最优化理论第一次作业

22336226

王泓沣

1. 证明：

$\because C \subseteq R^n$ 为一个凸集

$\therefore \forall x_1, x_2 \in C, \forall \theta_1, \theta_2 \in \mathbb{R}\ s.t.\ \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \geqslant 0$ 有 $\theta_1 x_1 + \theta_2 x_2 \in C$

$\therefore \exists u_1 \in C, u_1 = \theta_1 x_1 + \theta_2 x_2$

$\therefore \forall x_3 \in C, \forall \theta_1, \theta_2 \in \mathbb{R}\ s.t.\ \theta_1 + \theta_2 = 1 \therefore \theta_3 u_1 + \theta_4 x_3 \in C$

依此类推，得证

2. 证明：

$$
\begin{aligned}
&\forall x_1, x_2 \in C, \forall \theta_1, \theta_2 \in R\ s.t.\ \theta_1 + \theta_2 = 1, \theta_1, \theta_2 \geqslant 0 \\
&A(\theta_1 x_1 + \theta_2 x_2) = \theta_1 A x_1 + \theta_2 A x_2 = b_1 + b_2 \\
&\therefore \theta_1 x_1 + \theta_2 x_2 \in C \\
&\therefore C\text{为凸集}
\end{aligned}
$$

3. 证明：

$$
\begin{aligned}
&\forall x_1, x_2 \in \mathbb{R}^n, \forall \theta_1, \theta_2 \in \mathbb{R} \geqslant 0 \; s.t. \; \theta_1 + \theta_2 = 1, \\
\text{令} s =&(\theta_1 x_1 + \theta_2 x_2)^\top A(\theta_1 x_1 + \theta_2 x_2) + b^\top(\theta_1 x_1 + \theta_2 x_2) + c \\
=&\theta_1^2 x_1^\top A x_1 + \theta_1\theta_2 x_2^\top A x_1 + \theta_1\theta_2 x_1^\top A x_2 + \theta_2^2 x_2^\top A x_2 + \theta_1 b^\top x_1 + \theta_2 b^\top x_2 + c \\
&\because \; A \succeq 0 \\
&\therefore \; x_1^\top A x_2 = x_2^\top A x_1 \\
\text{令}\lambda_1 =&\theta_1(x_1^\top A x_1 + b^\top x_1 + c), \lambda_2 = \theta_2(x_2^\top A x_2 + b^\top x_2 + c) \\
s - \lambda_1 - \lambda_2 =& (\theta_1^2 - \theta_1) x_1^\top A x_1 + 2\theta_1\theta_2 x_1^\top A x_2 + (\theta_2^2 - \theta_2) x_2^\top A x_2 \\
=& -\theta_1\theta_2(x_1 - x_2)^\top A(x_1 - x_2) \\
&\because A \succeq 0 \\
&\therefore \; -\theta_1\theta_2(x_1 - x_2)^\top A(x_1 - x_2) \leqslant 0 \\
&\because \lambda_1 \leqslant 0, \lambda_2 \leqslant 0 \\
&\therefore s \leqslant 0 \\
&\therefore \theta_1 x_1 + \theta_2 x_2 \in C \\
&\therefore C\text{是凸集}
\end{aligned}
$$

4. 解：

(a)

$$
\begin{aligned}
&\bigtriangledown^2 f(x) = \sum_{i=1}^{n} \frac{1}{x_i} \\
&\because \; x_i \in \mathbb{R}_{++} \\
&\therefore \; \bigtriangledown^2 f(x) \succ 0 \\
&\therefore \; f(x)\text{为严格凸函数}
\end{aligned}
$$

(b)

$$\frac{\partial f}{\partial x_1} = x_2$$
$$\frac{\partial f}{\partial x_2} = x_1$$
$$\frac{\partial^2 f}{\partial x_1^2} = 0$$
$$\frac{\partial^2 f}{\partial x_2^2} = 0$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$
$$\therefore Hessian\text{矩阵为} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\therefore \lambda_1 = -1, \lambda_2 = 1$$
$$\therefore f(x_1, x_2)\text{为非凸非凹函数}$$

(c)

$$\frac{\partial f}{\partial x_1} = \frac{1}{x_2}$$
$$\frac{\partial f}{\partial x_2} = -\frac{x_1}{x_2^2}$$
$$\frac{\partial^2 f}{\partial x_1^2} = 0$$
$$\frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1}{x_2^3}$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}$$
$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{1}{x_2^2}$$
$$\therefore Hessian\text{矩阵为} \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$
$$\lambda^2 - \frac{2x_1}{x_2^3}\lambda - \frac{1}{x_2^4} = 0$$

$x_2 \qquad x_2$

$$\lambda_1, \lambda_2 = \frac{\frac{2x_1}{x_2^3} \pm \sqrt{\frac{4x_1^2}{x_2^6} + \frac{4}{x_2^4}}}{2}$$

$\therefore f(x_1, x_2)$是非凸非凹函数

5. 证明：

$$f(\theta x + (1-\theta)y) = h(g(\theta x + (1-\theta)y))$$

$\because g_i$为凹函数

$\therefore g(\theta x + (1-\theta)y) \geqslant \theta g(x) + (1-\theta)g(y)$

$\because h$关于其每个分量非增

$\therefore h(g(\theta x + (1-\theta)y)) \leqslant h(\theta g(x) + (1-\theta)g(y))$

$\because h$为凸函数

$\therefore h(\theta g(x) + (1-\theta)g(y)) \leqslant \theta h(g(x)) + (1-\theta)h(g(y))$

即$f := h \circ g$为凸函数