Chapter 4 Theory of motion and gravitation

4.1 Kepler's laws of planetary motion

In 1609 and 1619, Johannes Kepler (開普勒，1571-1630 AD)[1] deduced three empirical laws by analyzing the observational data of Tycho Brahe (第谷，1546-1601 AD).

Kepler′s three laws of planetary motion

- ***First law***: Planets move in *elliptical orbits* around the Sun, with the Sun is at one focus of the ellipse. This discovery marked the death of circularity.
- ***Second law***: A line from a planet to the Sun sweeps over *equal areas in equal intervals of time* (Fig. 5-1). Hence, planets move *faster* when *closer* to the Sun.
- ***Third law***: The square of a planet′s orbital period T is proportional to the cube of its average distance a from the Sun, i.e., $T^2 = Aa^3$, where A is a constant. It implies that planets *farther* from the Sun have *longer* orbital periods

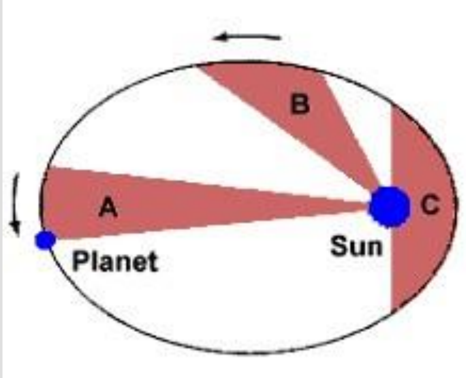

Fig. 4-1: Kepler′s second law states that a line joining the Sun and a planet sweeps out equal areas in equal time intervals. The three shaded areas (A, B, C) are equal, so that a planet moves faster when it is closer to the Sun.

Compared to the previous efforts, Kepler's laws predict the locations of the planets with much higher accuracy.[2] They provided strong evidences to the heliocentric model.[3]

4.2 Galileo's study of motion

✓ Galileo Galilei (伽利略，1565-1642 AD)[4] studied medicine at the University of Pisa, but then turned to study mathematics and natural philosophy (that is, science). In astronomy, he was a great defender of the Copernican model. He was the first one to

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[1] See http://galileo.rice.edu/sci/kepler.html for a biography of Johannes Kepler.

[2] Here the locations refer to those on the sky. In terms of the size of orbits (in the space), Kepler's estimation was about 1/6 of the current value. Kepler's three laws are still correct if you scale all the lengths with the same ratio.

[3] One important evidence of the heliocentric model finally arrived in the mid 19th century. With his famous experiment by using a large pendulum, Jean Foucault demonstrated the Coriolis effect due to the Earth's self-rotation. It explains the Sun's daily motion on the sky.

[4] See http://galileo.rice.edu/galileo.html for a biography of Galileo Galilei.

perform detailed observation using a telescope, and constructed theories based on the results of experiments and observations. Major discoveries in physics includes

Mass-independence of falling:

- ✓ Galileo regarded free-falling object is independent of mass. He invented a thought-experiment to look into this problem in a very intuitive way (Fig. 4-2). Suppose that more massive objects fall faster, 2 units of mass should fall faster than 1 unit. We use a string to connect them up. Since the 1 unit falls slower than the 2 units, the former will drag the latter and the latter will accelerate the former. As a result, their speed will be faster than that of 1 unit and slower than that of 2 units. However, they are together 3 units of mass, which should fall faster the 2 units. The statement "more massive objects fall faster" has internal inconsistence.

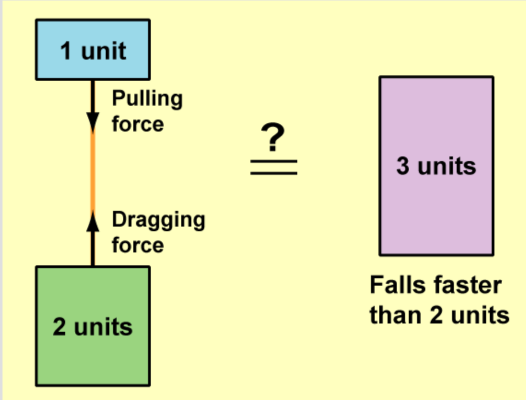


Fig. 4-2: A thought experiment invented by Galileo.

Horizontal motions are natural:

- ✓ He used graphs to represent motions. He found that objects climbing up an inclined plane decelerate while those sliding down accelerate. Both cases can be represented by straight lines in velocity against time graphs. From these graphs, he found that the straight lines approach horizontal as the inclined plane approaches horizontal. (Fig. 4-3)

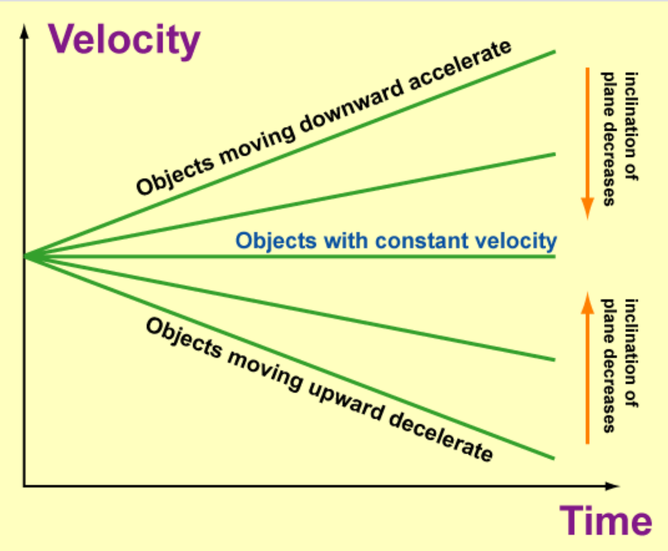


Fig. 4-3: The velocity of an object sliding on an inclined plane.

- ✓ He therefore concluded that the horizontal motion of objects is natural. In fact, this later became Newton's first law. We need not explain why it happens. On the contrary, we need to explain why horizontally moving objects eventually stop. In fact, it was later found that friction is the cause.

Inter-independence of horizontal and vertical motions

- ✓ He also found that the projectile motion is a combination of a horizontal motion in uniform velocity and a vertical falling motion. These two motions are independent of each other. (You may have applied this idea to tackle physics problems in high school.)

His major discoveries in astronomy include:

- ✓ He observed mountainous terrain on the Moon, the sunspots and their movement, and the rotation of the Sun. It contradicts the traditional view that heavenly bodies are perfect!
- ✓ Four satellites orbiting Jupiter, Jupiter is moving but does not leave its satellites behind. In other words, the universe could contain centres of motions. Similarly, the Earth *can* move and yet keep the Moon with it.
- ✓ He observed sunspots on the solar surface and the self-rotation of the Sun. Even the Sun is not a perfect body.

4.3 Newton's mechanical universe

The discoveries and theories of Kepler and Galileo laid the groundwork for Isaac Newton's theories, which unified the governing rules of the terrestrial and celestial motions.

Newton′s three laws of motion

Newton gave a precise definition of the "*motion*" of an object. It was referring to two general characteristics: ***speed*** and ***direction***. An object at rest is simply a special case of motion, with zero speed. The ***acceleration*** of an object is defined as the rate of change of velocity, which is speed with direction (the vector version of speed).

- ***Newton's first law* (the law of inertia):** A body remains its *motion* [5] unless acted upon by some net external force. For example, a block slides freely on a *smooth* table; a box remains stationary unless acted on by some force.
- ***Newton's second law*:** The acceleration ***a*** of a body is inversely proportional to its mass [6] m, directly proportional to the net force F, and in the same direction as the net force. (Fig. 4-4) It is the famous Newton's law which is commonly written as $\vec{F} = m\vec{a}$. A more

[5] Any motion has to be measured in a reference frame. An *inertial reference frame* is one in which Newton's first law holds true. A non-inertial frame is accelerated with respect to inertial frames. Example of non-inertial frame is a rotating frame, in which an object tends to move outward even without a real net force.
[6] The mass in the second law is the *inertia mass*, which is a measure of the inertia of an object.

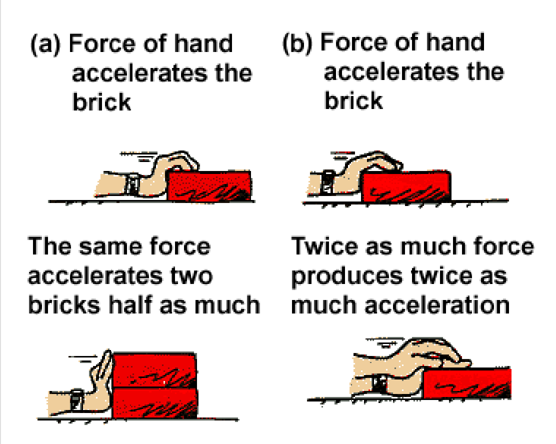


Fig. 4-4: An example of Newton's second law. (a) When the applied force is kept constant and the mass of the body is doubled, the acceleration of the body is halved. (b) When the mass is kept constant, doubling the force results in a doubling of acceleration.

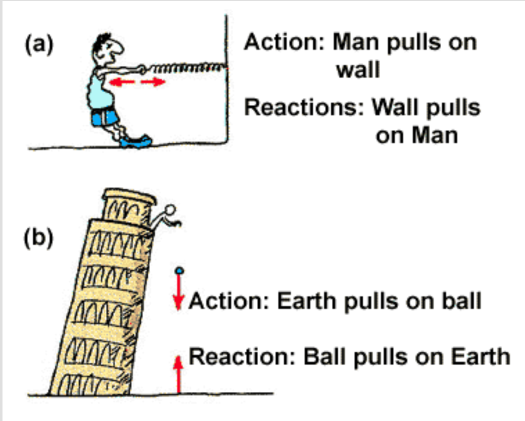


Fig. 4-5: Examples of Newton′s third law. (a) The wall produces a reaction force on the man as the man pulls the wall. (b) The ball attracts the Earth by the same force as the Earth attracts the ball. Note that action and reaction are always equal in magnitude, in opposite direction, and on different bodies.

general form is $\vec{F} = \frac{d\vec{p}}{dt}$, which states that the force on a body equals the rate of change of momentum $\vec{p} = m\vec{v}$ of the body.

- ***Newton's third law*** **(the law of action and reaction):** For every action, there is an equal and opposite reaction (Fig. 4-5). For example, when you push a block (apply a force on it), a force of the same *same magnitude* acts on your hand. Notice that actions and reactions are *opposite in direction*, and they act on *different bodies*.

Newton′s law of universal gravitation

- ***Law of Universal Gravitation***: [7] The attractive force F between any two point masses [8] M_1 and M_2 is directly proportional to the product of their masses and is inversely proportional to the square of their separation r, i.e., $F = -\frac{GM_1M_2}{r^2}$, where $G = 6.673\times10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$ is the gravitational constant. Any pair of bodies follows the same gravitation law, e.g., an apple and the Earth, the Sun and the Earth. For two spheres with uniform density, the

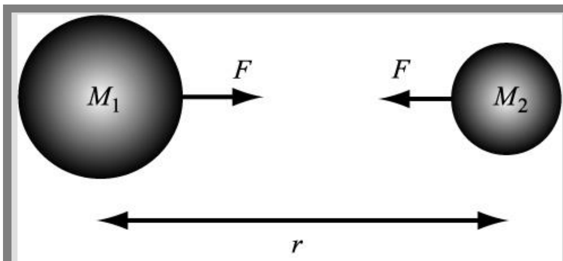


Fig. 4-6: **The gravitational force between two uniform spheres is directly proportional to the product of their masses and inversely proportional to the square of their centre-centre separation.**

[7] It has long been understood that the Inverse square law (referred to the dependence of r) was discovered by Isaac Newton (1643–1727). However, recent literatures show that Robert Hooke (1635–1703) proposed the law before Newton.

[8] The mass in the Law of Universal Gravitation is the *gravitational mass*, which is a property that leads to (and responses to) gravitational force.

distance r is the separation between their centres.[9] (Fig 4-6) The attractive force is *mutual*, i.e., M_1 attracts M_2, whereas M_2 attracts M_1 by a force of the same magnitude.

- ✓ No matter how far they are, the gravitational forces between two objects exist instantaneously.[10]
- ✓ Gravity is much *weaker* than electric force;[11] however, gravity dominates in astronomical scales[12] because celestial bodies are in general electrically neutral.

The motions of any celestial bodies can be calculated by using *Newton's laws of motion* plus *his law of universal gravitation*. The success of Newton's theories leads to the philosophical idea of the **mechanical universe** (also known as the **clockwork universe**).

4.4 Some applications of Newtonian mechanics

Galileo's mass-independence of falling

- ✓ Consider a free-falling object with mass m. The gravitational pull on the object by the Earth (mass M) is given by $F = \dfrac{GMm}{r^2}$. By Newton's second law, the acceleration of the object is $a = \dfrac{F}{m} = \dfrac{GM}{r^2}$, which is *independent* of the mass m of the object.

Orbits of space shuttle and Weightlessness

- ✓ A space shuttle tends to move in straight line due to inertia. The gravitational pull by the Earth results in a change in direction of motion. Even though the shuttle is free-falling, the fall is compensated by the radial component of its velocity. As a result, the shuttle orbits around the Earth (Fig. 4-7).

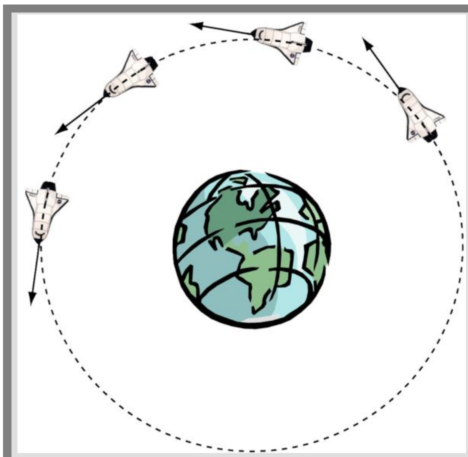


Fig. 4-7: In the absence of gravitational force, the shuttle would follow a straight-line path (dashed arrows). The path is curved because the Earth's gravity pulls the shuttle towards the Earth. (Note: there is no need for the engine to fire up in order for the shuttle to remain in orbit.)

[9] Actually, as long as the mass distribution is spherically symmetric, the *centre of mass* would locate right at the centre of the sphere. See Carroll & Ostlie 2007 pages 33-35 for proof.

[10] It is not compatible with relativity! More in Chapter 6.

[11] In the words of Richard Feynman: "If you were standing at arm's length from someone and each of you had one percent more electrons than protons, the repelling force would be incredible. How great? Enough to lift the Empire State building? No! To lift Mount Everest? No! The repulsion would be enough to lift a 'weight' equal to that of the entire Earth!"

[12] Strong force and weak force are of short range, and do not have any effect at astronomical scales.

✓ An astronaut inside the shuttle is free falling with the shuttle. The astronaut and shuttle have the *same acceleration* (falling is mass-independent). The normal force acted upon the astronaut by the ground disappears, and the astronaut has a feeling of weightlessness.[13] It is however different from the real weightlessness, which means no (net) gravitational force.

✓ Similarly, the orbit of the Moon is a result of free-falling of the Moon by the gravitational pull of the Earth.

Kepler's laws of planetary motion: Derived

The first law of planetary motion consists of two parts: (1) Planets move *around the Sun* and (2) in *elliptical orbits*. Over 99.8% of total mass in the entire Solar System concentrates in the Sun; hence, by Newton's laws, the inertia of the Sun is so huge that it hardly moves. Assume that the Sun is stationary. (We shall see that this assumption is not strictly correct. See "*Kepler's laws of planetary motion: Revisited*" below.)

Kepler's first law:

Let $\vec{r}$, $\vec{v}$ and $\vec{a} \equiv d\vec{v}/dt$ be the position vector, velocity, and acceleration of the planet respectively (Fig. 4-8). By Newton's second law and the law of gravitation, the equation of motion of a planet (mass m) due to the gravity of the Sun (mass M) is

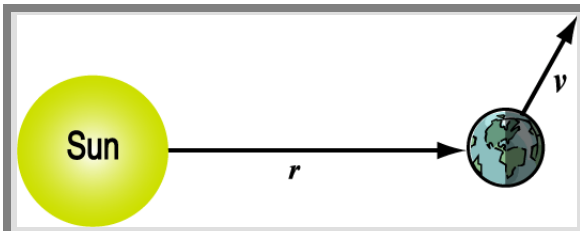


Fig. 4-8: The angular momentum of the planet is defined as $\vec{L} = m\vec{r} \times \vec{v}$, where m is the mass of the planet.

$$m\vec{a} = -\frac{GMm}{r^2}\hat{r}. \qquad \text{Eq. (1)}$$

The angular momentum $\vec{L}$ of the planet is defined as

$$\vec{L} = \vec{r} \times \vec{p} = mr\hat{r} \times \frac{d(r\hat{r})}{dt} = mr^2\hat{r} \times \frac{d\hat{r}}{dt} \qquad (\text{by } \hat{r} \times \hat{r} = 0)$$

$$\Rightarrow \vec{L} = mr^2\hat{r} \times \frac{d\hat{r}}{dt}. \qquad \text{Eq. (2)}$$

[13] The sensation of weight comes from our interaction with the environment. Image what happen if the astronaut stands on a weighing scale. Without normal force, the force that the astronaut acts on a scale is zero (Newton's third law), therefore the reading is zero.

Combining Eqs. (1) and (2), we have

$$\vec{a} \times \vec{L} = -GMm\hat{r} \times \left(\hat{r} \times \frac{d\hat{r}}{dt}\right) = -GMm\left[\left(\hat{r} \cdot \frac{d\hat{r}}{dt}\right)\hat{r} - \frac{(\hat{r} \cdot \hat{r})d\hat{r}}{dt}\right] = GMm\frac{d\hat{r}}{dt}, \qquad \text{Eq. (3)}$$

where we've used $\hat{r} \cdot (d\hat{r}/dt) = (1/2)[d(\hat{r} \cdot \hat{r})/dt] = 0$. Identifying the time-independent valuables,[14] we get

$$\frac{d}{dt}\left(\vec{v} \times \vec{L}\right) = \frac{d}{dt}(GMm\hat{r}) \;\Rightarrow\; \vec{v} \times \vec{L} = GMm\hat{r} + \vec{D}\,, \qquad \text{Eq. (4)}$$

Where $\vec{D}$ is a constant vector and directed toward *perihelion* (the nearest point to the Sun).

Then we take the dot product of $\vec{r}$ of Eq. (4), the two sides are

LHS $= \vec{r} \cdot \left(\vec{v} \times \vec{L}\right) = (\vec{r} \times \vec{v}) \cdot \vec{L} = L^2/m$, and

RHS $= GMmr\hat{r} \cdot \hat{r} + \vec{r} \cdot \vec{D} = GMmr + rD\cos\theta = GMmr(1 + e\cos\theta)$,

where $e = D/(GMm)$. Rearranging the variables to get an equation describing the orbit

$$r = \frac{L^2/GMm^2}{1 + e\cos\theta}. \qquad \text{Eq. (5)}$$

Eq. (5) represents the equation of a conic section in a polar form. See ***Box 4.1***.

In particular, for a closed orbit $L^2/GMm^2 = a(1 - e^2)$, i.e., the angular momentum of the planet is equal to

$$L = m\sqrt{GMa(1 - e^2)}. \qquad \text{Eq. (6)}$$

Kepler's second law:

The second law of planetary motion: In Fig. 4-9, the area $\Delta\vec{A}$ swept out in a very small time interval Δt is equal to $\Delta\vec{A} = (1/2)\vec{r} \times \vec{v}\Delta t$, then $d\vec{A}/dt = [1/(2m)]\vec{r} \times m\vec{v}$, or

$$\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m} \qquad \text{Eq. (7)}$$

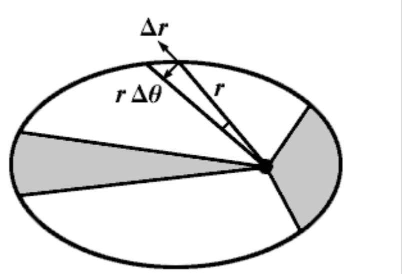


Fig. 4-9: The radius vector of a planet sweeps out a small area in Δt.

Therefore, $d\vec{A}/dt$ is constant as long as $\vec{L}$ is conserved. Hence, the second law is the conservation of angular momentum.

[14] Conservation of angular momentum of a system holds true in a central force field.

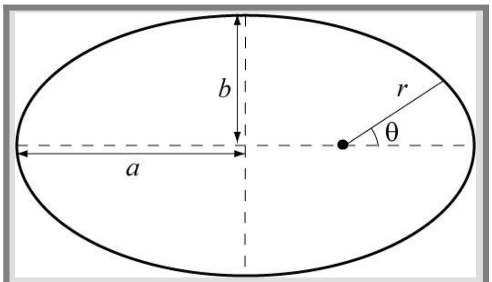
Box 4.1 Equations of conic sections

A conic section is the intersection of a plane and a conical surface (or the surface of two "tip-to-tip" cones). Depending on how the cone is sliced, the intersection can be an ellipse, a parabola or a hyperbola. A circle is a special case of an ellipse.

The equations of conic sections can be expressed in a polar form. For example, the equation of an ellipse is

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad (0 \le e < 1),$$

where a is the semi-major axis. It is left as an exercise to show that the total area of an ellipse is equal to $A = \pi ab$, where $b = a\sqrt{1-e^2}$ is called the semi-minor axis. The eccentricity e determines the shape of the ellipse. An ellipse with e = 0 is circular, and that with a large e is referred as highly elliptical.



An ellipse. The black dot is one of the two foci. Length *a* (*b*) is the length of semi-major (semi-minor) axis.

In polar form, the equation of a parabola can be written as

$$r = \frac{2r_s}{1+e\cos\theta} \qquad (e = 1),$$

with r_s being the shortest distance from the parabola to the only focus of the curve. The equation of a hyperbola is given by

$$r = \frac{a(e^2-1)}{1+e\cos\theta} \qquad (e > 1).$$

Kepler's third law:

Integrating Eq. (7), we get $\frac{A}{T} = \frac{L}{2m}$, where T is the orbital period of a planet. Together with Eq. (6) and the fact that the area of an ellipse equals $A = \pi a^2\sqrt{1-e^2}$, one can show without difficulty, $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$.

Kepler's laws of planetary motion: Revisited

The assumption of above derivation is that the Sun is immobile. However, this assumption is not strictly correct. The gravitational force between any two bodies is *mutual*, therefore the bodies are actually moving around their common ***centre of mass***.

✓ Suppose $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of m_1 and m_2 from their centre of mass, then $m_1\vec{r}_1 = -m_2\vec{r}_2$ (Fig. 4-10). Therefore,

$$\vec{r}_1 = -\frac{\mu}{m_1}\vec{r} \quad \text{and} \qquad \text{Eq. (8a)}$$

$$\vec{r}_2 = -\frac{\mu}{m_2}\vec{r}, \qquad \text{Eq. (8b)}$$

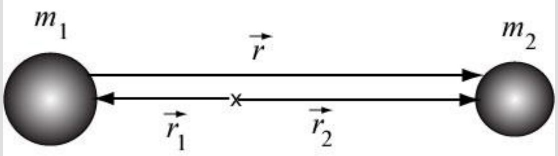


Fig. 4-10: The centre of mass of a two-body system.

with $\mu \equiv m_1 m_2/(m_1 + m_2)$ being the *reduced mass*.

✓ The total angular momentum is $\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2$. Using Eqs. (8a) and (8b), it is easy to show that

$$\vec{L} = \mu\vec{r} \times \vec{v}. \qquad \text{Eq. (9)}$$

The equations of motion of m_1 and m_2 under gravitational force, respectively, are

$$\frac{Gm_1m_2}{r^2}\hat{r} = m_1\vec{a}_1 \quad \text{and} \quad -\frac{Gm_1m_2}{r^2}\hat{r} = m_2\vec{a}_2,$$

which can be reduced to

$$-\frac{GM\mu}{r^2}\hat{r} = \mu\vec{a}, \qquad \text{Eq. (10)}$$

where $M = m_1 + m_2$ is the *total mass*. In general, the two-body problem may be treated as an equivalent one-body problem with the reduced mass μ moving about a fixed mass M at a distance r.

Kepler's first law: revisited

Since the two-body problem has been reduced to an equivalent one-body problem, it is natural that Eqs. (9) and (10) are identical to Eqs. (2) and (1), except for the different masses being used. Therefore, the locus $\vec{r}$ (i.e., the relative motion of m_2) is a conic section. Since $\vec{r}_1$ and $\vec{r}_2$ are proportional to $\vec{r}$, both objects in a binary orbit move about the centre of mass in a conic section, with the centre of mass occupying one focus of each path.

For example, the Sun m_1 is much more massive than the Earth m_2, and the centre of mass lies close to the solar centre. The result is that the Sun only ″wobbles″ a little, while the Earth appears to move around the Sun.

Kepler's second law: revisited

Again, the equations are identical, and the position vector $\vec{r}$ sweeps across equal areas in equal intervals of time.

Kepler's third law: revisited

In the two-body problem, the third law becomes $\frac{a^3}{T^2}=\frac{G(m_1+m_2)}{4\pi^2}$. Since each planet has different mass m_2, the square of the orbital period T actually is not proportional to the cube of the planet's average distance a from the Sun. The original version of the Kepler's third law is only a very good approximation because the Sun is massive compared to all planets.

Trajectory of projectile motion

At the perihelion of a closed (or bound) planetary orbit, $\theta=0$ and $r=a(1-e)\equiv r_p$, Eqs. (6) and (9) give

$$L=\mu\sqrt{GMa(1-e^2)} \quad \text{and} \quad L=\mu r_p v_p$$

$$\Rightarrow v_p^2=\frac{GM(1+e)}{a(1-e)}. \qquad \text{Eq. (11)}$$

Similar equations can be easily derived for an open (or unbound) orbit. In general, the eccentricity and thus the shape of an orbit is related to the speed at the nearest point. It also explains how the shapes of projectile trajectory change with launch speed (Fig. 4-11).

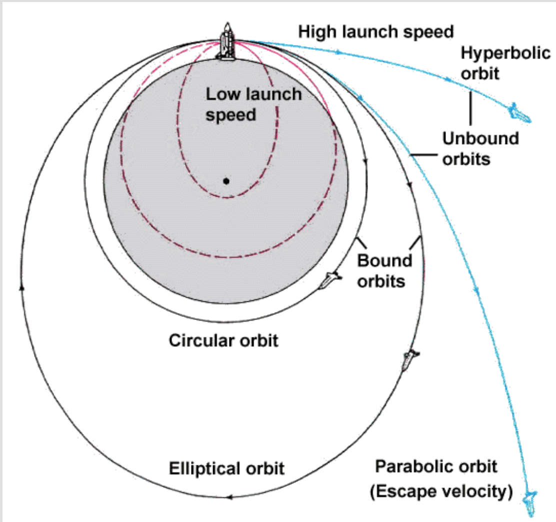


Fig. 4-11: The trajectories of projection depend on the initial velocity.

Energy of a two-body system

The total mechanical energy [15] of the system is equal to $E=\frac{1}{2}m_1{v_1}^2+\frac{1}{2}m_2{v_2}^2-\frac{Gm_1m_2}{r}$. In terms of the reduced mass, the energy can be written as

[15] We would not go into details about how energy is defined, and how we know that energy is conserved. Once you have established the idea of energy, the expression of kinetic energy K.E. can be found by integrating the work done $\vec{F}\cdot d\vec{r}$ using Newton's second law, the expression of potential energy P.E. can be found by integrating the work done using the Law of universal gravitation.

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} . \qquad \text{Eq. (12)}$$

Combining Eqs. (11) and (12), the energy at the perihelion is

$$E = -\frac{GM\mu}{2a} = -\frac{Gm_1 m_2}{2a} , \qquad \text{Eq. (13)}$$

That means the total energy in a two-body system is independent of the shape of the orbit, i.e., eccentricity *e*. In other words, the orbit of an object can change significantly without any additional energy. For example, an asteroid of nearly circular orbit can deform its orbit and becomes a highly elliptical orbit and overlapping with that of the Earth.

Be aware of the assumptions behind Eq. (13) when you apply it. For example, notice that we are only considering the potential energy and the translational kinetic energy in deriving Eq. (13). In the Moon-Earth system, the tides increase the Moon's energy by drawing from the rotational kinetic energy of the Earth's spin. Thus Eq. (13) is not applicable.

Question: What is the minimum speed required for an object to escape from the Earth's gravitational attraction?

Solution: For an object with just enough kinetic energy to overcome the gravitational potential energy, we have $\frac{1}{2}v_{esc}^2 - \frac{GM}{R} = 0$, where R is the radius of the Earth. Hence, the **escape velocity** [16] is $v_{esc} = \sqrt{\frac{2GM}{R}}$. It is actually identical to the shooting speed of the parabolic projectile in Fig. 5-9. Therefore from the point of view of the trajectory, it is the minimum speed to achieve an unbound orbit.

Substituting the constants in the equation, the escape velocity of the Earth is $v_{\mathrm{esc}} = 11.2\,\mathrm{km/s}$.

[16] Some aerospace science/engineering books call it the second cosmic velocity (第二宇宙速度), which means the escape velocity from the Earth. The term "cosmic velocity" is not as common in the fields of Physics and Astronomy, and we will not use the term in this course.

Box 4.2 The virial theorem

In the calculation of the energy in an orbit, you may notice that the total energy is exactly half of the gravitational potential energy. It is actually a general property of a gravitationally bound system in equilibrium. The time-averaged kinetic energy $\langle K \rangle$ is related to the time-averaged potential energy $\langle U \rangle$ by the **virial theorem**

$$2\langle K \rangle + \langle U \rangle = 0 .$$

Here only the simplest form of the virial theorem is shown. The theorem is very general, and it can be applied to a lot of astronomical systems.

4.5 Planetary Configurations

The orbits of many planets in our Solar System are *nearly coplanar*, *concentric circles*, so the simple Copernican model can explain many planetary phenomena [17] as seen on Earth.

- ✓ **Inferior planets:** Planets whose orbits lie inside that of the Earth (Mercury and Venus). As observed from the Earth, they go through *complete* phase cycles (from new to full). Inferior planets always appear close to the Sun, observed in the morning (morning stars) and evening (evening stars).
- ✓ **Superior planets:** Planets whose orbits lie outside that of the Earth (Mars, Jupiter, etc.)
- ✓ **Elongation** ψ is the angle between the Sun, the Earth and the planet, i.e., ∠ Sun-Earth-Planet; whereas the **phase angle** ϕ is the angle between the Sun, the planet and the Earth, i.e., ∠ Sun-Planet-Earth.

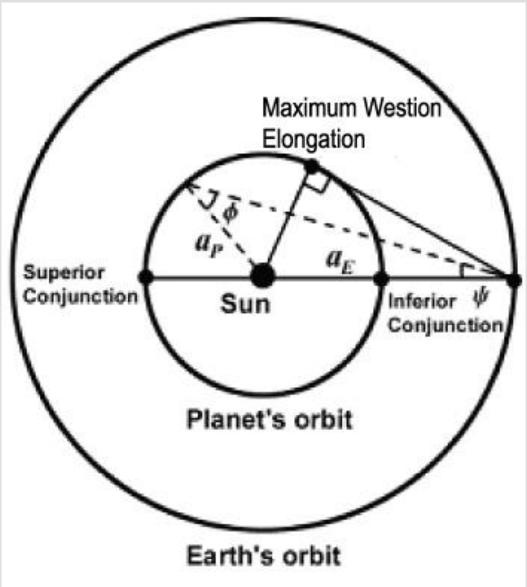


Fig. 4-12: Planetary configurations for an inferior planet.

By sine rule, we have $\sin\phi = \dfrac{\alpha_E}{a_P}\sin\psi$ for both inferior and superior planets.

In addition, there are some special configurations for *inferior planets* (Fig. 4-12), namely

[17] As explained, the retrograde motion of planet can be explained naturally in Copernican model. Discussion can be found in section 12.9 of Roy & Clark, *Astronomy: Principles and Practice (4th ed.)*, 2003.

- ✓ *Inferior conjunction* ($\psi = 0^o$, $\phi = 180^o$), solar transit may occur;
- ✓ *Superior conjunction* ($\psi = 0^o$, $\phi = 0^o$) and
- ✓ *Maximum elongation* ($\psi = \psi_{max}$, $\phi = 90^o$). **Maximum eastern elongation** - The planet is at its greatest angular separation east from the Sun. At an eastern elongation the planet sets after the Sun. It is thus an evening star. **Maximum western elongation** - The planet is at its greatest angular separation west from the Sun. At a western elongation the planet rises before the Sun. It is thus a morning star.

Special configurations for *superior planets* (Fig. 5-11):

- ✓ *Opposition* ($\psi = 180^o$, $\phi = 0^o$),
- ✓ *Conjunction* ($\psi = 0^o$, $\phi = 0^o$) and
- ✓ *Quadrature* ($\psi = 90^o$, $\phi = \phi_{max}$). **Eastern quadrature** - The planet is 90° east of the Sun which occurs a few months after opposition and is an evening object. **Western quadrature** - The planet is 90° west of the Sun which occurs a few months before opposition and is a morning object.

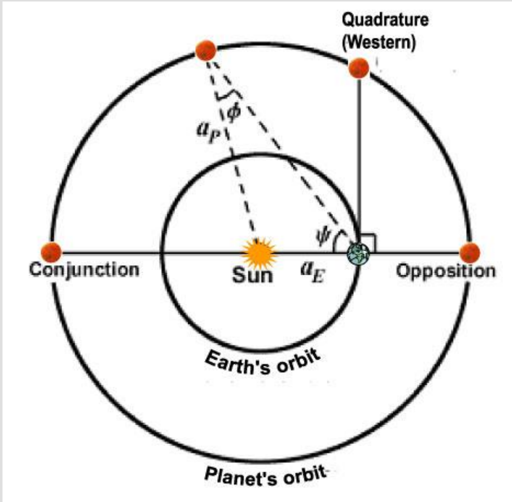


Fig. 5-11: Planetary configurations for a superior planet.

- ✓ **Synodic period:** The time interval between successive identical configuration of a planet, the Earth, and the Sun. In Fig. 5-12, Planet 1 is moving faster than Planet 2. In one *synodic period*, Planet 1 goes through $360^o + \theta$; whereas Planet 2 goes through θ.
- ✓ **Sidereal periods:** The orbital periods of the planets. Let T_1 and T_2 are, respectively, the orbital periods of planets of **1** and **2**. So the orbital angular speeds are, respectively, $\dfrac{360^o}{T_1}$ and $\dfrac{360^o}{T_2}$. After a synodic period S, planet **1** has moved an angle $360^o + \theta$; whereas planet **2** has moved an angle θ. Therefore,

$$\frac{360^o}{T_1}S - \frac{360^o}{T_2}S = 360^o, \quad \text{or} \quad \frac{1}{T_1} - \frac{1}{T_2} = \frac{1}{S}.$$

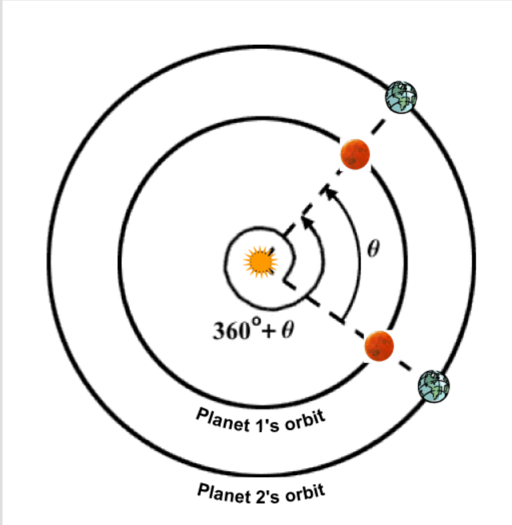


Fig. 5-12: After one synodic period, Planet 1 has moved an angle of 360° + θ; whereas Planet 2 moved an angle of θ.

If the orbital period of the Earth $= T_E$, then the synodic period S of another planet is given by $\boxed{\dfrac{1}{S} = \left|\dfrac{1}{T_E} - \dfrac{1}{T}\right|}$ for *both* inferior and superior planets.

How about if Planet 1 and 2 are moving in opposite direction?

✓ **Phase** q: the phase is defined as the fraction of the planet′s diameter illuminated as seen on Earth. In Fig. 5-13,

$$q = \frac{CF}{CD} = \frac{CP + PF}{CD}$$
$$= \frac{1}{2} + \frac{1}{2}\frac{PF}{PB}$$
$$= \frac{1}{2}(1 + \cos\phi)$$
$$q = \frac{1}{2}(1 + \cos\phi).$$

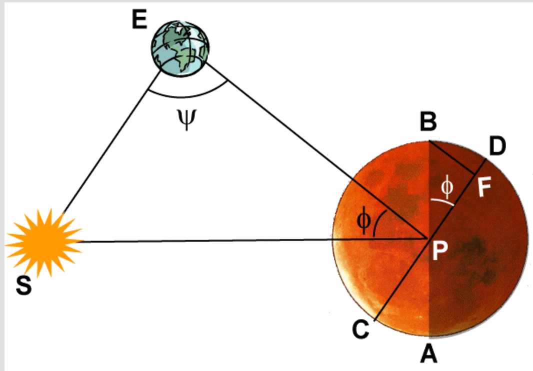


Fig. 5-13: The phase of a planet.

So the phase of a planet depends only on the phase angle.

For the inferior planets: At inferior conjunction, i.e., $\phi = 180^o$, then q = 0, it is ***new***; at superior conjunction, i.e., $\phi = 0^o$, then q = 1, it is ***full***; and for 0 < ϕ < 180º, 0 < q < 1, it is at ***intermediate phases***.

For the superior planets: At conjunction or opposition, i.e., $\phi = 0$, then q = 1, so it is ***full***; and for $\phi < 90^o$, then q > 1/2 it is ***gibbous***.

✓ **Solar Transit:** When the Sun, a planet and the Earth are almost in a straight line, you may see the shadow of the planet passes across the Sun in few hours. It is basically the planetary version of solar eclipse.[18] It occurs for inferior planets only.

Observation of the transit of Venus was historically an important tool for determining the astronomical unit (AU). Now it is superseded by more accurate method such as radar ranging.

Question: Estimate the *maximum* duration of transit of Venus across the Sun. Given that the apparent diameter of the Sun $= 32'$, the distance between Venus and the Sun = 0.723 AU, and the synodic period of Venus S = 584 days.

[18] In fact, the NASA Eclipse Web Site also provides the dates and other information of recent transits. See http://eclipse.gsfc.nasa.gov/transit/transit.html for details.

Solution: Assume the Earth's and Venus's orbit are circular. In Fig. 5-14, at time = t_1, the Earth led Venus by θ; at time = t_2, Venus led the Earth by θ. The maximum total time of transit is then $t = t_2 - t_1$. Since

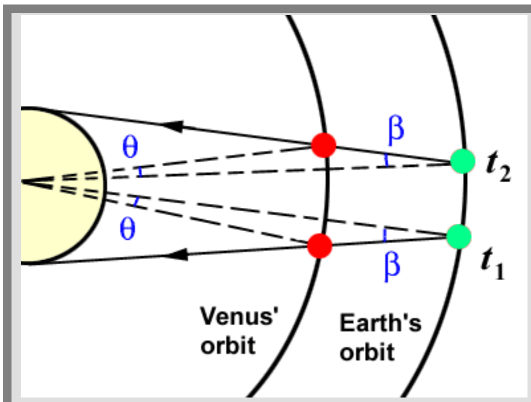


Fig. 5-14: Transit of Venus.

$$2\theta = \left(\frac{2\pi}{T_V} - \frac{2\pi}{T_E}\right)(t_2 - t_1) \Rightarrow 2\theta = \frac{2\pi t}{S}.$$

By sine rule, $\dfrac{\sin\beta}{\sin(\pi - \beta - \theta)} = \dfrac{0.723}{1}$ and $\sin\beta \approx \beta$ for small β. We then have

$$\frac{\beta}{\beta + \theta} \approx 0.723, \text{ and } 2\theta = \frac{2\beta}{0.723} - 2\beta.$$

$$\frac{2\pi t}{S} = \frac{2\beta}{0.723} - 2\beta, \text{ and } 2\beta = 32' = \frac{32}{60} \times \frac{\pi}{180}, \text{ so}$$

Therefore, $t = \dfrac{S}{2\pi}\left(\dfrac{2\beta}{0.723} - 2\beta\right) = 0.33\,\text{days} \approx 8\,\text{hrs}$

The maximum duration of transit of Venus across the Sun is about 8 hours.