, B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$ are lower triangular, and H is block diagonal and invertible. As before, our optimization problem is

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\|_2 \\ \text{subject to} \quad & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty) \end{aligned} \tag{4.3}$$

4.2 Main Results

Having established our notation and problem formulation, we will now present the optimal solution for (4.3). To this end, the following assumptions will be made throughout this chapter.

*A*1) (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable

*A*2) $D^T D > 0$ and $HH^T > 0$

*A*3) $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{T}$

We will discuss the need for these assumptions in the next section.

We state the main results here. The remaining sections will develop the proof of these results.

Theorem 14. *For the system in* (4.2)*, suppose assumptions A1–A3 hold. Let* X *and* Y *be the stabilizing solutions to the algebraic Riccati equations*

$$X = C^T C + A^T X A - (A^T X B + C^T D)(D^T D + B^T X B)^{-1}(B^T X A + D^T C) \quad (4.4)$$

$$Y = C_2^T C_2 + A_{22}^T Y A_{22} - (A_{22}^T Y B_{22} + C_2^T D_2)(D_2^T D_2 + B_{22}^T Y B_{22})^{-1}(B_{22}^T Y A_{22} + D_2^T C_2) \quad (4.5)$$

Define

$$K = (D^T D + B^T X B)^{-1}(B^T X A + D^T C) \quad (4.6)$$
$$J = (D_2^T D_2 + B_{22}^T Y B_{22})^{-1}(B_{22}^T Y A_{22} + D_2^T C_2) \quad (4.7)$$

and let

$$A_K = A - BK$$
$$A_J = A_{22} - B_{22} J$$

Then, there exists a unique optimal $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ *for* (2.5) *given by:*

- *Controller* 1 *has realization*

$$q_1(t+1) = (A_K)_{22} q_1(t) + (A_K)_{21} x_1(t)$$
$$u_1(t) = -K_{12} q_1(t) - K_{11} x_1(t)$$

- *Controller* 2 *has realization*

$$q_2(t+1) = (A_K)_{22} q_2(t) + (A_K)_{21} x_1(t)$$
$$u_2(t) = \big(J - K_{22}\big) q_2(t) - K_{21} x_1(t) - J x_2(t)$$

Note that Assumptions A1–A3 guarantees the existence of stabilizing solutions to the algebraic Riccati equations (4.4–4.5). This will be discussed in the following section. For a thorough discussion on algebraic Riccati equations, see [32].

Having established the form of the optimal controller, a number of remarks are in order.

With the inclusion of q_1 and q_2, the optimal controller is not a static gain, despite the fact that we have state feedback in each subsystem and player 2 has complete state information. Contrast this result with the classical LQR controller in which the optimal centralized controller would be the static gain K. In fact, both controllers have dynamics, and each has the same number of states as system 2.

It will be shown in Section 4.5 that q_1 and q_2 in the optimal controllers are in fact the minimum-mean square error estimate of x_2 given the history of x_1. Letting $\mathrm{E}(\cdot)$ denote expectation, if we define $\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(t), \ldots, x_1(0)\big)$, the optimal control policy can be written as

$$\begin{aligned} u_1(t) &= -K_{11}x_1(t) - K_{12}\eta(t) \\ u_2(t) &= -K_{21}x_1(t) - K_{22}\eta(t) + J\big(\eta(t) - x_2(t)\big) \end{aligned} \tag{4.8}$$

Thus, the optimal policy is, in fact, attempting to perform the optimal centralized policy, though using η instead of x_2. However, there is an additional term in u_2 which represents the error between x_2 and its estimate η. We also see that in the case where x_2 is deterministic, so that $\eta = x_2$, then the optimal distributed controller reduces to the optimal centralized solution, as it should.

4.3 Analysis

Before proving the results of the previous section, it is worth making a few remarks regarding the assumptions A1–A3.

For any control problem, it is important to establish when the system can be stabilized. The following lemma provides the necessary and sufficient conditions for the existence of any stabilizing controller.

Lemma 15. *There exists a controller* $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ *which stabilizes* P *in* (4.2) *if and only if* (A_{11}, B_{11}) *is stabilizable and* (A_{22}, B_{22}) *is stabilizable.*

Proof. ($\Leftarrow$) If (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable, then there exist matrices

F_1 and F_2 such that $A_{11} + B_{11}F_1$ and $A_{22} + B_{22}F_2$ are stable. Consequently, the controller

$$\mathcal{K} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$$

produces the closed loop system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11}F_1 & 0 \\ A_{21} + B_{21}F_1 & A_{22} + B_{22}F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which is clearly stable.

($\Rightarrow$) Suppose that (A_{11}, B_{11}) is not stabilizable. Then, there exists a transformation U such that

$$U^{-1}A_{11}U = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad U^{-1}B_{11} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

where a_{22} has at least one unstable eigenvalue λ, with corresponding eigenvector v. Then, it can be readily shown that with the initial condition $x_1(0) = U \begin{bmatrix} 0 \\ v \end{bmatrix}$, the state $x_1(t) \nrightarrow 0$ as $t \to \infty$, for any inputs u. A similar argument holds for the case where (A_{22}, B_{22}) is not stabilizable. ∎

As a result of Lemma 15, it is clear that Assumption A1 is required for the existence of any stabilizing controller.

Assumptions A2–A3 are standard assumptions, which guarantee existence and uniqueness of solutions. In particular, the spectral factorization approach, described in Chapter 3, requires us to find a factor $L \in \mathcal{RH}_\infty$, such that $P_{12}^* P_{12} = L^*L$ and $L^{-1} \in \mathcal{RH}_\infty$. The requirement that L has a stable inverse is a tricky issue, which must be dealt with carefully. It can be shown that this requires the existence of a stabilizing solution of an algebraic Riccati equation. Thus, we need to show when stabilizing solutions to the Riccati equations (4.4) and (4.5) exist. A well-known result to this end is as follows.

Lemma 16. *Suppose $D^TD > 0$. Then, there exists a unique $X \in \mathbb{R}^{n\times n}$ satisfying*

$$X = C^TC + A^TXA - (A^TXB + C^TD)(D^TD + B^TXB)^{-1}(B^TXA + D^TC)$$

such that $A - B(D^TD + B^TXB)^{-1}(B^TXA + D^TC)$ is stable, if and only if (A, B) is stabilizable and

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad \textit{has full column rank for all } \lambda \in \mathbb{T}$$

Proof. See [32, p.546] and [12] for proofs. ∎

With Lemma 16, Assumptions A1– A3 guarantee the existence of stabilizing solutions to the algebraic Riccati equation (4.4). To prove the existence of Y satisfying (4.5), we have the following lemma.

Lemma 17. *For the system in (4.2), suppose Assumptions A1–A3 hold. Then, there exists $Y \in \mathbb{R}^{n_2\times n_2}$ such that*

$$Y = C_2^TC_2 + A_{22}^TYA_{22} - (A_{22}^TYB_{22} + C_2^TD_2)(D_2^TD_2 + B_{22}^TYB_{22})^{-1}(B_{22}^TYA_{22} + D_2^TC_2)$$

and $A_{22} - B_{22}(D_2^TD_2 + B_{22}^TYB_{22})^{-1}(B_{22}^TYA_{22} + D_2^TC_2)$ is stable.

Proof. From Assumption A2, it is clear that $D_2^TD_2 > 0$. Using Lemma 16, we just need to prove that (A_{22}, B_{22}) is stabilizable and

$$\begin{bmatrix} A_{22} - \lambda I & B_{22} \\ C_2 & D_2 \end{bmatrix} \quad \text{has full column rank for all } \lambda \in \mathbb{T}$$

The stabilizability of (A_{22}, B_{22}) is given by Assumption A1. To prove the rank condition, Assumption A3 implies that

$$\begin{bmatrix} A_{11} - \lambda I & 0 & B_{11} & 0 \\ A_{21} & A_{22} - \lambda I & B_{21} & B_{22} \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{T}$. In particular, columns 2 and 4 must have full column rank, and the result follows. ∎

As a result, we see that Assumptions A1–A3 guarantee the existence of stabilizing solutions X and Y to the Riccati equations in (4.4) and (4.5).

When the system can be stabilized, by choosing stabilizing matrices F_1 and F_2, we can use the standard Youla parametrization to simplify our optimization problem.

Lemma 18. *Let $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$. Suppose (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable, and let F_1 and F_2 be matrices, such that $A_{11} + B_{11}F_1$ and $A_{22} + B_{22}F_2$ have stable eigenvalues. Let*

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$$

Then, the set of all stabilizing controllers $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ is parametrized by

$$\mathcal{K} = Q(I + MQ)^{-1} + F \qquad Q \in \mathcal{S}$$

where

$$M = \left[\begin{array}{c|c} A + BF & B \\ \hline I & 0 \end{array}\right]$$

Moreover, the set of stable closed-loop transfer functions satisfies

$$\{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty),\ \mathcal{K} \text{ stabilizing}\} = \{N_{11} + N_{12}QN_{21} \mid Q \in \mathcal{S}\}$$

where $N_{12} = z^{-1}((C + DF)(zI - (A + BF))^{-1}B + D)$ and

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} A + BF & H \\ \hline C + DF & 0 \\ A + BF & H \end{array}\right]$$

Proof. Notice in Figure 4.2 that we can create an equivalent feedback system by adding and subtracting the gain F. However, in doing so, we can create a new

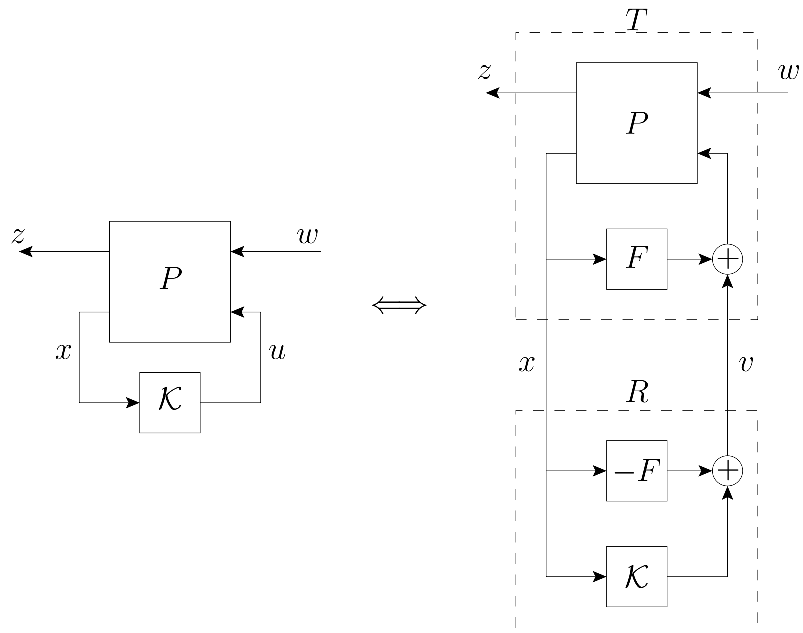


Figure 4.2: Equivalent Feedback Systems

feedback system with

$$\mathcal{F}(T, R) = \mathcal{F}(P, \mathcal{K})$$

where

$$T = \left[\begin{array}{c|cc} A + BF & H & B \\ \hline C + DF & 0 & D \\ I & 0 & 0 \end{array}\right] \qquad R = \mathcal{K} - F$$

By construction, it is clear that the states of the plant T are equal to the original states of the plant P. Similarly, the states of the controller R are equivalent to those of the original controller $\mathcal{K}$. Consequently, it is clear that R internally stabilizes T if and only if $\mathcal{K}$ internally stabilizes P.

Let us now define

$$Q = R(I - T_{22}R)^{-1}$$

Notice that the map $R \mapsto Q$ is bijective. As a result, we now have

$$\mathcal{F}(T, R) = T_{11} + T_{12}QT_{21}$$

Now, recalling our definition of internal stability from Section 2.4, suppose $Q \in S$. Since T and Q are stable, then the closed-loop system is internally stable. In addition, since both T_{22} and Q are lower triangular, then R is lower triangular, where

$$R = Q(I + T_{22}Q)^{-1}$$

Thus,

$$\mathcal{K} = R + F = Q(I + T_{22}Q)^{-1} + F \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$$

and $\mathcal{K}$ stabilizes P. As a result, we have

$$\{T_{11} + T_{12}QT_{21} \mid Q \in \mathcal{S}\} \subset \{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty),\ \mathcal{K} \text{ stabilizing}\}$$

Conversely, suppose that $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ stabilizes P. This implies that R stabilizes T in Figure 4.2. This, in turn, implies that $R(I - T_{22}R)^{-1}$ is stable. Consequently, we see that Q is stable. Moreover, since T_{22} and R are lower triangular, so is Q. Hence, $Q \in \mathcal{S}$. Thus, we've shown that

$$\{T_{11} + T_{12}QT_{21} \mid Q \in \mathcal{S}\} = \{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty),\ \mathcal{K} \text{ stabilizing}\}$$

The result follows by letting

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & M \end{bmatrix} = \begin{bmatrix} T_{11} & z^{-1}T_{12} \\ zT_{21} & T_{22} \end{bmatrix}$$

■

Lemma 18 is a simplification of the classical Youla parametrization, since there exists a static gain F which stabilizes the system. In the case where P_{22} is stable, we see that this parametrization reduces to the simple substitution, $Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1}$, as we suggested in previous chapters.

While the classic Youla parametrization reformulates (4.3) as a convex problem, we can make one additional simplification to our optimization problem with the following lemma.

Lemma 19. *For the system in* (4.2)*, let N be defined as in Lemma 18. Suppose $Q \in \mathcal{S}$ is optimal for*

$$\begin{aligned} &\textit{minimize} && \|N_{11} + N_{12}Q\|_2 \\ &\textit{subject to} && Q \in \mathcal{S} \end{aligned} \tag{4.9}$$

Then, there exists $\hat{Q} \in \mathcal{S}$, such that $Q = \hat{Q}N_{21}$, and $\hat{Q}$ is optimal for

$$\begin{aligned} &\textit{minimize} && \|N_{11} + N_{12}\hat{Q}N_{21}\|_2 \\ &\textit{subject to} && \hat{Q} \in \mathcal{S} \end{aligned} \tag{4.10}$$

Conversely, if $\hat{Q} \in \mathcal{S}$ is optimal for (4.10)*, then $Q = \hat{Q}N_{21}$ is optimal for* (4.9)*.*

Proof. By construction, it is clear that $N_{21} \in \mathcal{S}$. Also, we see that

$$N_{21}^{-1} = \left[\begin{array}{c|c} 0 & -I \\ \hline H^{-1}(A+BF) & H^{-1} \end{array}\right] = H^{-1}(I - z^{-1}(A+BF))$$

Thus, $N_{21}^{-1} \in \mathcal{S}$. As a result, $Q \in \mathcal{S}$ if and only if $\hat{Q} \in \mathcal{S}$. ■

It is important to note that Lemma 19 works because N_{21} is invertible in $\mathcal{S}$. This follows from the fact that we assume state feedback in this problem. We will weaken this assumption in our results of Chapter 6.

Applying Lemmas 18 and 19 to our problem, it is clear that (4.3) is equivalent to the optimization problem

$$\begin{aligned} &\text{minimize} && \|N_{11} + N_{12}Q\|_2 \\ &\text{subject to} && Q \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2) \end{aligned} \tag{4.11}$$

where N_{11} and N_{12} are defined in Lemma 18. In other words, the optimal Q for (4.11) corresponds to the optimal $\mathcal{K}$ for (4.3), and vice versa.

In order to solve the optimization problem in (4.11), we would like to solve an equivalent optimality condition, as we did in Chapter 3. This optimality condition is provided in the following lemma.

Lemma 20. *Let* $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$. *Suppose* $G_1, G_2 \in \mathcal{RH}_\infty$. *Then,* $Q \in \mathcal{S}$ *minimizes*

$$\begin{array}{ll} \textit{minimize} & \|G_1 + G_2 Q\|_2 \\ \textit{subject to} & Q \in \mathcal{S} \end{array}$$

if and only if

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{S}^\perp \tag{4.12}$$

Proof. The proof follows directly from Lemma 5 of Chapter 3. ∎

Consequently, $Q \in \mathcal{S}$ is optimal for (4.11) if and only if Q satisfies the optimality condition

$$N_{12}^* N_{11} + N_{12}^* N_{12} Q \in \mathcal{S}^\perp \tag{4.13}$$

While Lemma 20 provides necessary and sufficient conditions for optimality of a controller, it does not guarantee existence of such a controller. The existence of an optimal controller will be shown by explicit construction in the following sections.

4.4 Spectral Factorization

We are now ready to discuss the general form of (4.12), when G_1 and G_2 are matrix transfer functions. Once again, the approach taken here is very similar to our previous discussions. However, finding the spectral factorization of $G_2^* G_2$ is considerably more complicated than the scalar case. Fortunately, this process can shown by straightforward algebraic manipulations.

Lemma 21. *Suppose* $G_1, G_2 \in \mathcal{RH}_\infty$ *have the realizations*

$$\begin{aligned} G_1 &= C(zI - A)^{-1} H \\ G_2 &= z^{-1}(C(zI - A)^{-1} B + D) \end{aligned}$$

Suppose there exists a stabilizing solution X to the algebraic Riccati equation

$$X = C^TC + A^TXA - (A^TXB + C^TD)(D^TD + B^TXB)^{-1}(B^TXA + D^TC)$$

Let $W = D^TD + B^TXB$ and $K = W^{-1}(B^TXA + D^TC)$ and $L \in \mathcal{RH}_\infty$ satisfying

$$L = \left[\begin{array}{c|c} A & B \\ \hline W^{\frac{1}{2}}K & W^{\frac{1}{2}} \end{array}\right]$$

Then, $L^{-1} \in \mathcal{RH}_\infty$, $L^{-*} \in \mathcal{H}_\infty^-$, and

$$L^*L = G_2^*G_2$$

Moreover,

$$L^{-*}G_2^*G_1 = W^{-\frac{1}{2}}B^T(z^{-1}I - (A - BK)^T)^{-1}XH + zW^{\frac{1}{2}}K(zI - A)^{-1}H$$

Proof. This result follows from algebraic manipulations of the Riccati equation. A simple proof follows the approach in [10]. ∎

With the above spectral factorization, we can now solve the optimality condition (4.12) for the classical, centralized case, which will form the basis for our decentralized solution.

Lemma 22. *Let $G_1, G_2 \in \mathcal{RH}_\infty$ be defined as in Lemma 21. Suppose there exists a stabilizing solution X to the algebraic Riccati equation*

$$X = C^TC + A^TXA - (A^TXB + C^TD)(D^TD + B^TXB)^{-1}(B^TXA + D^TC)$$

Let K and L be defined as in Lemma 21. Then, the unique $Q \in \mathcal{RH}_2$ satisfying

$$G_2^*G_1 + G_2^*G_2Q \in \mathcal{H}_2^\perp$$

is given by

$$Q = -zK(zI - (A - BK))^{-1}H$$

Proof. From Lemma 21, we have the spectral factorization $G_2^*G_2 = L^*L$. Since we showed that $L^{-*} \in \mathcal{H}_\infty^-$, then $L^{-*}\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$. Hence, the optimality condition is equivalent to

$$L^{-*}G_2^*G_1 + LQ \in \mathcal{H}_2^\perp$$

Since $LQ \in \mathcal{RH}_2$, we can project the optimality condition onto $\mathcal{H}_2$ to obtain

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*}G_2^*G_1) + LQ = 0$$

From Lemma 21, we have

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*}G_2^*G_1) = zW^{\frac{1}{2}}K(zI - A)^{-1}H$$

Consequently, we have

$$\begin{aligned} Q &= -L^{-1}\mathrm{P}_{\mathcal{H}_2}(L^{-*}G_2^*G_1) \\ &= -zK(zI - (A - BK))^{-1}H \end{aligned}$$

■

This completes our discussion of the classical case; that is, when $\mathcal{S} = \mathcal{RH}_2$. As we will now show, the solution of the two-player problem relies on the spectral factorization approach that was developed for the classical case. To this end, we have the following result.

Lemma 23. *Let $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$, and suppose $G_1, G_2 \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ satisfies*

$$G_2^*G_1 + G_2^*G_2Q \in \mathcal{S}^\perp$$

if and only if the following two conditions both hold:

i) $(G_2^*G_1)_{22} + (G_2^*G_2)_{22}Q_{22} \in \mathcal{H}_2^\perp$

ii) $\begin{bmatrix} (G_2^*G_1)_{11} \\ (G_2^*G_1)_{21} \end{bmatrix} + G_2^*G_2 \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \in \mathcal{H}_2^\perp$

Proof. Let $G_2^* G_1 + G_2^* G_2 Q = \Lambda$ where $\Lambda \in \mathcal{S}^\perp$. Note that Λ is partitioned as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

where $\Lambda_{11}, \Lambda_{21}, \Lambda_{22} \in \mathcal{H}_2^\perp$. Consequently, (i) comes from the fact that $\Lambda_{22} \in \mathcal{H}_2^\perp$, and (ii) since $\begin{bmatrix} \Lambda_{11} \\ \Lambda_{21} \end{bmatrix} \in \mathcal{H}_2^\perp$. ∎

The important aspect of Lemma 23 is that it decomposes our optimality condition (4.13) over $\mathcal{S}^\perp$ into two separate conditions over $\mathcal{H}_2^\perp$. Each of these conditions can be solved via the spectral factorization approach of Lemmas 21 and 22.

It is worth noting here that an alternate formulation for Lemma 23 exists. To see this, it is straightforward to show that the optimization problem in (4.11) is equivalent to

$$\begin{aligned} \text{minimize} \quad & \left\| \begin{bmatrix} (N_{11})_{11} \\ (N_{11})_{21} \end{bmatrix} + N_{12} \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \right\|_2 + \|(N_{11})_{22} + (N_{12})_{22} Q_{22}\|_2 \\ \text{subject to} \quad & Q_{11}, Q_{21}, Q_{22} \in \mathcal{RH}_2 \end{aligned}$$

In other words, the norm decouples into two separate objectives which are functions of different variables. Hence, we can minimize each norm separately over variables in $\mathcal{RH}_2$. This amounts to two separate centralized problems, which is exactly the decomposition that we see in Lemma 23.

In any event, if we now want to apply this spectral factorization approach to our problem, our Riccati equations would be in terms of the pre-compensator F. However, this difficulty can be avoided with the following result.

Lemma 24. *Suppose $X \in \mathbb{R}^{n\times n}$, and $F \in \mathbb{R}^{m\times n}$. Then,*

$$X = C^T C + A^T X A - (A^T X B + C^T D)(D^T D + B^T X B)^{-1}(B^T X A + D^T C)$$

and $A - B(D^T D + B^T X B)^{-1}(B^T X A + D^T C)$ is stable, if and only if

$$X = C_F^T C_F + A_F^T X A_F - (A_F^T X B + C_F^T D)(D^T D + B^T X B)^{-1}(B^T X A_F + D^T C_F)$$

and $A_F - B(D^TD + B^TXB)^{-1}(B^TXA_F + D^TC_F)$ *is stable, where* $A_F = A + BF$ *and* $C_F = C + DF$.

Proof. By substitution of A_F and C_F, it can be readily shown that the two Riccati equations are equivalent. ∎

Thus, our ability to solve the optimality condition (4.13) is independent of our choice of the pre-compensator F.

For convenience, we define

$$E_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where the dimensions are defined by the context. We can now solve for the $Q \in \mathcal{S}$ satisfying the optimality condition (4.13).

Lemma 25. *For the system in* (4.2)*, suppose Assumptions A1–A3 hold. Let* F_1, F_2 *be matrices such that* $A_{11} + B_{11}F_1$ *and* $A_{22} + B_{22}F_2$ *have stable eigenvalues. Also, let* X *and* Y *be the stabilizing solutions to the algebraic Riccati equations* (4.4)–(4.5)*, and let* K*,* J *be given by* (4.6)–(4.7)*. Define*

$$\begin{aligned} A_F &= A + BF \\ A_K &= A - BK \\ A_J &= A_{22} - B_{22}J \end{aligned}$$

Finally, let N_{11} *and* N_{12} *be defined as in Lemma 18. Then, the unique optimal* $Q \in \mathcal{S}$ *for* (4.11) *is given by*

$$Q_{22} = \left[\begin{array}{c|c} A_J & A_J H_2 \\ \hline -J - F_2 & -(J + F_2)H_2 \end{array}\right] \tag{4.14}$$

$$\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \left[\begin{array}{c|c} A_K & A_K E_1 H_1 \\ \hline -K - F & -(K + F)E_1 H_1 \end{array}\right] \tag{4.15}$$

Proof. From Lemma 20, we know that the optimal $Q \in \mathcal{S}$ for (4.11) satisfies the optimality condition (4.12). Using Lemma 23, this can be solved as two separate problems. Condition (i) of the lemma can be solved via Lemma 22, where

$$\begin{aligned} G_1 &= (C_2 + D_2F_2)(zI - (A_F)_{22})^{-1}H_2 \\ G_2 &= z^{-1}\left((C_2 + D_2F_2)(zI - (A_F)_{22})^{-1}B_{22} + D_2\right) \end{aligned}$$

to obtain the optimal Q_{22} in (4.14). Note that (4.5) and Lemma 24 imply the existence of the required algebraic Riccati equation needed in Lemma 22, for whatever stabilizing F is chosen.

A similar argument is used to solve for Q_{11} and Q_{21} in condition (ii) of Lemma 23, via Lemma 22 where we let $F_1 = N_{11}E_1$ and $G_2 = N_{12}$. As we can see, the solution here requires the stabilizing solution (4.4) and A_K. ■

Having found the optimal $Q \in \mathcal{S}$, the following result provides the optimal controller for our decentralized two-player problem.

Theorem 26. *For the system in* (4.2)*, suppose Assumptions A1–A3 hold. Let* X, Y, K, J *be defined by the Riccati equations* (4.4)–(4.7)*, and let* $A_K = A - BK$*. Then, the unique optimal* $\mathcal{K} \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ *for* (4.3) *is*

$$\mathcal{K} = \begin{bmatrix} -K_{11} - K_{12}\Phi & 0 \\ -K_{21} - (K_{22} - J)\Phi & -J \end{bmatrix} \tag{4.16}$$

where

$$\Phi = (zI - (A_K)_{22})^{-1}(A_K)_{21}$$

Proof. From Lemma 25, the unique optimal $Q \in \mathcal{S}$ for (4.13) is given by (4.14) and (4.15). Lemma 19 then implies that $\hat{Q} = QN_{21}^{-1}$ is optimal for (4.10), where N_{21} is defined in Lemma 18. Using Lemma 18, the unique optimal $\mathcal{K}$ for (4.3) is given by

$$\mathcal{K} = \hat{Q}(I + M\hat{Q})^{-1} + F$$

with M defined in the lemma, and the result follows. ■

Proof of Theorem 14. The result follows directly from Theorem 26, where controller 1 is given by

$$\mathcal{K}_{11} = \left[\begin{array}{c|c} (A_K)_{22} & (A_K)_{21} \\ \hline -K_{12} & -K_{11} \end{array}\right]$$

and controller 2 is given by

$$\begin{bmatrix} \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} (A_K)_{22} & (A_K)_{21} & 0 \\ \hline -K_{22} + J & -K_{21} & -J \end{array}\right]$$

■

4.5 Estimation Structure

Having established our main result, the last step in our analysis of this problem is to discuss the structure of this optimal controller. Notice that the solution in (4.16) is given in terms of the transfer function Φ. While this answer is indeed what we need to run the controller in practice, some additional insight to the optimal policy can be obtained by further investigation of this transfer function. To this end, we let

$$\eta = \Phi x_1$$

This represents the following state-space system

$$\eta(t+1) = (A_K)_{22}\eta(t) + (A_K)_{21}x_1(t)$$

with initial condition $\eta(0) = 0$. As a result, the optimal policy can be written as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = -\begin{bmatrix} K_{11} & 0 & K_{12} \\ K_{21} & J & K_{22} - J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \eta(t) \end{bmatrix}$$

Combining this with the dynamics in (4.2), the closed-loop dynamics of the system become

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} (A_K)_{11} & (A_K)_{12} & 0 \\ (A_K)_{21} & (A_K)_{22} & 0 \\ (A_K)_{21} & (A_K)_{22} - A_J & A_J \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{4.17}$$

With this closed-loop system in mind, we now attempt to construct the minimum-mean square error estimator of $x_2(t)$ based on measurements of $x_1(0), \dots, x_1(t)$ and $\eta(0), \dots, \eta(t)$, given by the conditional mean

$$\mathrm{E}\big(x_2(t) \mid x_1(0), \dots, x_1(t), \eta(0), \dots, \eta(t)\big)$$

To this end, consider the following lemma.

Lemma 27. *Suppose x_1, x_2 represent the states of the following autonomous system driven by noise*

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{4.18}$$

where $x_1(0), x_2(0), w_1(t), w_2(t)$ are independent, zero-mean random variables for all $t \geq 0$. Define μ_t such that

$$\mu_t(z_0, \dots, z_t) = \mathrm{E}\big(x_2(t) \mid x_1(0) = z_0, \dots, x_1(t) = z_t\big)$$

Then, $\mu_0(z_0) = 0$, and for each $t \geq 0$,

$$\mu_{t+1}(z_0, \dots, z_{t+1}) = A_{22}\mu_t + A_{21} z_t$$

Proof. For each t, let q_t be the conditional probability density function

$$q_t(y) = p^{x_2(t) \mid x_1(0), \dots, x_1(t)}(y)$$

so that μ_t is the mean of this distribution. Since $x_1(0)$ and $x_2(0)$ are independent, then clearly $\mu_0(z_0) = 0$.

Now, using our definition for q_{t+1} and Bayes' law, we can show that

$$q_{t+1}(y) = \int_v g^{x_2(t+1)|x_1(t)x_2(t)}(y)\ q_t(v)\ dv$$

where $g^{x_2(t+1)|x_1(t)x_2(t)}$ is the transition pdf of x_2 defined by (4.18). Consequently, we can recursively compute the mean μ_{t+1} as

$$\mu_{t+1}(z_0, \dots, z_{t+1}) = A_{22}\mu_t(z_0, \dots, z_t) + A_{21}z_{\ t}$$

The result follows by induction. ∎

With this lemma, we obtain a very simple representation for the optimal controller.

Theorem 28. *Suppose x_1, x_2, η are the states of the autonomous system in* (4.17). *Then,*

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \dots, x_1(t)\big)$$

Proof. From (4.17), we see that the state transition matrix is lower triangular. Thus, we can use the results of Lemma 27 to get

$$\begin{aligned}
\mu_{t+1} &= \mathrm{E}\big(x_2(t+1) \mid x_1(0), \dots, x_1(t+1), \eta(0), \dots, \eta(t+1)\big) \\
&= A_J\mu_t + \begin{bmatrix} (A_K)_{21} & (A_K)_{22} - A_J \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \end{bmatrix} \\
&= \eta(t+1) + A_J\big(\mu_t - \eta(t)\big)
\end{aligned}$$

where we have used the definition of $\eta(t+1)$ in the last expression. Now, since $\mu_0 = \eta(0) = 0$, we inductively see that $\mu(t) = \eta(t)$ for all $t \geq 0$. Lastly, since $\eta(t)$ is simply a function of $x_1(0), \dots, x_1(t)$, we have

$$\begin{aligned}
\eta(t) &= \mathrm{E}\big(x_2(t) \mid x_1(0), \dots, x_1(t), \eta(0), \dots, \eta(t)\big) \\
&= \mathrm{E}\big(x_2(t) \mid x_1(0), \dots, x_1(t)\big)
\end{aligned}$$

as desired. ∎

4.6 Examples

We conclude our discussion of this two-player problem with a couple examples. The first example compares the optimal decentralized policy with a standard heuristic. The second example compares the centralized and decentralized policies for a particular system.

4.6.1 A Standard Heuristic

It is worth comparing the optimal decentralized solution in (4.8) with a standard heuristic solution to this problem. To motivate the heuristic, consider the centralized problem. In the state feedback case, the optimal policy is known to be the static gain

$$u(t) = -Kx(t)$$

where K is the same gain, given by (4.6). In this case, when the state is not known directly, but is instead estimated via some noisy output, the optimal policy becomes

$$u(t) = -Kx^{\text{est}}(t)$$

where x^{est} is the minimum mean square error estimate of x. In other words, the state x is replaced by its estimate in the optimal policy. This is the certainty equivalence principle.

Following this logic, in the two-player problem, the heuristic policy would be

$$\begin{aligned} u_1(t) &= -K_{11}x_1(t) - K_{12}x_2^{\text{est}}(t) \\ u_2(t) &= -K_{21}x_1(t) - K_{22}x_2(t) \end{aligned}$$

Since system 1 cannot directly measure state 2, it uses an estimate of x_2; this matches with the optimal decentralized solution. However, in the heuristic policy, system 2 has complete state information, so it should not need to estimate anything.

Consider the following simple two-player system.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0.1 & 0 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

As a result, the centralized gain is

$$K \approx \begin{bmatrix} 0 & -0.8 \\ 0 & -0.8 \end{bmatrix}$$

It is straightforward to show that, for the heuristic approach, the closed-loop dynamics for player 2 evolve as

$$x_2(t+1) = (2 - 0.8)x_2(t)$$

which is clearly unstable. Thus, the heuristic approach can destabilize a system.

The optimal decentralized solution provided in this chapter establishes the correct structure for the optimal policy. While the motivation for the heuristic is reasonable and intuitive, this intuition is misleading. A more accurate rationale for the optimal policy is that player 2 must correct for errors that player 1 makes in estimation.

4.6.2 Decentralized Policy

Our second example is a simulation of the following system of a simple two mass system. The dynamics are given by

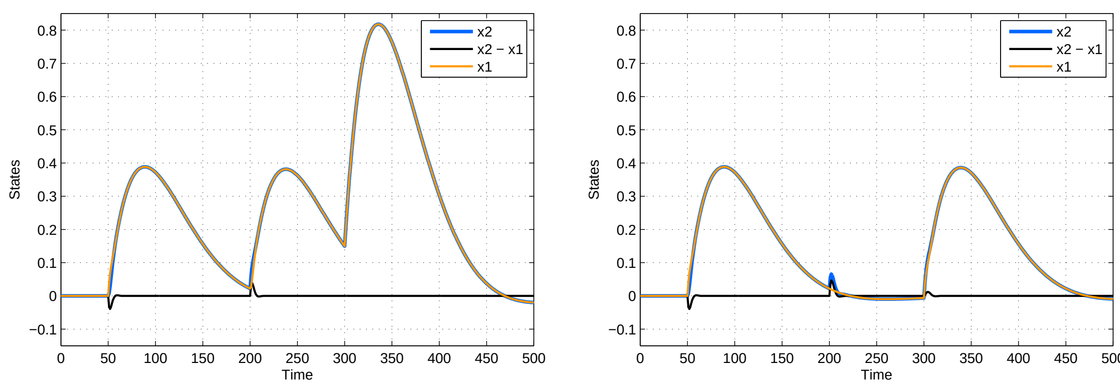
$$A = \begin{bmatrix} 1 & .05 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & .05 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \frac{1}{100} \times \begin{bmatrix} 0.1 & 0 \\ 5 & 0 \\ 0 & 0.1 \\ 0 & 5 \end{bmatrix} \qquad H = 0.01 \times I$$

with states $\begin{bmatrix} x_1^T & \dot{x}_1^T & x_2^T & \dot{x}_2^T \end{bmatrix}^T$. The cost objective has the form

$$100(x_1 - x_2)^2 + 0.001(x_1^2 + x_2^2 + \dot{x}_1^2 + \dot{x}_2^2) + \mu(u_1^2 + u_2^2)$$

and μ is a parameter that we will vary. Note that this cost penalizes the difference between the two masses' positions. We will use μ to trade off the cost of error with the control effort required. We see that this cost is principally concerned with minimizing the distance between the two masses. The second term above is simply there to drive the system back to zero.

Figure 4.3 shows the state trajectories in the case where $\mu = 0.01$. Here, we apply three impulses to the system: an impulse to system 1 at $t = 50$, an impulse to system 2 at $t = 200$, and an impulse to both systems at $t = 300$.



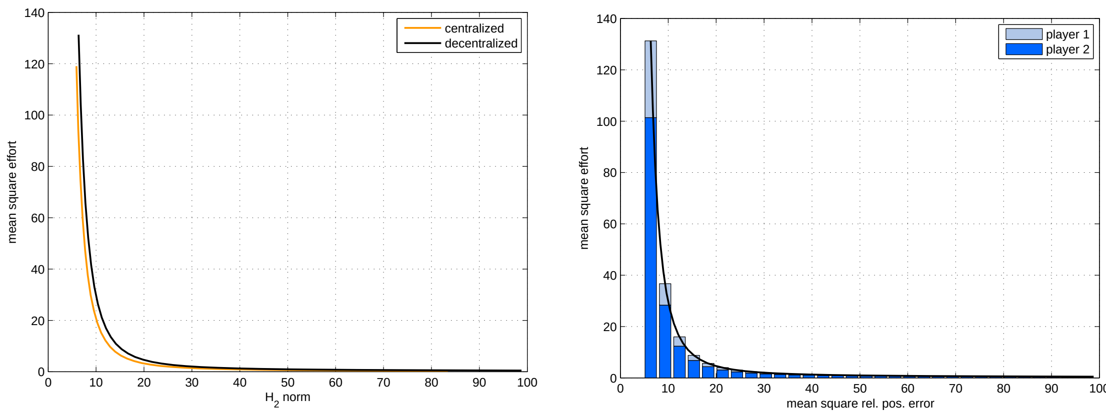
(a) Centralized (b) Decentralized

Figure 4.3: State Trajectories for Two-Player Example

Notice the difference between the two plots. For the centralized problem in Figure 4.3(a), the first two impulses are very similar since both players can observe the other's change in position. At the last impulse again, they can observe both impulses and move together.

For the decentralized case in Figure 4.3(b), there is not much difference at the first impulse since both systems observe the change in system 1's position. At the second impulse however, only player 2 can observe the position error, so it quickly returns to zero to avoid errors. At the third impulse, system 1 behaves as it did at the first impulse; recognizing this, system 2 moves to follow system 1.

Another way to view the optimal decentralized policy is by looking at the tradeoff between position error and control authority as we vary μ in Figure 4.4(a). Fortunately, the optimal decentralized policy is not significantly different from the centralized policy, indicating the value of this work.



(a) Cost Tradeoff

(b) Input Effort

Figure 4.4: Controller Effort for Two-Player Example

Figure 4.4(b) shows the relative control effort for both players in the decentralized policy. Notice that player 2 accounts for the vast majority of control authority used, since it is responsible for more information.

4.7 Summary

In this chapter, we focused on trying to solve a decentralized, two-player control problem. To this end, a number of steps were required. To begin, it was shown that stabilization of the whole system was possible if and only if each of the subsystems was individually stabilizable. With this in mind, we obtained a Youla parametrization for the optimization problem, which converted the problem into a convex one. To solve this problem, a spectral factorization approach, similar to our discussion in Chapter 3, was taken herein. A careful development of this method was provided. This approach admits an explicit state-space solution and provides significant insight into the optimal policies. In particular, it was shown that both players implement estimators, and the dimension of the optimal controllers is equal to the dimension of player two's state.

It should be noted that this two-player problem was originally solved for the finite-horizon case in [25]. While a spectral factorization approach was taken there and in this chapter, a dynamic programming method was also obtained for this problem in [24].

Chapter 5

The N-Player Problem

In the previous chapter, we used a spectral factorization approach to find an explicit state-space solution to the two-player decentralized problem. In this chapter, we extend these results to find the solution for more general decentralized systems defined over graphs.

5.1 Problem Formulation

We turn to solving the general N-player problem defined by graphs, as described in Chapter 2. As discussed there, we restrict our attention to the case where the system dynamics, defined by $\mathcal{G}^P$, satisfy the sparsity constraints of the controller, represented by $\mathcal{G}^K$. Without loss of generality, we will assume for this remainder of this chapter that $\mathcal{M}^{\mathcal{G}^P} = \mathcal{M}^{\mathcal{G}^K} = \mathcal{M}^{\mathcal{G}}$.

We briefly restate the problem formulation here. Connecting the plants according to $\mathcal{G}^P$, we obtain the following discrete time state-space system

$$x(t+1) = Ax(t) + Bu(t) + Hw(t) \tag{5.1}$$

where $A, B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$, and H is block diagonal and invertible. Our cost vector is

$$z(t) = \begin{bmatrix} C_1 & \cdots & C_N \end{bmatrix} x(t) + \begin{bmatrix} D_1 & \cdots & D_N \end{bmatrix} u(t)$$

where we assume that $D^T D > 0$. Consequently, our plant can be expressed as the matrix $P \in \mathcal{RL}_\infty$, where

$$P = \left[\begin{array}{c|cc} A & H & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array}\right] \tag{5.2}$$

Once again, state feedback is assumed throughout. Consequently, we are looking for controllers of the form

$$\begin{aligned} q_i(t+1) &= A_{K_i} q_i(t) + B_{K_i} x_{\mathcal{A}_i}(t) \\ u_i(t) &= C_{K_i} q_i(t) + D_{K_i} x_{\mathcal{A}_i}(t) \end{aligned} \tag{5.3}$$

for each $i \in \mathcal{V}$. Since no communication delays are assumed, decision u_i is a function of the history of x_j's, for each $j \in \mathcal{A}_i$. Equivalently, we want a transfer function $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$, which is optimal for

$$\begin{array}{ll} \text{minimize} & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\|_2 \\ \text{subject to} & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty) \end{array} \tag{5.4}$$

5.2 Main Results

Our presentation of the N-player solution will follow the same approach as the two-player solution of the previous chapter. To this end, the following assumptions will be made for this problem.

$A1)$ (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$

$A2)$ $D^T D > 0$ and $HH^T > 0$

$A3)$ $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{T}$

Note that these assumptions match closely with the assumptions needed in the two-player problem. We present the general solution here which will be developed in the following sections.

Theorem 29. *Suppose $\mathcal{G}$ is a directed acyclic graph, and let P be defined by (5.2). Furthermore, suppose Assumptions A1–A3 hold. For each $i \in \mathcal{V}$, let X_i be the stabilizing solution to the Riccati equation*

$$\begin{aligned} X_i = C_{\mathcal{D}_i}^T C_{\mathcal{D}_i} + A_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} - (A_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i} + C_{\mathcal{D}_i}^T D_{\mathcal{D}_i}) \\ \times (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i}) \end{aligned} \tag{5.5}$$

and let

$$K_i = (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i})$$

Also, define $A_K, B_K, C_K, D_K \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$ by

$$(A_K)_{ij} = \begin{cases} A_{\mathcal{D}_i'} - B_{\mathcal{D}_i'\mathcal{D}_i} K_i I_{\mathcal{D}_i \mathcal{D}_i'} & i = j \\ -(A_{\mathcal{D}_i' i} - B_{\mathcal{D}_i' \mathcal{D}_i} K_i I_{\mathcal{D}_i i}) I_{i \mathcal{D}_j'} & i \in \mathcal{D}_j' \\ 0 & \textit{otherwise} \end{cases}$$

$$(B_K)_{ij} = \begin{cases} A_{\mathcal{D}_i' i} - B_{\mathcal{D}_i' \mathcal{D}_i} K_i I_{\mathcal{D}_i i} & i = j \\ 0 & \textit{otherwise} \end{cases}$$

$$(C_K)_{\mathcal{V}j} = -I_{\mathcal{V}\mathcal{D}_j} K_j I_{\mathcal{D}_j \mathcal{D}_j'} + \sum_{k \in \mathcal{D}_j'} I_{\mathcal{V}\mathcal{D}_k} K_k I_{\mathcal{D}_k k} I_{k \mathcal{D}_j'}$$

$$(D_K)_{\mathcal{V}j} = -I_{\mathcal{V}\mathcal{D}_j} K_j I_{\mathcal{D}_j j}$$

Then, there exists a unique optimal $\mathcal{K} \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ for (5.4) given by

$$\mathcal{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{5.6}$$

Moreover, this controller corresponds to the policies:

$$\begin{aligned} q_i(t+1) &= (A_K)_{\mathcal{A}_i} q_i(t) + (B_K)_{\mathcal{A}_i} x_{\mathcal{A}_i}(t) \\ u_i(t) &= (C_K)_{i\mathcal{A}_i} q_i(t) + (D_K)_{i\mathcal{A}_i} x_{\mathcal{A}_i}(t) \end{aligned} \tag{5.7}$$

for each $i \in \mathcal{V}$*.*

Let us first compare the general solution provided by Theorem 29 with the two-player solution of the previous chapter. Using the above theorem and the fact that $\mathcal{D}_2' = \emptyset$, it is clear that the optimal $\mathcal{K}$ in the two-player case is given by (5.6) where

$$\begin{aligned} A_K &= (A - BK_1)_{22} \\ B_K &= \begin{bmatrix} (A - BK_1)_{21} & 0 \end{bmatrix} \\ C_K &= -K_1 I_{\mathcal{D}_1 2} + I_{\mathcal{V}2} K_2 = \begin{bmatrix} -(K_1)_{12} \\ -(K_1)_{22} + K_2 \end{bmatrix} \\ D_K &= \begin{bmatrix} -K_1 I_{\mathcal{D}_1 1} & -I_{\mathcal{V}2} K_2 \end{bmatrix} = \begin{bmatrix} -(K_1)_{11} & 0 \\ -(K_1)_{21} & -K_2 \end{bmatrix} \end{aligned}$$

Noting that K_1 and K_2 are equivalent to K and J in (4.6) and (4.7), respectively, it is clear that the optimal controller for the 2-player problem is consistent with the results for the N-player problem provided here.

While (5.7) provides a state space representation for each individual controller, it is interesting to construct the closed-loop system. To this end, let the state of $\mathcal{K}$ in (5.6) be η, so that

$$\begin{aligned} \eta(t+1) &= A_K \eta(t) + B_K x(t) \\ u(t) &= C_K \eta(t) + D_K x(t) \end{aligned}$$

Combining this with (5.1), we have states $\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}$. As will be shown later, we can use a transformation to get a new state $\zeta(t)$ such that

$$\zeta_i(t) = \begin{bmatrix} x_i(t) - \sum_{j \in \mathcal{A}_i'} I_{i\mathcal{D}_j'} \eta_j(t) \\ \eta_i(t) \end{bmatrix}$$

Under this transformation, the closed-loop system becomes

$$
\begin{aligned}
\begin{bmatrix} \zeta_1(t+1) \\ \vdots \\ \zeta_N(t+1) \end{bmatrix} &= \begin{bmatrix} A_{\mathcal{D}_1} - B_{\mathcal{D}_1} K_1 & & \\ & \ddots & \\ & & A_{\mathcal{D}_N} - B_{\mathcal{D}_N} K_N \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \vdots \\ \zeta_N(t) \end{bmatrix} \\
&\qquad + \begin{bmatrix} I_{\mathcal{D}_1 1} H_1 & & \\ & \ddots & \\ & & I_{\mathcal{D}_N N} H_N \end{bmatrix} w(t) \\
u(t) &= \begin{bmatrix} -I_{\mathcal{V}\mathcal{D}_1} K_1 & \cdots & -I_{\mathcal{V}\mathcal{D}_N} K_N \end{bmatrix} \zeta(t)
\end{aligned}
$$

In addition, we will show that the state $\zeta(t)$ can be represented as

$$
\zeta_i(t) = \mathrm{E}\big(x_{\mathcal{D}_i}(t) \;\big|\; x_{\mathcal{A}_i}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_i}(t) \;\big|\; x_{\mathcal{A}_i'}(0:t)\big)
$$

where $x(0 : t)$ is shorthand notation for $x(0), \ldots, x(t)$. This gives a slightly more intuitive view of the optimal controller; that is, it is based on the errors of estimating a player's descendants given its ancestors' states.

In addition, the optimal $\mathcal{K}$ in (5.6) establishes the order of the optimal controller. In particular, with each state x_i having dimension n_i, then the optimal state dimension σ of the controller is given by

$$
\sigma = \sum_{\substack{i \in \mathcal{V} \\ j \in \mathcal{D}_i'}} n_j
$$

Also note that, in principle, this optimal controller requires every node to run its own estimator for all of its descendant states conditioned on all of its ancestors, but also run all of the estimators that its ancestors' run. This is potentially a lot of estimation dynamics occurring simultaneously, and grows exponentially with the size of the graph. However, from above, it is clear that the important states are the ζ_i, whose dynamics are decoupled. Thus, it is much better for each player i to run the dynamics associated with ζ_i, and simply pass this variable to its descendants. With this policy, it is clear that the dynamics grow linearly with the graph size.

5.3 Analysis

Having established the class of graphs under consideration, it is important to discuss when stabilization of the overall system is possible. As in the two-player problem, it turns out that Assumption A1 is the necessary and sufficient condition for the existence of any stabilizing controller.

Lemma 30. *Suppose $\mathcal{G}$ is a directed acyclic graph. There exists a controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ which stabilizes P in (2.4) if and only if (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$.*

Proof. From Lemma 4, since $\mathcal{G}$ is acyclic, we can order the vertices in $\mathcal{V}$ such that $\mathcal{M}^{\mathcal{G}}$ is lower triangular. With this in mind, we can now prove the lemma.

($\Rightarrow$) If (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, then there exist matrices F_i such that $A_i + B_iF_i$ is stable. Consequently, the controller

$$\mathcal{K} = \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_N \end{bmatrix}$$

produces the closed-loop system

$$x(t+1) = (A + BF)x(t)$$

Since $A, B, F \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$, then $A + BF \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$. Thus, $A + BF$ is lower triangular, with diagonal entries equal to $A_i + B_iF_i$, for all $i \in \mathcal{V}$. Since every $A_i + B_iF_i$ is stable, then the closed-loop system is stable.

($\Leftarrow$) Suppose that (A_i, B_i) is not stabilizable, for some $i \in \mathcal{V}$. Then, there exists a transformation U such that

$$U^{-1}A_iU = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad U^{-1}B_i = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

where a_{22} has at least one unstable eigenvalue λ. Let v be the corresponding eigenvector of a_{22}, so that $a_{22}v = \lambda v$. Then, it can be readily shown that with the initial condition $x_i(0) = U \begin{bmatrix} 0 \\ v \end{bmatrix}$, the state $x_i(t) \nrightarrow 0$ as $t \to \infty$, for any inputs u, so that P cannot be stabilized. ■

Note that the stability considered in Lemma 30 is different from (A, B) stabilizable. Thus, this lemma shows that there exists a decentralized controller which stabilizes the overall system if and only if each individual subsystem can be stabilized.

As in the two-player problem of the previous chapter, the solution for the N-player case here requires the stabilizing solutions to multiple Riccati equations. In particular, we need stabilizing solutions X_i satisfying

$$\begin{aligned} X_i = C_{\mathcal{D}_i}^T C_{\mathcal{D}_i} + A_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} - (A_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i} + C_{\mathcal{D}_i}^T D_{\mathcal{D}_i}) \\ \times (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i}) \end{aligned} \quad (5.8)$$

for each $i \in \mathcal{V}$. Consequently, from Lemma 16, we will need $(A_{\mathcal{D}_i}, B_{\mathcal{D}_i})$ stabilizable, $D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} > 0$, and

$$\begin{bmatrix} A_{\mathcal{D}_i} - \lambda I & B_{\mathcal{D}_i} \\ C_{\mathcal{D}_i} & D_{\mathcal{D}_i} \end{bmatrix}$$

having full column rank for all $\lambda \in \mathbb{T}$, for all $i \in \mathcal{V}$. Checking these conditions for each $i \in \mathcal{V}$ can be computationally expensive in large, highly connected graphs. However, this can be simplified with the following lemma.

Lemma 31. *Suppose $D^T D > 0$, (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, and*

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad (5.9)$$

has full column rank for all $\lambda \in \mathbb{T}$. Then, for all $i \in \mathcal{V}$, $D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} > 0$, $(A_{\mathcal{D}_i}, B_{\mathcal{D}_i})$ is stabilizable, and

$$\begin{bmatrix} A_{\mathcal{D}_i} - \lambda I & B_{\mathcal{D}_i} \\ C_{\mathcal{D}_i} & D_{\mathcal{D}_i} \end{bmatrix} \quad (5.10)$$

has full column rank for all $\lambda \in \mathbb{T}$.

Proof. Clearly, $D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} > 0$ since

$$D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} = I_{\mathcal{V}\mathcal{D}_i}^T D^T D I_{\mathcal{V}\mathcal{D}_i} > 0$$

Since (A_i, B_i) is stabilizable, let F_i be matrices such that $A_i + B_i F_i$ is stable for all $i \in \mathcal{V}$. Define

$$F = \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_N \end{bmatrix}$$

Consequently, since we can order the vertices so that A and B are lower triangular, then $A_{\mathcal{D}_j} + B_{\mathcal{D}_j} F_{\mathcal{D}_j}$ is clearly stable for any $j \in \mathcal{V}$. This implies that $(A_{\mathcal{D}_j}, B_{\mathcal{D}_j})$ is stabilizable for any $j \in \mathcal{V}$.

For the last part of the proof, suppose that (5.9) has full column rank for all $\lambda \in \mathbb{T}$. This implies that

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} I_{\mathcal{V}\mathcal{D}_i} & 0 \\ 0 & I_{\mathcal{V}\mathcal{D}_i} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{V}\mathcal{D}_i} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathcal{D}_i} - \lambda I & B_{\mathcal{D}_i} \\ C_{\mathcal{D}_i} & D_{\mathcal{D}_i} \end{bmatrix}$$

has full column rank since

$$\begin{bmatrix} I_{\mathcal{V}\mathcal{D}_i} & 0 \\ 0 & I_{\mathcal{V}\mathcal{D}_i} \end{bmatrix}$$

has full column rank. Consequently,

$$\begin{bmatrix} A_{\mathcal{D}_i} - \lambda I & B_{\mathcal{D}_i} \\ C_{\mathcal{D}_i} & D_{\mathcal{D}_i} \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbb{T}$. ∎

While Lemma 31 provides sufficient conditions for the existence of stabilizing solutions to the Riccati equations (5.8), in general these conditions are not necessary. However, we note that (A_i, B_i) stabilizable is in fact necessary for the existence of a solution to the decentralized problem, from Lemma 30. As for the rank condition of (5.9), it will in fact be necessary in cases where there exists an $i \in \mathcal{V}$ such that

$\mathcal{D}_i = \mathcal{V}$. For more general graphs, a necessary condition would be that (5.10) has full column rank for all vertices $i \in \mathcal{V}$ satisfying $\mathcal{A}'_i = \emptyset$. In most cases though, it will still be computational cheaper to just check the column rank of (5.9). Consequently, it is clear that Assumptions A1–A3 guarantee the existence of stabilizing solutions to (5.8), for every $i \in \mathcal{V}$.

We now look for a convex parametrization of (5.4). Not surprisingly, the Youla parametrization for the N-player problem follows in the same manner as we showed for the two-player problem in the previous chapter. Thus, we state the results; the proofs follow directly from the analogous results of Chapter 4.

Lemma 32. *Suppose $\mathcal{G}$ is a directed acyclic graph, and let P be defined by* (5.2). *Suppose (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, and let F_i be matrices, such that $A_i + B_i F_i$ has stable eigenvalues. Furthermore, define $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$, and let*

$$F = \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_N \end{bmatrix}$$

Then, the set of all stabilizing controllers $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ is parametrized by

$$\mathcal{K} = Q(I + MQ)^{-1} + F \qquad Q \in \mathcal{S}$$

where

$$M = \left[\begin{array}{c|c} A + BF & B \\ \hline I & 0 \end{array}\right]$$

Moreover, the set of stable closed-loop transfer functions satisfies

$$\{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty),\ \mathcal{K} \textit{ stabilizing}\} = \{N_{11} + N_{12} Q N_{21} \mid Q \in \mathcal{S}\}$$

where $N_{12} = z^{-1}((C + DF)(zI - (A + BF))^{-1} B + D)$ and

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} A + BF & H \\ \hline C + DF & 0 \\ A + BF & H \end{array}\right]$$

Lemma 33. *For the system in* (5.2)*, let* N *be defined as in Lemma 32. Suppose* $Q \in \mathcal{S}$ *is optimal for*

$$\begin{array}{ll} \textit{minimize} & \|N_{11} + N_{12}Q\|_2 \\ \textit{subject to} & Q \in \mathcal{S} \end{array} \tag{5.11}$$

Then, there exists $\hat{Q} \in \mathcal{S}$*, such that* $Q = \hat{Q}N_{21}$*, and* $\hat{Q}$ *is optimal for*

$$\begin{array}{ll} \textit{minimize} & \|N_{11} + N_{12}\hat{Q}N_{21}\|_2 \\ \textit{subject to} & \hat{Q} \in \mathcal{S} \end{array} \tag{5.12}$$

Conversely, if $\hat{Q} \in \mathcal{S}$ *is optimal for* (5.12)*, then* $Q = \hat{Q}N_{21}$ *is optimal for* (5.11)*.*

Once again, state feedback allows us to group $\hat{Q}N_{21} \in \mathcal{S}$ in Lemma 33. Similarly, the optimality condition takes the same functional form as the two-player case.

Lemma 34. *Let* $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$*. Suppose* $G_1, G_2 \in \mathcal{RH}_\infty$*. Then,* $Q \in \mathcal{S}$ *minimizes*

$$\begin{array}{ll} \textit{minimize} & \|G_1 + G_2Q\|_2 \\ \textit{subject to} & Q \in \mathcal{S} \end{array}$$

if and only if

$$G_2^*G_1 + G_2^*G_2Q \in \mathcal{S}^\perp \tag{5.13}$$

5.4 Spectral Factorization

Our goal is now to find a solution $Q \in \mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$ which satisfies the optimality condition

$$N_{12}^*N_{11} + N_{12}^*N_{12}Q \in \mathcal{S}^\perp \tag{5.14}$$

To this end, we have the following result.

Lemma 35. *Let* $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$*, and suppose* $G_1, G_2 \in \mathcal{RH}_\infty$*. Then,* $Q \in \mathcal{S}$ *satisfies*

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{S}^\perp \tag{5.15}$$

if and only if

$$(G_2^* G_1)_{\mathcal{D}_i i} + (G_2^* G_2)_{\mathcal{D}_i} Q_{\mathcal{D}_i i} \in \mathcal{H}_2^\perp$$

for all $i \in \mathcal{V}$*.*

Proof. The optimality condition (5.15) can be equivalently written as

$$G_2^* G_1 + G_2^* G_2 Q = \Lambda \tag{5.16}$$

where $\Lambda \in \mathcal{S}^\perp$. Note that Λ satisfies

$$\Lambda_{ij} \in \begin{cases} \mathcal{H}_2^\perp & i \in \mathcal{D}_j \\ \mathcal{L}_2 & i \notin \mathcal{D}_j \end{cases}$$

Thus, (5.16) is satisfied if and only if it is satisfied for each (i, j) such that $i \in \mathcal{D}_j$. Breaking this up by column, we have for each $j \in \mathcal{V}$

$$\begin{aligned} I_{\mathcal{D}_j \mathcal{V}} (G_2^* G_1 + G_2^* G_2 Q) I_{\mathcal{V} j} &= I_{\mathcal{D}_j \mathcal{V}} \Lambda I_{\mathcal{V} j} \\ (G_2^* G_1)_{\mathcal{D}_j j} + (G_2^* G_2)_{\mathcal{D}_j} Q_{\mathcal{D}_j j} &= \Lambda_{\mathcal{D}_j j} \end{aligned}$$

Since $Q_{\mathcal{D}_j j} \in \mathcal{RH}_2$ and $\Lambda_{\mathcal{D}_j j} \in \mathcal{H}_2^\perp$, the result follows. ∎

Thus, to solve the decentralized optimality condition (5.14), we must apply our results from the centralized case N times, as seen in the following lemma.

Lemma 36. *Let* $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$*. Define* $A^F = A + BF$*, and* $C^F = C + DF$*, and*

$$\begin{aligned} N_{11} &= C^F (zI - A^F) H \\ N_{12} &= z^{-1} (C^F (zI - A^F)^{-1} B + D) \end{aligned}$$

Furthermore, for each $i \in \mathcal{V}$, suppose there exists a stabilizing solution X_i to the Riccati equation

$$\begin{aligned} X_i = (C_{\mathcal{D}_i})^T C_{\mathcal{D}_i} + (A_{\mathcal{D}_i})^T X_i A_{\mathcal{D}_i} - (A_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i} + C_{\mathcal{D}_i}^T D_{\mathcal{D}_i}) \\ \times (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i} + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i}) \end{aligned} \tag{5.17}$$

and define

$$K_i^F = (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i}^F + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i}^F) \tag{5.18}$$

Then, the unique $Q \in \mathcal{S}$ satisfying

$$N_{12}^* N_{11} + N_{12}^* N_{12} Q \in \mathcal{S}^\perp \tag{5.19}$$

is given by

$$Q = \sum_{i \in \mathcal{V}} I_{\mathcal{V}\mathcal{D}_i} Q_{\mathcal{D}_i i} I_{i\mathcal{V}}$$

where

$$Q_{\mathcal{D}_i i} = -z K_i^F (zI - (A_{\mathcal{D}_i}^F - B_{\mathcal{D}_i} K_i^F))^{-1} I_{\mathcal{D}_i i} H_i \tag{5.20}$$

Proof. From Lemma 35, we know that $Q \in \mathcal{S}$ if and only if

$$(N_{12}^* N_{11})_{\mathcal{D}_i i} + (N_{12}^* N_{12})_{\mathcal{D}_i} Q_{\mathcal{D}_i i} \in \mathcal{H}_2^\perp$$

for all $i \in \mathcal{V}$. This is equivalent to

$$G_2^* G_1 + G_2^* G_2 Q_{\mathcal{D}_i i} \in \mathcal{H}_2^\perp$$

where

$$\begin{aligned} G_1 &= C_{\mathcal{D}_i}^F (zI - A_{\mathcal{D}_i}^F) I_{\mathcal{D}_i i} H_i \\ G_2 &= z^{-1} (C_{\mathcal{D}_i}^F (zI - A_{\mathcal{D}_i}^F)^{-1} B_{\mathcal{D}_i} + D_{\mathcal{D}_i}) \end{aligned}$$

From Lemma 24, we see that (5.17) implies that X_i is also the stabilizing solution of

$$\begin{aligned} X_i = (C_{\mathcal{D}_i}^F)^T C_{\mathcal{D}_i}^F + (A_{\mathcal{D}_i}^F)^T X_i A_{\mathcal{D}_i}^F - ((A_{\mathcal{D}_i}^F)^T X_i B_{\mathcal{D}_i} + (C_{\mathcal{D}_i}^F)^T D_{\mathcal{D}_i}) \\ \times (D_{\mathcal{D}_i}^T D_{\mathcal{D}_i} + B_{\mathcal{D}_i}^T X_i B_{\mathcal{D}_i})^{-1} (B_{\mathcal{D}_i}^T X_i A_{\mathcal{D}_i}^F + D_{\mathcal{D}_i}^T C_{\mathcal{D}_i}^F) \end{aligned}$$

Applying the centralized solutions from Lemmas 21 and 22, it is straightforward to show that $Q_{\mathcal{D}_i i}$ satisfies (5.20). The result follows from concatenating the columns of Q given by $I_{\mathcal{V}\mathcal{D}_i} Q_{\mathcal{D}_i i}$. ■

As in the two-player case, the existence of a solution for the optimality condition (3.3) is independent of the pre-compensator F, as desired.

Now, to find the optimal $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$, a couple more steps are required. From Lemma 33, we first need to solve for $\hat{Q} = QN_{21}^{-1}$. Then, inverting the Youla parametrization in Lemma 32, we must compute

$$\begin{aligned} R &= (I + \hat{Q}M)^{-1}\hat{Q} \\ &= (I + QN_{21}^{-1}M)^{-1}QN_{21}^{-1} \end{aligned}$$

Lastly, we have $\mathcal{K} = R + F$. These computations are performed in the following results.

Lemma 37. *Let*

$$\begin{aligned} N_{21} &= z(zI - A^F)^{-1}H \\ M &= (zI - A^F)^{-1}B \end{aligned}$$

and suppose Q is defined as in Lemma 36. Lastly, define $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ as

$$\begin{aligned} \tilde{A}_{km} &= \begin{cases} A^F_{\mathcal{D}'_k} - B_{\mathcal{D}'_k \mathcal{D}_k} K^F_k I_{\mathcal{D}_k \mathcal{D}'_k} & k = m \\ -(A^F_{\mathcal{D}'_k k} - B_{\mathcal{D}'_k \mathcal{D}_k} K^F_k I_{\mathcal{D}_k k}) I_{k \mathcal{D}'_m} & k \in \mathcal{D}'_m \\ 0 & \textit{otherwise} \end{cases} \\ \tilde{B}_{km} &= \begin{cases} A^F_{\mathcal{D}'_k k} - B_{\mathcal{D}'_k \mathcal{D}_k} K^F_k I_{\mathcal{D}_k k} & k = m \\ 0 & \textit{otherwise} \end{cases} \\ \tilde{C}_{\mathcal{V}m} &= -I_{\mathcal{V}\mathcal{D}_m} K^F_m I_{\mathcal{D}_m \mathcal{D}'_m} + \sum_{p \in \mathcal{D}'_m} I_{\mathcal{V}\mathcal{D}_p} K^F_p I_{\mathcal{D}_p p} I_{p \mathcal{D}'_p} \\ \tilde{D}_{\mathcal{V}m} &= -I_{\mathcal{V}\mathcal{D}_m} K^F_m I_{\mathcal{D}_m m} \end{aligned}$$

Then,

$$R = (I + QN_{21}^{-1}M)^{-1}QN_{21}^{-1}$$

satisfies

$$R_{ij} = \left[\begin{array}{c|c} \tilde{A}_{\mathcal{A}_i\cap\mathcal{D}_j} & \tilde{B}_{\mathcal{A}_i\cap\mathcal{D}_j j} \\ \hline \tilde{C}_{i\mathcal{A}_i\cap\mathcal{D}_j} & \tilde{D}_{ij} \end{array}\right]$$

for each $i, j \in \mathcal{V}$.

Proof. To begin, we note that

$$N_{21}^{-1}M = z^{-1}H^{-1}B$$

Then,

$$\begin{aligned} I + QN_{21}^{-1}M &= I + \sum_{i\in\mathcal{V}} I_{\mathcal{V}\mathcal{D}_i}Q_{\mathcal{D}_i i}I_{i\mathcal{V}}z^{-1}H^{-1}B \\ &= \sum_{i\in\mathcal{V}} I_{\mathcal{V}\mathcal{D}_i}(I - K_i^F(zI - (A_{\mathcal{D}_i}^F - B_{\mathcal{D}_i}K_i^F))^{-1}I_{\mathcal{D}_i i}I_{i\mathcal{A}_i}B_{\mathcal{A}_i})I_{\mathcal{A}_i\mathcal{V}} \\ &= \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & I \end{array}\right] \end{aligned}$$

where

$$\hat{A} = \begin{bmatrix} A_{\mathcal{D}_1}^F - B_{\mathcal{D}_1}K_1^F & & \\ & \ddots & \\ & & A_{\mathcal{D}_N}^F - B_{\mathcal{D}_N}K_N^F \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} I_{\mathcal{D}_1\mathcal{A}_1}B_{\mathcal{A}_1}I_{\mathcal{A}_1\mathcal{V}} \\ \vdots \\ I_{\mathcal{D}_N\mathcal{A}_N}B_{\mathcal{A}_N}I_{\mathcal{A}_N\mathcal{V}} \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} -I_{\mathcal{V}\mathcal{D}_1}K_1^F & \cdots & -I_{\mathcal{V}\mathcal{D}_N}K_N^F \end{bmatrix}$$

The inverse of this transfer function is then

$$(I + QN_{21}^{-1}M)^{-1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}\hat{C} & \hat{B} \\ \hline -\hat{C} & I \end{array}\right]$$

Note that $\hat{A}, \hat{B}, \hat{C}$ are partitioned into blocks indexed by $\mathcal{V}$. Thus, we will use our subscripting notation for these matrices as well. To avoid confusion when using the identity matrix, we will use $\hat{I}$ to indicate the identity matrix corresponding to matrices with the larger blocks.

Notice that $\hat{A} - \hat{B}\hat{C} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$. Consequently, we can write

$$
\begin{aligned}
I_{i\mathcal{V}}(I + QN_{21}^{-1}M)^{-1} I_{\mathcal{V}\mathcal{D}_k} &= \left[\begin{array}{c|c} \hat{A} - \hat{B}\hat{C} & \hat{B} I_{\mathcal{V}\mathcal{D}_k} \\ \hline -I_{i\mathcal{V}}\hat{C} & I_{i\mathcal{D}_k} \end{array}\right] \\
&= \left[\begin{array}{c|c} \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_k} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k} I_{\mathcal{V}\mathcal{D}_k} \\ \hline -I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & I_{i\mathcal{D}_k} \end{array}\right]
\end{aligned}
$$

since

$$
\hat{B} I_{\mathcal{V}\mathcal{D}_k} = \hat{I}_{\mathcal{V}\mathcal{D}_k}\hat{B}_{\mathcal{D}_k} I_{\mathcal{V}\mathcal{D}_k} \qquad\qquad I_{i\mathcal{V}}\hat{C} = I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i}\hat{I}_{\mathcal{A}_i\mathcal{V}}
$$

Then,

$$
\begin{aligned}
& z^{-1} I_{i\mathcal{V}}(I + QN_{21}^{-1}M)^{-1} I_{\mathcal{V}\mathcal{D}_k} Q_{\mathcal{D}_k k} \\
&= \left[\begin{array}{cc|c} A^F_{\mathcal{D}_k} - B_{\mathcal{D}_k} K^F_k & 0 & I_{\mathcal{D}_k k} H_k \\ \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k} I_{\mathcal{V}\mathcal{D}_k} K^F_k & \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_k} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & 0 \\ \hline -I_{i\mathcal{D}_k} K^F_k & I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & 0 \end{array}\right]
\end{aligned}
$$

which, in turn, is equal to the expression in (5.21).

$$
\left[\begin{array}{ccc|c} A^F_{\mathcal{D}_k} - B_{\mathcal{D}_k} K^F_k & 0 & 0 & I_{\mathcal{D}_k k} H_k \\ I_{\mathcal{D}_k\mathcal{A}_k} B_{\mathcal{A}_k} I_{\mathcal{A}_k\mathcal{D}_k} K^F_k & A^F_{\mathcal{D}_k} - B_{\mathcal{D}_k} K^F_k + I_{\mathcal{D}_k\mathcal{A}_k} B_{\mathcal{A}_k} I_{\mathcal{A}_k\mathcal{D}_k} K^F_k & 0 & 0 \\ \hat{B}_{\mathcal{A}_i\cap\mathcal{D}'_k} I_{\mathcal{V}\mathcal{D}_k} K^F_k & -\hat{B}_{\mathcal{A}_i\cap\mathcal{D}'_k}\hat{C}_k & \hat{A}_{\mathcal{A}_i\cap\mathcal{D}'_k} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}'_k}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}'_k} & 0 \\ \hline -I_{i\mathcal{D}_k} K^F_k & -I_{i\mathcal{D}_k} K^F_k & I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}'_k} & 0 \end{array}\right] \tag{5.21}
$$

Using the state transformation matrix

$$T = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we see that we can remove an uncontrollable state to obtain

$$z^{-1} I_{i\mathcal{V}}(I + QN_{21}^{-1}M)^{-1} I_{\mathcal{V}\mathcal{D}_k} Q_{\mathcal{D}_k k} = \left[\begin{array}{c|c} \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_k} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_k k}(I_{\mathcal{D}_k k}H_k) \\ \hline I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & 0 \end{array}\right]$$

As a result, we have

$$\begin{aligned} R_{ij} &= \sum_k I_{i\mathcal{V}}(I + QN_{21}^{-1}M)^{-1} I_{\mathcal{V}\mathcal{D}_k} Q_{\mathcal{D}_k k} I_{k\mathcal{V}} N_{21}^{-1} I_{\mathcal{V}j} \\ &= \sum_{k\in\mathcal{A}_i\cap\mathcal{D}_j} \left[\begin{array}{c|c} \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_k} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_k}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_k k}(I_{\mathcal{D}_k k}I_{k\mathcal{D}_j}) \\ \hline I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_k} & 0 \end{array}\right] \times (zI - A^F_{\mathcal{D}_j})I_{\mathcal{D}_j j} \\ &= \left[\begin{array}{c|c} \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_j} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_j}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j} & \sum_{k\in\mathcal{A}_i\cap\mathcal{D}_j} \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j k}(I_{\mathcal{D}_k k}I_{k\mathcal{D}_j}) \\ \hline I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j} & 0 \end{array}\right] \times (zI - A^F_{\mathcal{D}_j})I_{\mathcal{D}_j j} \\ &= \left[\begin{array}{c|c} \hat{A}_{\mathcal{A}_i\cap\mathcal{D}_j} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_j}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j} & (\hat{A}_{\mathcal{A}_i\cap\mathcal{D}_j j} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_j}\hat{C}_j)I_{\mathcal{D}_j j} - \hat{E}_{\mathcal{A}_i\cap\mathcal{D}_j j} \\ \hline I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j} & -I_{i\mathcal{D}_j}K_j^F I_{\mathcal{D}_j j} \end{array}\right] \end{aligned}$$

where we've defined

$$\hat{E}_{\mathcal{V}j} = \begin{bmatrix} I_{\mathcal{D}_1 1}A^F_{1j} \\ \vdots \\ I_{\mathcal{D}_N N}A^F_{Nj} \end{bmatrix}$$

Note that the 'D term' above is equal to $\tilde{D}_{ij}$. Now, consider the state transformation matrix

$$T = I + \sum_{\substack{k,m \\ k\in\mathcal{D}'_m}} \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j k}(I_{\mathcal{D}_k k}I_{k\mathcal{D}_m})\hat{I}_{m\mathcal{A}_i\cap\mathcal{D}_j}$$

After some algebraic manipulations, we find that the new 'B term' is

$$\begin{aligned}
B_{\text{new}} &= T\big((\hat{A}_{\mathcal{A}_i\cap\mathcal{D}_j j} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_j}\hat{C}_j)I_{\mathcal{D}_j j} - \hat{E}_{\mathcal{A}_i\cap\mathcal{D}_j j}\big)\\
&= \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j j}\big(I_{\mathcal{D}_j\mathcal{D}'_j}(A^F_{\mathcal{D}'_j j} - B_{\mathcal{D}'_j\mathcal{D}_j}K^F_j I_{\mathcal{D}_j j})\big)
\end{aligned}$$

and the new 'A term' is

$$\begin{aligned}
A_{\text{new}} &= T(\hat{A}_{\mathcal{A}_i\cap\mathcal{D}_j} - \hat{B}_{\mathcal{A}_i\cap\mathcal{D}_j}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j})T^{-1}\\
&= \sum_p \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j p}(A^F_{\mathcal{D}_p} - I_{\mathcal{D}_p\mathcal{D}'_p}B_{\mathcal{D}'_p\mathcal{D}_p}K^F_p)\hat{I}_{p\mathcal{A}_i\cap\mathcal{D}_j}\\
&+ \sum_{\substack{k,m\\ k\in\mathcal{D}'_m}} \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j k}(I_{\mathcal{D}_k k}A^F_{km}I_{m\mathcal{D}_m} - I_{\mathcal{D}_k\mathcal{D}'_k}(A^F_{\mathcal{D}'_k k} - B_{\mathcal{D}'_k\mathcal{D}_k}K^F_k I_{\mathcal{D}_k k})I_{k\mathcal{D}_m})\hat{I}_{m\mathcal{A}_i\cap\mathcal{D}_j}
\end{aligned}$$

Lastly, suppose that

$$\begin{aligned}
S &= \operatorname{image}\Big(\sum_q \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j q}I_{\mathcal{D}_q\mathcal{D}'_q}\hat{I}_{q\mathcal{A}_i\cap\mathcal{D}_j}\Big)\\
&= \operatorname{image}\begin{bmatrix} I_{\mathcal{D}_{q_1}\mathcal{D}'_{q_1}} & & \\ & \ddots & \\ & & I_{\mathcal{D}_{q_n}\mathcal{D}'_{q_n}}\end{bmatrix} \qquad \text{where } \mathcal{A}_i\cap\mathcal{D}_j = \{q_1,\dots,q_n\}
\end{aligned}$$

It is straightforward to show that

$$A_{\text{new}}S \subset S \qquad B_{\text{new}} \subset S$$

indicating that there are uncontrollable states that can be removed. Using the projection matrix

$$P = \sum_q \hat{I}_{\mathcal{A}_i\cap\mathcal{D}_j q}I_{\mathcal{D}'_q\mathcal{D}_q}\hat{I}_{q\mathcal{A}_i\cap\mathcal{D}_j}$$

we can project out the uncontrollable states, obtaining

$$PA_{\text{new}}P^T = \tilde{A}_{\mathcal{A}_i\cap\mathcal{D}_j} \qquad\qquad PB_{\text{new}} = \tilde{B}_{\mathcal{A}_i\cap\mathcal{D}_j j}$$

and

$$C_{\text{new}}P^T = I_{i\mathcal{V}}\hat{C}_{\mathcal{A}_i\cap\mathcal{D}_j}T^{-1}P^T = \tilde{C}_{i\mathcal{A}_i\cap\mathcal{D}_j}$$

as desired. ■

We are now ready to prove Theorem 29.

Proof of Theorem 29. From Lemmas 16 and 31, there exist matrices X_i satisfying the Riccati equations (5.5). As a result, define K_i^F as in (5.18). Then, following Lemmas 32, 33, 34, 36, and 37, the optimal controller $\mathcal{K}$ is given by $\mathcal{K} = R + F$, with R given in Lemma 37.

Using the notation of Lemma 37, it is straightforward to see that

$$R = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

Notice that

$$\begin{aligned} A^F_{\mathcal{D}'_k} - B_{\mathcal{D}'_k\mathcal{D}_k}K^F_K I_{\mathcal{D}_k\mathcal{D}'_k} &= A_{\mathcal{D}'_k} - B_{\mathcal{D}'_k}F_{\mathcal{D}'_k} - B_{\mathcal{D}'_k\mathcal{D}_k}(K_k + F_{\mathcal{D}_k})I_{\mathcal{D}_k\mathcal{D}'_k} \\ &= A_{\mathcal{D}'_k} - B_{\mathcal{D}'_k\mathcal{D}_k}K_k I_{\mathcal{D}_k\mathcal{D}'_k} \end{aligned}$$

which implies that

$$\tilde{A} = A_K$$

Similarly, we can show that

$$\tilde{B} = B_K \qquad \tilde{C} = C_K$$

Lastly, note that

$$\begin{aligned} \tilde{D} + F &= F - \sum_j I_{\mathcal{V}\mathcal{D}_j}K_j^F I_{\mathcal{D}_j j}I_{j\mathcal{V}} \\ &= F - \sum_j I_{\mathcal{V}\mathcal{D}_j}(K_j + F_{\mathcal{D}_j})I_{\mathcal{D}_j j}I_{j\mathcal{V}} \\ &= -\sum_j I_{\mathcal{V}\mathcal{D}_j}K_j I_{\mathcal{D}_j j}I_{j\mathcal{V}} \\ &= D_K \end{aligned}$$

Thus, $\mathcal{K} = R + F$ satisfies (5.6). The individual controllers of (5.7) come from simply computing $u_i = \mathcal{K}_{i\mathcal{V}} x = \mathcal{K}_{i\mathcal{A}_i} x_{\mathcal{A}_i}$. ∎

5.5 Estimation Structure

While Theorem 29 provides the optimal controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$, the resulting controller is not particularly intuitive. Remember from the classical, centralized problem that the optimal controller is

$$u(t) = K\hat{x}(t)$$

where $\hat{x}(t)$ represents the minimum mean-square error (MMSE) estimate of the state. Thus, the dynamics of the optimal controller represents the Kalman filter estimator.

A similar result holds for our decentralized problem, and we will demonstrate this in this section. To begin, we find that the closed-loop system dynamics has a particularly nice representation.

Lemma 38. *Let P and $\mathcal{K}$ be defined by* (5.2) *and* (5.6)*, respectively. Then, the closed-loop state dynamics satisfy*

$$\zeta(t+1) = A_{cl}\zeta(t) + B_{cl}w(t) \tag{5.22}$$

where

$$A_{cl} = \begin{bmatrix} A_{\mathcal{D}_1} - B_{\mathcal{D}_1}K_1 & & \\ & \ddots & \\ & & A_{\mathcal{D}_N} - B_{\mathcal{D}_N}K_N \end{bmatrix} \qquad B_{cl} = \begin{bmatrix} I_{\mathcal{D}_1 1}H_1 & & \\ & \ddots & \\ & & I_{\mathcal{D}_N N}H_N \end{bmatrix}$$

Proof. The controller $\mathcal{K}$ can be represented by the following state-space system

$$\begin{aligned} \eta(t+1) &= A_K\eta(t) + B_Kx(t) \\ u(t) &= C_K\eta(t) + D_Kx(t) \end{aligned}$$

Combining this with the plant dynamics

$$x(t+1) = Ax(t) + Bu(t) + Hw(t)$$

the closed-loop dynamics are given by

$$\begin{bmatrix} x(t+1) \\ \eta(t+1) \end{bmatrix} = \begin{bmatrix} A + BD_K & BC_K \\ B_K & A_K \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} w(t)$$

Now, consider the state transformation matrix

$$T = \begin{bmatrix} \mathrm{I} & -\sum\limits_{\substack{i,j \\ i \in \mathcal{D}'_j}} I_{\mathcal{V}i} I_{i\mathcal{D}'_j} I_{j\mathcal{V}} \\ 0 & I \end{bmatrix} \tag{5.23}$$

After some simplifications, we obtain

$$\begin{aligned} &T \begin{bmatrix} A + BD_K & BC_K \\ B_K & A_K \end{bmatrix} T^{-1} \\ &\qquad = \begin{bmatrix} \sum_i I_{\mathcal{V}i}(A_i - B_{i\mathcal{D}_i} K_i I_{\mathcal{D}_i i}) I_{i\mathcal{V}} & \sum_i I_{\mathcal{V}i}(A_{i\mathcal{D}'_i} - B_{i\mathcal{D}_i} K_i I_{\mathcal{D}_i\mathcal{D}'_i} \hat{I}_{i\mathcal{V}} \\ \sum_i \hat{I}_{\mathcal{V}i}(A_{\mathcal{D}'_i i} - B_{\mathcal{D}'_i\mathcal{D}_i} K_i I_{\mathcal{D}_i i}) I_{i\mathcal{V}} & \sum_i \hat{I}_{\mathcal{V}i}(A_{\mathcal{D}'_i} - B_{\mathcal{D}'_i\mathcal{D}_i} K_i I_{\mathcal{D}_i\mathcal{D}'_i}) \hat{I}_{i\mathcal{V}} \end{bmatrix} \end{aligned}$$

Notice that each block is itself block diagonal. Thus, we can permute the states to get

$$A_{cl} = \mathrm{diag}\left(\begin{bmatrix} A_{\mathcal{D}_i} - B_{\mathcal{D}_i} K_i \end{bmatrix}\right)$$

Also, we have

$$T \begin{bmatrix} H \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ 0 \end{bmatrix}$$

Using the same permutation as above, we get

$$B_{cl} = \mathrm{diag}(I_{\mathcal{D}_i i} H_i)$$

■

Thus, from Lemma 38, we can rewrite the closed-loop dynamics in terms of the states ζ_i, whose dynamics are decoupled since A_{cl} and B_{cl} are block diagonal. Using the transformation T in (5.23) and permutation in the above proof, we see that this new state ζ satisfies

$$\zeta_i(t) = \begin{bmatrix} x_i(t) - \sum_{j \in \mathcal{A}'_i} I_{i\mathcal{D}'_j} \eta_j(t) \\ \eta_i(t) \end{bmatrix}$$

for each $i \in \mathcal{V}$. We can also rewrite the optimal controller in terms of these new states. Since

$$u(t) = \begin{bmatrix} D_K & C_K \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}$$

then

$$\begin{bmatrix} D_K & C_K \end{bmatrix} T^{-1} = \begin{bmatrix} -\sum_i I_{\mathcal{V}\mathcal{D}_i} K_i I_{\mathcal{D}_i i} I_{i\mathcal{V}} & -\sum_i I_{\mathcal{V}\mathcal{D}_i} K_i I_{\mathcal{D}_i \mathcal{D}'_i} \hat{I}_{i\mathcal{V}} \end{bmatrix}$$

Consequently, under the same permutation as above,

$$u(t) = \begin{bmatrix} -I_{\mathcal{V}\mathcal{D}_1} K_1 & \cdots & -I_{\mathcal{V}\mathcal{D}_N} K_N \end{bmatrix} \zeta(t)$$

This implies that each individual controller is

$$u_i(t) = -\sum_{j \in \mathcal{A}_i} I_{i\mathcal{D}_j} K_j \zeta_j(t)$$

While this is an amazingly clean result for the optimal control policy, we can gain significantly more intuition to it by further examination of the state ζ. Recall that in the classical, centralized problem, the optimal policy consists of a minimum mean-square error estimate of the state $x(t)$. A similar result holds here.

Theorem 39. *Suppose ζ satisfies the dynamics given in Lemma 38, with $w_i(t)$ independent random variables, and*

$$\zeta_i(t) = \begin{bmatrix} x_i(t) - \sum_{j \in \mathcal{A}'_i} I_{i\mathcal{D}'_j} \eta_j(t) \\ \eta_i(t) \end{bmatrix}$$

Then,

$$\zeta_i(t) = \mathrm{E}\big(x_{\mathcal{D}_i}(t) \mid x_{\mathcal{A}_i}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_i}(t) \mid x_{\mathcal{A}'_i}(0:t)\big)$$

Proof. From Lemma 38, it is clear that for any disjoint sets S_1, S_2

$$\mathrm{E}(\zeta_{S_1}(t) \mid \zeta_{S_2}(0:t)) = \mathrm{E}(\zeta_{S_1}(t)) = 0 \qquad S_1 \cap S_2 = \emptyset$$

since $w_i(t)$, and thus $\zeta_i(t)$, are independent. This implies that

$$\mathrm{E}(\eta_{S_1}(t) \mid \zeta_{S_2}(0:t)) = 0$$

Thus, since $\mathcal{D}_i \cap \mathcal{A}'_i = \emptyset$, we have

$$\mathrm{E}(\zeta_{\mathcal{D}_i}(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) = 0$$

As a result, we have

$$\begin{aligned}
\mathrm{E}(x_{\mathcal{D}_i}(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) &= \sum_{k\in\mathcal{D}_i} I_{\mathcal{D}_i k} \sum_{j\in\mathcal{A}'_k} I_{k\mathcal{D}'_j}\, \mathrm{E}(\eta_j(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) \\
&= \sum_{k\in\mathcal{D}_i\cap\mathcal{D}'_j} I_{\mathcal{D}_i k} I_{k\mathcal{D}'_j}\, \mathrm{E}(\eta_j(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) \\
&= \sum_{j\in\mathcal{A}'_i} I_{\mathcal{D}_i\mathcal{D}'_j} \eta_j(t)
\end{aligned}$$

where the last equality comes from the fact that

$$\mathrm{E}(\eta_j(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) = \begin{cases} \eta_j(t) & j \in \mathcal{A}'_i \\ 0 & \text{otherwise} \end{cases}$$

In particular, note that

$$\mathrm{E}(x_i(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) = \sum_{j\in\mathcal{A}'_i} I_{i\mathcal{D}'_j} \eta_j(t)$$

and

$$\mathrm{E}(x_{\mathcal{D}'_i}(t) \mid \zeta_{\mathcal{A}'_i}(0:t)) = \sum_{j\in\mathcal{A}'_i} I_{\mathcal{D}'_i\mathcal{D}'_j} \eta_j(t)$$

By the same construction, we can also show that

$$\begin{aligned}\mathrm{E}(x_{\mathcal{D}_i'}(t) \mid \zeta_{\mathcal{A}_i}(0:t)) &= \sum_{j \in \mathcal{A}_i} I_{\mathcal{D}_i'\mathcal{D}_j'} \eta_j(t) \\ &= \eta_i(t) + \sum_{j \in \mathcal{A}_i'} I_{\mathcal{D}_i'\mathcal{D}_j'} \eta_j(t)\end{aligned}$$

Lastly, since there is an invertible mapping between states $x, \eta \leftrightarrow \zeta$, note that conditioning on $\zeta(0:t)$ is equivalent to conditioning on $x(0:t), \eta(0:t)$. However, since η is simply a function of x, then conditioning on $x(0:t), \eta(0:t)$ is equivalent to conditioning on $x(0:t)$. Thus,

$$\begin{aligned}\zeta_i(t) &= \begin{bmatrix} x_i(t) - \mathrm{E}\big(x_i(t) \mid \zeta_{\mathcal{A}_i'}(0:t)\big) \\ \mathrm{E}\big(x_{\mathcal{D}_i'}(t) \mid \zeta_{\mathcal{A}_i}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_i'}(t) \mid \zeta_{\mathcal{A}_i'}(0:t)\big) \end{bmatrix} \\ &= \begin{bmatrix} x_i(t) - \mathrm{E}\big(x_i(t) \mid x_{\mathcal{A}_i'}(0:t)\big) \\ \mathrm{E}\big(x_{\mathcal{D}_i'}(t) \mid x_{\mathcal{A}_i}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_i'}(t) \mid x_{\mathcal{A}_i'}(0:t)\big) \end{bmatrix} \\ &= \mathrm{E}\big(x_{\mathcal{D}_i}(t) \mid x_{\mathcal{A}_i}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_i}(t) \mid x_{\mathcal{A}_i'}(0:t)\big)\end{aligned}$$

where we've used the fact that $x_i(t) = \mathrm{E}\big(x_i(t) \mid x_{\mathcal{A}_i}(0:t)\big)$. ■

With Theorem 39, it is clear that an alternate formulation for the optimal control policies can be given as

$$u_i(t) = -\sum_{j \in \mathcal{A}_i} I_{i\mathcal{D}_j} K_j \Big(\mathrm{E}\big(x_{\mathcal{D}_j}(t) \mid x_{\mathcal{A}_j}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_j}(t) \mid x_{\mathcal{A}_j'}(0:t)\big)\Big) \tag{5.24}$$

5.6 Examples

We will end our discussion of the general graph problem with a couple examples. The first section simply applies our analytic solution to a couple examples to more concretely demonstrate the form of the solution. In the second section, we consider the 3-player problem, and compare the optimal solutions for various graph structures.

5.6.1 Some Analytic Results

Given the solution from Chapter 4, it is important that our general N-player solution reduces to the optimal solution that was previously obtained for the 2-player case. For this case, we have

$$\mathcal{D}_1 = \{1,2\} \qquad \mathcal{D}_2 = \{2\}$$
$$\mathcal{A}_1 = \{1\} \qquad \mathcal{A}_2 = \{1,2\}$$

We will use the shorthand notation, $\hat{x}_{2|1}(t) = \mathrm{E}\big(x_2(t) \mid x_1(0:t)\big)$. Thus, applying the policy in (5.24) to this problem, we obtain

$$\begin{aligned}
u_1(t) &= -I_{1\mathcal{D}_1} K_1 \Big(\mathrm{E}\big(x_{\mathcal{D}_1}(t) \mid x_{\mathcal{A}_1}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_1}(t) \mid x_{\mathcal{A}_1'}(0:t)\big)\Big) \\
&= -I_{1\mathcal{D}_1} K_1 \begin{bmatrix} x_1(t) \\ \hat{x}_{2|1}(t) \end{bmatrix} \\
u_2(t) &= -I_{2\mathcal{D}_1} K_1 \Big(\mathrm{E}\big(x_{\mathcal{D}_1}(t) \mid x_{\mathcal{A}_1}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_1}(t) \mid x_{\mathcal{A}_1'}(0:t)\big)\Big) \\
&\qquad - I_{2\mathcal{D}_2} K_2 \Big(\mathrm{E}\big(x_{\mathcal{D}_2}(t) \mid x_{\mathcal{A}_2}(0:t)\big) - \mathrm{E}\big(x_{\mathcal{D}_2}(t) \mid x_{\mathcal{A}_2'}(0:t)\big)\Big) \\
&= -I_{2\mathcal{D}_1} K_1 \begin{bmatrix} x_1(t) \\ \hat{x}_{2|1}(t) \end{bmatrix} - K_2\big(x_2(t) - \hat{x}_{2|1}(t)\big)
\end{aligned}$$

Noting that $K_1 = K$ and $K_2 = J$ in the notation of the previous chapter, we see that the two solutions are indeed the same.

Consider now the simple extension to the 3-player chain, as seen in the following figure.

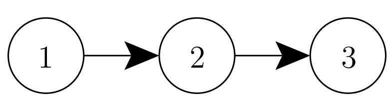


Figure 5.1: Three-Player Chain

For this problem, the graph properties are:

$$\mathcal{D}_1 = \{1,2,3\} \qquad \mathcal{D}_2 = \{2,3\} \qquad \mathcal{D}_3 = \{3\}$$
$$\mathcal{A}_1 = \{1\} \qquad \mathcal{A}_2 = \{1,2\} \qquad \mathcal{A}_3 = \{1,2,3\}$$

As above, we define $\hat{x}_{3|12}(t) = \mathrm{E}\big(x_3(t) \mid x_1(0:t), x_2(0:t)\big)$, and similarly for $\hat{x}_{2|1}$, and $\hat{x}_{3|1}$. As a result, the optimal policy can be written as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = -\begin{bmatrix} \\ K_1 \\ \\ \end{bmatrix}\begin{bmatrix} x_1(t) \\ \hat{x}_{2|1}(t) \\ \hat{x}_{3|1}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \begin{bmatrix} K_2 \end{bmatrix} \end{bmatrix}\begin{bmatrix} x_2(t) - \hat{x}_{2|1}(t) \\ \hat{x}_{3|12}(t) - \hat{x}_{3|1}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ K_3 \end{bmatrix}\big(x_3(t) - \hat{x}_{3|12}(t)\big)$$

where we've added brackets to the K_i to visually indicate their dimensions.

Compare this result to an alternate graph structure for the 3-player system, shown in Figure 5.2.

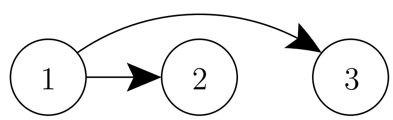


Figure 5.2: Another Three-Player Graph

This problem has the following graph properties:

$$\mathcal{D}_1 = \{1,2,3\} \qquad \mathcal{D}_2 = \{2\} \qquad \mathcal{D}_3 = \{3\}$$
$$\mathcal{A}_1 = \{1\} \qquad \mathcal{A}_2 = \{1,2\} \qquad \mathcal{A}_3 = \{1,3\}$$

Consequently, straightforward application of (5.24) yields the following optimal policy.

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = -\begin{bmatrix} \\ K_1 \\ \\ \end{bmatrix}\begin{bmatrix} x_1(t) \\ \hat{x}_{2|1}(t) \\ \hat{x}_{3|1}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ K_2 \\ 0 \end{bmatrix}\big(x_2(t) - \hat{x}_{2|1}(t)\big) - \begin{bmatrix} 0 \\ 0 \\ K_3 \end{bmatrix}\big(x_3(t) - \hat{x}_{3|1}(t)\big)$$

Comparing the optimal policies of these two 3-player systems, we see the graph structures of the controllers reflected in the structure of the matrix gains.

5.6.2 Communication Trade-off

In the 2-player case of the previous chapter, there are only two possible communication schemes (when the two subsystems are identical). These are the decentralized $1 \to 2$ graph considered there and the classical, centralized $1 \leftrightarrow 2$ graph. In Section 4.6.2, we plotted the optimal trade-off curves for both graphs.

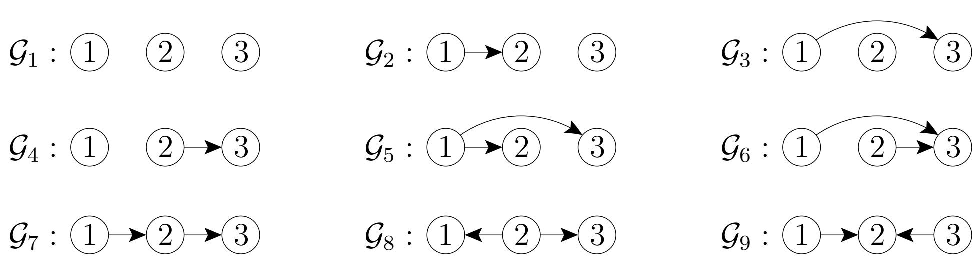
In the 3-player case, there are many more possible communication structures. In this section, we examine the trade-off curves for a number of different communication structures. To this end, we consider the subsystems, given by

$$\begin{bmatrix} x_i(t+1) \\ v_i(t+1) \end{bmatrix} = \begin{bmatrix} 0.98 & 0.45 \\ -0.09 & 0.80 \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \end{bmatrix} + \begin{bmatrix} 0.12 \\ 0.45 \end{bmatrix} u_i(t) + 0.1 w_i(t)$$

In order for all communication graph structures to admit tractable solutions, we assume that the dynamics of the subsystems are decoupled. In other words, the overall state-space matrices are block diagonal with the above matrices as the diagonal blocks. Hence, $P_{22} \in \text{Sparse}(I; \mathcal{RL}_\infty) \subset \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RL}_\infty)$, for any graph $\mathcal{G}^K$, so that Theorem 2 applies for any communication structure. The cost to minimize is given by

$$\sum_{i=1,2} (x_i(t) - x_{i+1}(t))^2 + \sum_{i=1,2,3} 0.01(x_i(t)^2 + v_i(t)^2) + \mu u_i(t)^2$$

Thus, we are essentially trading off the input effort with the positional errors between the masses. The following nine graph structures are considered.



Note that all of these graph structures are acyclic. Of course, adding cycles is possible, and there are many other possible graphs for this 3-player system. However, some interesting behavior is observed with the above acyclic structures.

By minimizing our cost over various μ, we obtain the following trade-off curves for these graph structures.

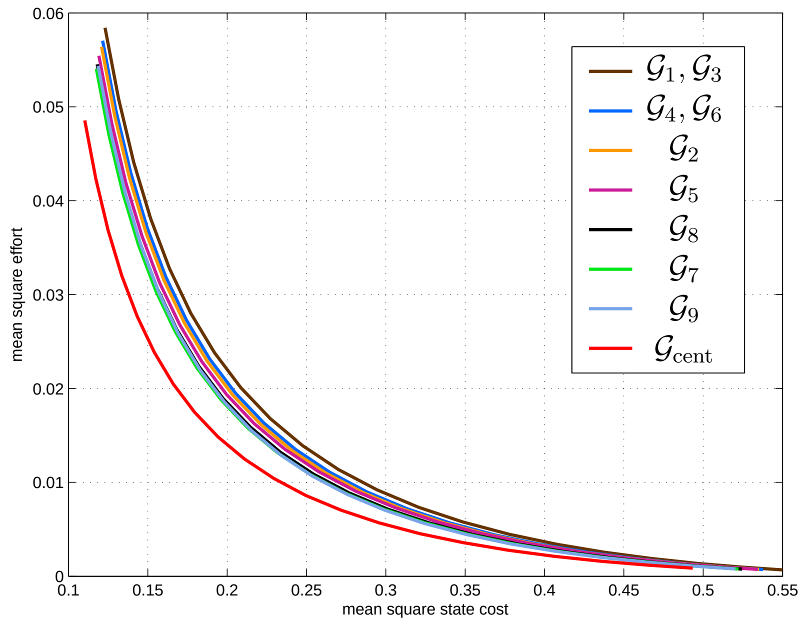


Figure 5.3: Three-Player Trade-off Curves

For comparison, the centralized solution is included as $\mathcal{G}_{\text{cent}}$. Also, the ordering in the legend is from graphs with higher cost to graphs with lowest cost, with $\mathcal{G}_{\text{cent}}$ having the lowest cost as expected.

Some interesting results can be obtained from Figure 5.3. First, $\mathcal{G}_1$ and $\mathcal{G}_3$ have the same cost, and $\mathcal{G}_4$ and $\mathcal{G}_6$ have the same, slightly lower cost. In other words, for both $\mathcal{G}_1$ and $\mathcal{G}_4$, there is no benefit to adding a line of communication $1 \to 3$. However, when adding the path $1 \to 3$ to graph $\mathcal{G}_2$, there is an added benefit. The remaining three graphs are also interesting. For all values of μ, graph $\mathcal{G}_8$ always has higher cost than $\mathcal{G}_7$. However, the relative ranking of $\mathcal{G}_9$ varies over μ. In particular, for low values of μ, $\mathcal{G}_9$ has higher cost than $\mathcal{G}_8$, while for higher values of μ, $\mathcal{G}_9$ has lower cost than $\mathcal{G}_7$.

This behavior is very interesting, and not immediately obvious from the graph structures alone. In particular, this analysis would be very useful in cases where communication itself is costly. For instance, if only one channel of communication can be afforded, it is clear that the non-trivial answer would be to use graph $\mathcal{G}_2$. Thus, there is a clear practical use for the analysis that our results provide. Moreover, our results are applicable to much more general settings than the simple system considered here.

5.7 Summary

This chapter extended the results of Chapter 4 to the general N-player graph problem. As we demonstrated herein, generalizing the spectral factorization approach to this case was straightforward, and simply required N applications of the classical, centralized result. Moreover, no additional assumptions were required for this problem. As a result, the complication associated with this extension was principally focused on the computation of the optimal controller from the corresponding optimal Youla parametrization. Though the resulting controller still had a complicated state-space realization, with some additional work, we provided greater insight into the optimal control policy by analyzing the estimation structure. In particular, the optimal decentralized policy reduced to linear combinations of the estimation errors at each node. Moreover, we established the order of the optimal decentralized controller.

Chapter 6

The 2-Player Problem: Partial Output Feedback

In the two-player problem of Chapter 4, the critical assumption that we made was that the controllers had state feedback. This again was assumed for the N-player graph. The benefit of this assumption was that our objective function could be parametrized as an affine function of the Youla parameter which is operated on by only a left multiplication operator. This fact allowed us to solve the resulting optimality condition column by column. Unfortunately, when output feedback is introduced, this is no longer possible, and all columns of the Youla parameter get coupled.

In this chapter, we return to the two-player problem, and we weaken this assumption by allowing for output feedback on player 2. Again, a clever spectral factorization approach is used to find the optimal solution, extending this methodology to this case. This work was originally presented in [27].

6.1 Problem Formulation

We consider the same two-player state-space system as in Chapter 4, restated here.

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

In general, noisy measurements y_1 and y_2 of each state are made. For this system, we consider a *partial output feedback* system. That is, player 1's state is measured directly, and player 2's measured output is

$$y_2(t) = C_{21}x_1(t) + C_{22}x_2(t) + H_v v(t)$$

Note that $w_1(t)$, $w_2(t)$, and $v(t)$ are independent, exogenous noise. Consequently, we are interested in finding controllers of the form

$$\begin{aligned} q_1(t+1) &= A_{K1}q_1(t) + B_{K1}x_1(t) \\ u_1(t) &= C_{K1}q_1(t) + D_{K1}x_1(t) \end{aligned}$$

and

$$\begin{aligned} q_2(t+1) &= A_{K2}q_2(t) + B_{K2}\begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix} \\ u_2(t) &= C_{K2}q_2(t) + D_{K2}\begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix} \end{aligned}$$

As before, player 1 makes decision u_1 based only on the history of his own state x_1. However, in this case, player 2 makes decision u_2 based on the histories of x_1 and y_2. Equivalently, we want a controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ such that

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix}$$

Our cost is the vector

$$z(t) = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = C_1 x(t) + Du(t)$$

so that the overall plant is given by

$$\begin{bmatrix} z \\ x \end{bmatrix} = \left[\begin{array}{c:c} P_{11} & P_{12} \\ \hdashline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} w \\ v \\ \hdashline u \end{array}\right]$$

where

$$P = \left[\begin{array}{cc|ccc:cc} A_{11} & 0 & H_1 & 0 & 0 & B_{11} & 0 \\ A_{21} & A_{22} & 0 & H_2 & 0 & B_{21} & B_{22} \\ \hline C_{11} & C_{12} & 0 & 0 & 0 & D_1 & D_2 \\ \hdashline I & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & H_v & 0 & 0 \end{array}\right] \tag{6.1}$$

The optimization problem is then the same as in Chapter 4.

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\|_2 \\ \text{subject to} \quad & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty) \end{aligned} \tag{6.2}$$

6.2 Main Results

We now solve the optimization problem in (6.2). To this end, the following assumptions will be made throughout this chapter.

*A*1) (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable

*A*2) (C_{22}, A_{22}) is detectable

*A*3) $D^T D > 0$ and $H_v H_v^T > 0$

*A*4) $\begin{bmatrix} A - \lambda I & B \\ C_1 & D \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{T}$

*A*5) $\begin{bmatrix} A_{22} - \lambda I & H_2 & 0 \\ C_{22} & 0 & H_v \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{T}$

Note that there exist additional assumptions in the partial output feedback case. These will be discussed in the next section. Nevertheless, when these assumptions hold, the optimal policy for (6.2) can be found with the following theorem.

Theorem 40. *For the system in* (6.1)*, suppose assumptions A1–A5 hold. Let X, Y, and S be the stabilizing solutions to the algebraic Riccati equations*

$$X = C_1^T C_1 + A^T X A - (A^T X B + C_1^T D)(D^T D + B^T X B)^{-1}(B^T X A + D^T C_1) \quad (6.3)$$

$$Y = C_{12}^T C_{12} + A_{22}^T Y A_{22} - (A_{22}^T Y B_{22} + C_{12}^T D_2)(D_2^T D_2 + B_{22}^T Y B_{22})^{-1}(B_{22}^T Y A_{22} + D_2^T C_{12}) \quad (6.4)$$

$$S = H_2 H_2^T + A_{22} S A_{22}^T - A_{22} S C_{22}^T (H_v H_v^T + C_{22} S C_{22}^T)^{-1} C_{22} S A_{22}^T \quad (6.5)$$

Define

$$K = (D^T D + B^T X B)^{-1}(B^T X A + D^T C_1) \quad (6.6)$$

$$J = (D_2^T D_2 + B_{22}^T Y B_{22})^{-1}(B_{22}^T Y A_{22} + D_2^T C_{12}) \quad (6.7)$$

$$N = S C_{22}^T (H_v H_v^T + C_{22} S C_{22}^T)^{-1} \quad (6.8)$$

and let

$$\begin{aligned} A_K &= A - BK \\ A_J &= A_{22} - B_{22} J \\ A_N &= (I - N C_{22}) A_J \end{aligned}$$

Then, there exists a unique optimal $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ for (6.2) *given by:*

- *Controller* 1 *has realization*

$$\begin{aligned} q_1(t+1) &= (A_K)_{22} q_1(t) + (A_K)_{21} x_1(t) \\ u_1(t) &= -K_{12} q_1(t) - K_{11} x_1(t) \end{aligned}$$

- *Controller 2 has realization*

$$q_2(t+1) = \begin{bmatrix} (A_K)_{22} & 0 \\ A_N N C_{22} & A_N \end{bmatrix} q_2(t) + \begin{bmatrix} (A_K)_{21} & 0 \\ A_N N C_{21} & -A_N N \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix}$$

$$u_2(t) = \begin{bmatrix} -K_{22} + JNC_{22} & J \end{bmatrix} q_2(t) + \begin{bmatrix} -K_{21} + JNC_{21} & -JN \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix}$$

In comparing this result with the two-player state feedback result of Chapter 4, we see that player 1 implements the same control policy. However, player 2's policy is now more complicated. In fact, player 1's controller has order equal to the state dimension of x_2, while player 2's controller now has order of twice that.

It will be shown that q_1 in the optimal controllers is in fact the minimum-mean square error estimate of x_2 given the history of x_1, as it was in the state feedback case. Letting $\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(t), \ldots, x_1(0)\big)$, the optimal control policy can be written as

$$\begin{aligned} u_1(t) &= -K_{11}x_1(t) - K_{12}\eta(t) \\ u_2(t) &= -K_{21}x_1(t) - K_{22}\eta(t) + J\big(q_{22}(t) - N(y_2(t) - C_{21}x_1(t) - C_{22}\eta(t))\big) \end{aligned}$$

As before, we see that the optimal policy is attempting to perform the optimal centralized policy, using η instead of x_2. However, the additional error term in the optimal policy here is slightly different.

6.3 Analysis

It is clear from Lemmas 16 and 17 that Assumptions A1–A5 guarantee the existence of the stabilizing solutions X, Y, S to the Riccati equations (6.3–6.5). The only new assumption that needs to be discussed is Assumption A2. As the following lemma shows, this assumption is necessary for the existence of any stabilizing controller.

Lemma 41. *There exists a controller* $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ *which stabilizes* P *in* (6.1) *only if* (C_{22}, A_{22}) *is detectable.*

Proof. We will prove this statement by showing that if (C_{22}, A_{22}) is not detectable, then it is not possible to stabilize P.

To this end, suppose that (C_{22}, A_{22}) is not detectable. Then, there exists a transformation T, such that

$$T^{-1}A_{22}T = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad C_{22}T = \begin{bmatrix} 0 & c_2 \end{bmatrix}$$

where a_{11} is unstable. Let $\eta(t) = T^{-1}x_2(t)$. Then, with $w(t) = v(t) = 0$, the dynamics of the system are equivalent to

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ T^{-1}A_{21} & \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ T^{-1}B_{21} & T^{-1}B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y_2(t) = \begin{bmatrix} C_{21} & \begin{bmatrix} 0 & c_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \end{bmatrix}$$

Now, suppose $x_1(0) = 0$ and consider two possible initial conditions

$$\hat{\eta}(0) = \begin{bmatrix} \hat{v} \\ 0 \end{bmatrix} \qquad \tilde{\eta}(0) = \begin{bmatrix} \tilde{v} \\ 0 \end{bmatrix}$$

Since $x_1(t)$ is the same for both initial conditions, then $u_1(t)$ must be the same for both. It is also straightforward to show that both of these initial conditions produce the same $y_2(t)$ for all t. Consequently, $u_2(t)$ is the same in both cases, for whatever policy is chosen. As a result, we see that the difference between the state evolutions of $\eta(t)$ for these two initial conditions is given by

$$e(t+1) = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} e(t)$$

where $e(t) = \hat{\eta}(t) - \tilde{\eta}(t)$. Thus, if we choose $\hat{v} - \tilde{v} = v$, where v is an unstable eigenvector of a_{11}, it is clear that $e(t)$ diverges.

If it were possible to stabilize P, then we could have chosen policies for u_1 and u_2 such that $x(t) \to 0$ for any initial condition. However, if this were true, then $e(t)$ must converge to zero as well, since both trajectories must converge to zero. Since our construction contradicts this, then it is not possible to stabilize P. ∎

Combining this result with Lemma 15, we see that Assumptions A1 and A2 are necessary for the existence of a stabilizing controller for P. That they are also sufficient will be proven by explicit construction of a stabilizing controller, as seen in the following lemma.

Lemma 42. *Let $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$. For the system in (6.1), suppose (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable and (C_{22}, A_{22}) is detectable. Let F_1, F_2, L_1, and L_2 be matrices such that $A_{11} + B_{11}F_1$, $A_{22} + B_{22}F_2$, $A_{11} + L_1$, and $A_{22} + L_2 C_{22}$ all have stable eigenvalues. Lastly, let $A_F = A + BF$, $C_F = C_1 + DF$, and $A_L = A + L\hat{C}$, where*

$$F = \begin{bmatrix} F_1 & \\ & F_2 \end{bmatrix} \qquad L = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} I & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

Then, the set of all stabilizing controllers $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ is parametrized by

$$\mathcal{K} = \left[\begin{array}{cc|c} A + BF + L\hat{C} + BD_Q\hat{C} & BC_Q & -L - BD_Q \\ B_Q\hat{C} & A_Q & -B_Q \\ \hline F + D_Q\hat{C} & C_Q & -D_Q \end{array}\right], \quad Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \in \mathcal{S} \tag{6.9}$$

Moreover, the set of stable closed-loop transfer functions satisfies

$$\{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty),\ \mathcal{K} \text{ stabilizing}\} = \{T_{11} + T_{12} Q T_{21} \mid Q \in \mathcal{S}\}$$

where

$$T_{11} = \left[\begin{array}{cc|cc} A_F & BF & H & 0 \\ 0 & A_L & -H & -LE_2H_v \\ \hline C_F & DF & 0 & 0 \end{array}\right] \qquad T_{12} = \left[\begin{array}{c|c} A_F & B \\ \hline C_F & D \end{array}\right]$$

$$T_{21} = \left[\begin{array}{c|cc} A_L & -H & -LE_2H_v \\ \hline \hat{C} & 0 & -E_2H_v \end{array}\right]$$

Proof. The result follows from the standard Youla parametrization for the problem. See, for example [5]. ∎

As a result of this Youla parametrization, the optimization problem in (6.2) is equivalent to

$$\begin{aligned} \text{minimize} \quad & \|T_{11} + T_{12}QT_{21}\|_2 \\ \text{subject to} \quad & Q \in \mathcal{S} \end{aligned} \tag{6.10}$$

Note that T_{21} is no longer invertible, as it was in the state feedback case. Thus, we cannot group QT_{21} as we did in that case. To understand how we will approach this problem, it is convenient to look at the corresponding optimality condition. From Lemma 5, this is clearly

$$T_{12}^*T_{11}T_{21}^* + T_{12}^*T_{12}QT_{21}T_{21}^* \in \mathcal{S}^\perp \tag{6.11}$$

Solving (6.11) requires a clever spectral factorization, described in the next section.

6.4 Spectral Factorization

Our goal is now to find a solution $Q \in \mathcal{S}$ which satisfies the optimality condition (6.11). Once again, the approach taken will be based on spectral factorization.

As in the state feedback case, where we had the precompensator F, if we want to apply our spectral factorization methods to our problem, our Riccati equations would be in terms of the pre-compensators F and L. The difficulty with F can be avoided with Lemma 24. A similar result holds for the gain L.

Lemma 43. *Suppose $S \in \mathbb{R}^{n\times n}$, and $L \in \mathbb{R}^{m\times n}$. Then,*

$$S = BB^T + ASA^T - (ASC^T + BD^T)(DD^T + CSC^T)^{-1}(CSA^T + DB^T)$$

and $A - (ASC^T + BD^T)(DD^T + CSC^T)^{-1}C$ is stable, if and only if

$$S = B_LB_L^T + A_LSA_L^T - (A_LSC^T + B_LD^T)(DD^T + CSC^T)^{-1}(CSA_L^T + DB_L^T)$$

and $A_L - (A_LSC^T + B_LD^T)(DD^T + CSC^T)^{-1}C$ is stable, where $A_L = A + LC$ and $B_L = B + LD$.

Proof. By substitution of A_L and B_L, it can be readily shown that the two Riccati equations are equivalent. ■

As mentioned above, since T_{21} is not invertible, we cannot group QT_{21} as a new variable $\hat{Q} \in \mathcal{S}$. In order to use a method like this, an additional step is required before such a parametrization is possible.

To this end, we have the following result.

Lemma 44. *For the system in* (6.1)*, suppose Assumptions A1– A5 hold. Let* $A_L = A + L\hat{C}$*. Then, there exists* $G \in \mathcal{S}$*, such that*

$$G^{-1} \in \mathcal{S} \qquad GG^* = T_{21}T_{21}^* \qquad \mathcal{S}^\perp G^{-*} \subset \mathcal{S}^\perp$$

Moreover, such a $G \in \mathcal{S}$ *is given by*

$$G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$$

where

$$\begin{aligned} G_{11} &= -(zI - (A_L)_{11})^{-1}(A_L)_{11}H_1 - H_1 \\ G_{21} &= -zC_2(zI - A_L)^{-1}HE_1 \\ G_{22} &= C_{22}(zI - (A_L)_{22})^{-1}(A_{22}N + L_2)V^{\frac{1}{2}} + V^{\frac{1}{2}} \end{aligned}$$

and

$$V = H_v H_v^T + C_{22}SC_{22}^T$$

with S *and* N *satisfying* (6.5) *and* (6.8)*, respectively. In addition, this* G *satisfies*

$$T_{21}^* G^{-*} = \begin{bmatrix} zI & 0 \\ 0 & (T_{21})_{22}^* G_{22}^{-*} \end{bmatrix}$$

and

$$\begin{aligned} &(zI - A_{22})^{-1}H_2E_1^T(T_{21})_{22}^*G_{22}^{-*} \\ &\qquad = z^{-1}S(z^{-1}I - (I - NC_{22})^T A_{22}^T)^{-1}C_{22}^T V^{-\frac{1}{2}} + (zI - A_{22})^{-1}A_{22}NV^{\frac{1}{2}} \end{aligned} \tag{6.12}$$

Proof. First, notice that G_{11} and G_{22} satisfy

$$\begin{aligned} G_{11}^{-1} &= -H_1^{-1} + z^{-1}H_1^{-1}(A_L)_{11} \\ G_{22}^{-1} &= V^{-\frac{1}{2}}(C_{22}(zI - A_{22}(I - NC_{22}))^{-1}(A_{22}N + L_2) + I) \end{aligned}$$

Thus, $G_{11}^{-1}, G_{22}^{-1} \in \mathcal{RH}_\infty$, so $G^{-1} \in \mathcal{S}$.

As a consequence of this, we see that $G_{11}^{-*}, G_{12}^{*}, G_{22}^{-*} \in \mathcal{H}_\infty^-$. Thus, for any $\Lambda \in \mathcal{S}^\perp$, we have

$$\Lambda G^{-*} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} G_{11}^{-*} & -G_{11}^{-*}G_{21}^{*}G_{22}^{-*} \\ 0 & G_{22}^{-*} \end{bmatrix}$$

Since $\Lambda_{11}, \Lambda_{21}, \Lambda_{22} \in \mathcal{H}_2^\perp$, it straightforward to see that the product of $\Lambda G^{-*} \in \mathcal{S}^\perp$.

Our last step is to show that $GG^* = T_{21}T_{21}^*$. To this end, note that

$$G_{11} = z(T_{21})_{11} \qquad G_{21} = z(T_{21})_{21}$$

Lastly, algebraic manipulations of the Riccati equation (6.5) shows that (6.12) holds and

$$G_{22}G_{22}^* = (T_{21})_{22}(T_{21})_{22}^*$$

This is a standard spectral factorization result and is the dual of Lemma 21. See [10] for a simple proof. As a result, we have

$$\begin{aligned} T_{21}T_{21}^* &= \begin{bmatrix} z^{-1}G_{11} & 0 \\ z^{-1}G_{21} & (T_{21})_{22} \end{bmatrix} \begin{bmatrix} zG_{11}^* & zG_{21}^* \\ 0 & (T_{21})_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} G_{11}G_{11}^* & G_{11}G_{21}^* \\ G_{21}G_{11}^* & G_{21}G_{21}^* + (T_{21})_{22}(T_{21})_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} G_{11}G_{11}^* & G_{11}G_{21}^* \\ G_{21}G_{11}^* & G_{21}G_{21}^* + G_{22}G_{22}^* \end{bmatrix} \\ &= GG^* \end{aligned}$$

∎

With Lemma 44, the optimality condition (6.11) is equivalent to

$$T_{12}^* T_{11} T_{21}^* G^{-*} + T_{12}^* T_{12} Q G \in \mathcal{S}^\perp$$

Consequently, since G is invertible in $\mathcal{S}$, then we can now group QG as a new variable $\hat{Q}$, as in Lemma 19, resulting in

$$T_{12}^* T_{11} T_{21}^* G^{-*} + T_{12}^* T_{12} \hat{Q} \in \mathcal{S}^\perp \tag{6.13}$$

As a result, we have once again decoupled the columns of the optimality condition. This allows for us to solve for $\hat{Q}$, as we did in the state feedback case.

Lemma 45. *For the system in* (6.1)*, suppose Assumptions A1– A5 hold. Let* X, Y, S *satisfy* (6.3)–(6.5)*, respectively. Also, define* K, J, N *as in* (6.6)–(6.8)*, and let* $G \in \mathcal{S}$ *be defined as in Lemma 44. Then, the unique* $Q \in \mathcal{S}$ *satisfying* (6.13) *is given by*

$$\begin{bmatrix} \hat{Q}_{11} \\ \hat{Q}_{21} \end{bmatrix} = -z(K+F)(zI - A_K)^{-1} E_1 H_1 + zF(zI - A_L)^{-1} E_1 H_1 \tag{6.14}$$

$$\hat{Q}_{22} = -(J + F_2)(zI - A_J)^{-1} A_J N V^{\frac{1}{2}} + J N V^{\frac{1}{2}} \tag{6.15}$$

$$- F_2(zI - (A_L)_{22})^{-1}(A_{22} N + L_2) V^{\frac{1}{2}} \tag{6.16}$$

Proof. Using Lemma 23, the optimality condition can be solved as two separate problems. Condition (i) of the lemma implies that we must find $\hat{Q} E_1 \in \mathcal{RH}_2$ satisfying

$$T_{12}^* T_{11} T_{21}^* G^{-*} E_1 + T_{12}^* T_{12} \hat{Q} E_1 \in \mathcal{H}_2^\perp \tag{6.17}$$

From Lemma 44, we have

$$T_{21}^* G^{-*} E_1 = z E_1$$

We define $\hat{L} \in \mathcal{RH}_\infty$ to factorize $T_{12}^* T_{12}$ as in Lemma 21, via (6.3). Since $\hat{L}^{-*} \in \mathcal{H}_\infty^-$, then $\hat{L}^{-*} \mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$, and (6.17) is equivalent to

$$z \hat{L}^{-*} T_{12}^* T_{11} E_1 + \hat{L} \hat{Q} E_1 \in \mathcal{H}_2^\perp$$

Consequently, $\hat{Q}E_1$ can be found by projecting this expression onto $\mathcal{H}_2$. Using Lemma 21, we obtain

$$\hat{Q}E_1 = -\hat{L}^{-1}\mathrm{P}_{\mathcal{H}_2}(z\hat{L}^{-*}T_{12}^*T_{11}E_1)$$

from which (6.14) follows.

For condition (ii), we must find $\hat{Q}_{22} \in \mathcal{H}_2^\perp$ satisfying

$$E_2^T T_{12}^* T_{11} T_{21}^* G^{-*} E_2 + E_2^T T_{12}^* T_{12} E_2 \hat{Q}_{22} \in \mathcal{H}_2^\perp \tag{6.18}$$

From Lemma 44, we now have

$$T_{21}^* G^{-*} E_2 = E_2 (T_{21})_{22}^* G_{22}^{-*}$$

Now define $\hat{L} \in \mathcal{RH}_\infty$ to factorize $E_2^T T_{12}^* T_{12} E_2$ according to Lemma 21, via (6.4). Then, (6.18) is equivalent to

$$\hat{L}^{-*}(T_{12})_{22}^*(T_{11})_{22}(T_{21})_{22}^* G_{22}^{-*} + \hat{L}\hat{Q}_{22} \in \mathcal{H}_2^\perp$$

Projecting this expression onto $\mathcal{H}_2$ follows from Lemmas 21 and 44, yielding

$$\mathrm{P}_{\mathcal{H}_2}(\hat{L}^{-*}(T_{12})_{22}^*(T_{11})_{22}(T_{21})_{22}^* G_{22}^{-*}) = \left[\begin{array}{cc|c} (A_F)_{22} & B_{22}F_2 & -A_{22}NV^{\frac{1}{2}} \\ 0 & (A_L)_{22} & (A_{22}N + L_2)V^{\frac{1}{2}} \\ \hline W_2^{\frac{1}{2}}(J+F_2) & W_2^{\frac{1}{2}}F_2 & -W_2^{\frac{1}{2}}JNV^{\frac{1}{2}} \end{array}\right]$$

The result follows by left-multiplying this expression with $\hat{L}^{-1}$. ∎

Having found the $\hat{Q} \in \mathcal{S}$ which satisfies (6.13), we can now prove our main results.

Proof of Theorem 40. From Lemma 42, $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RL}_\infty)$ is optimal for (6.2) if and only if $Q \in \mathcal{S}$ satisfies (6.11). Spectral factorizing $T_{21}T_{21}^*$ according to Lemma 44, the optimality condition is equivalent to (6.13), where $\hat{Q} = QG$. The unique $\hat{Q}$ satisfying this expression was found in Lemma 45. Consequently, the optimal $Q = \hat{Q}G^{-1}$ for (6.11) is found to be

$$Q = \left[\begin{array}{ccc|cc} A_K & 0 & 0 & A_K E_1 - E_1(A_L)_{11} & 0 \\ 0 & A_J & A_J N C_{22} & A_J N C_{21} & -A_J N \\ 0 & 0 & A_{22}(I - NC_{22}) & A_{21} - A_{22}NC_{21} & A_{22}N + L_2 \\ \hline E_1^T(K+F) & 0 & 0 & K_{11} & 0 \\ E_2^T(K+F) & -J - F_2 & -F_2 - JNC_{22} & K_{21} - JNC_{21} & JN \end{array}\right] \tag{6.19}$$

Thus, the optimal $\mathcal{K}$ is found by substitution of (6.19) into (6.9). After algebraic manipulations, we find that

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{K}_{11} &= -K_{11} - K_{12}\Phi \\ \mathcal{K}_{21} &= -K_{21} - K_{22}\Phi + zJ(zI - A_N)^{-1}N(C_{21} + C_{22}\Phi) \\ \mathcal{K}_{22} &= -zJ(zI - A_N)^{-1}N \end{aligned}$$

and

$$\Phi = (zI - (A_K)_{22})^{-1}(A_K)_{21}$$

In state-space this corresponds to the controllers in the theorem. ■

6.5 Estimation Structure

We end our analysis of this problem by discussing the structure of the optimal controller. To begin, we consider the order of the optimal controller. For player 1, the controller's state dimension is equal to the state dimension of player 2. This is reasonable since, as we will show next, the dynamics associated with player 1's controller is in estimating player 2's state. This also matches the results obtained for the full state feedback case in Chapter 4.

By comparison, player 2's controller requires a state dimension equal to twice player 2's state dimension. Part of these dynamics match the dynamics of player 1's controller. However, additional dynamics are required in the partial output feedback case. This is in contrast to both the full state feedback case and the classical LQG problem, in which the controller orders are equal to the dimension of the state being estimated.

Lastly, we show part of this estimation structure. To this end, we let

$$\eta = \Phi x_1 \qquad \lambda = (zI - A_N)^{-1} A_N N (C_{21} x_1 + C_{22}\eta - y_2)$$

This represents the following state-space dynamics

$$\begin{aligned} \eta(t+1) &= (A_K)_{22}\eta(t) + (A_K)_{21} x_1(t) \\ \lambda(t+1) &= A_N \lambda(t) + A_N N (C_{21} x_1(t) + C_{22}\eta(t) - y_2(t)) \end{aligned}$$

Combining this with the dynamics in (6.1), and letting $e = \eta - x_2$, the closed-loop dynamics of the system become

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \\ e(t+1) \\ \lambda(t+1) \end{bmatrix} = \hat{A} \begin{bmatrix} x_1(t) \\ \eta(t) \\ e(t) \\ \lambda(t) \end{bmatrix} + \hat{B} \begin{bmatrix} w_1(t) \\ w_2(t) \\ v(t) \end{bmatrix} \tag{6.20}$$

where

$$\hat{A} = \begin{bmatrix} (A_K)_{11} & (A_K)_{12} & 0 & 0 \\ (A_K)_{21} & (A_K)_{22} & 0 & 0 \\ 0 & 0 & A_{22} - B_{22} J N C_{22} & -B_{22} J \\ 0 & 0 & A_N N C_{22} & A_N \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -H_2 & B_{22} J N H_v \\ 0 & 0 & -A_N N H_v \end{bmatrix}$$

As a result, we obtain a very simple interpretation for η.

Lemma 46. *Suppose $x_1, \eta, \eta - x_2, \lambda$ are the states of the autonomous system in* (6.20), *and w_1, w_2, v are independent, zero mean random processes. Then,*

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big)$$

Proof. For notational convenience, let $x(0:t) = x(0), \ldots, x(t)$.

From the independence of the noises and the block diagonal structure of the dynamics, it is clear that x_1, η evolve independently of $\eta - x_2$. Consequently,

$$\mathrm{E}\big(\eta(t) - x_2(t) \mid x_1(0:t), \eta(0:t)\big) = \mathrm{E}\big(\eta(t) - x_2(t)\big) = 0$$

Thus,

$$\mathrm{E}\big(\eta(t) \mid x_1(0:t), \eta(0:t)\big) = \mathrm{E}\big(x_2(t) \mid x_1(0:t), \eta(0:t)\big) \tag{6.21}$$

Note that the expected value of $\eta(t)$, conditioned on itself, is just equal to $\eta(t)$. Also, by definition, η is a deterministic function of x_1. Thus, conditioning on $x_1(0:t)$ and $\eta(0:t)$ is equivalent to conditioning on just $x_1(0:t)$. As a result, (6.21) is equivalent to

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0:t)\big)$$

∎

6.6 Summary

This chapter provided an extension to the two-player decentralized problem considered in Chapter 4. In that chapter, state feedback for both players was assumed. In this work, we relaxed this condition to a partial output feedback structure, in which one player measures his state directly and the other player is limited to a noisy measurement of his own state.

The principle difference between these two problems was that T_{21} is not invertible in the partial output feedback case. Thus, an additional spectral factorization was required to first reformulate the problem with an invertible operator right multiplying the Youla parameter. Once this was achieved, the optimization problem decoupled in the same manner as the state feedback case.

Lastly, we examined the estimation structure of the optimal policy. This time, part of the controller dynamics represented the estimator from the state feedback case. Though unproven, it is believed that the remaining dynamics correspond to another state estimation conditioned on different information from the other estimator. As a result, we established that the order of the optimal controller was equal to twice the state dimension of player 2.

Chapter 7

Conclusions

In this work, we considered the problem of optimal decentralized control of systems connected over graphs. The decentralization constraints were imposed by sparsity of the overall controller. The results were obtained by a number of steps. The first of these was characterizing which decentralized systems are tractable. Since the resulting control system consists of two separate graphs, the tractability of these problems was based on the relationship between these graphs.

For systems which satisfy this tractability property, it was shown that an equivalent convex parametrization exists. However, though convex, this equivalent problem is still infinite-dimensional. In order to solve this problem, we employed the method of spectral factorization.

The simplest decentralized problems in this class is the two-player control problem, with communication allowed in only one direction. To start, we restricted attention to the state feedback case. This assumption allowed us to find the optimal control policy. This policy was shown to have state dimension equal to the size of player 2's state. Moreover, these dynamics corresponded to an estimation process of player 2's state given the history of player 1's state.

These results were extended to the general N-player state feedback problem. While the resulting optimal controller is rather complicated, we obtained significant insight into the policy by considering the estimation structure. In particular, the optimal policies were shown to involve linear combinations of the estimation errors of each player's descendant states conditioned on its ancestor states. Also, in cases where communication is costly, these results also provided the practical benefit of allowing us to compare the efficiency of different communication structures.

Lastly, we attempted to generalize these state feedback results to the output feedback case. To this end, we addressed a partial output feedback structure, in which one player measures his state directly and the other player is limited to a noisy measurement of his own state.

The optimal controller involves at least one estimation process. Additional dynamics are also required for one of the players. As a result, the order of the optimal controller was established and is equal to twice the state dimension of one of the subsystems.

Though not shown, it is straightforward to see that the two-player partial output feedback results extend to more general graphs. In particular, it is clear that the same approach will work for graphs in which all end vertices of the graph (vertices with no descendants) have output feedback while other vertices employ state feedback.

Unfortunately, the complete output feedback remains to be solved with this approach. As noted in this work, output feedback couples all elements of the Youla parameter. When state feedback, or partial output feedback, is assumed a nice decoupling of the parameters results.

Despite these difficulties for the general problem, some preliminary results, inspired by our work, have been obtained for other generalizations of these problems. In particular, similar results have been shown for nonlinear, Markov processes. Also, decentralized systems involving TCP-like packet drops is a developing area of work. It is anticipated that these results should extend naturally to systems involving delays, especially once the complete output feedback solution is obtained. It will also be interesting to see how well the results presented here extend to other norms.

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