Chapter 8 Structure of stars

8.1 The main sequence

A protostar becomes a main-sequence star when nuclear reactions ignite at its centre. The star settles on the *lower left side* of the main sequence, which is called the **zero-age main sequence (ZAMS)**. (Fig. 8-1)

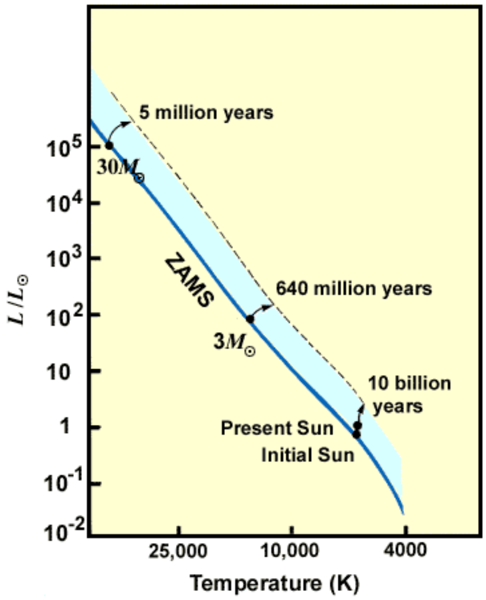

Fig. 8-1: As a main sequence star evolves, it will move to the upper-right the H-R diagram.

Main-sequence stars are very stable. Nuclear fusion of hydrogen generates energy steadily at their centres. The radiation and gas pressures counterbalance gravitational force, and the stars are in equilibrium. A star will spend about 90% of complete evaluation on the main sequence, so it is not surprised to find that most stars in the sky are main-sequence stars.

The main sequence in the Hertzsprung–Russell diagram (H-R diagram) is *not* a line but a *band* of finite width. As a main-sequence star evolves, it moves to the upper-right direction across the main sequence. (Fig. 8-1) The heuristic arguments are as follows:

As hydrogen fuel is consumed in the core of a main sequence star, the number of particles in the core decreases gradually, for example, a pp chain nuclear reaction converts 4 hydrogen nuclei into 1 helium nucleus (See "Physics of stars" below). The core is then decreased in pressure and hence shrinks. The contraction of the core leads to an increase in its temperature and density. As a result, the nuclear reactions in the core are even more violently and rapidly, and hence more energy is released, and the star becomes more luminous, i.e., the star moves upwards in the H-R diagram. In addition, high pushing-out radiation pressure causes the outer layers of the main sequence star to expand. Therefore, the star is getting larger, and its surface temperature decreases due to the expansion of the outer layers, i.e., the star moves towards the right in the H-R diagram.

In conclusion, a main-sequence star will move to the upper-right direction across the main sequence as it ages and evolves, and in addition, the core shrinks gradually but the outer layers expand.

8.2 Physics of stars

Virial theorem

The theorem states that for a gravitationally bound or a periodic system, the condition for the system in equilibrium is:[1]

$$2\langle K\rangle + \langle U\rangle = 0,$$

where $\langle K\rangle$ and $\langle U\rangle$ are, respectively, the time average kinetic energy and the time average gravitational potential energy of the system in concern. The virial theorem applies to a wide variety of systems, from an ideal gas to a cluster of galaxies. For instance, consider the case of a static star. In equilibrium a star must obey the virial theorem.

Now, assume that a nebula (a cloud of virial interstellar medium) is at quasi-hydrostatic equilibrium, i.e., it contracts so slowly that the theorem holds approximately. As the star is contracting, some potential energy is converted to kinetic energy. Mathematically, As the potential energy is decreasing, i.e., $\langle U\rangle$ will take more negative and its magnitude increased. Hence, the kinetic energy $\langle K\rangle = -\frac{1}{2}\langle U\rangle$ is getting larger, thereby heating up the nebula. On the other hand, the nebula loses energy since the total energy $\langle E\rangle = \langle K\rangle + \langle U\rangle = -\frac{1}{2}\langle U\rangle + \langle U\rangle = \frac{1}{2}\langle U\rangle$ decreases. That means, as the nebula is contracting, it will be heated up and radiating electromagnetic radiations.

[1] The proof of the virial theorem was demonstrated in Karttunen, K., et al., *Fundamental Astronomy* (New York: Springer, 2007), pp.124-125.

Next, it can be shown that the gravitational potential energy U of a uniform sphere of is written as $U = -\frac{3}{5}\frac{GM^2}{R}$, where M and R are, respectively, the mass and the radius of the sphere.[2] Applying the virial theorem, the total energy of the sphere is therefore

$E = \frac{1}{2}U = -\frac{3}{10}\frac{GM^2}{R}$. Suppose that the Sun was formed by a contracting spherical nebula and that the radius of the nebula is much larger than that of the Sun and hence the gravitational potential energy. Then, the energy radiated away during the collapse of the nebula would be $\Delta E = E_i - E_f \approx -E_f = \frac{3}{10}\frac{GM^2}{R} = 1.1 \times 10^{41}$ J, where E_i and E_f are the gravitational potential energy of the initial nebula and the Sun respectively. Further assume that the luminosity of the Sun has been roughly constant, $L = 3.8\times10^{26}$ W, throughout its lifetime, it could emit at that rate for approximately $t_{KH} = \frac{\Delta E}{L} \sim 10^7$ years. This is known as the Kelvin-Helmholtz timescale. However, based on radioactive dating techniques, the estimated age of rocks on the Moon's surface is over 4×10^9 years. It seems unlikely that the age of the Sun is less than the age of the Moon. Therefore, gravitational potential energy alone cannot account for the Sun's luminosity throughout its entire lifetime. This calls for another energy source, and a possible candidate is nuclear fusions

Nuclear fusion

Two major types of nuclear fusion in main sequence stars are introduced below, namely the **proton-proton (PP) chain** reaction and **carbon-nitrogen-oxygen (CNO) cycle**.

PP I involves a reaction sequence that ultimately results in

$4\ ^1H \rightarrow {}^2He + 2\ e^+ + 2\ \nu_e + 2\ \gamma$

through the intermediate production of deuterium (^2H) and helium-3 (^3He). The entire PP I reaction chain is

$^1H + {}^1H \rightarrow {}^2H + e^+ + \nu_e$

$^2H + {}^1H \rightarrow {}^3He + \gamma$

$^3He + {}^3He \rightarrow {}^4He + 2\ ^1H$

The production of helium-3 nuclei in the above chain also provides for the possibility of other interaction, resulting in branches of the proton-proton chain. Fig. 8-2 shows the three

[2] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), pp.296 - 297.

branches of the proton-proton (PP) chain, along with their branching ratios appropriate for the conditions in the core of the Sun.

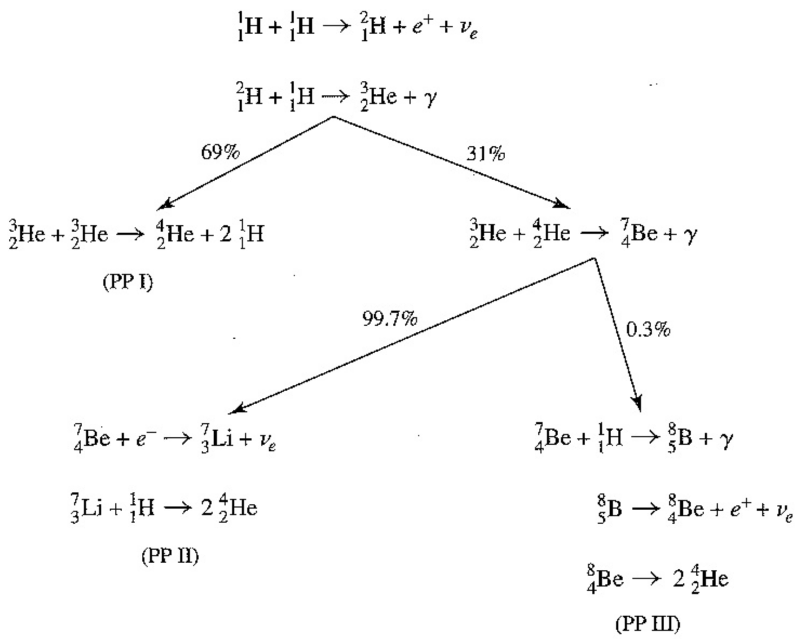

Fig. 8-2: The three branches of the pp chain, along with the branching ratios appropriate for conditions in the core of the Sun. (Adopt from Bradley, p.310)

According to the Einstein's formula for mass-energy equivalence, $E = \Delta mc^2$, some mass is converted to energy during nuclear fusion. The total mass of 4 ^{1}H nuclei is 6.693×10^{-27} kg and that of a ^{4}He nucleus is 6.645×10^{-27} kg. The mass loss during the fusion is $\Delta m = 0.048\times10^{-27}$ kg. Therefore, the energy release per reaction chain is

$E = \Delta mc^2 = 4.32\times10^{-12}$ J. In other words, about 0.7 % of the mass of hydrogen would be converted to energy in forming helium.

Is this source of nuclear energy sufficient to power the Sun during its lifetime? Assume that the Sun was originally 100% hydrogen and that only the inner 10% of the Sun's mass become hot enough to convert hydrogen into helium. The amount of energy available in the Sun would be $E_{nuclear} = 0.1 \times 0.007 \times M_0 \times c^2 = 1.26 \times 10^{44}$ J, where the solar mass $M_0 = 2 \times 10^{30}$ kg. In terms of solar luminosity L, this gives a nuclear timescale of approximately $t_{nuclear} = \frac{E_{nuclear}}{L} \sim 10^{10}$ years, more than enough time to account for the age of the Moon rocks. As a matter of fact, about 90 % of the solar energy is generated in this way. In the Sun, there are $m = \frac{L}{c^2} = 4.2 \times 10^9$ kg, equal to 4.2 million tons, of matter converted into energy per second.

Furthermore, the rate of pp chain is proportional to T^4 at a temperature near 1.5×10^7 K.[3] For instance, only 10 % increase in temperature will bring about a more vigorous rate by about 50 %.

Besides electromagnetic radiations, the pp chain also creates **positron** and **neutrino**. Positron (e^+) is an antiparticle of electron. A positron is identical to an electron in every aspect, except that it carries positive charge. When a positron annihilates with an electron, electromagnetic radiations are produced, i.e., $e^+ + e^- \rightarrow 2\gamma$. Another by-product of the fusion is neutrino (ν) which is a highly penetrating neutral particle and thus interacts with matter very weakly. Almost 10^{38} neutrinos produced every second at the solar core leave the Sun without colliding with any matter. Up until year 2002, the amount of solar neutrinos measured on Earth was about 1/3 of what was expected. The so-called *solar neutrino problem* has since been solved. The discrepancy was due to the fact that the solar neutrinos change from one type to another. The effect is known as neutrino oscillation. More details can be found in Kaufmann's *Universe*.[4]

Carbon-Nitrogen-Oxygen (CNO) cycle is an alternative series of reactions fusing 4 hydrogen nuclei into a helium nucleus:

$${}^{12}C + {}^{1}H \rightarrow {}^{13}N + \gamma$$

$${}^{13}N \rightarrow {}^{13}C + e^+ + \nu_e$$

$${}^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma$$

$${}^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma$$

$${}^{15}O \rightarrow {}^{15}N + e^+ + \nu_e$$

$${}^{15}N + {}^{1}H \rightarrow {}^{4}He + {}^{12}C$$

In the CNO cycle, carbon-12 serves as a catalyst to activate the reaction, and the reaction rate is highly sensitive to temperature and proportional to $T^{19.9}$ at a temperature near 1.5×10^7 K.[5] The rate will increase, for example, by more than 650 % if the temperature increases by 10 %

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[3] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), p.311.

[4] Kaufmann. J. William, and Freedman A. Roger, *Universe* (New York: W.H. Freeman and Company, 1999), pp.445 - 447.

[5] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), p.312.

only. As shown by the power dependence, the CNO cycle is much more strongly temperature-dependence than is the pp chain. This property implies that low-mass stars, which have smaller central temperature, are dominated by the pp chains during their "hydrogen burning" evolution, whereas more massive stars, with their higher central temperatures, convert hydrogen to helium by the CNO cycle. The transition in stellar mass between stars dominated by the pp chain and those dominated by the CNO cycle occurs for stars slightly more massive than the Sun.[6]

Hydrostatic equilibrium

Hydrostatic equilibrium represents one of fundamental equations of stellar structure for spherically symmetric objects under the assumption that accelerations are negligible. Consider a thin spherical shell of mass dm of radius r, the gravitational force on the shell by an enclosed mass M_r is $dF = \frac{GM_r}{r^2}dm$. Rewrite dm in terms of the density of the shell ρ, i.e., $dm = 4\pi r^2 \rho dr$. For a star at equilibrium, there must be some pressures in the star to counterbalance the gravitational force. (Fig. 8-3) The counterbalance force due to pressure difference at r and $r + dr$ is given by $[-P(r+dr) + P(r)] \times 4\pi r^2$. By equating the gravitational force and the pressure forces, we have

$$\frac{dP}{dr} = -\rho\frac{GM_r}{r^2}.$$

This is called the equation of hydrostatic equilibrium. The negative sign indicates that the pressure increases towards the core.

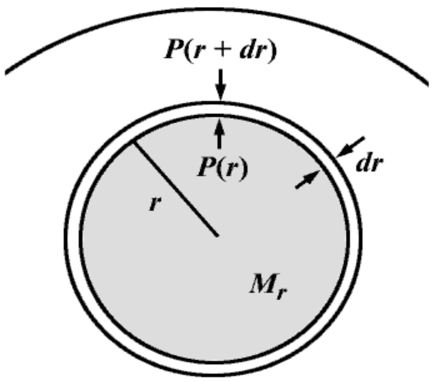

Fig. 8-3: Divide a star into many thin shells and consider the force balance on one of them.

[6] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), p.312.

A crude estimate of the pressure at the centre of the Sun can be obtained by applying the equation of hydrostatic equilibrium. Write $\frac{dP}{dr} \sim \frac{P_s - P_c}{R_s - 0} \sim -\frac{P_c}{R_s}$, where P_c is the central pressure, P_s and R_s are, respectively, the surface pressure and radius. Substituting into the equation of hydrostatic equilibrium and solving for the central pressure, we have $P_c \sim \frac{GM\rho}{R_s} \sim 2.7 \times 10^{14}$ N m^{-2}. To obtain a more accurate value, we need to integrate the hydrostatic equilibrium equation form the surface to the centre, taking into consideration the change in the interior mass M_r at each point, together with the variation of density ρ with radius, giving $\int_{P_s}^{P_c} dP = P_c = -\int_{R_s}^{R_c} \rho \frac{GM_r}{r^2} dr$. Actually, carrying out the integration requires functional forms of M_r and ρ. Such explicit expressions call for further development of relationships among these quantities. From a more rigorous calculation, a standard solar model gives a central pressure of about 2.23×10^{16} N m^{-2}. [7]

Ideal gas law

The core of a star is so hot that all atoms are completely ionized, in a state called **plasma**. The ions and electrons are in rapid random motions. The average kinetic energy of the ions and electrons is much greater than the coulomb energy among the charges. Therefore, it is reasonable to assume that the ideal gas law holds, and the empirical formula reads

$$PV = nRT,$$

where n is the number of moles of the particles, and the gas constant $R = 8.314$ J K^{-1} mol^{-1}. By applying the kinetic theory, the ideal gas law can be expressed in terms of the density of the particles ρ and the average mass of the particles $\bar{m}$, and read: [8]

$$P = \frac{\rho}{\bar{m}} kT,$$

where the Boltzmann constant $k = 1.380 \times 10^{-23}$ J K^{-1}. Note that if the star is spherically symmetric, then ρ, T, as well as P are functions of distance from the star's core only. In addition, the average kinetic energy per particle is given by

[7] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), p.287.

[8] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), pp.288 – 291, 294.

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT.$$

Note that $\overline{m}$ is the average mass of all particles. For example, the average mass of pure hydrogen atom is about the mass of a proton $m_p = 1.67 \times 10^{-27}$ kg. In cases where the hydrogen is completely ionized (in a state of plasma), each hydrogen atom contributes 1 hydrogen ion and 1 electron, hence $\overline{m} = \frac{m_p + m_e}{2} \approx 0.5m_p$. In addition, it has been shown that for completely ionized gas the average mass of all particles can be evaluated through: [9]

$$\frac{1}{\mu} \simeq 2X + \frac{3}{4}Y + \frac{1}{2}Z,$$

where $\mu \equiv \frac{\overline{m}}{m_p}$, m_p is the mass of proton, *X* and *Y* are, respectively, the mass fraction of hydrogen and helium, and *Z* is the mass fraction of all elements heavier than helium. For instance, the Sun is by mass composed of 71% of hydrogen, 27% of helium, and 2% of all elements heavier than helium. With *X* = 0.71, *Y* = 0.27, and *Z* = 0.02, then $\overline{m} \simeq 0.62m_p$.

Last, the central temperature can be estimated by applying the ideal gas law, we have $T = \frac{P\overline{m}}{\rho k} = \frac{2.7\times10^{14}\times0.62m_p}{1408k} \sim 1.44 \times 10^7$ K. This result is in reasonable agreement with more detailed calculations. One standard solar model gives a central temperature of 1.57×10^7 K.[10]

Role of quantum effect in producing nuclear energy

The pp chain nuclear reaction takes place in the core of the Sun. Since the nuclear size is of an order of 10^{-15} m, the positively charged protons must overcome a potential barrier one another at a distance of about 10^{-15} m, so that the protons are close enough to fuse together. Classically, the protons must have sufficient kinetic energy to overcome this barrier, i.e., $\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT_{classical} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$, it implies $T_{classical} = \frac{e^2}{6\pi\epsilon_0 kr} \sim 10^{10}$ K. However, the central temperature of the Sun is about 10^7 K only, much lower than required here. A very small fraction of particles has enough energy thermal energy to undergo fusion. Thanks is given to quantum mechanics, according to the uncertainly principle, a proton has a nonzero *probability*

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[9] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), pp.291 - 293.

[10] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), p.296.

to *tunnel* through the potential barrier even if it does not have enough energy. (Fig. 8-4) The complete derivation is out of scope of this course. As a crude estimate of the effect of tunnelling on the temperature necessary to sustain nuclear reactions, assume that a proton must be within approximately one de Broglie wavelength of its target in order to tunnel through the Coulomb barrier. [11] The wavelength of a particle such as proton is given by

$\lambda = \frac{h}{p}$, where λ and p are, respectively, the wavelength of the momentum of the particle, and the Planck constant $h = 6.626 \times 10^{-34}$ J s. Since the kinetic energy of the two-proton system can be expressed in terms of their momentum and hence their wavelengths as

$\frac{1}{2}m\overline{v^2} = \frac{p^2}{2m} = \frac{(h/\lambda)^2}{2m}$, where the reduced mass $m = m_p/2$, and m_p is proton mass. Solving $\frac{(h/\lambda)^2}{2m} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\lambda}$ for λ and then substituting $r = \lambda$ into $\frac{3}{2}kT_{quantum} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$. We have $T_{quantum} = \frac{e^4 m}{12\pi^2\epsilon_0^2 h^2 k} \sim 10^7$ K. This crude estimate is indeed consistent with the central temperature calculated by some sophisticated solar models.

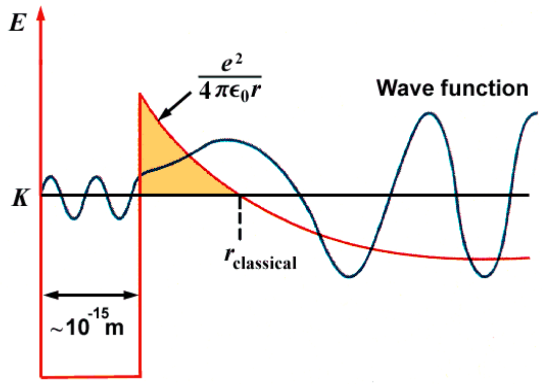

Fig. 8-4: Tunnelling of the Coulomb potential barrier.

Mass-luminosity relation

Years of careful astronomical observations of binaries have yielded the masses of many stars. For main-sequence stars, there is a direct correlation between mass and luminosity – the more massive a star, the more luminous it is. (Fig. 8-5) Measuring the slope of the curve yields **the mass-luminosity relation:**

[11] Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007), pp.300 - 302.

$$L = kM^{3.5},$$

where k is a constant. This relationship applies to main-sequence stars in the range 0.1 - 50 solar masses. [12]

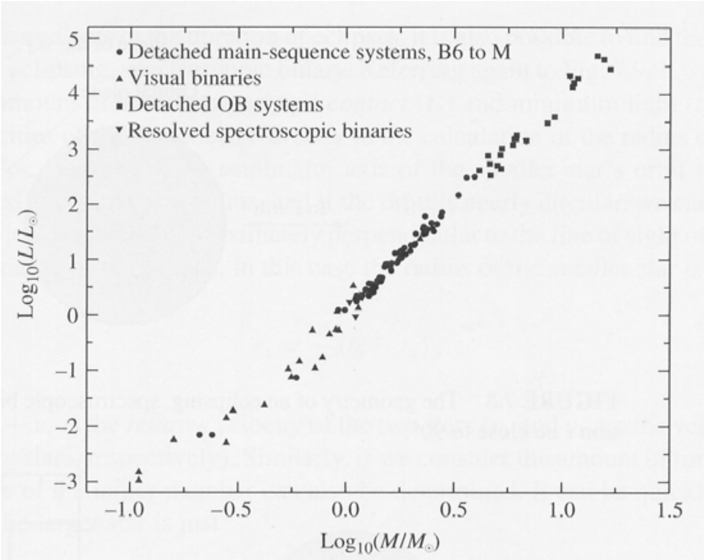

Fig. 8-5: The mass-luminosity relation of main-sequence stars. (Adapted from Popper, *Annu. Rev. Astron. Astrophys.*, **18**, 115, 1980)

Furthermore, a massive star consumes hydrogen fuel *much* more rapidly, so it would have *much* shorter life span on the main sequence. We shall expect to find more low-mass stars in space. By Einstein's equation of mass and energy equivalence, the lifetime of a star is given by $\tau = \frac{\Delta M c^2}{L} \propto \frac{M}{M^{3.5}} = M^{-2.5}$. In terms of solar mass, we have

$$\tau = \tau_0 \left(\frac{M_0}{M}\right)^{2.5},$$

where M_0 is the solar mass and $\tau_0 = 10^{10}$ years (10 billion years) is the solar lifespan. For example, the life span of a 10-solar-mass star on the main sequence is about 3×10^7 years. For a 20-solar-mass star, it only lives for a few million years. On the other hand, low-mass main-sequence stars (red dwarfs) live for about 200 to 300 billion years!

[12] *Wikipedia*, "Main sequence" (Retrieved: 13 May 2018, at 06:45).

Transportation of energy

The energy generated in the core of a star can be transported to the surface by **radiation** (energy carried by EM radiations), **convection** (gas rises and cooler gas sinks), and **conduction** (energy exchange during collisions of particles).

Stellar models show that the mechanism of energy transport from the inner core to the surface of the star depends on the mass of the star.[13] (Fig. 8-6) Greater mass of a main-sequence star implies greater pressure and temperature in the interior. Such a great temperature gradient causes convection deep in the star's interior. By contrast, a massive star's outer layers are of low density that energy flows through them more effectively by radiation than by convention. (Fig. 8-6 a) On the other hand, less massive main-sequence stars such as the Sun have an inner radiative zone and an outer convective zone. (Fig. 8-6 b)

The internal structure is different for main-sequence stars of very low mass. The interior temperature is not high enough to ionize the interior. The radiations can flow through easily, so energy is transported by convection throughout the volume of the star. (Fig. 8-6 c)

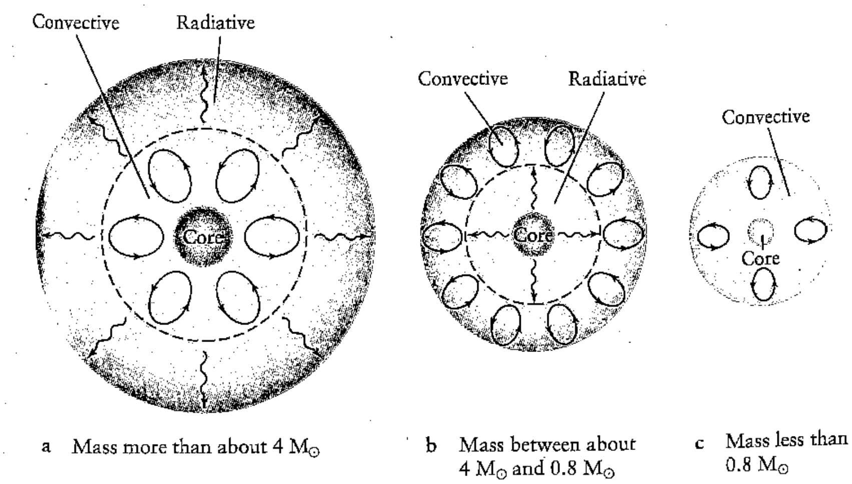

Fig. 8-6: Mechanism of energy transports. (a) In stars with masses in excess of about 4 solar masses, they have a convective core and a radiative outer-layer zone. (b) If the star's mass is the approximate range from 4 solar masses to 0.8 solar masses, it has a radiative core and a convective outer-layer zone. (c) Stars with masses below about 0.8 solar masses, energy is transported by convection throughout. (Adapted from Kaufmann, p.501)

[13] Kaufmann. J. William, and Freedman A. Roger, *Universe* (New York: W.H. Freeman and Company, 1999), p.501.

Pressure-temperature thermostat

It is a process to balance the contraction due to gravity and thermal pressure in a star. If the nuclear reactions slow down, the temperature will decrease, and so the thermal pressure will drop. The star contracts due to gravity. The compression results in higher temperature, which increases the rate of nuclear energy generation.

On the other hand, if the nuclear reactions are too fast, the temperature will increase, and so the thermal pressure will rise. Therefore, the stellar core expands. The expansion lowers the density of the core, and the core cools down. Thus, the nuclear reactions rate slows down.

Reference

Karttunen, K., et al., *Fundamental Astronomy* (New York: Springer, 2007).

Bradley W. Carroll, Dale A. Ostlie, *An Introduction to Modern Astrophysics* (San Francisco: Pearson, 2007).

Kaufmann. J. William, and Freedman A. Roger, *Universe* (New York: W.H. Freeman and Company, 1999).