Point Cloud Bundle Adjustment Based on Points Cluster

This is another direct derivation of Hessian and Jacobian matrix on one feature, which we used to fuse with other sensors. The difference is the pose update method. In this draft, the update method is

$$\mathbf{T}_i \boxplus \delta\mathbf{T}_i = (\mathbf{T}_i \exp\boldsymbol{\phi}_i{}^\wedge, \mathbf{t}_i + \delta\mathbf{t}_i)$$

i is the index of pose ($\mathbf{R}_i$, $\mathbf{t}_i$) and corresponding point cluster ($\mathbf{P}_i$, $\mathbf{v}_i$, n_i). N is the whole number of points. The Hessian matrix used in BA optimization is the sum of $\mathbf{H}_1$ and $\mathbf{H}_2$. "benchmark_realword.cpp" has the code of this draft.

I. Direct Derivation

The relationship between λ_k and $\mathbf{A}$ is

$$\lambda_k = \mathbf{u}_k^T\mathbf{A}\mathbf{u}_k$$

where

$$\begin{aligned}\mathbf{A} =& \frac{1}{N}\sum_{i=1}^{m}\widetilde{\mathbf{P}}_i - \frac{1}{N^2}\sum_{i=1}^{m}\widetilde{\mathbf{v}}_i\sum_{i=1}^{m}\widetilde{\mathbf{v}}_i^T\\ \widetilde{\mathbf{P}}_i =& \mathbf{R_i}\mathbf{P_i}\mathbf{R}_i^T + \mathbf{R_i}\mathbf{v_i}\mathbf{t}_i^T + \mathbf{t}_i(\mathbf{R_i}\mathbf{v_i})^T + n\mathbf{t_i}\mathbf{t_i}^T\\ \widetilde{\mathbf{v}}_i =& \mathbf{R_i}\mathbf{v_i} + n\mathbf{t}_i\end{aligned}$$

So,

$$\begin{aligned}&\lambda_k =\\ &\frac{1}{N}\sum_{i=1}^{m}\mathbf{u}_k^T(\mathbf{R_i}\mathbf{P_i}\mathbf{R}_i^T + \mathbf{R_i}\mathbf{v_i}\mathbf{t}_i^T + \mathbf{t}_i(\mathbf{R_i}\mathbf{v_i})^T + n_i\mathbf{t_i}\mathbf{t_i}^T)\mathbf{u}_k\\ &-\frac{1}{N^2}\sum_{i=1}^{m}\mathbf{u}_k^T(\mathbf{R_i}\mathbf{v_i}+n_i\mathbf{t}_i)\sum_{i=1}^{m}(\mathbf{R_i}\mathbf{v_i}+n_i\mathbf{t}_i)^T\mathbf{u}_k\end{aligned}$$

From BALM, we have

$$\frac{\partial\lambda_k}{\partial\mathbf{T}} = \mathbf{u}_k^T\frac{\partial\mathbf{A}}{\partial\mathbf{T}}\mathbf{u}_k = \frac{\partial\bar{\mathbf{u}}_k^T\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}}$$

Here, $\bar{\mathbf{u}}_k$ is equal to $\mathbf{u}_k$, but is a constant vector.

$$\begin{aligned}\Big(N\frac{\partial\lambda_k}{\partial\mathbf{R}_i}\Big)^T =& 2(\mathbf{P}_i^T\mathbf{R}_i^T\mathbf{u}_k)^\wedge(\mathbf{R}_i^T\mathbf{u}_k) + 2(\mathbf{v}_i\mathbf{t}_i^T\mathbf{u}_k)^\wedge(\mathbf{R}_i^T\mathbf{u}_k)\\ &-2\mathbf{v}_i^\wedge(\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T)\sum_{j=1}^{m}\frac{\mathbf{R}_j\mathbf{v}_j+n_j\mathbf{t}_j}{N}\\ \Big(N\frac{\partial\lambda_k}{\partial\mathbf{t}_i}\Big)^T =& 2\mathbf{u}_k\mathbf{u}_k^T\Big(\mathbf{R}_i\mathbf{v}_i + n_i\mathbf{t}_i - n_i\sum_{j=1}^{m}\frac{\mathbf{R}_j\mathbf{v}_j+n_j\mathbf{t}_j}{N}\Big)\end{aligned}$$

To get $\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}_i}$, we need to calculate $\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}_i}$

$$\begin{aligned}N\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{R}_i} =& \mathbf{R}_i\mathbf{P}_i(\mathbf{R}_i^T\mathbf{u}_k)^\wedge - \mathbf{R}_i(\mathbf{P}_i\mathbf{R}_i^T\mathbf{u}_k)^\wedge\\ &-\mathbf{R}_i(\mathbf{v}_i\mathbf{t}_i^T\mathbf{u}_k)^\wedge + \mathbf{t}_i\mathbf{v}_i^T(\mathbf{R}_i^T\mathbf{u}_k)^\wedge\\ &-\frac{\widetilde{\mathbf{v}}}{N}\mathbf{v}_i^T(\mathbf{R}_i^T\mathbf{u}_k)^\wedge + \mathbf{R}_i\mathbf{v}_i^\wedge\frac{\widetilde{\mathbf{v}}}{N}^T\mathbf{u}_k\\ =& (\mathbf{R}_i\mathbf{P}_i + \mathbf{t}_i\mathbf{v}_i^T - \frac{\widetilde{\mathbf{v}}}{N}\mathbf{v}_i^T)(\mathbf{R}_i^T\mathbf{u}_k)^\wedge\\ &-\mathbf{R}_i(\mathbf{P}_i\mathbf{R}_i^T\mathbf{u}_k)^\wedge - \mathbf{R}_i\mathbf{v}_i^\wedge(\mathbf{t}_i - \frac{\widetilde{\mathbf{v}}}{N})^T\mathbf{u}_k\\ N\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{t}_i} =& \mathbf{R}_i\mathbf{v}_i\mathbf{u}_k^T + (\mathbf{R}_i\mathbf{v}_i)^T\mathbf{u}_k + n_i\mathbf{t}_i\mathbf{u}_k^T + n_i\mathbf{t}_i^T\mathbf{u}_k\\ &-n_i\frac{\widetilde{\mathbf{v}}}{N}\mathbf{u}_k^T - n_i\frac{\widetilde{\mathbf{v}}}{N}^T\mathbf{u}_k\\ =& (\mathbf{R}_i\mathbf{v}_i + n_i\mathbf{t}_i - n_i\frac{\widetilde{\mathbf{v}}}{N})\mathbf{u}_k^T + (\mathbf{R}_i\mathbf{v}_i + n_i\mathbf{t}_i - n_i\frac{\widetilde{\mathbf{v}}}{N})^T\mathbf{u}_k\end{aligned}$$

Now let us discuss $\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}_i}$ and from BALM

$$\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}_i} = \mathbf{U}\mathbf{C}_{:,k}^{\mathbf{T}_i}$$

where

$$\mathbf{C}_{m,n}^{\mathbf{T}_i} = \begin{cases}\dfrac{1}{\lambda_n-\lambda_m}\mathbf{u}_m^T\dfrac{\partial\mathbf{A}}{\partial\mathbf{T}_i}\mathbf{u}_n, & m\neq n\\ 0 & ,m=n\end{cases}$$

Expand the equation

$$\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}_i} = \sum_{m=1,m\neq k}^{3}\frac{1}{\lambda_k-\lambda_m}\mathbf{u}_m\mathbf{u}_m^T\frac{\partial\mathbf{A}}{\partial\mathbf{T}_i}\mathbf{u}_k$$

where the $\frac{\partial\mathbf{A}}{\partial\mathbf{T}_i}\mathbf{u}_k = \frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}_i}$ and we have know it. Now it is the second derivative. It can be divided into two parts.

$$\mathbf{H} = \frac{\partial^2\lambda_k}{\partial\mathbf{T}^2} = \frac{\partial\mathbf{J}(\bar{\mathbf{u}}_k)}{\partial\mathbf{T}} + \frac{\partial\mathbf{J}(\mathbf{u}_k)}{\partial\mathbf{u}_k}\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}} = \mathbf{H}_1 + \mathbf{H}_2$$

In $\mathbf{H}_1$, we regard the $\mathbf{u}_k$ as the constant, only considering the part about $\mathbf{T}$. In $\mathbf{H}_2$, $\mathbf{u}_k$ is variable and we use derivative chain rule. Now let us consider $\mathbf{H}_2$ firstly. We have know the $\frac{\partial\mathbf{u}_k}{\partial\mathbf{T}}$, so

$$\begin{aligned}\frac{\partial\mathbf{J}(\mathbf{u}_k)}{\partial\mathbf{u}_k} = \frac{\partial}{\partial\mathbf{u}_k}(\frac{\mathbf{u}_k^T\partial\mathbf{A}\mathbf{u}_k}{\partial\mathbf{T}}) =& \Big(\frac{\partial}{\partial\mathbf{T}}(\frac{\mathbf{u}_k^T\partial\mathbf{A}\mathbf{u}_k}{\partial\mathbf{u}_k})\Big)^T\\ =& \Big(2\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}}\Big)^T\end{aligned}$$

So we can conclude

$$\mathbf{H}_2 = 2\Big(\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}}\Big)^T\Big(\sum_{m=1,m\neq k}^{3}\frac{1}{\lambda_k-\lambda_m}\mathbf{u}_m\mathbf{u}_m^T\Big)\frac{\partial\mathbf{A}\bar{\mathbf{u}}_k}{\partial\mathbf{T}}$$

Now let us discuss $\mathbf{H}_1$. We use $\mathbf{H}_1(\mathbf{T}_i, \mathbf{T}_j)$ to denote different part in the hessian matrix.

$$
\begin{aligned}
&N\mathbf{H}_1(\mathbf{R}_i, \mathbf{R}_i) = -\frac{1}{2}(N\frac{\partial \lambda_k}{\partial \mathbf{R}_i})^\wedge + \\
&2(\mathbf{P}_i\mathbf{R}_i^T\mathbf{u}_k)^\wedge(\mathbf{R}_i^T\mathbf{u}_k)^\wedge - 2(\mathbf{R}_i^T\mathbf{u}_k)^\wedge\mathbf{P}_i(\mathbf{R}_i^T\mathbf{u}_k)^\wedge + \\
&2\mathbf{v}_i^\wedge(\mathbf{R}_i^T\mathbf{u}_k)^\wedge\mathbf{u}_k^T(\mathbf{t}_i - \frac{\widetilde{\mathbf{v}}}{N}) + \frac{2}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_i\mathbf{v}_i^\wedge
\end{aligned}
$$

$$
\begin{aligned}
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_i) =& 2(1 - \frac{n_i}{N})\mathbf{v}_i^\wedge(\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T) \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{R}_i) =& 2(\frac{n_i}{N} - 1)\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_i\mathbf{v}_i^\wedge = N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_i)^T \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{t}_i) =& 2(n_i - \frac{n_i^2}{N})\mathbf{u}_k\mathbf{u}_k^T \\
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{R}_j) =& \frac{2}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_j\mathbf{v}_j^\wedge \\
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_j) =& -\frac{2n_j}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{R}_j) =& \frac{2n_i}{N}\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_j\mathbf{v}_j^\wedge \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{t}_j) =& -\frac{2n_in_j}{N}\mathbf{u}_k\mathbf{u}_k^T
\end{aligned}
$$

In conclude for programming,

$$
\begin{aligned}
N\frac{\partial \mathbf{A}\bar{\mathbf{u}}_k}{\partial \mathbf{R}_i} =& (\mathbf{R}_i\mathbf{P}_i + \mathbf{t}_i\mathbf{v}_i^T - \frac{\widetilde{\mathbf{v}}}{N}\mathbf{v}_i^T)(\mathbf{R}_i^T\mathbf{u}_k)^\wedge \\
& - \mathbf{R}_i\Big((\mathbf{P}_i\mathbf{R}_i^T\mathbf{u}_k)^\wedge + \mathbf{v}_i^\wedge\mathbf{u}_k^T(\mathbf{t}_i - \frac{\widetilde{\mathbf{v}}}{N})\Big) \\
N\frac{\partial \mathbf{A}\bar{\mathbf{u}}_k}{\partial \mathbf{t}_i} =& (\mathbf{R}_i\mathbf{v}_i + n_i\mathbf{t}_i - n_i\frac{\widetilde{\mathbf{v}}}{N})\mathbf{u}_k^T + \\
& (\mathbf{R}_i\mathbf{v}_i + n_i\mathbf{t}_i - n_i\frac{\widetilde{\mathbf{v}}}{N})^T\mathbf{u}_k \\
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{R}_i) =& -\frac{1}{2}(N\frac{\partial \lambda_k}{\partial \mathbf{R}_i})^\wedge + \frac{2}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_i\mathbf{v}_i^\wedge + \\
2\Big((\mathbf{P}_i\mathbf{R}_i^T\mathbf{u}_k)^\wedge + & \mathbf{v}_i^\wedge\mathbf{u}_k^T(\mathbf{t}_i - \frac{\widetilde{\mathbf{v}}}{N}) - (\mathbf{R}_i^T\mathbf{u}_k)^\wedge\mathbf{P}_i\Big)(\mathbf{R}_i^T\mathbf{u}_k)^\wedge \\
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_i) =& 2(1 - \frac{n_i}{N})\mathbf{v}_i^\wedge(\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T) \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{R}_i) =& 2(\frac{n_i}{N} - 1)\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_i\mathbf{v}_i^\wedge = N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_i)^T \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{t}_i) =& 2(n_i - \frac{n_i^2}{N})\mathbf{u}_k\mathbf{u}_k^T
\end{aligned}
$$

$$
\begin{aligned}
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{R}_j) =& \frac{2}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_j\mathbf{v}_j^\wedge \\
N\mathbf{H}_1(\mathbf{R}_i, \mathbf{t}_j) =& -\frac{2n_j}{N}\mathbf{v}_i^\wedge\mathbf{R}_i^T\mathbf{u}_k\mathbf{u}_k^T \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{R}_j) =& \frac{2n_i}{N}\mathbf{u}_k\mathbf{u}_k^T\mathbf{R}_j\mathbf{v}_j^\wedge \\
N\mathbf{H}_1(\mathbf{t}_i, \mathbf{t}_j) =& -\frac{2n_in_j}{N}\mathbf{u}_k\mathbf{u}_k^T
\end{aligned}
$$