Chapter 11 Relativity

11.1 Historical background: classical physics in crisis

- ✓ There were lots of scientific progresses between year 1600 and 1900. Kepler deduced a unified theory to describe the motion of all planets. Newton further unified the underlying principles behind the motions of all objects (i.e. both celestial and “earthly”). Theories of Classical Mechanics are well developed and also verified by experiment and observation.
- ✓ In the 1800s, Thomas Young showed that light exhibited interference, and thus clearly demonstrated the wave nature of light. In the 1860s to 70s, James Clerk Maxwell constructed a unified theory of electricity and magnetism. The speed of the electromagnetic (EM) wave as predicted in Maxwell’s equations was equal to the independently measured speed of light. With both experimental proof and theoretical understanding, the long-standing argument about the true nature of light was once and for all settled (or so it seemed).
- ✓ There were also significant improvements in other fields such as thermodynamics, atomic physics, and chemistry. It was clear that the end of scientific discovery was within reach! It is said that Lord Kelvin (right, *that* Kelvin) pronounced in 1900: “There is nothing new to be discovered in physics now. All that remains is more and more precise measurement.”
- ✓ Yet, several dark clouds over classical physics were looming on the horizon:
  - ✧ the Michelson-Morley experiment (1880s) found that the speed of light did not change [1] with the relative motion against *ether*, which was believed to be a universe-filling matter and the medium of light propagation;
  - ✧ the discovery of photoelectric effect (1880s) demonstrated light’s particle property;
  - ✧ the divergence of the classical Rayleigh-Jeans law of blackbody radiation, known as the ultraviolet catastrophe, remained unresolved.
- ✓ These cracks widened and led to two important fields of Physics - Relativity and Quantum Mechanics. As we have discussed before, scientific progress is a series of revolution. These two fields are great examples of paradigm shift.

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[1] If ether is the medium of EM wave propagation (just like how surface wave travels on water), there would be a difference between the measured speed of light in the direction of the Earth’s motion, and that perpendicular to the Earth’s motion. Michelson and Morley failed to detect any of such difference in their precise experiment (which, by the way, used the interference property of light). Also, no location-dependent, daily, or seasonal variations of the speed of light is found, despite what one would expect due to our relative motion against the ether background.

- ✓ Unfortunately we would not have time to talk about details of Quantum Mechanics. We shall however discuss the Relativity in this chapter.

11.2 Special Relativity

- ✓ In 1905, Albert Einstein proposed the **Special Theory of Relativity** [2], which was based on two postulates – the principle of relativity, and the constancy of the speed of light.
- **First postulate** – the principle of relativity: <u>The laws of physics are the same for any observer in uniform motion</u>.
- ✓ The observer is said to be in an inertial frame, where frame (or reference frame) is basically a coordinate system with a way to measure the time at every point.
- ✓ The principle of relativity goes way back to Galileo, who observed that the *mechanical* laws of physics are the same for any observer in uniform motion. Einstein extended the principle to *all* laws of physics.
- ✓ Note that being at rest is a uniform motion (with zero velocity).
- ✓ A direct consequence is that all uniform motions are relative. One cannot detect one's own uniform motion by performing any experiment [3], so the concept of absolute motion is meaningless for an inertial observer.
- ✓ Spaceships A and B drift past each other in uniform motion (Fig. 11-1). Astronauts on both spaceships may conclude that their own spaceship is at rest, while the other is moving. Neither of them is wrong.

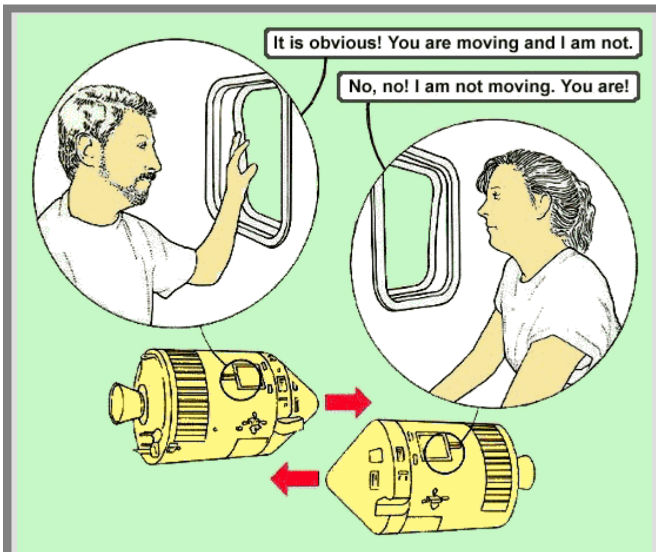

Fig. 11-1: Uniform motions are relative. We cannot decide which spaceship is moving.

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[2] In 1905, Einstein published several seminal papers that covered vastly different areas of Physics - the first one was about photoelectric effect (quantum mechanics, and property of light), the second one about Brownian motion (statistical mechanics, and atomic physics), the third and fourth about Special Relativity. Year 1905 is often referred as the annus mirabilis, or the miracle year. Einstein was awarded the Nobel Prize in Physics 1921 "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect".

[3] Say, you are aboard a fast-moving jet plane, and you pour a cup of coffee. Nothing strange should happen.

- **Second postulate**: The speed of light in vacuum is constant and is the same for any observer in uniform motion. [4]

✓ As shown in Maxwell's theory, light is the vibration of EM fields with propagation speed $c = 1/\sqrt{\varepsilon_0 \mu_0} \approx 3\times 10^8$ m/s. However, unlike other kinds of wave, light does not need to propagate through a medium.

✓ The speed of light c does not depend on the speed of an inertial observer. It is true even if the speed of the observer (relative to the light source) is comparable with the speed of light (Fig. 11-2).

✓ While the postulate is counter-intuitive, it is consistent with the outcomes of the Michelson-Morley experiment and other measurements of the speed of light.

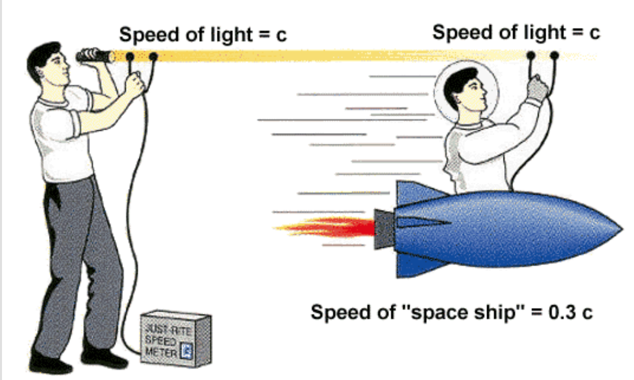

Fig. 11-2: The speed of light is absolute. The man on a spaceship measures the same value of c as the man on the ground.

Lorentz transformation

✓ Observer A′ is on a train that moves with uniform speed u relative to observer A on the platform (Fig. 11-3). These two observers refer to the same event with different coordinates.

✓ Both observers are inertial – they do not experience fictitious force that is resulted from the acceleration of their frames.

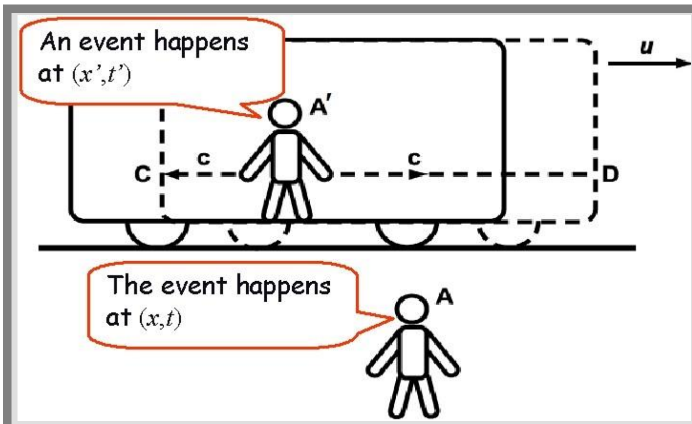

Fig. 11-3: An event happens at (x_1', t') according observer A′ in S'-frame, and at (x_1, t) according to observer A in S-frame.

✓ The coordinate transformation that is consistent with Special Relativity is called the Lorentz transformation.

[4] Because the constancy of the speed of light is so well established, since year 1983, the unit metre has been defined as the distance travelled by light in vacuum in 1/299792458 second.

✓ Without loss of generality, it is possible (and common) to assume that the two origins of the two frames S' and S coincides at $t'=0$ and $t=0$.

✓ If we further align the x'-axis and x-axis along the direction of u (Fig. 11-4), the Lorentz transformation is given by $t'=\gamma(t-ux/c^2)$, $x'=\gamma(x-ut)$, $y'=y$, $z'=z$, where $\gamma\equiv\dfrac{1}{\sqrt{1-u^2/c^2}}$ is the Lorentz factor (also known as the gamma factor).

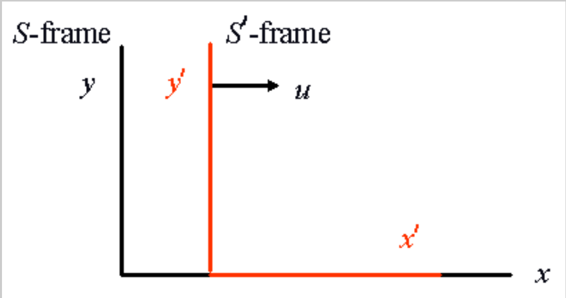

Fig. 11-4: Two inertial systems S and S'.

By defining $\beta\equiv u/c$ and doing some simple algebra (***Box 11.1***), we can express the Lorentz transformation in a more symmetric form:

$$\boxed{\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct-\beta x) \\ \gamma(-\beta ct + x) \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}}$$

✓ The inverse Lorentz transformation (from S to S') takes the same form except that the speed u has to be replaced by $-u$. By the principle of relativity, no inertial frame is special.

✓ When the speed u is small compared to c, the Lorentz transformation reduces to the Galilean transformation:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x-ut \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ -u & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

which relates the coordinates of two inertial frames in Newtonian mechanics.

Box 11.1 Derivation of the Lorentz transformation

We focus on t- and x-coordinates here. The general transformation is

$$t' = a_0 + a_1 t + a_2 x + a_3 t^2 + a_4 x^2 + \Lambda \qquad \text{(i)}$$

$$x' = b_0 + b_1 x + b_2 t + b_3 x^2 + b_4 t^2 + \Lambda \qquad \text{(ii)}$$

Coincidence of the frames' origins at $t = t' = 0$ results in $a_0 = 0$ and $b_0 = 0$.

All the quadratic or higher-order terms must vanish, such that Newton's first law ($x = vt$) has the same form under a coordinate transformation, as required by the principle of relativity (the first postulate). By tracking the origin of S' (which is always at $x' = 0$), Eq. (ii) implies $u = -b_2 / b_1$. Then the transformation becomes

$$t' = a_1 t + a_2 x \qquad \text{(i-a)}$$

$$x' = b_1 x - b_1 ut \qquad \text{(ii-a)}$$

A light pulse leaving the origin of S' at $t' = 0$ travels at the speed of light, so we have $(x')^2 - (ct')^2 = 0$. By using the transformation equations (i-a) and (ii-a),

$$(b_1^2 - a_2^2 c^2) x^2 - 2(a_1 a_2 c^2 + b_1^2 u) tx - (a_1^2 c^2 - b_1^2 u^2) t^2 = 0 \qquad \text{(iii)}$$

Due to the constancy of the speed of light (the second postulate), the light pulse travels at the same speed as observed in S, i.e. $x^2 - c^2 t^2 = 0$. Comparing with Eq. (iii), we get $b_1^2 - a_2^2 c^2 = 1$, $a_1 a_2 c^2 + b_1^2 u = 0$, and $a_1^2 c^2 - b_1^2 u^2 = c^2$. Having solved for the coefficients a_1, a_2 and b_1, we get the Lorentz transformation

$$\boxed{\begin{aligned} t' &= \gamma(t - \beta x/c) \\ x' &= \gamma(x - ut) \end{aligned}}, \text{ where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \text{ and } \beta = u/c.$$

Simultaneity

- ✓ Suppose two events at (x_1', t_1') and (x_2', t_2') happen simultaneously according to observer A′ on the train, then $\Delta t' \equiv t_2' - t_1' = 0$. An example is that A′ stands in the middle of the train, and send two light pulses in opposite direction (Fig. 11-5). The events are the arrivals of the pulses at the two walls.

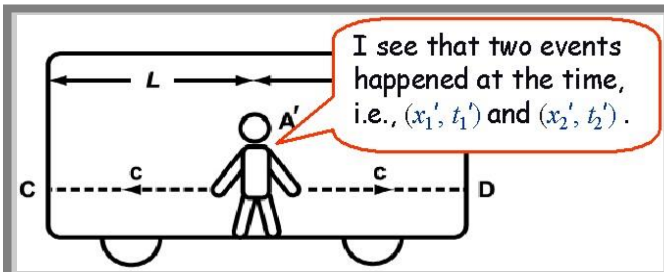

Fig. 11-5: The light pulses sent by observer A′ arrive the two walls at the same time according to him.

- ✓ Will the events happen at the same time according to observer A on the platform?

- ✓ The events happen at (x_1, t_1) and (x_2, t_2) according to A. By the Lorentz transformation, the time difference $t_2 - t_1 \equiv \Delta t = \gamma(\Delta t' + \beta \Delta x'/c) = \gamma\beta\Delta x'/c$ is not zero [5]. For a graphic demonstration, compare Figs. 11-3 and 11-5.
- ✓ Time is relative! In particular, simultaneity in time is relative! It depends on the frame of reference.

Time dilation

- ✓ On the train, there is a clock ticking once a second according to observer A′. What will observer A on the platform see? (Fig. 11-8)
- ✓ Consider two ticks at (x_1', t_1') and (x_2', t_2'), where obviously $\Delta x' \equiv x_2' - x_1' = 0$. By the Lorentz transformation, the time difference is $t_2 - t_1 \equiv \Delta t = \gamma(\Delta t' + \beta \Delta x'/c) = \gamma \Delta t'$.
- ✓ Instead of being constant, the time interval depends on frames of reference.
- ✓ The events happen at the same position according to A′. The time according to A′ is called the **proper time** τ. Since $\Delta t = \gamma \Delta t' \equiv \gamma \Delta \tau$ and $\gamma \geq 1$, the time interval measured in other frame is always longer than the proper time interval. In other words, time of a moving object flows slower. This effect is referred as time dilation.
- ✓ While observer A thinks that observer A′ slows down, observer A′ also thinks that observer A slows down. No inertial frame is special.
- ✓ Time is relative. In particular, the difference in time is relative! It depends on the frame of reference. For a moving inertial frame, there is a time dilation of $\boxed{\Delta t = \gamma \Delta \tau}$.

Lorentz contraction

One should measure the distance between the two ends of the object *simultaneously* to determine its length. Since **simultaneity** depends on reference frames, it leads to the fact that lengths also depend on the frames of reference.

- ✓ The length of an object can be found by measuring the two ends simultaneously. Since simultaneity is relative, lengths also depend on frames of reference.
- ✓ Suppose a rod of length L' is at rest on the moving train. Is the length of the rod the same as measured by A on the platform?

[5] Unless the train is at rest relative to the platform ($u = 0$), or the two events are at the same location ($\Delta x = 0$).

- ✓ Observer A has to measure both ends of the rod at the same time, with events (x_1,t) and (x_2,t). By the Lorentz transformation, $L' \equiv x'_2 - x'_1 \equiv \Delta x' = \gamma(\Delta x - u\Delta t) = \gamma\Delta x \equiv \gamma L$. The rod is at rest in S'-frame, and the length measured by A′ is called the **proper length** L_0.

  Question: the two events happen simultaneously in S-frame, but not in S'-frame. Then why do we define $x'_2 - x'_1$ as length L'?

- ✓ Space is relative! The difference in space depends on the frame of reference! For a moving inertial frame, there is a Lorentz contraction of $\boxed{L = L_0/\gamma}$.

- ✓ Any object will contract along its direction of motion with the same factor. So one would say that the contraction is space itself. And again, space is relative!

Relativistic effects in daily lives

- ✓ The speeds that we encounter in our daily lives are much smaller than the speed of light $c = 3\times10^8\ \mathrm{ms^{-1}}$. For example, a train moves at *u* = 100 km/h or 28 m/s (only $\sim 10^{-7}$ the speed of light), the Lorentz factor $\gamma = 1/\sqrt{1-30^2/(3\times10^8)^2} \approx 1$, and thus the effects of time dilation and Lorentz contraction are too small to be observed.

- ✓ Such effects become noticeable for particles close to the light speed *c*, e.g., in particle accelerators, cosmic rays, etc. Numerous experiments have verified the accuracy of special relativity.

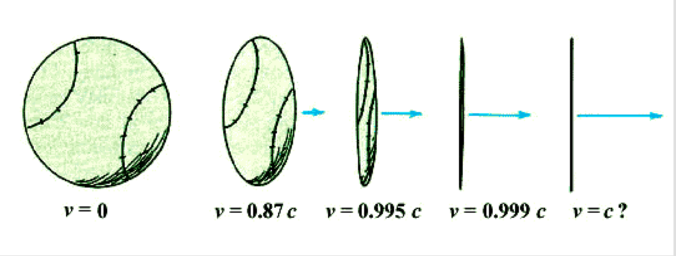

Fig. 11-6: Lorentz contraction is apparent when the speed *v* of an object is close to the speed of light.

Space-time

- ✓ In the Galilean transformation, time coordinate and the three space coordinates are very different. Yet in the Lorentz transformation, there is a certain kind of symmetry [6] between the time and space coordinates. In fact, a paradigm shift brought by Special Relativity is that we have to treat space and time on equal footing. Also, space and time are no longer separate entities. Together they are referred as **space-time**. [7]

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[6] Yet time and space are not identical. We cannot go back in time, and we cannot even stay put or go forward at a different “rate”.

[7] Minkowski came up with the concept of space-time in 1907. Later Einstein used it in general relativity.

- ✓ One may define a space-time distance between two events, namely (x_1, t_1) and (x_2, t_2), as $\Delta S^2 \equiv \Delta x^2 - c^2 \Delta t^2$. While the spatial distance Δx and "temporal distance" Δt are relative, the space-time distance of the two events is invariant [8] under the Lorentz transformation [9]. So $\Delta S'^2$ (of the same events) according to an observer in another inertial frame is identical to ΔS^2. (Try proving it! Just cost you couple minutes.)

Paradox of Special Relativity

- ✓ A paradox, according to the Oxford Dictionary, is "a seemingly absurd or contradictory statement or proposition which when investigated may prove to be well founded or true."
- ✓ The predications of Special Relativity are very different what we experience in daily lives. Therefore, ever since the publication of Special Relativity, people have been proposing various paradoxes. However, so far none of the common paradoxes points out any real inconsistency (if there's any) in the theory.
- ✓ See ***Box 11.2*** for details about the Twin Paradox.

Velocity Addition

Suppose A′ on a train throws a ball with speed v' horizontally. (Fig. 11-7) What is the ball's speed according to A? Is it simply $v = u + v'$?

According to the Lorentz transformation, $\Delta x = \gamma(\Delta x' + u\Delta t')$ and $\Delta t = \gamma(\Delta t' + u\Delta x'/c^2)$. The speed according to observer A is

$$\boxed{v = \frac{\Delta x}{\Delta t} = \frac{v' + u}{1 + uv'/c^2}}$$

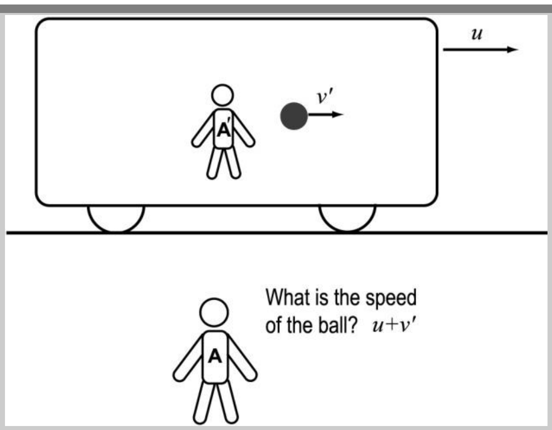

Fig. 11-7: Velocity addition under relativity.

In general, the ball may not move in the direction of u. There will be a relativistic correction even for the perpendicular components of the velocity.

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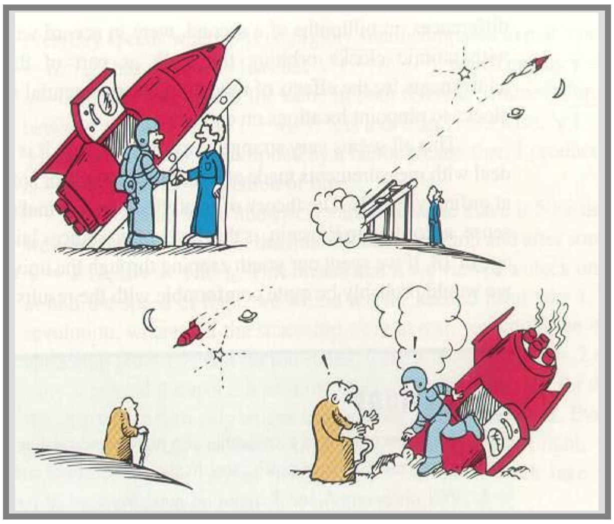
[8] Similarly, the spatial distance and time difference are both invariant under the Galilean transformation.

[9] Some people refer the Lorentz transformation as rotation in 4-dimensional space-time. While there is some similarity, this idea could be confusing or even misleading if one does not understand the formulae well enough.

Box 11.2 Twin Paradox (雙生子佯謬)



Travis and Eddie are identical twins. Travis is an astronaut who takes a high-speed round-trip journey while Eddie stays home on Earth. When Travis returns from his trip, should he be older or younger than Eddie?

Explanation 1: Due to time dilation, the fast motion of Travis would slow down his aging. Therefore Travis is younger than Eddie, who ages normally on Earth.

Explanation 2: From Travis' point of view, Eddie and the Earth are moving with high-speed. Due to time dilation, Eddie should be younger when they meet again.

It is not possible that both outcomes are correct (well, though it's possible that both are wrong). The seemingly contradicting outcomes results from misapplying Special Relativity. While motion is indeed relative, time dilation can be directly applied only if both frames are inertial. In order to come back to the Earth, Travis' spaceship has to accelerate (direction change is acceleration). It breaks the symmetry. A careful analysis leads to the conclusion that Eddie is older.

Probably due to the impracticality of the setting, or the unethicality of human experimentation, the experiment has not been done for real. The effect is however measureable by comparing atomic clocks on the ground and aircraft or satellite.

Momentum and Mass

Is momentum conserved under the theory of relativity? Let's check. Suppose two balls are moving with the same speed but in opposite direction. (Fig. 11-8a) After a totally inelastic collision, two balls stick together and are at rest. The total momentum is conserved.

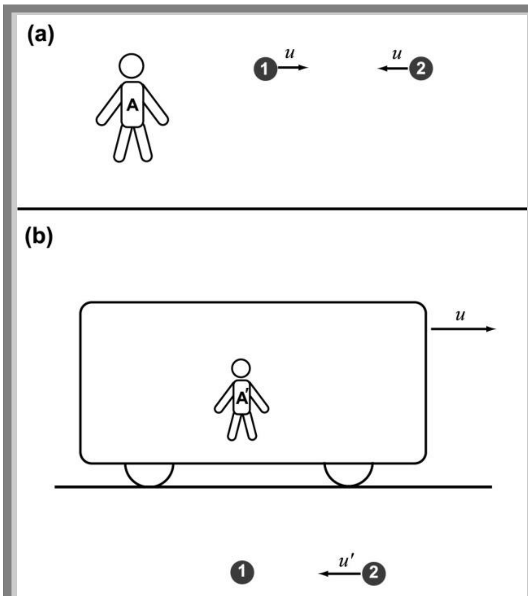


Fig. 11-8: Is momentum conserved in the both frames of reference?

In Fig. 11-8b, before collision, A′ inside the moving train observes that ball 1 is at rest, and ball 2 moves with $u' = \dfrac{2u}{1+u^2/c^2}$ (velocity addition). After collision, the stuck-together balls move with speed *u* according to A′. The momentum is not conserved in this frame of reference.

In order to preserve momentum conserved under relativity, one could define the relativistic momentum of a particle as $p = \gamma m_0 v$, where $\gamma = 1\big/\sqrt{1 - v^2/c^2}$, *v* is the speed of the object and m_0 is the **rest mass** of the object. Some people refer the effective mass γm_0 as the "relativistic mass", which increases with speed because the effective mass (i.e. effective inertia) increases as speed approaches the speed of light.

Speed of light as an upper limit

As the speed of an object increases and approaches the speed of light, the Lorentz factor γ increases and approaches infinity. Apart from infinite temporal dilation and indefinite spatial contraction, its effective mass also increases and tends to infinity. Since one needs an infinitely large force in this case, object with nonzero mass cannot travel faster than light. We will state without proof that particles with zero mass (e.g. photon) must travel with the speed of light in vacuum. In fact, in fact, neither *material objects* nor *information* can cross the barrier of light speed.

Equivalence of Mass and Energy

Suppose an object is accelerated by a force *F*, the kinetic energy of the particle can be expressed as $K.E. = \int F ds$, where force $F = \dfrac{dp}{dt}$ and relativistic momentum $p = \gamma m_0 v$. Now,

$$\begin{aligned} K.E. &= \int F ds = \int \frac{dp}{dt} ds = \int u dp \\ &= up - \int p du \\ &= \gamma m_0 u^2 + \frac{1}{\gamma} m_0 c^2 - m_0 c^2 \\ &= \gamma m_0 c^2 - m_0 c^2 \end{aligned}$$

This leads to Einstein's famous mass-energy equation $\boxed{E = mc^2}$. The rest mass energy is $m_0 c^2$, and the total energy is $E = m_0 c^2 + K.E.$, i.e., $E = \gamma m_0 c^2$.

✓ It is then easy to prove that $\boxed{E^2 = p^2 c^2 + m_0^2 c^4}$. In particular, zero-mass particle such as photon has energy $E = pc$ and thus has momentum.

Question: Now you are able to able to proof that momentum is conserved in the aforementioned example. Not very simple, but worth doing!

- ✓ Consequently, a change of mass Δm can be converted to energy ΔE and vice versa as according to $\Delta E = \Delta mc^2$. Fig. 11-9 shows a nuclear power plant, which uses the energy for electricity generation.

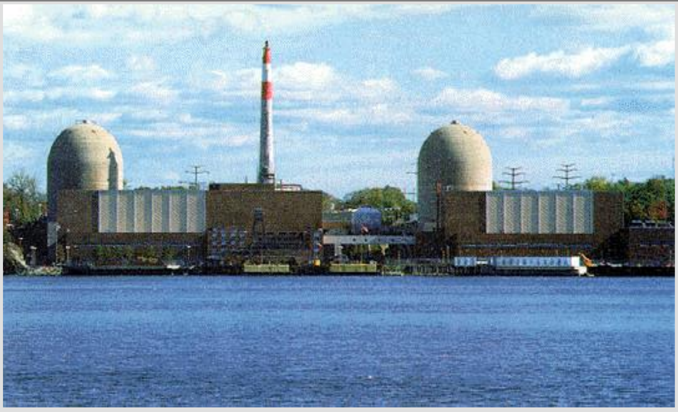


Fig. 11-9: Nuclear plant is operated according to $E = mc^2$.

- ✓ When $u^2 << c^2$, we have $\gamma = (1 - u^2/c^2)^{-1/2} \approx 1 + (u^2/c^2)/2$ and $E \approx m_0c^2 + \frac{1}{2}m_0u^2$. Thus the Newtonian kinetic energy $K.E. = m_0u^2/2$ is a low-speed approximation.

No rigid body (剛體)

- ✓ A rigid body is a solid with unchangeable shape. It does not change its shape even a bit when being compressed, squeezed, twisted, etc. A diamond is very hard but it is still not strictly rigid, because a rigid body is an infinitely hard body. The concept of rigid body is actually inconsistent with Special Relativity.
- ✓ Let us imagine that there is a rigid rod. If we push on end of the rod, the other will move *immediately*! What is so strange? The strange thing is that it takes *no time* for the signal of pushing to reach the other end. This infinite signal speed is not allowed in Special Relativity. Thus, **there is no rigid body**.

Time reversal

- ✓ In principle, if we could travel faster than light, we could record the past signals and hence the past event could be seen again. Such a *time reversal* violates Special Relativity, and thus is believed to be impossible. Most physicists conclude that the only *direction of time is forward*.
- ✓ From time to time, there have been claims that faster-than-light speed is measured. So far none of the claims has been verified.

Beyond Special Relativity

- ✓ As should be clear by now, Newtonian kinematics is consistent with Special Relativity when the speed is much lower than the speed of light.
- ✓ However, Special Relativity is a theory of physics in inertial frames. It thus could not explain the fictitious acceleration arisen in a non-inertial frame. In fact, it also only describes the motion of objects, instead of the force (in particular, gravity). One needs a relativistic theory of gravitation to explain other parts of Newtonian mechanics.
- ✓ Actually, even with Special Relativity, it is clear that Newton's theory of gravitation is not satisfactory. In Newton's theory, the gravitational force of interaction between bodies is assumed to *transmit instantaneously*. Such an infinite speed again is not allowed in Special Relativity.

11.3 General Relativity

- ✓ In 1915, Albert Einstein proposed the **General Theory of Relativity**, which, in addition to the previous two postulates, was mainly [10] based on the **principle of equivalence**: [11]

  Observers cannot distinguish locally between inertial forces due to acceleration and uniform gravitational forces due to the presence of a massive body.

- ✓ In Fig. 11-10, the observer locates in a rocket far from any gravitational source. If the rocket does not accelerate, the observer E is not weightless. On the other hand, if the rocket accelerates uniformly upwards, the floor (or the scale) pushes on the observer, and provides him with a sense of weight. In fact, he cannot do any experiment (e.g. by dropping an object) to determine whether he is in an accelerating system, or being stationary on Earth.

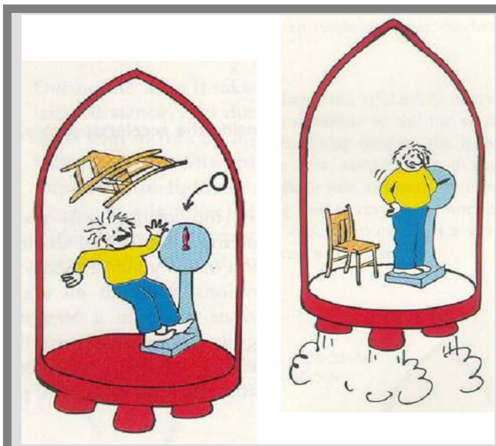


Fig. 11-10: (Left) Observer E is weightless if the rocket does not accelerate. (Right) He cannot distinguish if he is in an accelerating rocket, or standing on Earth.

[10] The other less discussed principles are the principle of general covariance and the principle of consistency. See Lambourne 10 for details, or see the brief discussion in Collier's *A Most Incomprehensible Thing*.

[11] This is now known as the weak equivalence principle. Another way of putting it is that gravitational mass and inertial mass are the same. It leads to Galileo's theory of the mass independence of falling. Therefore, the weak equivalence principle also called the principle of universality of free fall.

Bending of light by gravity

- ✓ According to the principle of equivalence, a beam of light will be deflected under the influence of gravity. Let us imagine that we are now travelling in an accelerating lift in the space. A light beam is passing through the lift. From the viewpoint of an external stationary observer, the beam is straight (Fig. 11-11a).
- ✓ However, the observer in the lift will observe that the beam is bent (Fig. 11-11b). Remember that the observer in the lift cannot distinguish between inertial acceleration and gravity (the principle of equivalence). Hence he concludes that gravity could cause the bending of the light beam.

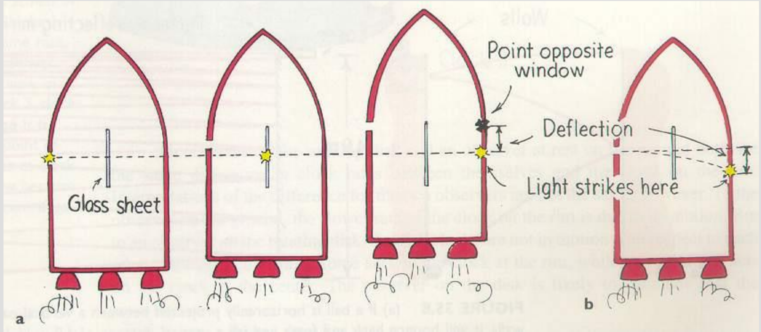


Fig. 11-11: (a) An outside observer sees that light travels horizontally in a straight line. (b) To an inside observer, the light bends as if it responds to gravity.

- ✓ Light has no mass, so classical theory predicts that a light beam is straight, no matter whether there is gravity. Therefore this phenomenon cannot be explained by Newton's theory of gravity. In Einstein's theory, the light beam bends because it travels in a space-time geometry that *is bent* — a **curved space-time**. [12]

Curved space-time

- ✓ According to general relativity, mass causes space-time to curve. Objects in a curved space-time move in curved paths, such as circles. It seems that they are pulled by a force, which we call *"gravity"*. It is this gravity that pulls the Earth towards the Sun. A more massive star causes a larger space-time curvature. (Fig. 11-12)

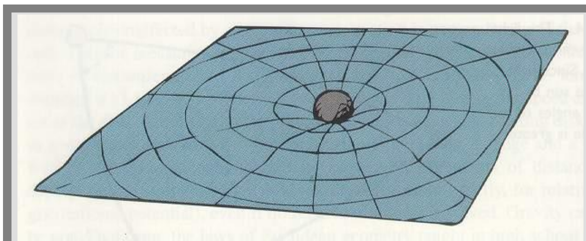


Fig. 11-12: A two-dimensional analogy of the four-dimensional warped space-time.

[12] In the words of John Wheeler: "Mass tells space-time how to curve, and space-time tells mass how to move." The sentence neatly sums up General Relativity – Einstein's theory of gravity.

✓ It should be emphasized that a free body always takes the straight or "shortest" path (known as *geodesic*). However, since space-time is curved, a geodesic appears curved. This can be easily understood by considering the surface of a sphere. In Fig. 11-13, the string indicates the shortest path between two points on a curved (two-dimensional) space. Note that an ant walk along the line, being not able to feel the curvature, regards its path as a straight line.

Fig. 11-13: A sphere as an illustration of a curved space.

✓ Other than a convex surface (such as a spherical surface), there are other possibilities of the curvature of space-time. As shown in Fig. 11-14, the sum of interior angles of a triangle in the two-dimensional curved space may be greater or less than 180 degrees.

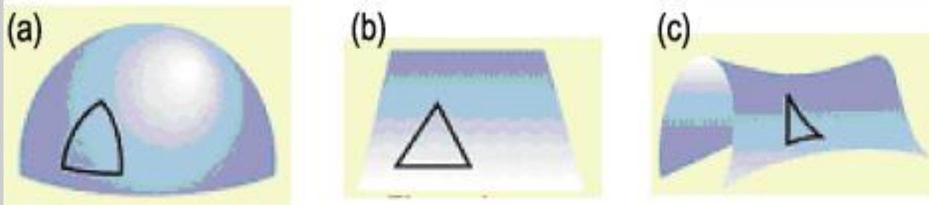


Fig. 11-14: Two-dimensional analogy. Under the influence of gravity, the sum of interior angles may be (a) greater, (b) equal to, or (c) less than 180 degrees.

Gravitational redshift

✓ In Fig. 11-15a, an observer A in a uniform field g (downward) receiving light signal from a static source on ground level. Is there any change in frequency as observed by A, according to the theory of general relativity?

✓ By the principle of equivalence, it is equivalent to a situation that B receives light in a frame with an acceleration g upward (Fig. 11-15b). Light takes $\Delta t = h/c$ for the pulse to reach B, and

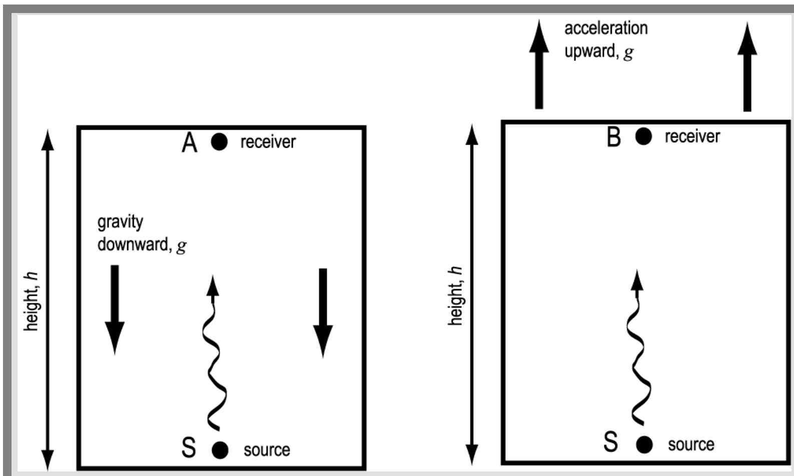


Fig. 11-15: (a) A (in a uniform downward gravity g) receives a light signal. (b) B (in a lift with uniform upward acceleration g) receives a light signal. The two cases are indistinguishable according to the principle of equivalence.

during Δt the speed of B has increased by $v = g\Delta t = gh/c$. As a result, B receives the signal as if he is moving away from the source at a speed of *v*. In the limit of small velocity $v << c$, according to Doppler effects, B will receive the signal with frequency red-shifted, and the fractional change is equal to $\Delta f / f_0 = -v/c = -gh/c^2$, where f_0 is the frequency of the signal emitted from the source.

- ✓ Equivalently, observer A sees the received light red-shifted (to smaller frequency) in a gravitational field by $\boxed{\Delta f / f_0 = -gh/c^2}$ or $f = \left(1 - gh/c^2\right) f_0$. Notice that this expression is valid only for a light travelling in constant gravity *g*. However, in terms of gravitational potential change $\Delta\phi$ one may consider $gh \approx \Delta\phi$ approximately. In conclusion, *light will be red-shifted when it moves against gravitational field*. On the other hand, light that moves in the direction of a gravitational field will be *blue-shifted*.
- ✓ For example, for light emitted from the surface of the Sun (to the Earth), the change in gravitational potential roughly equals $\Delta\phi / c^2 = GM / rc^2 \approx 2.1\times 10^{-6}$, i.e., the fractional change in frequency is in the order of 10^{-6}. This small change is very difficult to detect and leads to the fact that measurement was not successful until 1960s to within 5%.

Gravitational time dilation

- ✓ Suppose two synchronised clocks in position S (deeper gravitational potential) measure the emitted light with frequency f_0 and hence period $\Delta t_0 = 1/f_0$. Then, one of the clocks is now placed in position A (shallower gravitational potential) and receives a red-shifted light with frequency *f* and hence period $\Delta t = 1/f$. (Fig. 11-15a) One can show that $\Delta t = \Delta t_0 \left(1 - gh/c^2\right)^{-1}$ without difficulty. Consequences, one would conclude that the two clocks keep time at different rates. In other words, the clock at source (deeper gravitational potential) runs slower compared with the clock at A (shallower gravitational potential). *Clocks close to a massive body run slow compared to ones that are farther away*. The effect is known as **gravitational time dilation.**
- ✓ In particular, the time interval Δt measured at the surface of a star of mass *M* and radius *r* and that $\Delta\tau$ measured in outer space (zero gravitational potential) could be related by $\Delta t^2 = \Delta\tau^2 \left(1 - \frac{2GM}{rc^2}\right)$. The derivation is beyond the scope of this course. [13]

[13] See M. S. Longair, *Theoretical concepts in physics*, pp.287-288 for a naive derivation.

✓ Besides, in the example of twin paradox (see *Box 11.2*), Travis has experienced an acceleration (or equivalently, a gravity) of the spaceship in turning around two different frames of reference. Consequences, his clock ran slower when he was going into a deeper gravity, and then less time passed for Travis than for his twin on Earth. This is no more than gravitational time dilation.

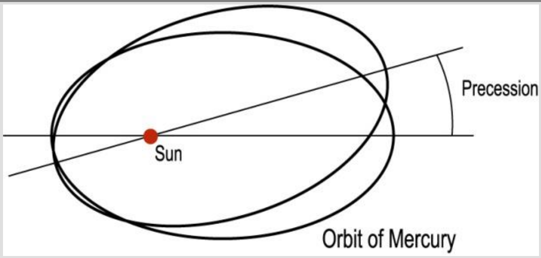


Fig. 11-16: Mercury′s orbit precession is ~5600″ per century from an Earth-based laboratory. There is an excess precession of 43″ faster than that predicted by Newtonian theory.

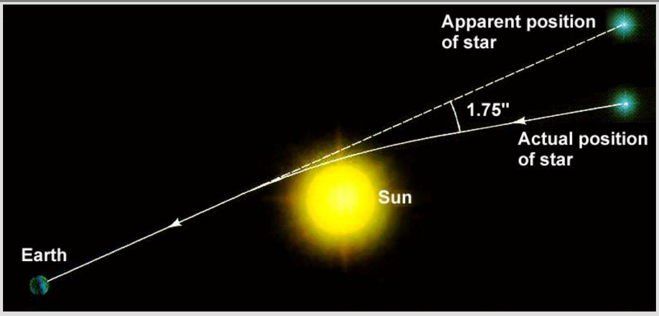


Fig. 11-17: Bending of starlight by the Sun depicted as a consequence of the curvature of space near the Sun. Ray of light pursues geodesic, but the geometry in which it travels is curved (actual travel takes place in space-time rather than space alone).

Experimental Evidences

✓ **Precession of Mercury orbits:** In 1859, the French astronomer Urbain Le Verrier [14] had analysed an anomaly in the orbit of Mercury and found the precession of 574 arcseconds per century. (Fig. 11-16) Le Verrier used Newton's theory to explain the anomaly by taking into account the small gravitational attraction by other planets, but it could account for only 531 arcseconds. On the other hand, Einstein was able to explain the excess 43 arcseconds per century by the theory of general relativity.

✓ **Deflection of light by gravity:** Because of the gravity of the Sun, Einstein predicted that a star appearing at the edge of the Sun should be deflected by 1.74 arcseconds, which was about twice the deflection predicted by Newton. During the solar eclipse on May 29, 1919, it was found that the gravitational deflection caused by the Sun was 1.61±0.3 arcseconds. This meant that Einstein's prediction by the general theory was in agreement with the actual measurement. (Fig. 11-17).

✓ Nowadays, there are a lot of experimental tests of Einstein's general relativity [15]. So far it passes all the tests with flying colours.

✓ The theory also explains phenomena associated with black holes, distant galaxies and quasars. It even explains the expansion of the universe and hence the big bang theory.

[14] Urbain Le Verrier (勒威耶，1811-1877). He also discovered Neptune (海王星) in 1846.
[15] Details can be found in the Living Reviews article *"The Confrontation between General Relativity and Experiment"* by C.M. Will: http://relativity.livingreviews.org/Articles/lrr-2014-4/