Efficient and Consistent Bundle Adjustment on Lidar Point Clouds (Supplementary)

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Please note that equation numbers and section numbers from the main manuscript are labelled in this letter in red.

I. Lemmas

Lemma 1. *For a scalar $x \in \mathbb{R}$ and a matrix $\mathbf{A} \in \mathbb{S}^{3\times 3}$ which depends on x, we have the two following conclusions.*

$$\frac{\partial \lambda_l(x)}{\partial x} = \mathbf{u}_l(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x) \tag{1}$$

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k(x)\mathbf{u}_k(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x) \tag{2}$$

where λ_l ($l = 1, 2, 3$) denotes the l-th largest eigenvalue and $\mathbf{u}_l$ is the corresponding eigenvector.

Proof. Since the matrix $\mathbf{A}(x)$ is symmetric, its singular value decomposition is,

$$\mathbf{A}(x) = \mathbf{U}(x)\mathbf{\Lambda}(x)\mathbf{U}(x)^T \tag{3}$$

where $\mathbf{\Lambda}(x) = \operatorname{diag}(\lambda_1(x), \lambda_2(x), \lambda_3(x))$ consists of all the eigenvalues and $\mathbf{U}(x) = \begin{bmatrix}\mathbf{u}_1(x) & \mathbf{u}_2(x) & \mathbf{u}_3(x)\end{bmatrix}$ is an orthonormal matrix consisting of the eigenvectors. Therefore,

$$\mathbf{\Lambda}(x) = \mathbf{U}(x)^T\mathbf{A}(x)\mathbf{U}(x) \tag{4}$$

Both sides take the derivative of x,

$$\begin{aligned}\frac{\partial \mathbf{\Lambda}(x)}{\partial x} &= \mathbf{U}(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{U}(x) + \mathbf{U}(x)^T\mathbf{A}(x)\frac{\partial \mathbf{U}(x)}{\partial x} \\ &\quad + \left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T \mathbf{A}(x)\mathbf{U}(x)\end{aligned} \tag{5}$$

Since $\mathbf{U}(x)^T\mathbf{A}(x) = \mathbf{\Lambda}(x)\mathbf{U}(x)^T$ and $\mathbf{A}(x)\mathbf{U}(x) = \mathbf{U}(x)\mathbf{\Lambda}(x)$, the equation is

$$\begin{aligned}\frac{\partial \mathbf{\Lambda}(x)}{\partial x} &= \mathbf{U}(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{U}(x) + \mathbf{\Lambda}(x)\underbrace{\mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}}_{\mathbf{D}(x)} \\ &\quad + \underbrace{\left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T \mathbf{U}(x)}_{\mathbf{D}^T(x)}\mathbf{\Lambda}(x)\end{aligned} \tag{6}$$

Denote $\mathbf{D}(x) \triangleq \mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}$. Since $\mathbf{U}(x)\mathbf{U}(x)^T = \mathbf{I}$, differentiating both sides with respect to x leads to,

$$\begin{aligned}&\mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x} + \left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T\mathbf{U}(x) = \mathbf{0} \\ \Rightarrow\ &\mathbf{D}(x) + \mathbf{D}^T(x) = \mathbf{0}\end{aligned}$$

It is seen that $\mathbf{D}(x)$ is a skew symmetric matrix whose diagonal elements are zeros. Moreover, since $\mathbf{\Lambda}(x)$ is diagonal, the last two items of the right side of (6) sum to zero on diagonal positions. Only considering the diagonal elements in (6) leads to

$$\frac{\partial \lambda_l(x)}{\partial x} = \mathbf{u}_l(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), l \in \{1, 2, 3\} \tag{7}$$

which yields the first conclusion. Now we aims to prove the second one. In (6), $\frac{\partial \mathbf{\Lambda}(x)}{\partial x}$ is diagonal matrix and thus for the off-diagonal, k-th row, l-th column, element ($k \neq l$),

$$0 = \mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x) + \lambda_k D_x^{k,l} - D_x^{k,l}\lambda_l \tag{8}$$

where $D_x^{k,l}$ is the k-th row, l-th column element in the skew symmetric $\mathbf{D}(x)$ and satisfy $D_x^{k,l} = -D_x^{l,k}$. From (8), we can solve $D_x^{k,l}$

$$D_x^{k,l} = \begin{cases}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), k \neq l \\ 0 \qquad\qquad\qquad\qquad , k = l\end{cases} \tag{9}$$

Since $\mathbf{D}(x) \triangleq \mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}$, we have $\frac{\partial \mathbf{U}(x)}{\partial x} = \mathbf{U}(x)\mathbf{D}(x)$. Taking the l-th column on both sides leads to

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \mathbf{U}(x)\mathbf{D}_x^{:,l}, \tag{10}$$

where $\mathbf{D}_x^{:,l} \in \mathbb{R}^3$ represents the l-th column of $\mathbf{D}(x)$. Finally, substituting $\mathbf{U}(x) = [\mathbf{u}_1(x) \quad \mathbf{u}_2(x) \quad \mathbf{u}_3(x)]$ and (9) into (10), we obtain

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k(x)\mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), \tag{11}$$

which yields the second conclusion.

□

Lemma 2. *Given*

1. *Matrices* $\mathbf{C}_j = \begin{bmatrix}\mathbf{P}_j & \mathbf{v}_j \\ \mathbf{v}_j^T & N_j\end{bmatrix} \in \mathbb{S}^{4\times 4}, j = 1, \cdots, M_p$;
2. *Poses* $\mathbf{T}_j \in SE(3), j = 1, \cdots, M_p$;
3. *A matrix* $\mathbf{C} = \begin{bmatrix}\mathbf{P} & \mathbf{v} \\ \mathbf{v}^T & N\end{bmatrix} \triangleq \sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T \in \mathbb{S}^{4\times 4}$ *and a matrix function* $\mathbf{A}(\mathbf{C}) \triangleq \frac{1}{N}\mathbf{P} - \frac{1}{N^2}\mathbf{v}\mathbf{v}^T$;
4. *Two constant eigenvectors* $\mathbf{u}_k, \mathbf{u}_l \in \mathbb{R}^3$,

then the first and second order derivatives of $\mathbf{u}_k^T\mathbf{A}(\mathbf{T})\mathbf{u}_l$ w.r.t. $\mathbf{T}$ are:

$$\mathbf{g}_{kl} \triangleq \frac{\partial \mathbf{u}_k^T \mathbf{A}(\delta \mathbf{T}) \mathbf{u}_l}{\partial \delta \mathbf{T}} = \begin{bmatrix} \cdots & \mathbf{g}_{kl}^{\mathrm{q}} & \cdots \end{bmatrix} \in \mathbb{R}^{1 \times 6M_p}, \quad (12)$$

$$\mathbf{H}_{kl} \triangleq \frac{\partial^2 \mathbf{u}_k^T \mathbf{A}(\delta \mathbf{T}) \mathbf{u}_l}{\partial (\delta \mathbf{T})^2} = \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{H}_{kl}^{\mathrm{p,q}} & \cdots \\ & \vdots & \end{bmatrix} \in \mathbb{R}^{6M_p \times 6M_p}, \quad (13)$$

where $\mathbf{g}_{kl}^{\mathrm{q}} \in \mathbb{R}^{1\times 6}$, $\mathbf{H}_{kl}^{\mathrm{p,q}} \in \mathbb{R}^{6\times 6}, \forall \mathrm{p,q} \in \{1, \cdots, M_p\}$, *are*

$$\mathbf{g}_{kl}^{\mathrm{q}} = \frac{1}{N} \mathbf{u}_l^T \mathbf{S}_{\mathbf{P}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F}) \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_k^T + \frac{1}{N} \mathbf{u}_k^T \mathbf{S}_{\mathbf{P}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F}) \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T \quad (14)$$

$$\mathbf{H}_{kl}^{\mathrm{p,q}} = -\frac{2}{N^2} \mathbf{U}_k \mathbf{T}_{\mathrm{p}} \mathbf{D}_{\mathrm{p,q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \mathbb{1}_{\mathrm{p=q}} \Big(\frac{2}{N} \mathbf{U}_k \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \begin{bmatrix} \mathbf{K}_{kl}^{\mathrm{q}} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \Big) \quad (15)$$

$$\mathbf{K}_{kl}^{\mathrm{q}} = \frac{1}{N} \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_k \rfloor \lfloor \mathbf{u}_l \rfloor + \frac{1}{N} \lfloor \mathbf{u}_k \rfloor \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor \quad (16)$$

where

$$\mathbf{U}_l = \begin{bmatrix} -\lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_l \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \quad (17)$$

$$\mathbf{S}_{\mathbf{P}} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1} \end{bmatrix} \quad \mathbf{D}_{\mathrm{p,q}} = \mathbf{C}_{\mathrm{p}} \mathbf{F} \mathbf{C}_{\mathrm{q}}, \quad (18)$$

and

$$\mathbf{g}_{kl}^{\mathrm{q}} = \mathbf{0}_{1\times 6}, \quad \mathbf{H}_{kl}^{\mathrm{p,q}} = \mathbf{0}_{6\times 6}, \forall k, l, \quad \text{if p or q} \in \boldsymbol{\mathcal{I}} \triangleq \{j | 1 \le j \le M_p, \mathbf{C}_j = \mathbf{0}\}. \quad (19)$$

Proof. Partition the matrix $\mathbf{C}$ as

$$\mathbf{C} = \begin{bmatrix} \mathbf{P} & \mathbf{v} \\ \mathbf{v}^T & N \end{bmatrix} = \sum_{j=1}^{M_p} \mathbf{T}_j \mathbf{C}_j \mathbf{T}_j^T, \quad (20)$$

then

$$\mathbf{P} = \mathbf{S}_{\mathbf{P}} \mathbf{C} \mathbf{S}_{\mathbf{P}}^T = \sum_{j=1}^{M_p} \left(\mathbf{R}_j \mathbf{P}_j \mathbf{R}_j^T + \mathbf{R}_j \mathbf{v}_j \mathbf{t}_j^T + \mathbf{t}_j \mathbf{v}_j^T \mathbf{R}_j^T + N_j \mathbf{t}_j \mathbf{t}_j^T \right), \quad (21)$$

$$\mathbf{v} = \mathbf{S}_{\mathbf{P}} \mathbf{C} \mathbf{S}_{\mathbf{v}}^T = \sum_{j=1}^{M_p} (\mathbf{R}_j \mathbf{v}_j + N_j \mathbf{t}_j), N = \sum_{j=1}^{M_p} N_j, \quad (22)$$

$$\mathbf{S}_{\mathbf{P}} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1} \end{bmatrix} \in \mathbb{R}^{3\times 4}, \quad (23)$$

$$\mathbf{S}_{\mathbf{v}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{1\times 4}. \quad (24)$$

Therefore,

$$\mathbf{A} \left(\sum_{j=1}^{M_p} \mathbf{T}_j \mathbf{C}_j \mathbf{T}_j^T \right) = \frac{1}{N} \mathbf{P} - \frac{1}{N^2} \mathbf{v} \mathbf{v}^T \quad (25)$$

$$= \frac{1}{N} \mathbf{S}_{\mathbf{P}} \left(\mathbf{C} - \frac{1}{N} \mathbf{C} \mathbf{S}_{\mathbf{v}}^T \mathbf{S}_{\mathbf{v}} \mathbf{C}^T \right) \mathbf{S}_{\mathbf{P}}^T \quad (26)$$

$$= \frac{1}{N} \mathbf{S}_{\mathbf{P}} \left(\sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T - \frac{1}{N} \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{C}_{\mathrm{p}} \mathbf{T}_{\mathrm{p}}^T \mathbf{S}_{\mathbf{v}}^T \mathbf{S}_{\mathbf{v}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \right) \mathbf{S}_{\mathbf{P}}^T. \quad (27)$$

Since $\mathbf{T}_{\mathrm{p}}^T \mathbf{S}_{\mathbf{v}}^T \mathbf{S}_{\mathbf{v}} \mathbf{T}_{\mathrm{q}} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$, we obtain

$$\mathbf{A} = \frac{1}{N} \mathbf{S}_{\mathbf{P}} \left(\sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T - \frac{1}{N} \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{D}_{\mathrm{p,q}} \mathbf{T}_{\mathrm{q}}^T \right) \mathbf{S}_{\mathbf{P}}^T, \quad (28)$$

$$\mathbf{D}_{\mathrm{p,q}} = \mathbf{C}_{\mathrm{p}} \mathbf{F} \mathbf{C}_{\mathrm{q}}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \quad (29)$$

where we omitted the input argument of $\mathbf{A}$ for the sake of notation simplicity. Since $N = \sum_{j=1}^{M_p} N_j$ is a constant number that is irrelevant to the pose $\mathbf{T}$, perturbing the input of $\mathbf{A}$, which is the pose vector $\mathbf{T}$, by $\delta \mathbf{T}$ yields

$$\mathbf{u}_k^T \mathbf{A}(\delta \mathbf{T}) \mathbf{u}_l = \frac{1}{N} \mathbf{u}_k^T \mathbf{S}_{\mathbf{P}} \left(\sum_{\mathrm{q}=1}^{M_p} (\mathbf{T}_{\mathrm{q}} \boxplus \delta \mathbf{T}_{\mathrm{q}}) \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} \boxplus \delta \mathbf{T}_{\mathrm{q}})^T - \frac{1}{N} \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} (\mathbf{T}_{\mathrm{p}} \boxplus \delta \mathbf{T}_{\mathrm{p}}) \mathbf{D}_{\mathrm{p,q}} (\mathbf{T}_{\mathrm{q}} \boxplus \delta \mathbf{T}_{\mathrm{q}})^T \right) \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l. \quad (30)$$

Based on the definition of $\boxplus$ on $SE(3)$ in (27), we define,

$$\mathbf{w}_{\mathrm{q}l}(\delta \mathbf{T}_{\mathrm{q}}) \triangleq (\mathbf{T}_{\mathrm{q}} \boxplus \delta \mathbf{T}_{\mathrm{q}})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l = \begin{bmatrix} \mathbf{R}_{\mathrm{q}}^T \exp^T(\lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor) \mathbf{u}_l \\ \mathbf{t}_{\mathrm{q}}^T \exp^T(\lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor) \mathbf{u}_l + \mathbf{u}_l^T \delta \mathbf{t}_{\mathrm{q}}. \end{bmatrix} \quad (31)$$

When $\delta \boldsymbol{\phi}_{\mathrm{q}}$ is small, which is indeed the case for the purpose of derivative computation, we have

$$\exp(\lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor) \approx \mathbf{I} + \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor + \frac{1}{2} \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2 \quad (32)$$

Substituting (32) into $\mathbf{w}_{\mathrm{q}l}(\delta \mathbf{T}_{\mathrm{q}})$, we obtain

$$\mathbf{w}_{\mathrm{q}l}(\delta \mathbf{T}_{\mathrm{q}}) \approx \begin{bmatrix} \mathbf{R}_{\mathrm{q}}^T (\mathbf{I} - \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor + \frac{1}{2} \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2) \mathbf{u}_l \\ \mathbf{t}_{\mathrm{q}}^T (\mathbf{I} - \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor + \frac{1}{2} \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2) \mathbf{u}_l + \mathbf{u}_l^T \delta \mathbf{t}_{\mathrm{q}} \end{bmatrix}$$
$$\approx \underbrace{\begin{bmatrix} \mathbf{R}_{\mathrm{q}}^T \mathbf{u}_l \\ \mathbf{t}_{\mathrm{q}}^T \mathbf{u}_l \end{bmatrix}}_{\bar{\mathbf{w}}_{\mathrm{q}l}} + \underbrace{\begin{bmatrix} \mathbf{R}_{\mathrm{q}}^T \lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times 3} \\ \mathbf{t}_{\mathrm{q}}^T \lfloor \mathbf{u}_l \rfloor & \mathbf{u}_l^T \end{bmatrix}}_{\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} \in \mathbb{R}^{4\times 6}} \underbrace{\begin{bmatrix} \delta \boldsymbol{\phi}_{\mathrm{q}} \\ \delta \mathbf{t}_{\mathrm{q}} \end{bmatrix}}_{\delta \mathbf{T}_{\mathrm{q}}} + \underbrace{\begin{bmatrix} \frac{1}{2} \mathbf{R}_{\mathrm{q}}^T \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2 \mathbf{u}_l \\ \frac{1}{2} \mathbf{t}_{\mathrm{q}}^T \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2 \mathbf{u}_l \end{bmatrix}}_{\delta \boldsymbol{\Phi}_{\mathrm{q}l}} \quad (33)$$

where $\bar{\mathbf{w}}_{\mathrm{q}l}$, $\delta \boldsymbol{\Phi}_{\mathrm{q}l}$ and $\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}}$ can be simplified

$$\bar{\mathbf{w}}_{\mathrm{q}l} = (\mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}})^T \mathbf{u}_l \quad \delta \boldsymbol{\Phi}_{\mathrm{q}l} = \frac{1}{2} (\mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}})^T \lfloor \delta \boldsymbol{\phi}_{\mathrm{q}} \rfloor^2 \mathbf{u}_l$$
$$\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} = \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T \quad \mathbf{U}_l = \begin{bmatrix} -\lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_l \end{bmatrix} \quad (34)$$

Substituting (33) into (30) and keeping terms up to the second order lead to

$$\mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = \frac{1}{N}\sum_{\mathrm{q}=1}^{M_p}\mathbf{w}_{\mathrm{q}k}^T(\delta\mathbf{T}_\mathrm{q})\mathbf{C}_\mathrm{q}\mathbf{w}_{\mathrm{q}l}(\delta\mathbf{T}_\mathrm{q}) - \frac{1}{N^2}\sum_{\mathrm{p}=1}^{M_p}\sum_{\mathrm{q}=1}^{M_p}\mathbf{w}_{\mathrm{p}k}^T(\delta\mathbf{T}_\mathrm{p})\mathbf{D}_{\mathrm{p,q}}\mathbf{w}_{\mathrm{q}l}(\delta\mathbf{T}_\mathrm{q}) \quad (35)$$

$$= \frac{1}{N}\sum_{\mathrm{q}=1}^{M_p}\Big((\bar{\mathbf{w}}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} + \bar{\mathbf{w}}_{\mathrm{q}l}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}k}})\delta\mathbf{T}_\mathrm{q} + \bar{\mathbf{w}}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\bar{\mathbf{w}}_{\mathrm{q}l} + \delta\mathbf{T}_\mathrm{q}^T\mathbf{J}_{\mathbf{w}_{\mathrm{q}k}}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}}\delta\mathbf{T}_\mathrm{q} + \underbrace{\bar{\mathbf{w}}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\delta\mathbf{\Phi}_{\mathrm{q}l} + \delta\mathbf{\Phi}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\bar{\mathbf{w}}_{\mathrm{q}l}}_{\frac{1}{2}\delta\mathbf{T}_\mathrm{q}^T\mathbf{Q}_{kl}^\mathrm{q}\delta\mathbf{T}_\mathrm{q}}\Big)$$
$$- \frac{1}{N^2}\sum_{\mathrm{p}=1}^{M_p}\sum_{\mathrm{q}=1}^{M_p}\Big(\delta\mathbf{T}_\mathrm{p}^T\mathbf{J}_{\mathbf{w}_{\mathrm{p}k}}^T\mathbf{D}_{\mathrm{p,q}}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}}\delta\mathbf{T}_\mathrm{q} + \bar{\mathbf{w}}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}}\delta\mathbf{T}_\mathrm{q} + \delta\mathbf{T}_\mathrm{p}^T\mathbf{J}_{\mathbf{w}_{\mathrm{p}k}}^T\mathbf{D}_{\mathrm{p,q}}\bar{\mathbf{w}}_{\mathrm{q}l} + \bar{\mathbf{w}}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\bar{\mathbf{w}}_{\mathrm{q}l} + \underbrace{\bar{\mathbf{w}}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\delta\mathbf{\Phi}_{\mathrm{q}l}}_{\frac{1}{2}\delta\mathbf{T}_\mathrm{q}^T\mathbf{N}_{kl}^{\mathrm{p,q}}\delta\mathbf{T}_\mathrm{q}} + \underbrace{\delta\mathbf{\Phi}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\bar{\mathbf{w}}_{\mathrm{q}l}}_{\frac{1}{2}\delta\mathbf{T}_\mathrm{p}^T\mathbf{M}_{kl}^{\mathrm{p,q}}\delta\mathbf{T}_\mathrm{p}}\Big). \quad (36)$$

To compute $\mathbf{Q}_{kl}^\mathrm{q}$ in $\frac{1}{2}\delta\mathbf{T}_\mathrm{q}^T\mathbf{Q}_{kl}^\mathrm{q}\delta\mathbf{T}_\mathrm{q}$, note that $\mathbf{a}^T\lfloor\delta\boldsymbol{\phi}\rfloor^2\mathbf{b} = \delta\boldsymbol{\phi}^T\lfloor\mathbf{a}\rfloor\lfloor\mathbf{b}\rfloor\delta\boldsymbol{\phi}, \forall\mathbf{a},\mathbf{b},\delta\boldsymbol{\phi}\in\mathbb{R}^3$, we have

$$\bar{\mathbf{w}}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\delta\mathbf{\Phi}_{\mathrm{q}l} = \frac{1}{2}\mathbf{u}_k^T\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\lfloor\delta\boldsymbol{\phi}_\mathrm{q}\rfloor^2\mathbf{u}_l = \frac{1}{2}\delta\boldsymbol{\phi}_\mathrm{q}^T\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_\mathrm{q}. \quad (37)$$

Similarly,

$$\delta\mathbf{\Phi}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\bar{\mathbf{w}}_{\mathrm{q}l} = \frac{1}{2}\delta\boldsymbol{\phi}_\mathrm{q}^T\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_\mathrm{q}. \quad (38)$$

Summing (37) and (38) and extending the $\delta\boldsymbol{\phi}$ into $\delta\mathbf{T}$:

$$\mathbf{Q}_{kl}^\mathrm{q} = \begin{bmatrix}\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor + \lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \quad (39)$$

For $\frac{1}{2}\delta\mathbf{T}_\mathrm{q}^T\mathbf{N}_{kl}^{\mathrm{p,q}}\delta\mathbf{T}_\mathrm{q}$ and $\frac{1}{2}\delta\mathbf{T}_\mathrm{p}^T\mathbf{M}_{kl}^{\mathrm{p,q}}\delta\mathbf{T}_\mathrm{p}$,

$$\bar{\mathbf{w}}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\delta\mathbf{\Phi}_{\mathrm{q}l} = \frac{1}{2}\mathbf{u}_k^T\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\lfloor\delta\boldsymbol{\phi}_\mathrm{q}\rfloor^2\mathbf{u}_l = \frac{1}{2}\delta\boldsymbol{\phi}_\mathrm{q}^T\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{D}_{\mathrm{q,p}}\mathbf{T}_\mathrm{p}^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_\mathrm{q} \quad (40)$$

$$\delta\mathbf{\Phi}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\bar{\mathbf{w}}_{\mathrm{q}l} = \frac{1}{2}\mathbf{u}_k^T\lfloor\delta\boldsymbol{\phi}_\mathrm{p}\rfloor^2\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_l = \frac{1}{2}\delta\boldsymbol{\phi}_\mathrm{p}^T\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_\mathrm{p} \quad (41)$$

where $\mathbf{D}_{\mathrm{q,p}} = \mathbf{D}_{\mathrm{p,q}}^T$. Thus, extending the $\delta\boldsymbol{\phi}$ into $\delta\mathbf{T}$, we obtain

$$\mathbf{N}_{kl}^{\mathrm{p,q}} = \begin{bmatrix}\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{D}_{\mathrm{q,p}}\mathbf{T}_\mathrm{p}^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \quad (42)$$

$$\mathbf{M}_{kl}^{\mathrm{p,q}} = \begin{bmatrix}\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{S_P}^T\mathbf{u}_l\rfloor & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \quad (43)$$

It can be seen that (36) is quadratic w.r.t. $\delta\mathbf{T}$, so we cast it into the following standard form

$$\mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = \frac{1}{2}\delta\mathbf{T}^T\cdot\mathbf{H}_{kl}\cdot\delta\mathbf{T} + \mathbf{g}_{kl}\cdot\delta\mathbf{T} + r_{kl}, \quad (44)$$

where $\mathbf{g}_{kl}$ is

$$\mathbf{g}_{kl} = \begin{bmatrix}\cdots & \mathbf{g}_{kl}^\mathrm{q} & \cdots\end{bmatrix} \in \mathbb{R}^{1\times6M_p} \quad (45)$$

$$\mathbf{g}_{kl}^\mathrm{q} = \frac{1}{N}(\bar{\mathbf{w}}_{\mathrm{q}k}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} + \bar{\mathbf{w}}_{\mathrm{q}l}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}k}}) - \frac{1}{N^2}\sum_{\mathrm{p}=1}^{M_p}(\bar{\mathbf{w}}_{\mathrm{p}k}^T\mathbf{D}_{\mathrm{p,q}}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} + \bar{\mathbf{w}}_{\mathrm{p}l}^T\mathbf{D}_{\mathrm{p,q}}\mathbf{J}_{\mathbf{w}_{\mathrm{q}k}}) \quad (46)$$

$$= \frac{1}{N}(\mathbf{u}_k^T\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T + \mathbf{u}_l^T\mathbf{S_P}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_k^T) - \frac{1}{N^2}\sum_{\mathrm{p}=1}^{M_p}(\mathbf{u}_k^T\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{C}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T + \mathbf{u}_l^T\mathbf{S_P}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}}\mathbf{T}_\mathrm{q}^T\mathbf{U}_k^T)$$

$$= \frac{1}{N}\mathbf{u}_l^T\mathbf{S_P}(\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q} - \frac{1}{N}\sum_{\mathrm{p}=1}^{M_p}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}})\mathbf{T}_\mathrm{q}^T\mathbf{U}_k^T + \frac{1}{N}\mathbf{u}_k^T\mathbf{S_P}(\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q} - \frac{1}{N}\sum_{\mathrm{p}=1}^{M_p}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}})\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T \quad (47)$$

where $\mathbf{D}_{\mathrm{p,q}} = \mathbf{C}_\mathrm{p}\mathbf{F}\mathbf{C}_\mathrm{q}$ and $\mathbf{F} = \begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}$.

Since $\mathbf{v} = \sum_{j=1}^{M_p}(\mathbf{R}_j\mathbf{v}_j + N_j\mathbf{t}_j)$,

$$\sum_{\mathrm{p}=1}^{M_p}\mathbf{T}_\mathrm{p}\mathbf{C}_\mathrm{p}\mathbf{F} = \sum_{\mathrm{p}=1}^{M_p}\begin{bmatrix}\mathbf{R}_\mathrm{p} & \mathbf{t}_\mathrm{p}\\ \mathbf{0} & 1\end{bmatrix}\begin{bmatrix}\mathbf{P}_\mathrm{p} & \mathbf{v}_\mathrm{p}\\ \mathbf{v}_\mathrm{p}^T & N_\mathrm{p}\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix} = \sum_{\mathrm{p}=1}^{M_p}\begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{R}_\mathrm{p}\mathbf{v}_\mathrm{p} + N_\mathrm{p}\mathbf{t}_\mathrm{p}\\ \mathbf{0}_{1\times3} & N_\mathrm{p}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{v}\\ \mathbf{0} & N_\mathrm{p}\end{bmatrix} = \mathbf{CF}. \quad (48)$$

The common part in (47) is

$$\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q} - \frac{1}{N}\sum_{\mathrm{p}=1}^{M_p}\mathbf{T}_\mathrm{p}\mathbf{D}_{\mathrm{p,q}} = \mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q} - \frac{1}{N}\underbrace{\sum_{\mathrm{p}=1}^{M_p}\mathbf{T}_\mathrm{p}\mathbf{C}_\mathrm{p}\mathbf{F}}_{\mathbf{CF}}\mathbf{C}_\mathrm{q} = (\mathbf{T}_\mathrm{q} - \frac{1}{N}\mathbf{CF})\mathbf{C}_\mathrm{q}. \quad (49)$$

Therefore,

$$\mathbf{g}_{kl}^\mathrm{q} = \frac{1}{N}\mathbf{u}_l^T\mathbf{S_P}(\mathbf{T}_\mathrm{q} - \frac{1}{N}\mathbf{CF})\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_k^T + \frac{1}{N}\mathbf{u}_k^T\mathbf{S_P}(\mathbf{T}_\mathrm{q} - \frac{1}{N}\mathbf{CF})\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T. \quad (50)$$

Additionally, it is seen that if $\mathbf{C}_\mathrm{q} = \mathbf{0}_{4\times4}$, $\mathbf{g}_{kl}^\mathrm{q} = \mathbf{0}_{1\times6}, \forall k,l$.

For $\mathbf{H}_{kl}$ in (44), partition it as

$$\mathbf{H}_{kl} = \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{H}_{kl}^{\mathrm{p,q}} & \cdots\\ & \vdots & \end{bmatrix} \in \mathbb{R}^{6M_p\times6M_p} \quad (51)$$

$$\mathbf{H}_{kl}^{\mathrm{p,q}} = -\frac{2}{N^2}\mathbf{J}_{\mathbf{w}_{\mathrm{p}k}}^T\mathbf{D}_{\mathrm{p,q}}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} + \mathbb{1}_{\mathrm{p}=\mathrm{q}}\Big(\frac{2}{N}\mathbf{J}_{\mathbf{w}_{\mathrm{q}k}}^T\mathbf{C}_\mathrm{q}\mathbf{J}_{\mathbf{w}_{\mathrm{q}l}} + \frac{1}{N}\mathbf{Q}_{kl}^\mathrm{q} - \frac{1}{N^2}\sum_{\nu=1}^{M_p}(\mathbf{N}_{kl}^{\nu,\mathrm{q}} + \mathbf{M}_{kl}^{\mathrm{q},\nu})\Big) \quad (52)$$

where

$$\mathbf{J}_{\mathbf{w}_{pk}}^T \mathbf{D}_{p,q} \mathbf{J}_{\mathbf{w}_{ql}} = \mathbf{U}_k \mathbf{T}_p \mathbf{D}_{p,q} \mathbf{T}_q^T \mathbf{U}_l^T \tag{53}$$

$$\mathbf{J}_{\mathbf{w}_{qk}}^T \mathbf{C}_q \mathbf{J}_{\mathbf{w}_{ql}} = \mathbf{U}_k \mathbf{T}_q \mathbf{C}_q \mathbf{T}_q^T \mathbf{U}_l^T \tag{54}$$

$$\sum_{\nu=1}^{M_p} (\mathbf{N}_{kl}^{\nu,q} + \mathbf{M}_{kl}^{q,\nu}) = \sum_{\nu=1}^{M_p} \begin{bmatrix} \lfloor \mathbf{S_P} \mathbf{T}_q \mathbf{D}_{q,\nu} \mathbf{T}_\nu^T \mathbf{S_P}^T \mathbf{u}_k \rfloor \lfloor \mathbf{u}_l \rfloor + \lfloor \mathbf{u}_k \rfloor \lfloor \mathbf{S_P} \mathbf{T}_q \mathbf{D}_{q,\nu} \mathbf{T}_\nu^T \mathbf{S_P}^T \mathbf{u}_l \rfloor & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{55}$$

By using similar method in (48), we have

$$\sum_{\nu=1}^{M_p} \mathbf{D}_{q,\nu} \mathbf{T}_\nu^T = \sum_{\nu=1}^{M_p} \mathbf{C}_q \mathbf{F} \mathbf{C}_\nu \mathbf{T}_\nu^T = \mathbf{C}_q \mathbf{F} \mathbf{C} \tag{56}$$

The value of $\mathbf{Q}_{kl}^q$ is in (39). Thus, the matrix $\mathbf{H}_{kl}^{p,q}$ is

$$\mathbf{H}_{kl}^{p,q} = -\frac{2}{N^2} \mathbf{U}_k \mathbf{T}_p \mathbf{D}_{p,q} \mathbf{T}_q^T \mathbf{U}_l^T + \mathbb{1}_{p=q} \Big(\frac{2}{N} \mathbf{U}_k \mathbf{T}_q \mathbf{C}_q \mathbf{T}_q^T \mathbf{U}_l^T + \begin{bmatrix} \mathbf{K}_{kl}^q & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \Big) \tag{57}$$

$$\begin{aligned} \mathbf{K}_{kl}^q &= \frac{1}{N} \mathbf{Q}_{kl}^q - \frac{1}{N^2} \sum_{\nu=1}^{M_p} (\mathbf{N}_{kl}^{\nu,q} + \mathbf{M}_{kl}^{q,\nu}) \\ &= \frac{1}{N} \lfloor \mathbf{S_P} \mathbf{T}_q \mathbf{C}_q (\mathbf{T}_q - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S_P}^T \mathbf{u}_k \rfloor \lfloor \mathbf{u}_l \rfloor \\ &\quad + \frac{1}{N} \lfloor \mathbf{u}_k \rfloor \lfloor \mathbf{S_P} \mathbf{T}_q \mathbf{C}_q (\mathbf{T}_q - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S_P}^T \mathbf{u}_l \rfloor, \end{aligned} \tag{58}$$

which yields the solution. Additionally, it can be seen that if $\mathbf{C}_q = \mathbf{0}_{4\times4}$ or $\mathbf{C}_p = \mathbf{0}_{4\times4}$, we have $\mathbf{D}_{p,q} = \mathbf{C}_p \mathbf{F} \mathbf{C}_q = \mathbf{0}_{4\times4}$, $\mathbf{K}_{kl}^q = \mathbf{0}_{3\times3}$ and then $\mathbf{H}_{kl}^{p,q} = \mathbf{0}_{6\times6}, \forall k, l$. □

II. Proof of Theorems

A. Proof of formula (6) and (8)

Proof. The variable to be optimized is $\boldsymbol{\pi}_i = (\mathbf{n}_i, \mathbf{q}_i)$ and the cost function is

$$c_i = \min_{\boldsymbol{\pi}_i = (\mathbf{n}_i, \mathbf{q}_i)} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \big\| \mathbf{h}_i^T (\mathbf{p}_{ijk} - \mathbf{q}_i) \big\|_2^2 \Big) \tag{59}$$

where $\mathbf{h}_i = \mathbf{n}_i$ for plane feature and $\mathbf{h}_i = (\mathbf{I} - \mathbf{n}_i \mathbf{n}_i^T)$ for edge feature. The dimensions of $\mathbf{h}_i$ may be different for these two features but it has no influence on following derivation.

$$\begin{aligned} c_i &= \min_{\mathbf{n}_i} \min_{\mathbf{q}_i} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \big\| \mathbf{h}_i^T (\mathbf{p}_{ijk} - \mathbf{q}_i) \big\|_2^2 \Big) \\ &= \min_{\mathbf{n}_i} \left(\min_{\mathbf{q}_i} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \mathbf{q}_i)^T \mathbf{h}_i \mathbf{h}_i^T (\mathbf{p}_{ijk} - \mathbf{q}_i) \Big) \right). \end{aligned} \tag{60}$$

As can be seen, the inner optimization on $\mathbf{q}_i$ is a standard quadratic optimization problem. So, the optimum $\mathbf{q}_i^\star$ can be solved by setting the derivative to zero:

$$\begin{aligned} 2 \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \mathbf{h}_i \mathbf{h}_i^T (\mathbf{p}_{ijk} - \mathbf{q}_i) = \mathbf{0} \implies \\ 2 \mathbf{h}_i \mathbf{h}_i^T \left(\sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \mathbf{p}_{ijk} - N_i \mathbf{q}_i \right) = \mathbf{0} \end{aligned} \tag{61}$$

where $N_i = \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} 1 = \sum_{j=1}^{M_p} N_{ij}$. This equation does not lead to a unique solution of $\mathbf{q}_i$, one particular optimum solution is $\mathbf{q}_i^\star = \frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \mathbf{p}_{ijk} = \bar{\mathbf{p}}_i$ as is defined in (7).

Now, substituting the optimum solution $\mathbf{q}_i^\star = \bar{\mathbf{p}}_i$ into (60) leads to:

$$c_i = \min_{\mathbf{n}_i} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \big\| \mathbf{h}_i^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \big\|_2^2 \Big) \tag{62}$$

To solve for the optimal parameter $\mathbf{n}_i$ in the above optimization problem, we discuss the case of plane and edge features separately, as follows.

1) Plane feature: $\mathbf{h}_i = \mathbf{n}_i$.

$$\begin{aligned} c_i &= \min_{\|\mathbf{n}_i\|_2 = 1} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \big\| \mathbf{n}_i^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \big\|_2^2 \Big) \\ &= \min_{\|\mathbf{n}_i\|_2 = 1} \Big(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \big(\mathbf{n}_i^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)(\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T \mathbf{n}_i \big) \Big) \\ &= \min_{\mathbf{n}_i} \mathbf{n}_i^T \mathbf{A}_i \mathbf{n}_i, \end{aligned} \tag{63}$$

where $\mathbf{A}_i$ is defined in (7) and is a symmetric matrix. Performing Singular Value Decomposition (SVD) of $\mathbf{A}_i$

$$\mathbf{A}_i = \mathbf{U}_i \boldsymbol{\Lambda}_i \mathbf{U}_i^T \tag{64}$$

where

$$\mathbf{U}_i = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \quad \boldsymbol{\Lambda}_i = \mathrm{diag}(\lambda_1 \quad \lambda_2 \quad \lambda_3) \tag{65}$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\mathbf{U}_i^T \mathbf{U}_i = \mathbf{I}$.

Denote $\mathbf{m} = \mathbf{U}_i \mathbf{n}_i = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T$, $\|\mathbf{m}\|_2 = \sqrt{\mathbf{n}_i^T \mathbf{U}_i^T \mathbf{U}_i \mathbf{n}_i} = 1$, then (63) reduces to

$$\begin{aligned} c_i &= \min_{\|\mathbf{n}_i\|_2 = 1} (\mathbf{n}_i^T \mathbf{U}_i \boldsymbol{\Lambda}_i \mathbf{U}_i^T \mathbf{n}_i) = \min_{\|\mathbf{m}\|_2 = 1} (\mathbf{m}^T \boldsymbol{\Lambda}_i \mathbf{m}) \\ &= \min_{\|\mathbf{m}\|_2 = 1} (\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2) \\ &\geq \min_{\|\mathbf{m}\|_2 = 1} (\lambda_3 m_1^2 + \lambda_3 m_2^2 + \lambda_3 m_3^2) = \lambda_3, \end{aligned} \tag{66}$$

where the minimum value λ_3 is reached when $m_3 = 1$, i.e., $\mathbf{m}^\star = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\mathbf{n}_i^\star = \mathbf{U}_i \mathbf{m}^\star = \mathbf{u}_3$.

Therefore, the optimal cost is $\lambda_3(\mathbf{A}_i)$ and the optimum solution is $\mathbf{n}^\star = \mathbf{u}_3(\mathbf{A}_i)$ and $\mathbf{q}^\star = \bar{\mathbf{p}}_i$.

2) Edge feature: $\mathbf{h}_i = \mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T$, then $\mathbf{h}_i\mathbf{h}_i^T = \mathbf{h}_i$ and

$$\begin{aligned} c_i =& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \left\| (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)(\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right\|_2^2 \right) \\ =& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)(\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right) \\ =& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right. \\ & \left. -\mathbf{n}_i^T \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right) \mathbf{n}_i \right) \\ =& \min_{\mathbf{n}_i} (\operatorname{trace}(\mathbf{A}_i) - \mathbf{n}_i^T \mathbf{A}_i \mathbf{n}_i) \\ =& \lambda_1 + \lambda_2 + \lambda_3 - \underbrace{\max_{\mathbf{n}_i} \mathbf{n}_i^T \mathbf{A}_i \mathbf{n}_i}_{=\lambda_1, \text{ when } , \mathbf{n}_i^\star = \mathbf{u}_1} \\ =& \lambda_2 + \lambda_3. \end{aligned} \tag{67}$$

Therefore, the optimal cost is $\lambda_2(\mathbf{A}_i) + \lambda_3(\mathbf{A}_i)$ and the optimum solution is $\mathbf{n}^\star = \mathbf{u}_1(\mathbf{A}_i)$ and $\mathbf{q}^\star = \bar{\mathbf{p}}_i$.

□

B. Proof of Theorem 1

For the point collections $\boldsymbol{\mathcal{C}} = \{\mathbf{p}_k \in \mathbb{R}^3 | k = 1, \cdots, n\}$, its point cluster is

$$\mathfrak{R}(\boldsymbol{\mathcal{C}}) = \sum_{k=1}^{n} \begin{bmatrix} \mathbf{p}_k \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k^T & 1 \end{bmatrix} \tag{68}$$

The rigid transformation of $\boldsymbol{\mathcal{C}}$ by pose $\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix}$ is

$$\mathbf{T} \circ \boldsymbol{\mathcal{C}} = \{\mathbf{R}\mathbf{p}_k + \mathbf{t} \in \mathbb{R}^3 | k = 1, \cdots, n\} \tag{69}$$

whose point cluster is

$$\begin{aligned} \mathfrak{R}(\mathbf{T} \circ \boldsymbol{\mathcal{C}}) &= \sum_{k=1}^{n} \begin{bmatrix} \mathbf{R}\mathbf{p}_k + \mathbf{t} \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{R}\mathbf{p}_k + \mathbf{t})^T & 1 \end{bmatrix} \\ &= \sum_{k=1}^{n} \mathbf{T} \begin{bmatrix} \mathbf{p}_k \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k^T & 1 \end{bmatrix} \mathbf{T}^T = \mathbf{T}\mathfrak{R}(\boldsymbol{\mathcal{C}})\mathbf{T}^T \end{aligned} \tag{70}$$

which yields the solution. □

C. Proof of Theorem 2

For two point collections $\boldsymbol{\mathcal{C}}_1 = \{\mathbf{p}_k^1 \in \mathbb{R}^3 | k = 1, \cdots, n_1\}$ and $\boldsymbol{\mathcal{C}}_2 = \{\mathbf{p}_k^2 \in \mathbb{R}^3 | k = 1, \cdots, n_2\}$, their point clusters are

$$\mathfrak{R}(\boldsymbol{\mathcal{C}}_i) = \sum_{k=1}^{n_l} \begin{bmatrix} \mathbf{p}_k^l \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^l)^T & 1 \end{bmatrix} \quad l = 1, 2 \tag{71}$$

The merge of $\boldsymbol{\mathcal{C}}_1$ and $\boldsymbol{\mathcal{C}}_2$ is

$$\boldsymbol{\mathcal{C}}_1 \oplus \boldsymbol{\mathcal{C}}_2 = \{\mathbf{p}_k^l \in \mathbb{R}^3 | l = 1, 2; k = 1, \cdots, n_l\} \tag{72}$$

whose point cluster is

$$\begin{aligned} \mathfrak{R}(\boldsymbol{\mathcal{C}}_1 \oplus \boldsymbol{\mathcal{C}}_2) &= \sum_{k=1}^{n_1} \begin{bmatrix} \mathbf{p}_k^1 \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^1)^T & 1 \end{bmatrix} + \sum_{k=1}^{n_2} \begin{bmatrix} \mathbf{p}_k^2 \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^2)^T & 1 \end{bmatrix} \\ &= \mathfrak{R}(\boldsymbol{\mathcal{C}}_1) + \mathfrak{R}(\boldsymbol{\mathcal{C}}_2) \end{aligned} \tag{73}$$

which yields the solution. □

D. Proof of Theorem 3

Let $\mathbf{T}_0 = \begin{bmatrix} \mathbf{R}_0 & \mathbf{t}_0 \\ 0 & 1 \end{bmatrix}$ and $\bar{\mathbf{C}} = \mathbf{T}_0\mathbf{C}\mathbf{T}_0^T = \begin{bmatrix} \bar{\mathbf{P}} & \bar{\mathbf{v}} \\ \bar{\mathbf{v}}^T & N \end{bmatrix}$, then

$$\bar{\mathbf{P}} = \mathbf{R}_0\mathbf{P}_0\mathbf{R}_0^T + \mathbf{R}_0\mathbf{v}_0\mathbf{t}_0^T + \mathbf{t}_0\mathbf{v}_0^T\mathbf{R}_0^T + N\mathbf{t}_0\mathbf{t}_0^T, \tag{74}$$

$$\bar{\mathbf{v}} = \mathbf{R}_0\mathbf{v}_0 + N\mathbf{t}_0, \tag{75}$$

$$\mathbf{A}\left(\mathbf{T}_0\mathbf{C}\mathbf{T}_0^T\right) = \frac{1}{N}\bar{\mathbf{P}} - \frac{1}{N^2}\bar{\mathbf{v}}\bar{\mathbf{v}}^T = \mathbf{R}_0\mathbf{A}(\mathbf{C})\mathbf{R}_0^T. \tag{76}$$

Since $\mathbf{A}\left(\mathbf{T}_0\mathbf{C}\mathbf{T}_0^T\right)$ and $\mathbf{A}(\mathbf{C})$ are similar by transformation $\mathbf{R}_0$, they have the same eigenvalue. □

E. Proof of Theorem 4

Denote λ_l the l-th largest eigenvalue of $\mathbf{A}$ and $\mathbf{u}_l$ the corresponding vector, i.e., $\lambda_l\mathbf{u}_l = \mathbf{A}\mathbf{u}_l$. Since $\mathbf{A}$ is symmetric, $\mathbf{u}_l$ is an orthonormal vector. Multiplying both sides of $\lambda_l\mathbf{u}_l = \mathbf{A}\mathbf{u}_l$ by $\mathbf{u}_l^T$ leads to

$$\lambda_l = \mathbf{u}_l^T\mathbf{A}\mathbf{u}_l. \tag{77}$$

Note that in the above equation, $\lambda_l, \mathbf{u}_l$ and $\mathbf{A}_l$ all depend on the pose $\mathbf{T}$. To avoid any confusion, we write them as explicit functions of $\mathbf{T}$:

$$\lambda_l(\mathbf{T}) = \mathbf{u}_l^T(\mathbf{T})\mathbf{A}(\mathbf{T})\mathbf{u}_l(\mathbf{T}). \tag{78}$$

Parameterizing the pose $\mathbf{T}$ by $\delta\mathbf{T}$ leads to

$$\lambda_l(\delta\mathbf{T}) = \mathbf{u}_l^T(\delta\mathbf{T})\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l(\delta\mathbf{T}). \tag{79}$$

From the first conclusion in Lemma 1, we know that for a vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \in \mathbb{R}^m$ that the matrix $\mathbf{A}$ depends on, we have

$$\frac{\partial\lambda_l(\mathbf{x})}{\partial x_i} = \frac{\partial\left(\mathbf{u}_l^T(\mathbf{x})\mathbf{A}(\mathbf{x})\mathbf{u}_l(\mathbf{x})\right)}{\partial x_i} = \mathbf{u}_l^T(\mathbf{x})\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x}), \quad \forall i = 1, \cdots, m, x_i \in \mathbb{R} \tag{80}$$

To avoid the use of a tensor when applying this derivative to the complete parameter vector $\mathbf{x}$, we fix the vector $\mathbf{u}_l(\mathbf{x})$ at its current value and lump it with the matrix $\mathbf{A}(\mathbf{x})$ within the derivative, i.e.,

$$\mathbf{u}_l^T(\mathbf{x})\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x}) = \frac{\partial\mathbf{u}_l^T\mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i} \tag{81}$$

where on the right hand side, $\mathbf{u}_l$ is fixed (so we remove its argument $\mathbf{x}$) and the derivative is only applied on the component $\mathbf{A}(\mathbf{x})$ (so we keep its argument $\mathbf{x}$).

Now, the input parameter is the poses parameterized by $\delta\mathbf{T}$, setting $\mathbf{x}$ to $\delta\mathbf{T}$) in (81) leads to

$$\frac{\partial\lambda_l(\delta\mathbf{T})}{\partial\delta\mathbf{T}} = \frac{\partial\mathbf{u}_l^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial\delta\mathbf{T}}. \tag{82}$$

Recalling the results from Lemma 2 with $k = l$, we obtain:

$$\mathbf{u}_l^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = \frac{1}{2}\delta\mathbf{T}^T \cdot \mathbf{H}_{ll} \cdot \delta\mathbf{T} + \mathbf{g}_{ll} \cdot \delta\mathbf{T} + r_{ll}. \tag{83}$$

Therefore, the first order derivative of $\lambda_l(\mathbf{T})$ w.r.t. $\mathbf{T}$ is

$$\mathbf{J} \triangleq \frac{\partial\lambda_l(\delta\mathbf{T})}{\partial\delta\mathbf{T}} = \frac{\partial\mathbf{u}_l^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial\delta\mathbf{T}} = \mathbf{g}_{ll}. \tag{84}$$

Now, we derive the second order derivative. From (80), we have,

$$\frac{\partial \lambda_l(\mathbf{x})}{\partial x_i} = \mathbf{u}_l^T(\mathbf{x})\frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x}), \forall x_i \in \mathbb{R}, \tag{85}$$

Differentiating it w.r.t. the second parameter $x_j \in \mathbb{R}$ leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\mathbf{x})}{\partial x_j \partial x_i} &= \frac{\partial}{\partial x_j}\left(\mathbf{u}_l^T(\mathbf{x})\frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x})\right)\\ &= \left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial x_j}\right)^T\left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \mathbf{u}_l^T\mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right)\\ &+ \left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right)^T\left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial x_j}\right)\end{aligned} \tag{86}$$

Applying the above results to each elements x_i, x_j leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\mathbf{x})}{\partial \mathbf{x}^2} &= \left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right) \qquad (87)\\ &+ \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{u}_l^T\mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}\right).\end{aligned}$$

To compute $\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}$, we apply the second conclusion in Lemma 1 to all components of $\mathbf{x}$ and use the notation trick similar to (81):

$$\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}} = \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{u}_k^T\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}} \tag{88}$$

Now, the input parameter is the pose vector parameterized by $\delta\mathbf{T}$, substituting $\mathbf{x} = \delta\mathbf{T}$ into (87) leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}^2} &= \underbrace{\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)}_{\mathbf{H}_\mathrm{l}}\\ &+ \underbrace{\frac{\partial^2 \mathbf{u}_l^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}^2}}_{\mathbf{H}_{ll}} + \underbrace{\left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right)}_{\mathbf{H}_\mathrm{l}^T},\end{aligned} \tag{89}$$

where term $\mathbf{H}_{ll}$ is from (83). To obtain the term $\mathbf{H}_\mathrm{l}$, we substitute $\mathbf{x} = \delta\mathbf{T}$ into (88):

$$\begin{aligned}\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}} &= \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{u}_k^T\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}} &(90)\\ &= \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\frac{\partial \mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}, &(91)\\ &= \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{g}_{kl}. &(92)\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{H}_\mathrm{l} &= \left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right) &(93)\\ &= \left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{g}_{kl} &(94)\\ &= \sum_{k=1,k\neq l}^{3}\left(\frac{\partial \mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\frac{1}{\lambda_l - \lambda_k}\mathbf{g}_{kl} &(95)\\ &= \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{g}_{kl}^T\mathbf{g}_{kl}. &(96)\end{aligned}$$

Therefore, the second order derivative of $\lambda_l(\mathbf{T})$ w.r.t. $\mathbf{T}$ is

$$\begin{aligned}\mathbf{H} &\triangleq \frac{\partial^2 \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}^2} = \mathbf{H}_\mathrm{l} + \mathbf{H}_{ll} + \mathbf{H}_\mathrm{l}^T\\ &= \mathbf{H}_{ll} + \sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l - \lambda_k}\mathbf{g}_{kl}^T\mathbf{g}_{kl}.\end{aligned} \tag{97}$$

Next, we show that

$$\mathbf{J}\delta\mathbf{T} = 0, \delta\mathbf{T}^T\mathbf{H}\delta\mathbf{T} = 0, \delta\mathbf{T} = \begin{bmatrix}\mathbf{w}\\ \vdots\\ \mathbf{w}\end{bmatrix}, \forall \mathbf{w} \in \mathbb{R}^6 \tag{98}$$

From (84), we can obtain

$$\mathbf{J}\delta\mathbf{T} = \mathbf{g}_{ll}\delta\mathbf{T} = \sum_{\mathrm{q}=1}^{M_p}\mathbf{g}_{ll}^{\mathrm{q}}\mathbf{w} \tag{99}$$

where, from (14), we know

$$\begin{aligned}\mathbf{g}_{ll}^{\mathrm{q}} &= \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_\mathrm{q} - \frac{1}{N}\mathbf{C}\mathbf{F})\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T &(100)\\ \mathbf{U}_l &= \begin{bmatrix}-\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{u}_l\end{bmatrix} \quad \mathbf{F} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix} &(101)\\ \mathbf{S}_\mathbf{P} &= \begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{0}_{3\times1}\end{bmatrix} \quad \mathbf{C} = \sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T &(102)\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{\mathrm{q}=1}^{M_p}\mathbf{g}_{ll}^{\mathrm{q}}\mathbf{w} &= \frac{2}{N}\sum_{\mathrm{q}=1}^{M_p}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_\mathrm{q} - \frac{1}{N}\mathbf{C}\mathbf{F})\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T\mathbf{U}_l^T\mathbf{w}\\ &= \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}\left(\sum_{\mathrm{q}=1}^{M_p}(\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T - \frac{1}{N}\mathbf{C}\mathbf{F}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T)\right)\mathbf{U}_l^T\mathbf{w}.\end{aligned}$$

Since $\sum_{\mathrm{q}=1}^{M_p}\mathbf{T}_\mathrm{q}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T = \mathbf{C}$ and $\sum_{\mathrm{q}=1}^{M_p}\mathbf{F}\mathbf{C}_\mathrm{q}\mathbf{T}_\mathrm{q}^T = \mathbf{F}\mathbf{C}$ from equation (48), we have

$$\sum_{\mathrm{q}=1}^{M_p}\mathbf{g}_{ll}^{\mathrm{q}}\mathbf{w} = \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{C} - \frac{1}{N}\mathbf{C}\mathbf{F}\mathbf{C})\mathbf{U}_l^T\mathbf{w}.$$

Recalling $\mathbf{A} = \frac{1}{N}\mathbf{P} - \frac{1}{N^2}\mathbf{v}\mathbf{v}^T$,

$$\begin{aligned}&\mathbf{C} - \frac{1}{N}\mathbf{C}\mathbf{F}\mathbf{C}\\ &= \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} - \frac{1}{N}\begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} - \begin{bmatrix}\frac{1}{N}\mathbf{v}\mathbf{v}^T & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} = \begin{bmatrix}N\mathbf{A} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix}\end{aligned} \tag{103}$$

Thus,

$$\sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{ll}^{\mathrm{q}} \mathbf{w} = \frac{2}{N} \mathbf{u}_l^T \mathbf{S}_{\mathbf{P}} \begin{bmatrix} N\mathbf{A} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix} \mathbf{U}_l^T \mathbf{w} \tag{104}$$

$$= 2 \begin{bmatrix} \mathbf{u}_l^T \mathbf{A} & \mathbf{0}_{3\times1} \end{bmatrix} \begin{bmatrix} -\lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{u}_l \end{bmatrix}^T \mathbf{w} \tag{105}$$

$$= 2 \begin{bmatrix} \mathbf{u}_l^T \mathbf{A} \lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times1} \end{bmatrix} \mathbf{w} \tag{106}$$

Since $\mathbf{A}\mathbf{u}_l = \lambda_l \mathbf{u}_l$,

$$\mathbf{u}_l^T \mathbf{A} \lfloor \mathbf{u}_l \rfloor = \lambda_l \mathbf{u}_l^T \lfloor \mathbf{u}_l \rfloor = \mathbf{0}_{1\times3}. \tag{107}$$

Therefore,

$$\mathbf{J}\delta\mathbf{T} = \sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{ll}^{\mathrm{q}} \mathbf{w} = 0 \tag{108}$$

For the proof of $\delta\mathbf{T}^T \mathbf{H} \delta\mathbf{T} = 0$, from (97)

$$\delta\mathbf{T}^T \mathbf{H} \delta\mathbf{T} = \delta\mathbf{T}^T (\mathbf{H}_{ll} + \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} \mathbf{g}_{kl}^T \mathbf{g}_{kl}) \delta\mathbf{T}$$
$$= \mathbf{w}^T \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \left(\mathbf{H}_{ll}^{\mathrm{p,q}} + \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} (\mathbf{g}_{kl}^{\mathrm{p}})^T \mathbf{g}_{kl}^{\mathrm{q}} \right) \mathbf{w} \tag{109}$$

where

$$\mathbf{H}_{ll}^{\mathrm{p,q}} = -\frac{2}{N^2} \mathbf{U}_l \mathbf{T}_{\mathrm{p}} \mathbf{D}_{\mathrm{p,q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \mathbb{1}_{\mathrm{p=q}} \Big(\frac{2}{N} \mathbf{U}_l \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \begin{bmatrix} \mathbf{K}_{ll}^{\mathrm{q}} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \Big) \tag{110}$$

$$\mathbf{K}_{ll}^{\mathrm{q}} = \frac{1}{N} \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor \lfloor \mathbf{u}_l \rfloor + \frac{1}{N} \lfloor \mathbf{u}_l \rfloor \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor, \tag{111}$$

$$\mathbf{g}_{kl}^{\mathrm{q}} = \frac{1}{N} \mathbf{u}_l^T \mathbf{S}_{\mathbf{P}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F}) \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_k^T + \frac{1}{N} \mathbf{u}_k^T \mathbf{S}_{\mathbf{P}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F}) \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T \tag{112}$$

We will divide (109) into two parts to discuss. For the first part,

$$\sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{H}_{ll}^{\mathrm{p,q}} = -\frac{2}{N^2} \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{U}_l \mathbf{T}_{\mathrm{p}} \mathbf{D}_{\mathrm{p,q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \frac{2}{N} \sum_{\mathrm{q}=1}^{M_p} \mathbf{U}_l \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T \mathbf{U}_l^T + \begin{bmatrix} \sum_{\mathrm{q}=1}^{M_p} \mathbf{K}_{ll}^{\mathrm{q}} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \tag{113}$$

Since $\sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T = \mathbf{C}$, $\sum_{\mathrm{q}=1}^{M_p} \mathbf{F}\mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T = \mathbf{F}\mathbf{C}$ and $\sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{F} = \mathbf{C}\mathbf{F}$ from (48), we have

$$\sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{D}_{\mathrm{p,q}} \mathbf{T}_{\mathrm{q}}^T = \sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{C}_{\mathrm{p}} \mathbf{F} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T$$
$$= \sum_{\mathrm{p}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{C}_{\mathrm{p}} \sum_{\mathrm{q}=1}^{M_p} \mathbf{F} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T = \sum_{\mathrm{p}=1}^{M_p} \mathbf{T}_{\mathrm{p}} \mathbf{C}_{\mathrm{p}} \mathbf{F}\mathbf{C} = \mathbf{C}\mathbf{F}\mathbf{C}, \tag{114}$$

and

$$\sum_{\mathrm{q}=1}^{M_p} \mathbf{K}_{ll}^{\mathrm{q}} = \frac{1}{N} \sum_{\mathrm{q}=1}^{M_p} \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor \lfloor \mathbf{u}_l \rfloor + \frac{1}{N} \sum_{\mathrm{q}=1}^{M_p} \lfloor \mathbf{u}_l \rfloor \lfloor \mathbf{S}_{\mathbf{P}} \mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} (\mathbf{T}_{\mathrm{q}} - \frac{1}{N} \mathbf{C}\mathbf{F})^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor$$
$$= \frac{1}{N} \lfloor \mathbf{S}_{\mathbf{P}} (\mathbf{C} - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}) \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor \lfloor \mathbf{u}_l \rfloor + \frac{1}{N} \lfloor \mathbf{u}_l \rfloor \lfloor \mathbf{S}_{\mathbf{P}} (\mathbf{C} - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}) \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_l \rfloor. \tag{115}$$

Then, from (103) and $\mathbf{A}\mathbf{u}_l = \lambda_l \mathbf{u}_l$,

$$\sum_{\mathrm{q}=1}^{M_p} \mathbf{K}_{ll}^{\mathrm{q}} = \lfloor \mathbf{A}\mathbf{u}_l \rfloor \lfloor \mathbf{u}_l \rfloor + \lfloor \mathbf{u}_l \rfloor \lfloor \mathbf{A}\mathbf{u}_l \rfloor = 2\lambda_l \lfloor \mathbf{u}_l \rfloor^2. \tag{116}$$

Now, substituting the results in (114) and (116) into (113):

$$\sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \mathbf{H}_{ll}^{\mathrm{p,q}} = -\frac{2}{N^2} \mathbf{U}_l \mathbf{C}\mathbf{F}\mathbf{C} \mathbf{U}_l^T + \frac{2}{N} \mathbf{U}_l \mathbf{C} \mathbf{U}_l^T + \begin{bmatrix} 2\lambda_l \lfloor \mathbf{u}_l \rfloor^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = 2\mathbf{U}_l \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \mathbf{U}_l^T + \begin{bmatrix} 2\lambda_l \lfloor \mathbf{u}_l \rfloor^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$
$$= 2 \begin{bmatrix} \lambda_l \lfloor \mathbf{u}_l \rfloor^2 - \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{117}$$

For the second part in (109),

$$\sum_{\mathrm{p}=1}^{M_p} \sum_{\mathrm{q}=1}^{M_p} \left(\sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} (\mathbf{g}_{kl}^{\mathrm{p}})^T \mathbf{g}_{kl}^{\mathrm{q}} \right) = \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} \left(\sum_{\mathrm{p}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{p}} \right)^T \left(\sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{q}} \right) \tag{118}$$

where

$$\sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{q}} = \frac{1}{N} \mathbf{u}_l^T \mathbf{S}_{\mathbf{P}} \sum_{\mathrm{q}=1}^{M_p} (\mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T) \mathbf{U}_k^T + \frac{1}{N} \mathbf{u}_k^T \mathbf{S}_{\mathbf{P}} \sum_{\mathrm{q}=1}^{M_p} (\mathbf{T}_{\mathrm{q}} \mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}_{\mathrm{q}} \mathbf{T}_{\mathrm{q}}^T) \mathbf{U}_l^T$$
$$= \frac{1}{N} \mathbf{u}_l^T \mathbf{S}_{\mathbf{P}} (\mathbf{C} - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}) \mathbf{U}_k^T + \frac{1}{N} \mathbf{u}_k^T \mathbf{S}_{\mathbf{P}} (\mathbf{C} - \frac{1}{N} \mathbf{C}\mathbf{F}\mathbf{C}) \mathbf{U}_l^T$$
$$= \mathbf{u}_l^T \begin{bmatrix} \mathbf{A} & \mathbf{0}_{3\times1} \end{bmatrix} \mathbf{U}_k^T + \mathbf{u}_k^T \begin{bmatrix} \mathbf{A} & \mathbf{0}_{3\times1} \end{bmatrix} \mathbf{U}_l^T$$
$$= \begin{bmatrix} \mathbf{u}_l^T \mathbf{A} \lfloor \mathbf{u}_k \rfloor & \mathbf{0}_{3\times1} \end{bmatrix} + \begin{bmatrix} \mathbf{u}_k^T \mathbf{A} \lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times1} \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_l \mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor + \lambda_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times1} \end{bmatrix} \tag{119}$$

Therefore,

$$\sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} \left(\sum_{\mathrm{p}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{p}} \right)^T \left(\sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{q}} \right) = \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} \begin{bmatrix} \mathbf{r}_{kl} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix} \tag{120}$$

where due to $\lfloor \mathbf{u}_k \rfloor \mathbf{u}_l = -\lfloor \mathbf{u}_l \rfloor \mathbf{u}_k$ and $\mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor = -\mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor$

$$\begin{aligned}\mathbf{r}_{kl} &= \left(\lambda_l \mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor + \lambda_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor)\right)^T \left(\lambda_l \mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor + \lambda_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor)\right) \\ &= -\lambda_l^2 \lfloor \mathbf{u}_k \rfloor \mathbf{u}_l \mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor - \lambda_l \lambda_k \lfloor \mathbf{u}_k \rfloor \mathbf{u}_l \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor \\ &\quad - \lambda_k^2 \lfloor \mathbf{u}_l \rfloor \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor - \lambda_l \lambda_k \lfloor \mathbf{u}_l \rfloor \mathbf{u}_k \mathbf{u}_l^T \lfloor \mathbf{u}_k \rfloor \\ &= -(\lambda_l - \lambda_k)^2 \lfloor \mathbf{u}_l \rfloor \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor. \end{aligned} \tag{121}$$

Thus,

$$\begin{aligned}&\sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} \left(\sum_{\mathrm{p}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{p}}\right)^T \left(\sum_{\mathrm{q}=1}^{M_p} \mathbf{g}_{kl}^{\mathrm{q}}\right) \\ &= 2 \sum_{k=1,k\neq l}^{3} \begin{bmatrix} (\lambda_k - \lambda_l) \lfloor \mathbf{u}_l \rfloor \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \end{aligned} \tag{122}$$

Now from (117) and (122), the equation (109) is turned into

$$\delta \mathbf{T}^T \mathbf{H} \delta \mathbf{T} = \mathbf{w}^T \begin{bmatrix} 2\mathbf{L}_l & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \mathbf{w}, \tag{123}$$

where

$$\begin{aligned}\mathbf{L}_l &= \lambda_l \lfloor \mathbf{u}_l \rfloor^2 - \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor + \sum_{k=1,k\neq l}^{3} (\lambda_k - \lambda_l) \lfloor \mathbf{u}_l \rfloor \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor \\ &= \lambda_l \lfloor \mathbf{u}_l \rfloor^2 - \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor \\ &\quad + \lfloor \mathbf{u}_l \rfloor \sum_{k=1,k\neq l}^{3} \lambda_k \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor - \lambda_l \lfloor \mathbf{u}_l \rfloor \sum_{k=1,k\neq l}^{3} \mathbf{u}_k \mathbf{u}_k^T \lfloor \mathbf{u}_l \rfloor. \end{aligned} \tag{124}$$

Since $\mathbf{u}_k$ ($k = 1, 2, 3$) is the eigenvector (with eigenvalue λ_k) of matrix $\mathbf{A}$, which is symmetric, we have the following two conditions from the singular value decomposition of $\mathbf{A}$:

$$\mathbf{A} = \sum_{k=1}^{3} \lambda_k \mathbf{u}_k \mathbf{u}_k^T, \quad \mathbf{I} = \sum_{k=1}^{3} \mathbf{u}_k \mathbf{u}_k^T, \tag{125}$$

which imply

$$\sum_{k=1,k\neq l}^{3} \lambda_k \mathbf{u}_k \mathbf{u}_k^T = \mathbf{A} - \lambda_l \mathbf{u}_l \mathbf{u}_l^T, \qquad \sum_{k=1,k\neq l}^{3} \mathbf{u}_k \mathbf{u}_k^T = \mathbf{I} - \mathbf{u}_l \mathbf{u}_l^T. \tag{126}$$

Substituting the above results into $\mathbf{L}_l$ in (124):

$$\begin{aligned}\mathbf{L}_l &= \lambda_l \lfloor \mathbf{u}_l \rfloor^2 - \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor \\ &+ \lfloor \mathbf{u}_l \rfloor (\mathbf{A} - \lambda_l \mathbf{u}_l \mathbf{u}_l^T) \lfloor \mathbf{u}_l \rfloor - \lambda_l \lfloor \mathbf{u}_l \rfloor (\mathbf{I} - \mathbf{u}_l \mathbf{u}_l^T) \lfloor \mathbf{u}_l \rfloor \\ &= \lambda_l \lfloor \mathbf{u}_l \rfloor^2 - \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor + \lfloor \mathbf{u}_l \rfloor \mathbf{A} \lfloor \mathbf{u}_l \rfloor - \lambda_l \lfloor \mathbf{u}_l \rfloor^2 = \mathbf{0}. \end{aligned} \tag{127}$$

As a result,

$$\delta \mathbf{T}^T \mathbf{H} \delta \mathbf{T} = \mathbf{w}^T \begin{bmatrix} 2\mathbf{L}_l & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \mathbf{w} = 0. \tag{128}$$

Finally, for the results in (40), if p or q $\in \mathcal{I} \triangleq \{j | 1 \le j \le M_p, \mathbf{C}_j = \mathbf{0}\}$, we have $\mathbf{g}_{kl}^{\mathrm{p}} = \mathbf{0}_{1\times 6}$ (or $\mathbf{g}_{kl}^{\mathrm{q}} = \mathbf{0}_{1\times 6}$) and $\mathbf{H}_{kl}^{\mathrm{p,q}} = \mathbf{0}_{6\times 6}, \forall k, l$, from (19). As a result, $\mathbf{J}^{\mathrm{p}} = \mathbf{g}_{kl}^{\mathrm{p}} = \mathbf{0}$ and $\mathbf{H}^{\mathrm{p,q}} = \mathbf{H}_{ll}^{\mathrm{p,q}} + \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k} (\mathbf{g}_{kl}^{\mathrm{p}})^T \mathbf{g}_{kl}^{\mathrm{q}} = \mathbf{0}$. □

F. Derivation of pose covariance

The quantity $\delta \mathbf{C}_{f_{ij}}$ can be obtained by substituting (43) into the definition of $\mathbf{C}_{f_{ij}}^{\mathrm{gt}}$ in (44) and retaining only the first order items:

$$\delta \mathbf{C}_{f_{ij}} = \begin{bmatrix} \delta \mathbf{P}_{f_{ij}} & \delta \mathbf{v}_{f_{ij}} \\ \delta \mathbf{v}_{f_{ij}}^T & 0 \end{bmatrix}, \quad \text{where} \tag{129}$$

$$\delta \mathbf{P}_{f_{ij}} = \sum_{k=1}^{N_{ij}} (\mathbf{p}_{f_{ijk}} \delta \mathbf{p}_{f_{ijk}}^T + \delta \mathbf{p}_{f_{ijk}} \mathbf{p}_{f_{ijk}}^T), \tag{130}$$

$$\delta \mathbf{v}_{f_{ij}} = \sum_{k=1}^{N_{ij}} \delta \mathbf{p}_{f_{ijk}}. \tag{131}$$

To derive $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f} \delta \mathbf{C}_f$ with involving any tensor, we parameterize the matrix $\mathbf{C}_{f_{ij}}$ by a column vector $\mathbf{c}_{f_{ij}}$, which consists of the independent elements in $\mathbf{C}_{f_{ij}}$:

$$\begin{aligned}\mathbf{c}_{f_{ij}} = \mathrm{vec}(\mathbf{C}_{f_{ij}}) \triangleq \big[&\mathbf{e}_1^T \mathbf{C}_{f_{ij}} \mathbf{e}_1 \quad \mathbf{e}_1^T \mathbf{C}_{f_{ij}} \mathbf{e}_2 \quad \mathbf{e}_1^T \mathbf{C}_{f_{ij}} \mathbf{e}_3 \\ &\mathbf{e}_2^T \mathbf{C}_{f_{ij}} \mathbf{e}_2 \quad \mathbf{e}_2^T \mathbf{C}_{f_{ij}} \mathbf{e}_3 \quad \mathbf{e}_3^T \mathbf{C}_{f_{ij}} \mathbf{e}_3 \\ &\mathbf{e}_1^T \mathbf{C}_{f_{ij}} \mathbf{e}_4 \quad \mathbf{e}_2^T \mathbf{C}_{f_{ij}} \mathbf{e}_4 \quad \mathbf{e}_3^T \mathbf{C}_{f_{ij}} \mathbf{e}_4\big]^T \in \mathbb{R}^9 \end{aligned} \tag{132}$$

where $\mathrm{vec}(\cdot) : \mathbb{S}^{4\times 4} \mapsto \mathbb{R}^9$ maps a symmetric matrix to its column vector representation, $\mathbf{e}_l \in \mathbb{R}^4$ ($l \in \{1, 2, 3, 4\}$) is a vector with all zero elements except for the l-th element being one. Note that the constant N in the 4-th row, 4-th column of $\mathbf{C}_{f_{ij}}$ is not contained in $\mathbf{c}_{f_{ij}}$ since it is a constant number independent of the noise. Correspondingly, noises in $\mathbf{C}_{f_{ij}}$ becomes the noise of $\mathbf{c}_{f_{ij}}$ as below:

$$\begin{aligned}\delta \mathbf{c}_{f_{ij}} &= \big[\mathbf{e}_1^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_1 \quad \mathbf{e}_1^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_2 \quad \mathbf{e}_1^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_3 \\ &\quad \mathbf{e}_2^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_2 \quad \mathbf{e}_2^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_3 \quad \mathbf{e}_3^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_3 \\ &\quad \mathbf{e}_1^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_4 \quad \mathbf{e}_2^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_4 \quad \mathbf{e}_3^T \delta \mathbf{C}_{f_{ij}} \mathbf{e}_4\big]^T \end{aligned} \tag{133}$$

$$= \sum_{k=1}^{N_{ij}} \begin{bmatrix} 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{11} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{12} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{13} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{22} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{23} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{33} \mathbf{S}_{\mathrm{P}}^T \delta \mathbf{p}_{f_{ijk}} \\ \delta \mathbf{p}_{f_{ijk}} \end{bmatrix} = \sum_{k=1}^{N_{ij}} \mathbf{B}_{f_{ijk}} \delta \mathbf{p}_{f_{ijk}}, \tag{134}$$

where $\mathbf{S}_{\mathrm{P}} = [\mathbf{I}_{3\times 3}, 0_{3\times 1}]$, $\mathbf{E}_{kl} = \mathbf{e}_k \mathbf{e}_l^T + \mathbf{e}_l \mathbf{e}_k^T \in \mathbb{S}^{4\times 4}$, $k, l \in \{1, 2, 3, 4\}$, and

$$\mathbf{B}_{f_{ijk}} = \begin{bmatrix} 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{11} \mathbf{S}_{\mathrm{P}}^T \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{12} \mathbf{S}_{\mathrm{P}}^T \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{13} \mathbf{S}_{\mathrm{P}}^T \\ 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{22} \mathbf{S}_{\mathrm{P}}^T \\ \mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{23} \mathbf{S}_{\mathrm{P}}^T \\ 2\mathbf{p}_{f_{ijk}}^T \mathbf{S}_{\mathrm{P}} \mathbf{E}_{33} \mathbf{S}_{\mathrm{P}}^T \\ \mathbf{I}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{9\times 3}. \tag{135}$$

With the column representation of each $\mathbf{C}_{f_{ijk}}$ contained in $\mathbf{C}_f$, $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f} \delta \mathbf{C}_f$ can now be computed as

$$\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f} \delta \mathbf{C}_f = \sum_{i=1}^{M_f} \sum_{j=1}^{M_p} \frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}} \delta \mathbf{c}_{f_{ij}}. \tag{136}$$

To derive the quantity $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}}$, we give two lemmas which are useful for subsequent deduction.

Lemma 3. *For $\mathbf{w} \in \mathbb{R}^4, \mathbf{C} \in \mathbb{S}^{4\times 4}$ and its vector form $\mathbf{c} = \text{vec}(\mathbf{C})$, we have*

$$\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{c}} = \mathbf{g}_1(\mathbf{w}) \in \mathbb{R}^{4\times 9},$$

where

$$\mathbf{g}_1(\mathbf{w}) = \begin{bmatrix}\mathbf{E}_{11}\mathbf{w} & \mathbf{E}_{12}\mathbf{w} & \mathbf{E}_{13}\mathbf{w} & \mathbf{E}_{22}\mathbf{w} & \mathbf{E}_{23}\mathbf{w} \\ & \mathbf{E}_{33}\mathbf{w} & \mathbf{E}_{14}\mathbf{w} & \mathbf{E}_{24}\mathbf{w} & \mathbf{E}_{34}\mathbf{w}\end{bmatrix},$$

where $\mathbf{e}_l \in \mathbb{R}^4$ ($l \in \{1, 2, 3, 4\}$) is a vector with all zero elements except for the l-th element being one, $\mathbf{E}_{kl} \in \mathbb{S}^{4\times 4}$, $k, l \in \{1, 2, 3, 4\}$.

$$\mathbf{E}_{kl} = \begin{cases} \mathbf{e}_k\mathbf{e}_l^T + \mathbf{e}_l\mathbf{e}_k^T & (k \neq l) \\ \mathbf{e}_l\mathbf{e}_l^T & (k = l) \end{cases} \tag{137}$$

Proof. For the k-th row, l-th column element of $\mathbf{C}$, denoted by $\mathbf{C}_{k,l}$, we have

$$\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{k,l}} = \mathbf{E}_{kl}\mathbf{w} \in \mathbb{R}^4.$$

Hence,

$$\begin{aligned}\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{c}} &= \Big[\tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,1}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,2}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,3}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,2}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,3}} \\ &\qquad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{3,3}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,4}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,4}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{3,4}}\Big] \\ &= \big[\mathbf{E}_{11}\mathbf{w} \quad \mathbf{E}_{12}\mathbf{w} \quad \mathbf{E}_{13}\mathbf{w} \quad \mathbf{E}_{22}\mathbf{w} \quad \mathbf{E}_{23}\mathbf{w} \\ &\qquad \mathbf{E}_{33}\mathbf{w} \quad \mathbf{E}_{14}\mathbf{w} \quad \mathbf{E}_{24}\mathbf{w} \quad \mathbf{E}_{34}\mathbf{w}\big].\end{aligned}$$

□

Lemma 4. *For $\mathbf{u}_l \in \mathbb{R}^3$ and*

$$\mathbf{w} = \begin{bmatrix}\mathbf{w}_{1:3} \\ w_4\end{bmatrix} \in \mathbb{R}^4, \quad \mathbf{U}_l = \begin{bmatrix}-\lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_l\end{bmatrix} \in \mathbb{R}^{6\times 4}$$

where $\mathbf{w}_{1:3}$ represents the first three elements in $\mathbf{w}$, we have

$$\frac{\partial \mathbf{U}_l\mathbf{w}}{\partial \mathbf{u}_l} = \mathbf{g}_2(\mathbf{w}) = \begin{bmatrix}\lfloor \mathbf{w}_{1:3} \rfloor \\ w_4\mathbf{I}_{3\times 3}\end{bmatrix} \in \mathbb{R}^{6\times 3}.$$

Proof.

$$\mathbf{U}_l\mathbf{w} = \begin{bmatrix}-\lfloor \mathbf{u}_l \rfloor & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_l\end{bmatrix}\begin{bmatrix}\mathbf{w}_{1:3} \\ w_4\end{bmatrix} = \begin{bmatrix}\lfloor \mathbf{w}_{1:3} \rfloor \mathbf{u}_l \\ w_4\mathbf{u}_l\end{bmatrix} = \begin{bmatrix}\lfloor \mathbf{w}_{1:3} \rfloor \\ w_4\mathbf{I}_{3\times 3}\end{bmatrix}\mathbf{u}_l$$

Thus,

$$\frac{\partial \mathbf{U}_l\mathbf{w}}{\partial \mathbf{u}_l} = \begin{bmatrix}\lfloor \mathbf{w}_{1:3} \rfloor \\ w_4\mathbf{I}_{3\times 3}\end{bmatrix} = \mathbf{g}_2(\mathbf{w})$$

□

With these two lemmas, next we will continue to derive $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}}$. Theorem 4 gives the Jacobian for one cost item, to distinguish the different cost item, we add a subscript ν to (30) to denotes the ν-th item and replace $\mathbf{C}_j$ with the actual point cluster notation $\mathbf{C}_{f_{\nu j}}$ corresponding to the ν-th item, leading to:

$$\begin{aligned}\mathbf{J}_\nu &= \begin{bmatrix}\cdots & \mathbf{J}_\nu^{\mathrm{p}} & \cdots\end{bmatrix} \in \mathbb{R}^{1\times 6M_p} \\ \mathbf{J}_\nu^{\mathrm{p}} &= \frac{2}{N}\mathbf{u}_{\nu l}^T\mathbf{S_P}\left(\mathbf{T}_{\mathrm{p}} - \frac{1}{N}\mathbf{C}_\nu\mathbf{F}\right)\mathbf{C}_{f_{\nu \mathrm{p}}}\mathbf{T}_{\mathrm{p}}^T\mathbf{U}_{\nu l}^T \in \mathbb{R}^{1\times 6}.\end{aligned} \tag{138}$$

The total Jacobian is hence

$$\mathbf{J} = \sum_{\nu=1}^{M_f}\mathbf{J}_\nu = \begin{bmatrix}\cdots & \mathbf{J}^{\mathrm{p}} & \cdots\end{bmatrix} \in \mathbb{R}^{1\times 6M_p}, \tag{139}$$

$$\begin{aligned}\mathbf{J}^{\mathrm{p}} &= \sum_{\nu=1}^{M_f}\mathbf{J}_\nu^{\mathrm{p}} \in \mathbb{R}^{1\times 6}, \\ &= \sum_{\nu=1}^{M_f}\left(\frac{2}{N}\mathbf{u}_{\nu l}^T\mathbf{S_P}\left(\mathbf{T}_{\mathrm{p}} - \frac{1}{N}\mathbf{C}_\nu\mathbf{F}\right)\mathbf{C}_{f_{\nu \mathrm{p}}}\mathbf{T}_{\mathrm{p}}^T\mathbf{U}_{\nu l}^T\right).\end{aligned} \tag{140}$$

Next, we calculate the partial derivative $\frac{\partial (\mathbf{J}^{\mathrm{p}})^T}{\partial \mathbf{c}_{f_{ij}}}$. Note that in the summation of (140), only the i-th summation term (i.e., $\nu = i$) is related to $\mathbf{c}_{f_{ij}}$, hence,

$$\frac{\partial (\mathbf{J}^{\mathrm{p}})^T}{\partial \mathbf{c}_{f_{ij}}} = \frac{\partial}{\partial \mathbf{c}_{f_{ij}}}\left(\frac{2}{N_i}\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}\right). \tag{141}$$

Since $\mathbf{C}_i = \sum_{\nu=1}^{M_p}\mathbf{T}_\nu\mathbf{C}_{f_{i\nu}}\mathbf{T}_\nu^T$, $\mathbf{u}_{il}$ is the eigenvector associated to the l-th largest eigenvalue of matrix $\mathbf{A}(\mathbf{C}_i)$, and $\mathbf{U}_{il} = \begin{bmatrix}-\lfloor \mathbf{u}_{il} \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_{il}\end{bmatrix}$, the derivative with respect to $\mathbf{c}_{f_{ij}}$ on the right hand side of (141) consists of four terms from $\mathbf{U}_{il}$, $\mathbf{C}_{f_{i\mathrm{p}}}$ (only when $\mathrm{p} = j$), $\mathbf{C}_i$, and $\mathbf{u}_{il}$, respectively. Combining with Lemma 3 and Lemma 4, we have

$$\mathbf{L}_{ij}^{\mathrm{p}} \triangleq \frac{\partial (\mathbf{J}^{\mathrm{p}})^T}{\partial \mathbf{c}_{f_{ij}}} \in \mathbb{R}^{6\times 9} \tag{142}$$

$$\begin{aligned}&= \frac{2}{N_i}\Big(\mathbf{g}_2(\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}) \\ &+ \mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\Big)\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} \\ &- \frac{2}{N_i^2}\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}\mathbf{F}\mathbf{T}_j\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &+ \frac{2}{N_i}\mathbb{1}_{\mathrm{p}=j}\left(\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{g}_1\left((\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}\right)\right)\end{aligned} \tag{143}$$

Only the term $\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}}$ is unknown in this formula. To compute it, we apply the second conclusion in Lemma 1 to all components of $\mathbf{c}_{f_{ij}}$ and use the notation trick similar to (81):

$$\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} = \sum_{k=1, k\neq l}^{3}\frac{1}{\lambda_{il} - \lambda_{ik}}\mathbf{u}_{ik}\mathbf{u}_{ik}^T\frac{\partial \mathbf{A}(\mathbf{C}_i)\mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} \tag{144}$$

From Lemma 2,

$$\mathbf{u}_{ik}^T\mathbf{A}(\mathbf{C}_i)\mathbf{u}_{il} = \frac{1}{N_i}\mathbf{u}_{ik}^T\mathbf{S_P}$$

$$\left(\sum_{\mu=1}^{M_p}\mathbf{T}_\mu\mathbf{C}_{f_{i\mu}}\mathbf{T}_\mu^T - \frac{1}{N_i}\sum_{\mu=1}^{M_p}\sum_{\nu=1}^{M_p}\mathbf{T}_\mu\mathbf{C}_{f_{i\mu}}\mathbf{F}\mathbf{C}_{f_{i\nu}}\mathbf{T}_\nu^T\right)\mathbf{S_P}^T\mathbf{u}_{il}$$

Denote

$$\mathbf{G}_{kl}^{ij} \triangleq \frac{\partial \mathbf{u}_{ik}^T \mathbf{A}(\mathbf{C}_{f_i})\mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} = \frac{1}{N_i}\mathbf{u}_{ik}^T \mathbf{S}_{\mathbf{P}} \mathbf{T}_j \mathbf{g}_1(\mathbf{T}_j^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il})$$
$$- \frac{1}{N_i^2}\mathbf{u}_{ik}^T \mathbf{S}_{\mathbf{P}} \sum_{\mu=1}^{M_p} \mathbf{T}_\mu \mathbf{C}_{f_{i\mu}} \mathbf{F}\mathbf{g}_1(\mathbf{T}_j^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il})$$
$$- \frac{1}{N_i^2}\mathbf{u}_{ik}^T \mathbf{S}_{\mathbf{P}} \mathbf{T}_j \mathbf{g}_1\Big(\sum_{\nu=1}^{M_p} \mathbf{F}\mathbf{C}_{f_{i\nu}} \mathbf{T}_\nu^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il}\Big) \in \mathbb{R}^{1\times 9}$$
$$= \frac{1}{N_i}\mathbf{u}_{ik}^T \mathbf{S}_{\mathbf{P}}\Big(\mathbf{T}_j \mathbf{g}_1(\mathbf{T}_j^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il}) - \frac{1}{N_i}\mathbf{C}_i \mathbf{F}\mathbf{g}_1(\mathbf{T}_j^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il})$$
$$- \frac{1}{N_i}\mathbf{T}_j \mathbf{g}_1(\mathbf{F}\mathbf{C}_i \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il})\Big) \tag{145}$$

Substitute it into $\mathbf{L}_{ij}^{\mathrm{p}}$:

$$\mathbf{L}_{ij}^{\mathrm{p}} = \frac{2}{N_i}\Big(\mathbf{g}_2(\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il}) +$$
$$\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N}\mathbf{F}\mathbf{C}_i)\mathbf{S}_{\mathbf{P}}^T\Big)\left(\sum_{k=1,k\neq l}^{3} \frac{\mathbf{u}_{ik}\mathbf{G}_{kl}^{ij}}{\lambda_{il} - \lambda_{ik}}\right)$$
$$- \frac{2}{N_i^2}\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}\mathbf{F}\mathbf{T}_j \mathbf{g}_1(\mathbf{T}_j^T \mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il})$$
$$+ \frac{2}{N_i}\mathbb{1}_{\mathrm{p}=j}\left(\mathbf{U}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{g}_1\Big((\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S}_{\mathbf{P}}^T \mathbf{u}_{il}\Big)\right), \tag{146}$$

$$\mathbf{L}_{ij} = \frac{\partial \mathbf{J}^T\left(\mathbf{T}^\star, \mathbf{C}_{f_{ij}}\right)}{\partial \mathbf{c}_{f_{ij}}} = \begin{bmatrix} \vdots \\ \mathbf{L}_{ij}^{\mathrm{p}} \\ \vdots \end{bmatrix} \in \mathbb{R}^{6M_p \times 9}, \tag{147}$$

and

$$\delta \mathbf{T}^\star = -\mathbf{H}^{-1}\left(\sum_{i=1}^{M_f}\sum_{j=1}^{M_p} \mathbf{L}_{ij}\delta \mathbf{c}_{f_{ij}}\right) \in \mathbb{R}^{6M_p}, \tag{148}$$

$$\mathbf{\Sigma}_{\delta \mathbf{T}^\star} = \mathbf{H}^{-1}\left(\sum_{i=1}^{M_f}\sum_{j=1}^{M_p} \mathbf{L}_{ij}\mathbf{\Sigma}_{\mathbf{c}_{f_{ij}}}\mathbf{L}_{ij}^T\right)\mathbf{H}^{-1}, \tag{149}$$

where

$$\mathbf{\Sigma}_{\mathbf{c}_{f_{ij}}} = \sum_{k=1}^{N_{ij}} \mathbf{B}_{f_{ijk}} \mathbf{\Sigma}_{\mathbf{p}_{f_{ijk}}} \mathbf{B}_{f_{ijk}}^T, \tag{150}$$

which can be computed beforehand without enumerating each raw point in the run time.