Efficient and Consistent Bundle Adjustment on Lidar Point Clouds (Supplementary)

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Please note that equation numbers and section numbers from the main manuscript are labelled in this letter in red.

I. Lemmas

Lemma 1. *For a scalar $x \in \mathbb{R}$ and a matrix $\mathbf{A} \in \mathbb{S}^{3\times 3}$ which depends on x, we have the two following conclusions.*

$$\frac{\partial \lambda_l(x)}{\partial x} = \mathbf{u}_l(x)^T \frac{\partial \mathbf{A}(x)}{\partial x} \mathbf{u}_l(x) \tag{1}$$

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k} \mathbf{u}_k(x)\mathbf{u}_k(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x) \tag{2}$$

where λ_l ($l = 1,2,3$) denotes the l-th largest eigenvalue and $\mathbf{u}_l$ is the corresponding eigenvector.

Proof. Since the matrix $\mathbf{A}(x)$ is symmetric, its singular value decomposition is,

$$\mathbf{A}(x) = \mathbf{U}(x)\mathbf{\Lambda}(x)\mathbf{U}(x)^T \tag{3}$$

where $\mathbf{\Lambda}(x) = \text{diag}\left(\lambda_1(x), \lambda_2(x), \lambda_3(x)\right)$ consists of all the eigenvalues and $\mathbf{U}(x) = \begin{bmatrix}\mathbf{u}_1(x) & \mathbf{u}_2(x) & \mathbf{u}_3(x)\end{bmatrix}$ is an orthonormal matrix consisting of the eigenvectors. Therefore,

$$\mathbf{\Lambda}(x) = \mathbf{U}(x)^T\mathbf{A}(x)\mathbf{U}(x) \tag{4}$$

Both sides take the derivative of x,

$$\frac{\partial \mathbf{\Lambda}(x)}{\partial x} = \mathbf{U}(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{U}(x) + \mathbf{U}(x)^T\mathbf{A}(x)\frac{\partial \mathbf{U}(x)}{\partial x} + \left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T \mathbf{A}(x)\mathbf{U}(x) \tag{5}$$

Since $\mathbf{U}(x)^T\mathbf{A}(x) = \mathbf{\Lambda}(x)\mathbf{U}(x)^T$ and $\mathbf{A}(x)\mathbf{U}(x) = \mathbf{U}(x)\mathbf{\Lambda}(x)$, the equation is

$$\frac{\partial \mathbf{\Lambda}(x)}{\partial x} = \mathbf{U}(x)^T \frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{U}(x) + \mathbf{\Lambda}(x)\underbrace{\mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}}_{\mathbf{D}(x)} + \underbrace{\left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T \mathbf{U}(x)}_{\mathbf{D}^T(x)}\mathbf{\Lambda}(x) \tag{6}$$

Denote $\mathbf{D}(x) \triangleq \mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}$. Since $\mathbf{U}(x)\mathbf{U}(x)^T = \mathbf{I}$, differentiating both sides with respect to x leads to,

$$\mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x} + \left(\frac{\partial \mathbf{U}(x)}{\partial x}\right)^T\mathbf{U}(x) = \mathbf{0}$$
$$\Rightarrow \mathbf{D}(x) + \mathbf{D}^T(x) = \mathbf{0}$$

It is seen that $\mathbf{D}(x)$ is a skew symmetric matrix whose diagonal elements are zeros. Moreover, since $\mathbf{\Lambda}(x)$ is diagonal, the last two items of the right side of (6) sum to zero on diagonal positions. Only considering the diagonal elements in (6) leads to

$$\frac{\partial \lambda_l(x)}{\partial x} = \mathbf{u}_l(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), l \in \{1,2,3\} \tag{7}$$

which yields the first conclusion. Now we aims to prove the second one. In (6), $\frac{\partial \mathbf{\Lambda}(x)}{\partial x}$ is diagonal matrix and thus for the off-diagonal, k-th row, l-th column, element ($k \neq l$),

$$0 = \mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x) + \lambda_k D_x^{k,l} - D_x^{k,l}\lambda_l \tag{8}$$

where $D_x^{k,l}$ is the k-th row, l-th column element in the skew symmetric $\mathbf{D}(x)$ and satisfy $D_x^{k,l} = -D_x^{l,k}$. From (8), we can solve $D_x^{k,l}$

$$D_x^{k,l} = \begin{cases} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), k \neq l \\ 0 \qquad\qquad\qquad , k = l\end{cases} \tag{9}$$

Since $\mathbf{D}(x) \triangleq \mathbf{U}(x)^T\frac{\partial \mathbf{U}(x)}{\partial x}$, we have $\frac{\partial \mathbf{U}(x)}{\partial x} = \mathbf{U}(x)\mathbf{D}(x)$. Taking the l-th column on both sides leads to

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \mathbf{U}(x)\mathbf{D}_x^{:,l}, \tag{10}$$

where $\mathbf{D}_x^{:,l} \in \mathbb{R}^3$ represents the l-th column of $\mathbf{D}(x)$. Finally, substituting $\mathbf{U}(x) = [\mathbf{u}_1(x) \quad \mathbf{u}_2(x) \quad \mathbf{u}_3(x)]$ and (9) into (10), we obtain

$$\frac{\partial \mathbf{u}_l(x)}{\partial x} = \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k(x)\mathbf{u}_k(x)^T\frac{\partial \mathbf{A}(x)}{\partial x}\mathbf{u}_l(x), \tag{11}$$

which yields the second conclusion. □

Lemma 2. *Given*

1. *Matrices* $\mathbf{C}_j = \begin{bmatrix}\mathbf{P}_j & \mathbf{v}_j \\ \mathbf{v}_j^T & N_j\end{bmatrix} \in \mathbb{S}^{4\times 4}, j = 1, \cdots, M_p$;
2. *Poses* $\mathbf{T}_j \in SE(3), j = 1, \cdots, M_p$;
3. *A matrix* $\mathbf{C} = \begin{bmatrix}\mathbf{P} & \mathbf{v} \\ \mathbf{v}^T & N\end{bmatrix} \triangleq \sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T \in \mathbb{S}^{4\times 4}$, *which is the aggregation of* $\mathbf{C}_j$, *and a matrix function* $\mathbf{A}(\mathbf{C}) \triangleq \frac{1}{N}\mathbf{P} - \frac{1}{N^2}\mathbf{v}\mathbf{v}^T \in \mathbb{S}^{3\times 3}$;
4. *Two constant vectors* $\mathbf{u}_k, \mathbf{u}_l \in \mathbb{R}^3$,

then the first and second order derivatives of $\mathbf{u}_k^T\mathbf{A}(\mathbf{T})\mathbf{u}_l$ *w.r.t.* $\mathbf{T}$ *are:*

$$\mathbf{g}_{kl} \triangleq \frac{\partial \mathbf{u}_k^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}} = \begin{bmatrix} \cdots & \mathbf{g}_{kl}^j & \cdots \end{bmatrix} \in \mathbb{R}^{1\times 6M_p}, \tag{12}$$

$$\mathbf{Q}_{kl} \triangleq \frac{\partial^2 \mathbf{u}_k^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial(\delta\mathbf{T})^2} = \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{Q}_{kl}^{ij} & \cdots \\ & \vdots & \end{bmatrix} \in \mathbb{R}^{6M_p\times 6M_p}, \tag{13}$$

where $\mathbf{g}_{kl}^j \in \mathbb{R}^{1\times 6}, \mathbf{Q}_{kl}^{ij} \in \mathbb{R}^{6\times 6}, \forall i,j \in \{1,\cdots,M_p\}$, *are block elements of* $\mathbf{g}_{kl}$ *and* $\mathbf{Q}_{kl}$ *defined as below*

$$\mathbf{g}_{kl}^j = \frac{1}{N}\mathbf{u}_l^T \mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_k^T + \frac{1}{N}\mathbf{u}_k^T \mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T \tag{14}$$

$$\mathbf{Q}_{kl}^{ij} = -\frac{2}{N^2}\mathbf{V}_k\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \mathbb{1}_{i=j}\cdot\left(\frac{2}{N}\mathbf{V}_k\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \begin{bmatrix}\mathbf{K}_{kl}^j & \mathbf{0}_{3\times 3}\\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}\end{bmatrix}\right) \tag{15}$$

$$\mathbf{K}_{kl}^j = \frac{1}{N}\lfloor \mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_k\rfloor\lfloor \mathbf{u}_l\rfloor + \frac{1}{N}\lfloor \mathbf{u}_k\rfloor\lfloor \mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor \tag{16}$$

where

$$\mathbf{V}_l = \begin{bmatrix} -\lfloor \mathbf{u}_l\rfloor & \mathbf{0}_{3\times 1}\\ \mathbf{0}_{3\times 3} & \mathbf{u}_l\end{bmatrix} \quad \mathbf{F} = \begin{bmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1}\\ \mathbf{0}_{1\times 3} & 1\end{bmatrix} \tag{17}$$

$$\mathbf{S}_\mathbf{P} = \begin{bmatrix}\mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1}\end{bmatrix} \qquad \mathbb{1}_{i=j} = \begin{cases} 1, & i=j\\ 0, & i\neq j\end{cases}. \tag{18}$$

Additionally, the block elements $\mathbf{g}_{kl}^j, \mathbf{Q}_{kl}^{ij}$ *satisfy that* $\forall \mathbf{u}_k, \mathbf{u}_l$,

$$\mathbf{g}_{kl}^j = \mathbf{0}_{1\times 6},\ \textit{if } \mathbf{C}_j = 0, \tag{19}$$

$$\mathbf{Q}_{kl}^{ij} = \mathbf{0}_{6\times 6},\ \textit{if } \mathbf{C}_i = 0 \textit{ or } \mathbf{C}_j = 0. \tag{20}$$

Proof. Partition the matrix $\mathbf{C}$ as

$$\mathbf{C} = \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} = \sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T, \tag{21}$$

then

$$\mathbf{P} = \mathbf{S}_\mathbf{P}\mathbf{C}\mathbf{S}_\mathbf{P}^T = \sum_{j=1}^{M_p}\left(\mathbf{R}_j\mathbf{P}_j\mathbf{R}_j^T + \mathbf{R}_j\mathbf{v}_j\mathbf{t}_j^T + \mathbf{t}_j\mathbf{v}_j^T\mathbf{R}_j^T + N_j\mathbf{t}_j\mathbf{t}_j^T\right), \tag{22}$$

$$\mathbf{v} = \mathbf{S}_\mathbf{P}\mathbf{C}\mathbf{S}_\mathbf{v}^T = \sum_{j=1}^{M_p}(\mathbf{R}_j\mathbf{v}_j + N_j\mathbf{t}_j), \tag{23}$$

$$N = \sum_{j=1}^{M_p} N_j, \tag{24}$$

where

$$\mathbf{S}_\mathbf{P} = \begin{bmatrix}\mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1}\end{bmatrix} \in \mathbb{R}^{3\times 4}, \tag{25}$$

$$\mathbf{S}_\mathbf{v} = \begin{bmatrix}\mathbf{0}_{1\times 3} & 1\end{bmatrix} \in \mathbb{R}^{1\times 4}. \tag{26}$$

Therefore,

$$\mathbf{A}\left(\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\right) = \frac{1}{N}\mathbf{P} - \frac{1}{N^2}\mathbf{v}\mathbf{v}^T \tag{27}$$

$$= \frac{1}{N}\mathbf{S}_\mathbf{P}\left(\mathbf{C} - \frac{1}{N}\mathbf{C}\mathbf{S}_\mathbf{v}^T\mathbf{S}_\mathbf{v}\mathbf{C}^T\right)\mathbf{S}_\mathbf{P}^T \tag{28}$$

$$= \frac{1}{N}\mathbf{S}_\mathbf{P}\left(\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T - \frac{1}{N}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{T}_i^T\mathbf{S}_\mathbf{v}^T\mathbf{S}_\mathbf{v}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\right)\mathbf{S}_\mathbf{P}^T. \tag{29}$$

Since $\mathbf{T}_i^T\mathbf{S}_\mathbf{v}^T\mathbf{S}_\mathbf{v}\mathbf{T}_j = \begin{bmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1}\\ \mathbf{0}_{1\times 3} & 1\end{bmatrix} \triangleq \mathbf{F}$, we obtain

$$\mathbf{A} = \frac{1}{N}\mathbf{S}_\mathbf{P}\left(\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T - \frac{1}{N}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\right)\mathbf{S}_\mathbf{P}^T, \tag{30}$$

where we omitted the input argument of $\mathbf{A}$ for the sake of notation simplicity. Since $N = \sum_{j=1}^{M_p} N_j$ is a constant number that is irrelevant to the pose $\mathbf{T}$, perturbing the pose $\mathbf{T}$ (i.e., the input of $\mathbf{A}$) by $\delta\mathbf{T}$ yields

$$\mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = \frac{1}{N}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}\left(\sum_{j=1}^{M_p}(\mathbf{T}_j \boxplus \delta\mathbf{T}_j)\mathbf{C}_j(\mathbf{T}_j \boxplus \delta\mathbf{T}_j)^T - \frac{1}{N}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}(\mathbf{T}_i \boxplus \delta\mathbf{T}_i)(\mathbf{C}_i\mathbf{F}\mathbf{C}_j)(\mathbf{T}_j \boxplus \delta\mathbf{T}_j)^T\right)\mathbf{S}_\mathbf{P}^T\mathbf{u}_l. \tag{31}$$

Based on the definition of $\boxplus$ on $SE(3)$ in (27), we define,

$$\mathbf{w}_{jl}(\delta\mathbf{T}_j) \triangleq (\mathbf{T}_j \boxplus \delta\mathbf{T}_j)^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l = \begin{bmatrix}\mathbf{R}_j^T\exp^T(\lfloor\delta\boldsymbol{\phi}_j\rfloor)\mathbf{u}_l\\ \mathbf{t}_j^T\exp^T(\lfloor\delta\boldsymbol{\phi}_j\rfloor)\mathbf{u}_l + \mathbf{u}_l^T\delta\mathbf{t}_j\end{bmatrix}. \tag{32}$$

When $\delta\boldsymbol{\phi}_j$ is small, which is indeed the case for the purpose of derivative computation, we have

$$\exp(\lfloor\delta\boldsymbol{\phi}_j\rfloor) \approx \mathbf{I} + \lfloor\delta\boldsymbol{\phi}_j\rfloor + \frac{1}{2}\lfloor\delta\boldsymbol{\phi}_j\rfloor^2 \tag{33}$$

Substituting (33) into $\mathbf{w}_{jl}(\delta\mathbf{T}_j)$, we obtain

$$\mathbf{w}_{jl}(\delta\mathbf{T}_j) \approx \begin{bmatrix}\mathbf{R}_j^T(\mathbf{I} - \lfloor\delta\boldsymbol{\phi}_j\rfloor + \frac{1}{2}\lfloor\delta\boldsymbol{\phi}_j\rfloor^2)\mathbf{u}_l\\ \mathbf{t}_j^T(\mathbf{I} - \lfloor\delta\boldsymbol{\phi}_j\rfloor + \frac{1}{2}\lfloor\delta\boldsymbol{\phi}_j\rfloor^2)\mathbf{u}_l + \mathbf{u}_l^T\delta\mathbf{t}_j\end{bmatrix}$$

$$\approx \underbrace{\begin{bmatrix}\mathbf{R}_j^T\mathbf{u}_l\\ \mathbf{t}_j^T\mathbf{u}_l\end{bmatrix}}_{\bar{\mathbf{w}}_{jl}} + \underbrace{\begin{bmatrix}\mathbf{R}_j^T\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times 3}\\ \mathbf{t}_j^T\lfloor\mathbf{u}_l\rfloor & \mathbf{u}_l^T\end{bmatrix}}_{\mathbf{J}_{\mathbf{w}_{jl}}\in\mathbb{R}^{4\times 6}}\underbrace{\begin{bmatrix}\delta\boldsymbol{\phi}_j\\ \delta\mathbf{t}_j\end{bmatrix}}_{\delta\mathbf{T}_j} + \underbrace{\begin{bmatrix}\frac{1}{2}\mathbf{R}_j^T\lfloor\delta\boldsymbol{\phi}_j\rfloor^2\mathbf{u}_l\\ \frac{1}{2}\mathbf{t}_j^T\lfloor\delta\boldsymbol{\phi}_j\rfloor^2\mathbf{u}_l\end{bmatrix}}_{\delta\boldsymbol{\Phi}_{jl}} \tag{34}$$

where $\bar{\mathbf{w}}_{jl}$, $\delta\boldsymbol{\Phi}_{jl}$ and $\mathbf{J}_{\mathbf{w}_{jl}}$ can be simplified as

$$\bar{\mathbf{w}}_{jl} = (\mathbf{S}_\mathbf{P}\mathbf{T}_j)^T\mathbf{u}_l, \quad \delta\boldsymbol{\Phi}_{jl} = \frac{1}{2}(\mathbf{S}_\mathbf{P}\mathbf{T}_j)^T\lfloor\delta\boldsymbol{\phi}_j\rfloor^2\mathbf{u}_l$$

$$\mathbf{J}_{\mathbf{w}_{jl}} = \mathbf{T}_j^T\mathbf{V}_l^T, \quad \text{where } \mathbf{V}_l = \begin{bmatrix}-\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times 1}\\ \mathbf{0}_{3\times 3} & \mathbf{u}_l\end{bmatrix} \tag{35}$$

Substituting (34) into (31) and keeping terms up to the second order lead to

$$
\begin{aligned}
\mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l &= \frac{1}{N}\sum_{j=1}^{M_p}\mathbf{w}_{jk}^T(\delta\mathbf{T}_j)\mathbf{C}_j\mathbf{w}_{jl}(\delta\mathbf{T}_j) \\
&\quad - \frac{1}{N^2}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{w}_{ik}^T(\delta\mathbf{T}_i)\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{w}_{jl}(\delta\mathbf{T}_j) \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{j=1}^{M_p}\Big((\bar{\mathbf{w}}_{jk}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} + \bar{\mathbf{w}}_{jl}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jk}})\delta\mathbf{T}_j + \bar{\mathbf{w}}_{jk}^T\mathbf{C}_j\bar{\mathbf{w}}_{jl} \\
&+ \delta\mathbf{T}_j^T\mathbf{J}_{\mathbf{w}_{jk}}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}}\delta\mathbf{T}_j + \underbrace{\bar{\mathbf{w}}_{jk}^T\mathbf{C}_j\delta\boldsymbol{\Phi}_{jl} + \delta\boldsymbol{\Phi}_{jk}^T\mathbf{C}_j\bar{\mathbf{w}}_{jl}}_{\frac{1}{2}\delta\mathbf{T}_j^T\mathbf{Y}_{kl}^j\delta\mathbf{T}_j}\Big) \\
&- \frac{1}{N^2}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\Big(\delta\mathbf{T}_i^T\mathbf{J}_{\mathbf{w}_{ik}}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}}\delta\mathbf{T}_j \quad (37) \\
&+ \bar{\mathbf{w}}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}}\delta\mathbf{T}_j + \delta\mathbf{T}_i^T\mathbf{J}_{\mathbf{w}_{ik}}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\bar{\mathbf{w}}_{jl} \\
&+ \bar{\mathbf{w}}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\bar{\mathbf{w}}_{jl} + \underbrace{\bar{\mathbf{w}}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\delta\boldsymbol{\Phi}_{jl}}_{\frac{1}{2}\delta\mathbf{T}_j^T\mathbf{N}_{kl}^{ij}\delta\mathbf{T}_j} + \underbrace{\delta\boldsymbol{\Phi}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\bar{\mathbf{w}}_{jl}}_{\frac{1}{2}\delta\mathbf{T}_i^T\mathbf{M}_{kl}^{ij}\delta\mathbf{T}_i}\Big). \\
& \quad (38)
\end{aligned}
$$

To compute $\mathbf{Y}_{kl}^j$ in $\frac{1}{2}\delta\mathbf{T}_j^T\mathbf{Y}_{kl}^j\delta\mathbf{T}_j$, note that $\mathbf{a}^T\lfloor\delta\boldsymbol{\phi}\rfloor^2\mathbf{b} = \delta\boldsymbol{\phi}^T\lfloor\mathbf{a}\rfloor\lfloor\mathbf{b}\rfloor\delta\boldsymbol{\phi}, \forall\mathbf{a},\mathbf{b},\delta\boldsymbol{\phi}\in\mathbb{R}^3$, we have

$$
\begin{aligned}
\bar{\mathbf{w}}_{jk}^T\mathbf{C}_j\delta\boldsymbol{\Phi}_{jl} &= \frac{1}{2}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\lfloor\delta\boldsymbol{\phi}_j\rfloor^2\mathbf{u}_l \\
&= \frac{1}{2}\delta\boldsymbol{\phi}_j^T\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_j. \quad (39)
\end{aligned}
$$

Similarly,

$$
\delta\boldsymbol{\Phi}_{jk}^T\mathbf{C}_j\bar{\mathbf{w}}_{jl} = \frac{1}{2}\delta\boldsymbol{\phi}_j^T\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_j. \quad (40)
$$

Summing (39) and (40) and extending the $\delta\boldsymbol{\phi}$ into $\delta\mathbf{T}$:

$$
\mathbf{Y}_{kl}^j = \begin{bmatrix}\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor + \lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} \quad (41)
$$

For $\frac{1}{2}\delta\mathbf{T}_j^T\mathbf{N}_{kl}^{ij}\delta\mathbf{T}_j$ and $\frac{1}{2}\delta\mathbf{T}_i^T\mathbf{M}_{kl}^{ij}\delta\mathbf{T}_i$,

$$
\begin{aligned}
\bar{\mathbf{w}}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\delta\boldsymbol{\Phi}_{jl} &= \frac{1}{2}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\lfloor\delta\boldsymbol{\phi}_j\rfloor^2\mathbf{u}_l \\
&= \frac{1}{2}\delta\boldsymbol{\phi}_j^T\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{F}\mathbf{C}_i\mathbf{T}_i^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_j \quad (42) \\
\delta\boldsymbol{\Phi}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\bar{\mathbf{w}}_{jl} &= \frac{1}{2}\mathbf{u}_k^T\lfloor\delta\boldsymbol{\phi}_i\rfloor^2\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l \\
&= \frac{1}{2}\delta\boldsymbol{\phi}_i^T\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\delta\boldsymbol{\phi}_i \quad (43)
\end{aligned}
$$

Thus, extending the $\delta\boldsymbol{\phi}$ into $\delta\mathbf{T}$, we obtain

$$
\mathbf{N}_{kl}^{ij} = \begin{bmatrix}\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{F}\mathbf{C}_i\mathbf{T}_i^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \quad (44)
$$

$$
\mathbf{M}_{kl}^{ij} = \begin{bmatrix}\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \quad (45)
$$

It can be seen that (38) is quadratic w.r.t. $\delta\mathbf{T}$, so we cast it into the following standard form

$$
\mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = r_{kl} + \mathbf{g}_{kl}\cdot\delta\mathbf{T} + \frac{1}{2}\delta\mathbf{T}^T\cdot\mathbf{Q}_{kl}\cdot\delta\mathbf{T}, \quad (46)
$$

where $\mathbf{g}_{kl}$ and $\mathbf{Q}_{kl}$ are partitioned as

$$
\mathbf{g}_{kl} = \begin{bmatrix}\cdots & \mathbf{g}_{kl}^j & \cdots\end{bmatrix} \in \mathbb{R}^{1\times6M_p} \quad (47)
$$

$$
\mathbf{Q}_{kl} = \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{Q}_{kl}^{ij} & \cdots \\ & \vdots & \end{bmatrix} \in \mathbb{R}^{6M_p\times6M_p}. \quad (48)
$$

For $\mathbf{g}_{kl}^j \in \mathbb{R}^{1\times6}, \forall j\in\{1,\cdots,M_p\}$, the j-th column block of $\mathbf{g}_{kl}$ in (47), it is

$$
\begin{aligned}
\mathbf{g}_{kl}^j &= \frac{1}{N}(\bar{\mathbf{w}}_{jk}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} + \bar{\mathbf{w}}_{jl}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jk}}) \\
&\quad - \frac{1}{N^2}\sum_{i=1}^{M_p}(\bar{\mathbf{w}}_{ik}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} + \bar{\mathbf{w}}_{il}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jk}}) \quad (49) \\
&= \frac{1}{N}(\mathbf{u}_k^T\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \mathbf{u}_l^T\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_k^T) \\
&- \frac{1}{N^2}\sum_{i=1}^{M_p}(\mathbf{u}_k^T\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \mathbf{u}_l^T\mathbf{S}_\mathbf{P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_k^T) \\
&= \frac{1}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j\mathbf{C}_j - \frac{1}{N}\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j)\mathbf{T}_j^T\mathbf{V}_k^T \\
&\quad + \frac{1}{N}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j\mathbf{C}_j - \frac{1}{N}\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j)\mathbf{T}_j^T\mathbf{V}_l^T \quad (50)
\end{aligned}
$$

Since $\mathbf{v} = \sum_{j=1}^{M_p}(\mathbf{R}_j\mathbf{v}_j + N_j\mathbf{t}_j)$ from (23) and $\mathbf{F} = \begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1\end{bmatrix}$, we have

$$
\begin{aligned}
\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F} &= \sum_{i=1}^{M_p}\begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1\end{bmatrix}\begin{bmatrix}\mathbf{P}_i & \mathbf{v}_i \\ \mathbf{v}_i^T & N_i\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1\end{bmatrix} \\
&= \sum_{i=1}^{M_p}\begin{bmatrix}\mathbf{0}_{3\times1} & \mathbf{R}_i\mathbf{v}_i + N_i\mathbf{t}_i \\ \mathbf{0}_{1\times3} & N_i\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{v} \\ \mathbf{0} & N_i\end{bmatrix} = \mathbf{CF}. \quad (51)
\end{aligned}
$$

Hence, the part $\left(\mathbf{T}_j\mathbf{C}_j - \frac{1}{N}\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\right)$ in (50) is

$$
\begin{aligned}
\mathbf{T}_j\mathbf{C}_j - \frac{1}{N}\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j &= \mathbf{T}_j\mathbf{C}_j - \frac{1}{N}\underbrace{\left(\sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\right)}_{\mathbf{CF}}\mathbf{C}_j \\
&= (\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j. \quad (52)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{g}_{kl}^j &= \frac{1}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_k^T \\
&\quad + \frac{1}{N}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T, \quad (53)
\end{aligned}
$$

which yields the result in (14). Additionally, it is seen that if $\mathbf{C}_j = \mathbf{0}_{4\times4}$, $\mathbf{g}_{kl}^j = \mathbf{0}_{1\times6}, \forall\mathbf{u}_k,\mathbf{u}_l$.

For $\mathbf{Q}_{kl}^{ij} \in \mathbb{R}^{6\times6}, \forall i,j \in \{1,\cdots,M_p\}$, the i-th row, j-th column block of $\mathbf{Q}_{kl}$ in (48), it is

$$\mathbf{Q}_{kl}^{ij} = -\frac{2}{N^2}\mathbf{J}_{\mathbf{w}_{ik}}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} + \mathbb{1}_{i=j}\cdot\Big(\frac{2}{N}\mathbf{J}_{\mathbf{w}_{jk}}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} + \frac{1}{N}\mathbf{Y}_{kl}^j - \frac{1}{N^2}\sum_{\nu=1}^{M_p}(\mathbf{N}_{kl}^{\nu j}+\mathbf{M}_{kl}^{i\nu})\Big) \tag{54}$$

where

$$\mathbb{1}_{i=j} = \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{55}$$

$$\mathbf{J}_{\mathbf{w}_{ik}}^T\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} = \mathbf{V}_k\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T \tag{56}$$

$$\mathbf{J}_{\mathbf{w}_{jk}}^T\mathbf{C}_j\mathbf{J}_{\mathbf{w}_{jl}} = \mathbf{V}_k\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T \tag{57}$$

$$\sum_{\nu=1}^{M_p}(\mathbf{N}_{kl}^{\nu j}+\mathbf{M}_{kl}^{i\nu}) = \sum_{\nu=1}^{M_p}\begin{bmatrix}\lfloor\mathbf{S_P}\mathbf{T}_j\mathbf{C}_j\mathbf{F}\mathbf{C}_\nu\mathbf{T}_\nu^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \sum_{\nu=1}^{M_p}\begin{bmatrix}\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_\nu\mathbf{T}_\nu^T\mathbf{S_P}^T\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{58}$$

By using similar method in (51), we have

$$\sum_{\nu=1}^{M_p}\mathbf{F}\mathbf{C}_\nu\mathbf{T}_\nu^T = \mathbf{F}\mathbf{C} \tag{59}$$

and

$$\sum_{\nu=1}^{M_p}(\mathbf{N}_{kl}^{\nu j}+\mathbf{M}_{kl}^{i\nu}) = \begin{bmatrix}\lfloor\mathbf{S_P}\mathbf{T}_j\mathbf{C}_j\mathbf{F}\mathbf{C}\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor + \lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}\mathbf{S_P}^T\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{60}$$

Thus, the matrix $\mathbf{Q}_{kl}^{ij}$ is

$$\mathbf{Q}_{kl}^{ij} = -\frac{2}{N^2}\mathbf{V}_k\mathbf{T}_i\mathbf{C}_i\mathbf{F}\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \mathbb{1}_{i=j}\cdot\Big(\frac{2}{N}\mathbf{V}_k\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \begin{bmatrix}\mathbf{K}_{kl}^j & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix}\Big) \tag{61}$$

$$\begin{aligned}\mathbf{K}_{kl}^j &= \frac{1}{N}\mathbf{Y}_{kl}^j - \frac{1}{N^2}\sum_{\nu=1}^{M_p}(\mathbf{N}_{kl}^{\nu j}+\mathbf{M}_{kl}^{i\nu}), \quad \text{with } i=j\\ &= \frac{1}{N}\lfloor\mathbf{S_P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})^T\mathbf{S_P}^T\mathbf{u}_k\rfloor\lfloor\mathbf{u}_l\rfloor\\ &\quad + \frac{1}{N}\lfloor\mathbf{u}_k\rfloor\lfloor\mathbf{S_P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})^T\mathbf{S_P}^T\mathbf{u}_l\rfloor,\end{aligned} \tag{62}$$

which yields the results in (15) and (16). Additionally, it can be seen that if $\mathbf{C}_i = \mathbf{0}_{4\times4}$ or $\mathbf{C}_j = \mathbf{0}_{4\times4}$, we have $\mathbf{C}_i\mathbf{F}\mathbf{C}_j = \mathbf{0}_{4\times4}$, $\mathbf{K}_{kl}^j = \mathbf{0}_{3\times3}$ and then $\mathbf{Q}_{kl}^{ij} = \mathbf{0}_{6\times6}, \forall\mathbf{u}_k,\mathbf{u}_l$. □

II. Proof of Theorems

A. Proof of formula (6) and (8)

Proof. The variable to be optimized is $\boldsymbol{\pi}_i = (\mathbf{n}_i,\mathbf{q}_i)$ and the cost function is

$$c_i = \min_{\boldsymbol{\pi}_i=(\mathbf{n}_i,\mathbf{q}_i)}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\left\|\mathbf{h}_i^T(\mathbf{p}_{ijk}-\mathbf{q}_i)\right\|_2^2\Big) \tag{63}$$

where $\mathbf{h}_i = \mathbf{n}_i$ for plane feature and $\mathbf{h}_i = (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)$ for edge feature. The dimensions of $\mathbf{h}_i$ may be different for these two features but it has no influence on following derivation.

$$\begin{aligned}c_i &= \min_{\mathbf{n}_i}\min_{\mathbf{q}_i}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\left\|\mathbf{h}_i^T(\mathbf{p}_{ijk}-\mathbf{q}_i)\right\|_2^2\Big)\\ &= \min_{\mathbf{n}_i}\left(\min_{\mathbf{q}_i}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}(\mathbf{p}_{ijk}-\mathbf{q}_i)^T\mathbf{h}_i\mathbf{h}_i^T(\mathbf{p}_{ijk}-\mathbf{q}_i)\Big)\right).\end{aligned} \tag{64}$$

As can be seen, the inner optimization on $\mathbf{q}_i$ is a standard quadratic optimization problem. So, the optimum $\mathbf{q}_i^\star$ can be solved by setting the derivative to zero:

$$\begin{aligned}&2\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\mathbf{h}_i\mathbf{h}_i^T(\mathbf{p}_{ijk}-\mathbf{q}_i) = \mathbf{0} \implies\\ &2\mathbf{h}_i\mathbf{h}_i^T\left(\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\mathbf{p}_{ijk} - N_i\mathbf{q}_i\right) = \mathbf{0}\end{aligned} \tag{65}$$

where $N_i = \sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}1 = \sum_{j=1}^{M_p}N_{ij}$. This equation does not lead to a unique solution of $\mathbf{q}_i$, one particular optimum solution is $\mathbf{q}_i^\star = \frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\mathbf{p}_{ijk} = \bar{\mathbf{p}}_i$ as is defined in (7).

Now, substituting the optimum solution $\mathbf{q}_i^\star = \bar{\mathbf{p}}_i$ into (64) leads to:

$$c_i = \min_{\mathbf{n}_i}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\left\|\mathbf{h}_i^T(\mathbf{p}_{ijk}-\bar{\mathbf{p}}_i)\right\|_2^2\Big) \tag{66}$$

To solve for the optimal parameter $\mathbf{n}_i$ in the above optimization problem, we discuss the case of plane and edge features separately, as follows.

1) Plane feature: $\mathbf{h}_i = \mathbf{n}_i$.

$$\begin{aligned}c_i &= \min_{\|\mathbf{n}_i\|_2=1}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\left\|\mathbf{n}_i^T(\mathbf{p}_{ijk}-\bar{\mathbf{p}}_i)\right\|_2^2\Big)\\ &= \min_{\|\mathbf{n}_i\|_2=1}\Big(\frac{1}{N_i}\sum_{j=1}^{M_p}\sum_{k=1}^{N_{ij}}\big(\mathbf{n}_i^T(\mathbf{p}_{ijk}-\bar{\mathbf{p}}_i)(\mathbf{p}_{ijk}-\bar{\mathbf{p}}_i)^T\mathbf{n}_i\big)\Big)\\ &= \min_{\mathbf{n}_i}\mathbf{n}_i^T\mathbf{A}_i\mathbf{n}_i,\end{aligned} \tag{67}$$

where $\mathbf{A}_i$ is defined in (7) and is a symmetric matrix. Performing Singular Value Decomposition (SVD) of $\mathbf{A}_i$

$$\mathbf{A}_i = \mathbf{U}_i\boldsymbol{\Lambda}_i\mathbf{U}_i^T \tag{68}$$

where

$$\mathbf{U}_i = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \quad \boldsymbol{\Lambda}_i = \operatorname{diag}(\lambda_1 \quad \lambda_2 \quad \lambda_3) \tag{69}$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\mathbf{U}_i^T\mathbf{U}_i = \mathbf{I}$.

Denote $\mathbf{m} = \mathbf{U}_i\mathbf{n}_i = [m_1 \quad m_2 \quad m_3]^T$, $\|\mathbf{m}\|_2 = \sqrt{\mathbf{n}_i^T\mathbf{U}_i^T\mathbf{U}_i\mathbf{n}_i} = 1$, then (67) reduces to

$$\begin{aligned}c_i &= \min_{\|\mathbf{n}_i\|_2=1}(\mathbf{n}_i^T\mathbf{U}_i\boldsymbol{\Lambda}_i\mathbf{U}_i^T\mathbf{n}_i) = \min_{\|\mathbf{m}\|_2=1}(\mathbf{m}^T\boldsymbol{\Lambda}_i\mathbf{m})\\ &= \min_{\|\mathbf{m}\|_2=1}(\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2)\\ &\geq \min_{\|\mathbf{m}\|_2=1}(\lambda_3 m_1^2 + \lambda_3 m_2^2 + \lambda_3 m_3^2) = \lambda_3,\end{aligned} \tag{70}$$

where the minimum value λ_3 is reached when $m_3 = 1$, i.e., $\mathbf{m}^\star = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\mathbf{n}_i^\star = \mathbf{U}_i \mathbf{m}^\star = \mathbf{u}_3$.

Therefore, the optimal cost is $\lambda_3(\mathbf{A}_i)$ and the optimum solution is $\mathbf{n}^\star = \mathbf{u}_3(\mathbf{A}_i)$ and $\mathbf{q}^\star = \bar{\mathbf{p}}_i$.

2) Edge feature: $\mathbf{h}_i = \mathbf{I} - \mathbf{n}_i \mathbf{n}_i^T$, then $\mathbf{h}_i \mathbf{h}_i^T = \mathbf{h}_i$ and

$$\begin{aligned}
c_i =& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} \left\| (\mathbf{I} - \mathbf{n}_i \mathbf{n}_i^T)(\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right\|_2^2 \right) \\
=& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{I} - \mathbf{n}_i \mathbf{n}_i^T)(\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right) \\
=& \min_{\mathbf{n}_i} \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right. \\
& \left. - \mathbf{n}_i^T \left(\frac{1}{N_i} \sum_{j=1}^{M_p} \sum_{k=1}^{N_{ij}} (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i)^T (\mathbf{p}_{ijk} - \bar{\mathbf{p}}_i) \right) \mathbf{n}_i \right) \\
=& \min_{\mathbf{n}_i} (\text{trace}(\mathbf{A}_i) - \mathbf{n}_i^T \mathbf{A}_i \mathbf{n}_i) \\
=& \lambda_1 + \lambda_2 + \lambda_3 - \underbrace{\max_{\mathbf{n}_i} \mathbf{n}_i^T \mathbf{A}_i \mathbf{n}_i}_{=\lambda_1, \text{ when } , \mathbf{n}_i^\star = \mathbf{u}_1} \\
=& \lambda_2 + \lambda_3.
\end{aligned} \tag{71}$$

Therefore, the optimal cost is $\lambda_2(\mathbf{A}_i) + \lambda_3(\mathbf{A}_i)$ and the optimum solution is $\mathbf{n}^\star = \mathbf{u}_1(\mathbf{A}_i)$ and $\mathbf{q}^\star = \bar{\mathbf{p}}_i$.

$\square$

B. Proof of Theorem 1

For the point collections $\mathcal{C} = \{\mathbf{p}_k \in \mathbb{R}^3 | k = 1, \cdots, n\}$, its point cluster is

$$\mathfrak{R}(\mathcal{C}) = \sum_{k=1}^{n} \begin{bmatrix} \mathbf{p}_k \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k^T & 1 \end{bmatrix} \tag{72}$$

The rigid transformation of $\mathcal{C}$ by pose $\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_{3\times 1} & 1 \end{bmatrix}$ is

$$\mathbf{T} \circ \mathcal{C} = \{\mathbf{R}\mathbf{p}_k + \mathbf{t} \in \mathbb{R}^3 | k = 1, \cdots, n\} \tag{73}$$

whose point cluster is

$$\begin{aligned}
\mathfrak{R}(\mathbf{T} \circ \mathcal{C}) &= \sum_{k=1}^{n} \begin{bmatrix} \mathbf{R}\mathbf{p}_k + \mathbf{t} \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{R}\mathbf{p}_k + \mathbf{t})^T & 1 \end{bmatrix} \\
&= \sum_{k=1}^{n} \mathbf{T} \begin{bmatrix} \mathbf{p}_k \\ 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k^T & 1 \end{bmatrix} \mathbf{T}^T = \mathbf{T} \mathfrak{R}(\mathcal{C}) \mathbf{T}^T
\end{aligned} \tag{74}$$

which yields the solution. $\square$

C. Proof of Theorem 2

For two point collections $\mathcal{C}_1 = \{\mathbf{p}_k^1 \in \mathbb{R}^3 | k = 1, \cdots, n_1\}$ and $\mathcal{C}_2 = \{\mathbf{p}_k^2 \in \mathbb{R}^3 | k = 1, \cdots, n_2\}$ in the same reference frame, their point clusters are respectively

$$\mathfrak{R}(\mathcal{C}_l) = \sum_{k=1}^{n_l} \begin{bmatrix} \mathbf{p}_k^l \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^l)^T & 1 \end{bmatrix}, \quad l = 1, 2 \tag{75}$$

The merge of $\mathcal{C}_1$ and $\mathcal{C}_2$ is

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\mathbf{p}_k^l \in \mathbb{R}^3 | l = 1, 2; k = 1, \cdots, n_l\} \tag{76}$$

whose point cluster is

$$\begin{aligned}
\mathfrak{R}(\mathcal{C}_1 \oplus \mathcal{C}_2) &= \sum_{k=1}^{n_1} \begin{bmatrix} \mathbf{p}_k^1 \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^1)^T & 1 \end{bmatrix} + \sum_{k=1}^{n_2} \begin{bmatrix} \mathbf{p}_k^2 \\ 1 \end{bmatrix} \begin{bmatrix} (\mathbf{p}_k^2)^T & 1 \end{bmatrix} \\
&= \mathfrak{R}(\mathcal{C}_1) + \mathfrak{R}(\mathcal{C}_2)
\end{aligned} \tag{77}$$

which yields the solution. $\square$

D. Proof of Theorem 3

Let $\mathbf{T}_0 = \begin{bmatrix} \mathbf{R}_0 & \mathbf{t}_0 \\ 0 & 1 \end{bmatrix}$ and $\bar{\mathbf{C}} = \mathbf{T}_0 \mathbf{C} \mathbf{T}_0^T = \begin{bmatrix} \bar{\mathbf{P}} & \bar{\mathbf{v}} \\ \bar{\mathbf{v}}^T & N \end{bmatrix}$, then

$$\bar{\mathbf{P}} = \mathbf{R}_0 \mathbf{P}_0 \mathbf{R}_0^T + \mathbf{R}_0 \mathbf{v}_0 \mathbf{t}_0^T + \mathbf{t}_0 \mathbf{v}_0^T \mathbf{R}_0^T + N \mathbf{t}_0 \mathbf{t}_0^T, \tag{78}$$

$$\bar{\mathbf{v}} = \mathbf{R}_0 \mathbf{v}_0 + N \mathbf{t}_0, \tag{79}$$

$$\mathbf{A}\left(\mathbf{T}_0 \mathbf{C} \mathbf{T}_0^T\right) = \frac{1}{N} \bar{\mathbf{P}} - \frac{1}{N^2} \bar{\mathbf{v}} \bar{\mathbf{v}}^T = \mathbf{R}_0 \mathbf{A}(\mathbf{C}) \mathbf{R}_0^T. \tag{80}$$

Since $\mathbf{A}\left(\mathbf{T}_0 \mathbf{C} \mathbf{T}_0^T\right)$ and $\mathbf{A}(\mathbf{C})$ are similar by transformation $\mathbf{R}_0$, they have the same eigenvalue. $\square$

E. Proof of Theorem 4

Denote λ_l the l-th largest eigenvalue of $\mathbf{A}$ and $\mathbf{u}_l$ the corresponding vector, i.e., $\lambda_l \mathbf{u}_l = \mathbf{A} \mathbf{u}_l$. Since $\mathbf{A}$ is symmetric, $\mathbf{u}_l$ is an orthonormal vector. Multiplying both sides of $\lambda_l \mathbf{u}_l = \mathbf{A} \mathbf{u}_l$ by $\mathbf{u}_l^T$ leads to

$$\lambda_l = \mathbf{u}_l^T \mathbf{A} \mathbf{u}_l. \tag{81}$$

Note that in the above equation, λ_l, $\mathbf{u}_l$ and $\mathbf{A}_l$ all depend on the pose $\mathbf{T}$. To avoid any confusion, we write them as explicit functions of $\mathbf{T}$:

$$\lambda_l(\mathbf{T}) = \mathbf{u}_l^T(\mathbf{T}) \mathbf{A}(\mathbf{T}) \mathbf{u}_l(\mathbf{T}). \tag{82}$$

Parameterizing the pose $\mathbf{T}$ by $\delta\mathbf{T}$ leads to

$$\lambda_l(\delta\mathbf{T}) = \mathbf{u}_l^T(\delta\mathbf{T}) \mathbf{A}(\delta\mathbf{T}) \mathbf{u}_l(\delta\mathbf{T}). \tag{83}$$

From the first conclusion of Lemma 1 (i.e., (1)), we know that for a vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \in \mathbb{R}^m$ that the matrix $\mathbf{A}$ depends on, we have

$$\begin{aligned}
\frac{\partial \lambda_l(\mathbf{x})}{\partial x_i} = \frac{\partial \left(\mathbf{u}_l^T(\mathbf{x}) \mathbf{A}(\mathbf{x}) \mathbf{u}_l(\mathbf{x}) \right)}{\partial x_i} &= \mathbf{u}_l^T(\mathbf{x}) \frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i} \mathbf{u}_l(\mathbf{x}), \\
\forall i = 1, \cdots, m, x_i &\in \mathbb{R}
\end{aligned} \tag{84}$$

Directly applying this result to the entire vector $\mathbf{x}$ leads to the notation of $\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}$, which is a tensor. To avoid it, we fix the vector $\mathbf{u}_l(\mathbf{x})$ at its current value and lump it with the matrix $\mathbf{A}(\mathbf{x})$ within the derivative, i.e.,

$$\mathbf{u}_l^T(\mathbf{x}) \frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i} \mathbf{u}_l(\mathbf{x}) := \frac{\partial \mathbf{u}_l^T \mathbf{A}(\mathbf{x}) \mathbf{u}_l}{\partial x_i} \tag{85}$$

where on the right hand side, $\mathbf{u}_l$ is fixed (so we remove its argument $\mathbf{x}$) and the derivative is only applied on the component $\mathbf{A}(\mathbf{x})$ (so we keep its argument $\mathbf{x}$). If applying (85) to the entire vector $\mathbf{x} \in \mathbb{R}^m$, the result would be $\frac{\partial \mathbf{u}_l^T \mathbf{A}(\mathbf{x}) \mathbf{u}_l}{\partial \mathbf{x}}$, which is now a row vector of dimension m.

Since the input parameter is the poses parameterized by $\delta\mathbf{T}$, setting $\mathbf{x}$ to $\delta\mathbf{T}$ in (84) and applying the notation trick in (85) lead to

$$\frac{\partial \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}} = \frac{\partial \mathbf{u}_l^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}. \tag{86}$$

Recalling Lemma 2 with $k = l$, we obtain:

$$\mathbf{u}_l^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l = \frac{1}{2}\delta\mathbf{T}^T \cdot \mathbf{Q}_{ll} \cdot \delta\mathbf{T} + \mathbf{g}_{ll} \cdot \delta\mathbf{T} + r_{ll}. \tag{87}$$

where $\mathbf{g}_{ll}$ and $\mathbf{Q}_{ll}$ are defined in (12) and (13) with $k = l$, respectively.

Therefore, the first order derivative of $\lambda_l(\mathbf{T})$ w.r.t. $\mathbf{T}$ is

$$\mathbf{J}_l \triangleq \frac{\partial \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}} = \frac{\partial \mathbf{u}_l^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}} = \mathbf{g}_{ll}, \tag{88}$$

which is the result of (30) in Theorem 4.

Next, we derive the second order derivative of $\lambda_l(\mathbf{T})$ w.r.t. $\mathbf{T}$. From (84), we have,

$$\frac{\partial \lambda_l(\mathbf{x})}{\partial x_i} = \mathbf{u}_l^T(\mathbf{x})\frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x}), \forall x_i \in \mathbb{R}, \tag{89}$$

Differentiating it w.r.t. the second parameter $x_j \in \mathbb{R}$ leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\mathbf{x})}{\partial x_j \partial x_i} &= \frac{\partial}{\partial x_j}\left(\mathbf{u}_l^T(\mathbf{x})\frac{\partial \mathbf{A}(\mathbf{x})}{\partial x_i}\mathbf{u}_l(\mathbf{x})\right) \\ &= \left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial x_j}\right)^T\left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \mathbf{u}_l^T \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right) \\ &+ \left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial x_i}\right)^T\left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial x_j}\right) \end{aligned} \tag{90}$$

Applying the above results to each elements x_i, x_j leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\mathbf{x})}{\partial \mathbf{x}^2} &= \left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right) \\ &+ \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{u}_l^T \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}\right). \end{aligned} \tag{91}$$

To compute $\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}}$, we apply the second conclusion in Lemma 1 (i.e., (2)) to all components of $\mathbf{x}$ and use the notation trick similar to (85):

$$\frac{\partial \mathbf{u}_l(\mathbf{x})}{\partial \mathbf{x}} = \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{u}_k^T\frac{\partial \mathbf{A}(\mathbf{x})\mathbf{u}_l}{\partial \mathbf{x}} \tag{92}$$

Now, the input parameter is the pose vector parameterized by $\delta\mathbf{T}$, substituting $\mathbf{x} = \delta\mathbf{T}$ into (91) leads to

$$\begin{aligned}\frac{\partial^2 \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}^2} &= \underbrace{\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)}_{\mathbf{D}_l^T} \\ &+ \underbrace{\frac{\partial^2 \mathbf{u}_l^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}^2}}_{\mathbf{Q}_{ll}} + \underbrace{\left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right)}_{\mathbf{D}_l}, \end{aligned} \tag{93}$$

where term $\mathbf{Q}_{ll}$ is from (87), which is further from (13) with $k = l$. To obtain the term $\mathbf{D}_l$, we substitute $\mathbf{x} = \delta\mathbf{T}$ into (92):

$$\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}} = \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{u}_k^T\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}} \tag{94}$$

$$= \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\frac{\partial \mathbf{u}_k^T\mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}. \tag{95}$$

From Lemma 2, we have

$$\frac{\partial \mathbf{u}_k^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}} = \mathbf{g}_{kl} \tag{96}$$

with $\mathbf{g}_{kl}$ defined in (12). Hence,

$$\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}} = \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{g}_{kl} \tag{97}$$

and

$$\mathbf{D}_l = \left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T\left(\frac{\partial \mathbf{u}_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}}\right) \tag{98}$$

$$= \left(\frac{\partial \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{u}_k\mathbf{g}_{kl} \tag{99}$$

$$= \sum_{k=1,k\neq l}^{3} \left(\frac{\partial \mathbf{u}_k^T \mathbf{A}(\delta\mathbf{T})\mathbf{u}_l}{\partial \delta\mathbf{T}}\right)^T \frac{1}{\lambda_l - \lambda_k}\mathbf{g}_{kl} \tag{100}$$

$$= \sum_{k=1,k\neq l}^{3} \frac{1}{\lambda_l - \lambda_k}\mathbf{g}_{kl}^T\mathbf{g}_{kl}. \tag{101}$$

Therefore, the second order derivative of $\lambda_l(\mathbf{T})$ w.r.t. $\mathbf{T}$ is

$$\begin{aligned}\mathbf{H}_l \triangleq \frac{\partial^2 \lambda_l(\delta\mathbf{T})}{\partial \delta\mathbf{T}^2} &= \mathbf{D}_l^T + \mathbf{Q}_{ll} + \mathbf{D}_l \\ &= \mathbf{W}_l + \sum_{k=1,k\neq l}^{3} \frac{2}{\lambda_l - \lambda_k}\mathbf{g}_{kl}^T\mathbf{g}_{kl}. \end{aligned} \tag{102}$$

where $\mathbf{W}_l = \mathbf{Q}_{ll}$ defined in (13) with $k = l$. (102) gives the result of (31) in Theorem 4. □

F. Proof of Corollary 4.1

First, we show that

$$\mathbf{J}\delta\mathbf{T} = 0, \delta\mathbf{T}^T\mathbf{H}\delta\mathbf{T} = 0, \delta\mathbf{T} = \begin{bmatrix}\mathbf{w}\\ \vdots \\ \mathbf{w}\end{bmatrix}, \forall \mathbf{w} \in \mathbb{R}^6 \tag{103}$$

From (30) in Theorem 4, we can obtain

$$\mathbf{J}_l\delta\mathbf{T} = \mathbf{g}_{ll}\delta\mathbf{T} = \sum_{j=1}^{M_p} \mathbf{g}_{ll}^j\mathbf{w} \tag{104}$$

where $\mathbf{g}_{ll}^j$ is defined in (14) with $k = l$:

$$\mathbf{g}_{ll}^j = \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{C}\mathbf{F})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T \tag{105}$$

$$\mathbf{V}_l = \begin{bmatrix} -\lfloor \mathbf{u}_l \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_l \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{106}$$

$$\mathbf{S}_\mathbf{P} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 1} \end{bmatrix} \quad \mathbf{C} = \sum_{j=1}^{M_p} \mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T \tag{107}$$

Thus,

$$\sum_{j=1}^{M_p}\mathbf{g}_{ll}^j\mathbf{w} = \frac{2}{N}\sum_{j=1}^{M_p}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T\mathbf{w}$$
$$= \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}\left(\sum_{j=1}^{M_p}(\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T - \frac{1}{N}\mathbf{CFC}_j\mathbf{T}_j^T)\right)\mathbf{V}_l^T\mathbf{w}.$$

Since $\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T \triangleq \mathbf{C}$ and $\sum_{j=1}^{M_p}\mathbf{FC}_j\mathbf{T}_j^T = \mathbf{FC}$ from equation (51), we have

$$\sum_{j=1}^{M_p}\mathbf{g}_{ll}^j\mathbf{w} = \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{C} - \frac{1}{N}\mathbf{CFC})\mathbf{V}_l^T\mathbf{w}.$$

Partition $\mathbf{C} = \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix}$ and recalling $\mathbf{A} = \frac{1}{N}\mathbf{P} - \frac{1}{N^2}\mathbf{v}\mathbf{v}^T$,

$$\begin{aligned}&\mathbf{C} - \frac{1}{N}\mathbf{CFC}\\ &= \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} - \frac{1}{N}\begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{P} & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} - \begin{bmatrix}\frac{1}{N}\mathbf{v}\mathbf{v}^T & \mathbf{v}\\ \mathbf{v}^T & N\end{bmatrix} = \begin{bmatrix}N\mathbf{A} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix}\end{aligned} \tag{108}$$

Thus,

$$\sum_{j=1}^{M_p}\mathbf{g}_{ll}^j\mathbf{w} = \frac{2}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}\begin{bmatrix}N\mathbf{A} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix}\mathbf{V}_l^T\mathbf{w} \tag{109}$$
$$= 2\begin{bmatrix}\mathbf{u}_l^T\mathbf{A} & \mathbf{0}_{3\times1}\end{bmatrix}\begin{bmatrix}-\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1}\\ \mathbf{0}_{3\times3} & \mathbf{u}_l\end{bmatrix}^T\mathbf{w} \tag{110}$$
$$= 2\begin{bmatrix}\mathbf{u}_l^T\mathbf{A}\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1}\end{bmatrix}\mathbf{w} \tag{111}$$

Since $\mathbf{A}\mathbf{u}_l = \lambda_l\mathbf{u}_l$,

$$\mathbf{u}_l^T\mathbf{A}\lfloor\mathbf{u}_l\rfloor = \lambda_l\mathbf{u}_l^T\lfloor\mathbf{u}_l\rfloor = \mathbf{0}_{1\times3}. \tag{112}$$

Therefore,

$$\mathbf{J}_l\delta\mathbf{T} = \sum_{j=1}^{M_p}\mathbf{g}_{ll}^j\mathbf{w} = 0. \tag{113}$$

For the proof of $\delta\mathbf{T}^T\mathbf{H}_l\delta\mathbf{T} = 0$, from (31) in Theorem 4,

$$\delta\mathbf{T}^T\mathbf{H}_l\delta\mathbf{T} = \delta\mathbf{T}^T(\mathbf{W}_l + \sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}\mathbf{g}_{kl}^T\mathbf{g}_{kl})\delta\mathbf{T}$$
$$= \mathbf{w}^T\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\left(\mathbf{W}_l^{ij} + \sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}(\mathbf{g}_{kl}^i)^T\mathbf{g}_{kl}^j\right)\mathbf{w} \tag{114}$$

where

$$\mathbf{W}_l^{ij} = -\frac{2}{N^2}\mathbf{V}_l\mathbf{T}_i\mathbf{C}_i\mathbf{FC}_j\mathbf{T}_j^T\mathbf{V}_l^T + \mathbb{1}_{i=j}\cdot\left(\frac{2}{N}\mathbf{V}_l\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \begin{bmatrix}\mathbf{K}_l^j & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix}\right), \tag{115}$$
$$\mathbf{K}_l^j = \frac{1}{N}\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\lfloor\mathbf{u}_l\rfloor + \frac{1}{N}\lfloor\mathbf{u}_l\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor, \tag{116}$$
$$\mathbf{g}_{kl}^j = \frac{1}{N}\mathbf{u}_l^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_k^T + \frac{1}{N}\mathbf{u}_k^T\mathbf{S}_\mathbf{P}(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T, \tag{117}$$

We will divide (114) into two parts to discuss. For the first part,

$$\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{W}_l^{ij} = -\frac{2}{N^2}\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{V}_l\mathbf{T}_i\mathbf{C}_i\mathbf{FC}_j\mathbf{T}_j^T\mathbf{V}_l^T + \frac{2}{N}\sum_{j=1}^{M_p}\mathbf{V}_l\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\mathbf{V}_l^T + \begin{bmatrix}\sum_{j=1}^{M_p}\mathbf{K}_l^j & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \tag{118}$$

Since $\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T \triangleq \mathbf{C}$, $\sum_{j=1}^{M_p}\mathbf{FC}_j\mathbf{T}_j^T = \mathbf{FC}$ and $\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{F} = \mathbf{CF}$ from (51), we have

$$\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{FC}_j\mathbf{T}_j^T = \sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\sum_{j=1}^{M_p}\mathbf{FC}_j\mathbf{T}_j^T = \sum_{i=1}^{M_p}\mathbf{T}_i\mathbf{C}_i\mathbf{FC} = \mathbf{CFC}, \tag{119}$$

and

$$\begin{aligned}\sum_{j=1}^{M_p}\mathbf{K}_l^j &= \frac{1}{N}\sum_{j=1}^{M_p}\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\lfloor\mathbf{u}_l\rfloor\\ &+ \frac{1}{N}\sum_{j=1}^{M_p}\lfloor\mathbf{u}_l\rfloor\lfloor\mathbf{S}_\mathbf{P}\mathbf{T}_j\mathbf{C}_j(\mathbf{T}_j - \frac{1}{N}\mathbf{CF})^T\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\\ &= \frac{1}{N}\lfloor\mathbf{S}_\mathbf{P}(\mathbf{C} - \frac{1}{N}\mathbf{CFC})\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor\lfloor\mathbf{u}_l\rfloor\\ &+ \frac{1}{N}\lfloor\mathbf{u}_l\rfloor\lfloor\mathbf{S}_\mathbf{P}(\mathbf{C} - \frac{1}{N}\mathbf{CFC})\mathbf{S}_\mathbf{P}^T\mathbf{u}_l\rfloor.\end{aligned} \tag{120}$$

Then, from (108) and $\mathbf{A}\mathbf{u}_l = \lambda_l\mathbf{u}_l$,

$$\sum_{j=1}^{M_p}\mathbf{K}_l^j = \lfloor\mathbf{A}\mathbf{u}_l\rfloor\lfloor\mathbf{u}_l\rfloor + \lfloor\mathbf{u}_l\rfloor\lfloor\mathbf{A}\mathbf{u}_l\rfloor = 2\lambda_l\lfloor\mathbf{u}_l\rfloor^2. \tag{121}$$

Now, substituting the results in (119) and (121) into (118):

$$\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\mathbf{W}_l^{ij}$$
$$=-\frac{2}{N^2}\mathbf{V}_l\mathbf{CFCV}_l^T+\frac{2}{N}\mathbf{V}_l\mathbf{CV}_l^T+\begin{bmatrix}2\lambda_l\lfloor\mathbf{u}_l\rfloor^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{122}$$
$$=\frac{2}{N}\mathbf{V}_l\underbrace{\left(\mathbf{C}-\frac{1}{N}\mathbf{CFC}\right)}_{\text{Recall (108)}}\mathbf{V}_l^T+\begin{bmatrix}2\lambda_l\lfloor\mathbf{u}_l\rfloor^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{123}$$
$$=2\mathbf{V}_l\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix}\mathbf{V}_l^T+\begin{bmatrix}2\lambda_l\lfloor\mathbf{u}_l\rfloor^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$$
$$=2\begin{bmatrix}\lambda_l\lfloor\mathbf{u}_l\rfloor^2-\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{124}$$

For the second part in (114),

$$\sum_{i=1}^{M_p}\sum_{j=1}^{M_p}\left(\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}(\mathbf{g}_{kl}^i)^T\mathbf{g}_{kl}^j\right)$$
$$=\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}\left(\sum_{i=1}^{M_p}\mathbf{g}_{kl}^i\right)^T\left(\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j\right) \tag{125}$$

where

$$\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j=\frac{1}{N}\mathbf{u}_l^T\mathbf{S_P}\sum_{j=1}^{M_p}(\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T-\frac{1}{N}\mathbf{CFC}_j\mathbf{T}_j^T)\mathbf{V}_k^T$$
$$+\frac{1}{N}\mathbf{u}_k^T\mathbf{S_P}\sum_{j=1}^{M_p}(\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T-\frac{1}{N}\mathbf{CFC}_j\mathbf{T}_j^T)\mathbf{V}_l^T. \tag{126}$$

Again, since $\sum_{j=1}^{M_p}\mathbf{T}_j\mathbf{C}_j\mathbf{T}_j^T\triangleq\mathbf{C}$ and $\sum_{j=1}^{M_p}\mathbf{FC}_j\mathbf{T}_j^T=\mathbf{FC}$ from (51), we have

$$\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j=\frac{1}{N}\mathbf{u}_l^T\mathbf{S_P}(\mathbf{C}-\frac{1}{N}\mathbf{CFC})\mathbf{V}_k^T$$
$$+\frac{1}{N}\mathbf{u}_k^T\mathbf{S_P}(\mathbf{C}-\frac{1}{N}\mathbf{CFC})\mathbf{V}_l^T \tag{127}$$

Further, from (108), $\mathbf{Au}_l=\lambda_l\mathbf{u}_l$, and $\mathbf{Au}_k=\lambda_k\mathbf{u}_k$, we have

$$\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j=\mathbf{u}_l^T\begin{bmatrix}\mathbf{A} & \mathbf{0}_{3\times1}\end{bmatrix}\mathbf{V}_k^T+\mathbf{u}_k^T\begin{bmatrix}\mathbf{A} & \mathbf{0}_{3\times1}\end{bmatrix}\mathbf{V}_l^T$$
$$=\begin{bmatrix}\mathbf{u}_l^T\mathbf{A}\lfloor\mathbf{u}_k\rfloor & \mathbf{0}_{3\times1}\end{bmatrix}+\begin{bmatrix}\mathbf{u}_k^T\mathbf{A}\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1}\end{bmatrix}$$
$$=\begin{bmatrix}\lambda_l\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor+\lambda_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1}\end{bmatrix} \tag{128}$$

Therefore,

$$\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}\left(\sum_{i=1}^{M_p}\mathbf{g}_{kl}^i\right)^T\left(\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j\right)$$
$$=\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}\begin{bmatrix}\mathbf{r}_{kl} & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix} \tag{129}$$

where due to $\lfloor\mathbf{u}_k\rfloor\mathbf{u}_l=-\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k$ and $\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor=-\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor$

$$\begin{aligned}\mathbf{r}_{kl}&=\left(\lambda_l\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor+\lambda_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor\right)^T\left(\lambda_l\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor+\lambda_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor\right)\\&=-\lambda_l^2\lfloor\mathbf{u}_k\rfloor\mathbf{u}_l\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor-\lambda_l\lambda_k\lfloor\mathbf{u}_k\rfloor\mathbf{u}_l\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor\\&\quad-\lambda_k^2\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor-\lambda_l\lambda_k\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k\mathbf{u}_l^T\lfloor\mathbf{u}_k\rfloor\\&=-(\lambda_l-\lambda_k)^2\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor.\end{aligned} \tag{130}$$

Thus,

$$\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}\left(\sum_{i=1}^{M_p}\mathbf{g}_{kl}^i\right)^T\left(\sum_{j=1}^{M_p}\mathbf{g}_{kl}^j\right)$$
$$=2\sum_{k=1,k\neq l}^{3}\begin{bmatrix}(\lambda_k-\lambda_l)\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix} \tag{131}$$

Now, from (124) and (131), the equation (114) is turned into

$$\delta\mathbf{T}^T\mathbf{H}_l\delta\mathbf{T}=\mathbf{w}^T\begin{bmatrix}2\mathbf{L}_l & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix}\mathbf{w}, \tag{132}$$

where

$$\begin{aligned}\mathbf{L}_l&=\lambda_l\lfloor\mathbf{u}_l\rfloor^2-\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor+\sum_{k=1,k\neq l}^{3}(\lambda_k-\lambda_l)\lfloor\mathbf{u}_l\rfloor\mathbf{u}_k\mathbf{u}_k^T\lfloor\mathbf{u}_l\rfloor\\&=\lambda_l\lfloor\mathbf{u}_l\rfloor^2-\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor\\&\quad+\lfloor\mathbf{u}_l\rfloor\left(\sum_{k=1,k\neq l}^{3}\lambda_k\mathbf{u}_k\mathbf{u}_k^T\right)\lfloor\mathbf{u}_l\rfloor-\lambda_l\lfloor\mathbf{u}_l\rfloor\left(\sum_{k=1,k\neq l}^{3}\mathbf{u}_k\mathbf{u}_k^T\right)\lfloor\mathbf{u}_l\rfloor.\end{aligned} \tag{133}$$

Since $\mathbf{u}_k$ $(k=1,2,3)$ is the eigenvector (with eigenvalue λ_k) of matrix $\mathbf{A}$, which is symmetric, we have the following two conditions from the singular value decomposition of $\mathbf{A}$:

$$\mathbf{A}=\sum_{k=1}^{3}\lambda_k\mathbf{u}_k\mathbf{u}_k^T,\quad\mathbf{I}=\sum_{k=1}^{3}\mathbf{u}_k\mathbf{u}_k^T, \tag{134}$$

which imply

$$\sum_{k=1,k\neq l}^{3}\lambda_k\mathbf{u}_k\mathbf{u}_k^T=\mathbf{A}-\lambda_l\mathbf{u}_l\mathbf{u}_l^T,\quad\sum_{k=1,k\neq l}^{3}\mathbf{u}_k\mathbf{u}_k^T=\mathbf{I}-\mathbf{u}_l\mathbf{u}_l^T. \tag{135}$$

Substituting the above results into $\mathbf{L}_l$ in (133):

$$\begin{aligned}\mathbf{L}_l&=\lambda_l\lfloor\mathbf{u}_l\rfloor^2-\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor\\&+\lfloor\mathbf{u}_l\rfloor\left(\mathbf{A}-\lambda_l\mathbf{u}_l\mathbf{u}_l^T\right)\lfloor\mathbf{u}_l\rfloor-\lambda_l\lfloor\mathbf{u}_l\rfloor(\mathbf{I}-\mathbf{u}_l\mathbf{u}_l^T)\lfloor\mathbf{u}_l\rfloor\\&=\lambda_l\lfloor\mathbf{u}_l\rfloor^2-\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor+\lfloor\mathbf{u}_l\rfloor\mathbf{A}\lfloor\mathbf{u}_l\rfloor-\lambda_l\lfloor\mathbf{u}_l\rfloor^2=\mathbf{0}.\end{aligned} \tag{136}$$

As a result,

$$\delta\mathbf{T}^T\mathbf{H}\delta\mathbf{T}=\mathbf{w}^T\begin{bmatrix}2\mathbf{L}_l & \mathbf{0}_{3\times1}\\ \mathbf{0}_{1\times3} & 0\end{bmatrix}\mathbf{w}=0. \tag{137}$$

Finally, if $\mathbf{C}_j=\mathbf{0}$, we have $\mathbf{g}_{kl}^j=\mathbf{0}_{1\times6}$ from (19) of Lemma 2, hence $\mathbf{J}^j=\mathbf{g}_{ll}^j=\mathbf{0}$, the result (40) of Corollary 4.1; If $\mathbf{C}_i=\mathbf{0}$ or $\mathbf{C}_j=\mathbf{0}$, we have $\mathbf{Q}_{kl}^{ij}=\mathbf{0}_{6\times6}$ from (20) of Lemma 2, hence $\mathbf{W}_l^{ij}=\mathbf{Q}_{ll}^{ij}=\mathbf{0}_{6\times6}$ and $\mathbf{H}_l^{ij}=\mathbf{W}_l^{ij}+\sum_{k=1,k\neq l}^{3}\frac{2}{\lambda_l-\lambda_k}(\mathbf{g}_{kl}^j)^T\mathbf{g}_{kl}^j=\mathbf{0}$, the result (41) of Corollary 4.1. □

G. Derivation of pose covariance

The quantity $\delta\mathbf{C}_{f_{ij}}$ can be obtained by substituting (45) into the definition of $\mathbf{C}^{\text{gt}}_{f_{ij}}$ in (46) and retaining only the first order items:

$$\delta\mathbf{C}_{f_{ij}} = \begin{bmatrix} \delta\mathbf{P}_{f_{ij}} & \delta\mathbf{v}_{f_{ij}} \\ \delta\mathbf{v}^T_{f_{ij}} & 0 \end{bmatrix}, \quad \text{where} \tag{138}$$

$$\delta\mathbf{P}_{f_{ij}} = \sum_{k=1}^{N_{ij}} (\mathbf{p}_{f_{ijk}}\delta\mathbf{p}^T_{f_{ijk}} + \delta\mathbf{p}_{f_{ijk}}\mathbf{p}^T_{f_{ijk}}), \tag{139}$$

$$\delta\mathbf{v}_{f_{ij}} = \sum_{k=1}^{N_{ij}} \delta\mathbf{p}_{f_{ijk}}. \tag{140}$$

To derive $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f}\delta\mathbf{C}_f$ without involving any tensor, we parameterize the matrix $\mathbf{C}_{f_{ij}}$ by a column vector $\mathbf{c}_{f_{ij}}$, which consists of the independent elements in $\mathbf{C}_{f_{ij}}$:

$$\begin{aligned}\mathbf{c}_{f_{ij}} = \text{vec}(\mathbf{C}_{f_{ij}}) \triangleq \big[&\mathbf{e}_1^T\mathbf{C}_{f_{ij}}\mathbf{e}_1 \quad \mathbf{e}_1^T\mathbf{C}_{f_{ij}}\mathbf{e}_2 \quad \mathbf{e}_1^T\mathbf{C}_{f_{ij}}\mathbf{e}_3 \\ &\mathbf{e}_2^T\mathbf{C}_{f_{ij}}\mathbf{e}_2 \quad \mathbf{e}_2^T\mathbf{C}_{f_{ij}}\mathbf{e}_3 \quad \mathbf{e}_3^T\mathbf{C}_{f_{ij}}\mathbf{e}_3 \\ &\mathbf{e}_1^T\mathbf{C}_{f_{ij}}\mathbf{e}_4 \quad \mathbf{e}_2^T\mathbf{C}_{f_{ij}}\mathbf{e}_4 \quad \mathbf{e}_3^T\mathbf{C}_{f_{ij}}\mathbf{e}_4\big]^T \in \mathbb{R}^9\end{aligned} \tag{141}$$

where $\text{vec}(\cdot) : \mathbb{S}^{4\times4} \mapsto \mathbb{R}^9$ maps a symmetric matrix to its column vector representation, $\mathbf{e}_l \in \mathbb{R}^4$ $(l \in \{1,2,3,4\})$ is a vector with all zero elements except for the l-th element being one. Note that the constant N in the 4-th row, 4-th column of $\mathbf{C}_{f_{ij}}$ is not contained in $\mathbf{c}_{f_{ij}}$ since it is a constant number independent of the noise. Correspondingly, noises in $\mathbf{C}_{f_{ij}}$ becomes the noise of $\mathbf{c}_{f_{ij}}$ as below:

$$\begin{aligned}\delta\mathbf{c}_{f_{ij}} = \big[&\mathbf{e}_1^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_1 \quad \mathbf{e}_1^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_2 \quad \mathbf{e}_1^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_3 \\ &\mathbf{e}_2^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_2 \quad \mathbf{e}_2^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_3 \quad \mathbf{e}_3^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_3 \\ &\mathbf{e}_1^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_4 \quad \mathbf{e}_2^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_4 \quad \mathbf{e}_3^T\delta\mathbf{C}_{f_{ij}}\mathbf{e}_4\big]^T\end{aligned} \tag{142}$$

$$= \sum_{k=1}^{N_{ij}} \begin{bmatrix} 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{11}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{12}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{13}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{22}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{23}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{33}\mathbf{S}_{\text{P}}^T\delta\mathbf{p}_{f_{ijk}} \\ \delta\mathbf{p}_{f_{ijk}} \end{bmatrix} = \sum_{k=1}^{N_{ij}} \mathbf{B}_{f_{ijk}}\delta\mathbf{p}_{f_{ijk}}, \tag{143}$$

where $\mathbf{S}_{\text{P}} = [\mathbf{I}_{3\times3}, 0_{3\times1}]$, $\mathbf{E}_{kl} = \mathbf{e}_k\mathbf{e}_l^T + \mathbf{e}_l\mathbf{e}_k^T \in \mathbb{S}^{4\times4}$, $k,l \in \{1,2,3,4\}$, and

$$\mathbf{B}_{f_{ijk}} = \begin{bmatrix} 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{11}\mathbf{S}_{\text{P}}^T \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{12}\mathbf{S}_{\text{P}}^T \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{13}\mathbf{S}_{\text{P}}^T \\ 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{22}\mathbf{S}_{\text{P}}^T \\ \mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{23}\mathbf{S}_{\text{P}}^T \\ 2\mathbf{p}^T_{f_{ijk}}\mathbf{S}_{\text{P}}\mathbf{E}_{33}\mathbf{S}_{\text{P}}^T \\ \mathbf{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{9\times3}. \tag{144}$$

With the column representation of each $\mathbf{C}_{f_{ij}}$ contained in $\mathbf{C}_f$, $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f}\delta\mathbf{C}_f$ can now be computed as

$$\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_f)}{\partial \mathbf{C}_f}\delta\mathbf{C}_f = \sum_{i=1}^{M_f}\sum_{j=1}^{M_p} \frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}}\delta\mathbf{c}_{f_{ij}}. \tag{145}$$

To derive the quantity $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}}$, we give two lemmas which are useful for subsequent derivations.

Lemma 3. *For $\mathbf{w} \in \mathbb{R}^4, \mathbf{C} \in \mathbb{S}^{4\times4}$ and its vector form $\mathbf{c} = \text{vec}(\mathbf{C})$, we have*

$$\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{c}} = \mathbf{g}_1(\mathbf{w}) \in \mathbb{R}^{4\times9},$$

where

$$\begin{aligned}\mathbf{g}_1(\mathbf{w}) = \big[&\mathbf{E}_{11}\mathbf{w} \quad \mathbf{E}_{12}\mathbf{w} \quad \mathbf{E}_{13}\mathbf{w} \quad \mathbf{E}_{22}\mathbf{w} \quad \mathbf{E}_{23}\mathbf{w} \\ &\mathbf{E}_{33}\mathbf{w} \quad \mathbf{E}_{14}\mathbf{w} \quad \mathbf{E}_{24}\mathbf{w} \quad \mathbf{E}_{34}\mathbf{w}\big],\end{aligned}$$

where $\mathbf{E}_{kl} \in \mathbb{S}^{4\times4}$, $k,l \in \{1,2,3,4\}$, is

$$\mathbf{E}_{kl} = \begin{cases} \mathbf{e}_k\mathbf{e}_l^T + \mathbf{e}_l\mathbf{e}_k^T & (k \neq l) \\ \mathbf{e}_l\mathbf{e}_l^T & (k = l) \end{cases} \tag{146}$$

where $\mathbf{e}_l \in \mathbb{R}^4$ $(l \in \{1,2,3,4\})$ is a vector with all zero elements except for the l-th element being one.

Proof. For the k-th row, l-th column element of $\mathbf{C}$, denoted by $\mathbf{C}_{k,l}$, we have

$$\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{k,l}} = \mathbf{E}_{kl}\mathbf{w} \in \mathbb{R}^4.$$

Hence,

$$\begin{aligned}\frac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{c}} &= \Big[\tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,1}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,2}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,3}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,2}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,3}} \\ &\qquad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{3,3}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{1,4}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{2,4}} \quad \tfrac{\partial \mathbf{C}\mathbf{w}}{\partial \mathbf{C}_{3,4}}\Big] \\ &= \big[\mathbf{E}_{11}\mathbf{w} \quad \mathbf{E}_{12}\mathbf{w} \quad \mathbf{E}_{13}\mathbf{w} \quad \mathbf{E}_{22}\mathbf{w} \quad \mathbf{E}_{23}\mathbf{w} \\ &\qquad \mathbf{E}_{33}\mathbf{w} \quad \mathbf{E}_{14}\mathbf{w} \quad \mathbf{E}_{24}\mathbf{w} \quad \mathbf{E}_{34}\mathbf{w}\big].\end{aligned}$$

□

Lemma 4. *For $\mathbf{w} \in \mathbb{R}^4, \mathbf{u}_l \in \mathbb{R}^3$ and*

$$\mathbf{V}_l = \begin{bmatrix} -\lfloor\mathbf{u}_l\rfloor & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{u}_l \end{bmatrix} \in \mathbb{R}^{6\times4}$$

we have

$$\frac{\partial \mathbf{V}_l\mathbf{w}}{\partial \mathbf{u}_l} = \mathbf{g}_2(\mathbf{w}) = \begin{bmatrix} \lfloor\mathbf{w}_{1:3}\rfloor \\ w_4\mathbf{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{6\times3}.$$

where $\mathbf{w}_{1:3}$ represents the first three elements in $\mathbf{w}$ and w_4 is the 4-th element of $\mathbf{w}$.

Proof.

$$\mathbf{V}_l\mathbf{w} = \begin{bmatrix} -\lfloor\mathbf{u}_l\rfloor & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_l \end{bmatrix}\begin{bmatrix} \mathbf{w}_{1:3} \\ w_4 \end{bmatrix} = \begin{bmatrix} \lfloor\mathbf{w}_{1:3}\rfloor\mathbf{u}_l \\ w_4\mathbf{u}_l \end{bmatrix} = \begin{bmatrix} \lfloor\mathbf{w}_{1:3}\rfloor \\ w_4\mathbf{I}_{3\times3} \end{bmatrix}\mathbf{u}_l$$

Thus,

$$\frac{\partial \mathbf{V}_l\mathbf{w}}{\partial \mathbf{u}_l} = \begin{bmatrix} \lfloor\mathbf{w}_{1:3}\rfloor \\ w_4\mathbf{I}_{3\times3} \end{bmatrix} = \mathbf{g}_2(\mathbf{w})$$

□

With these two lemmas, next we will continue to derive $\frac{\partial \mathbf{J}^T(\mathbf{T}^\star, \mathbf{C}_{f_{ij}})}{\partial \mathbf{c}_{f_{ij}}}$ in (145). Theorem 4 gives the Jacobian for one cost item while the required Jacobian consists of items from all features, to distinguish the different cost item, we add a subscript ν to (30) to denote the ν-th item and replace $\mathbf{C}_j$

with the actual point cluster notation $\mathbf{C}_{f_{\nu j}}$ corresponding to the ν-th item, leading to:

$$\mathbf{J} = \sum_{\nu=1}^{M_f} \mathbf{J}_\nu \tag{147}$$

$$\mathbf{J}_\nu = \begin{bmatrix} \cdots & \mathbf{J}_\nu^{\mathrm{p}} & \cdots \end{bmatrix} \in \mathbb{R}^{1\times 6M_p} \tag{148}$$

where $\mathbf{J}_\nu^{\mathrm{p}} \in \mathbb{R}^{1\times 6}$ is the p-th column block of $\mathbf{J}_\nu$ as shown in (32) of Theorem 4 (with $j = \mathrm{p}$):

$$\mathbf{J}_\nu^{\mathrm{p}} = \frac{2}{N_\nu}\mathbf{u}_{\nu l}^T\mathbf{S_P}\left(\mathbf{T}_{\mathrm{p}} - \frac{1}{N_\nu}\mathbf{C}_\nu\mathbf{F}\right)\mathbf{C}_{f_{\nu\mathrm{p}}}\mathbf{T}_{\mathrm{p}}^T\mathbf{V}_{\nu l}^T \in \mathbb{R}^{1\times 6}.$$

where $\mathbf{u}_{\nu l}$ is the eigenvector associated to the l-th largest eigenvalue of the covariance matrix of the ν-th feature (or cost item), N_ν is the total number of points of the ν-th feature, $\mathbf{C}_\nu$ is the aggregation of all point clusters of the ν-th feature, and $\mathbf{C}_{f_{\nu\mathrm{p}}}$ is the point cluster contributed by the p-th pose to the ν-th feature.

The total Jacobian is hence

$$\mathbf{J} = \sum_{\nu=1}^{M_f}\mathbf{J}_\nu = \begin{bmatrix} \cdots & \mathbf{J}^{\mathrm{p}} & \cdots \end{bmatrix} \in \mathbb{R}^{1\times 6M_p}, \tag{149}$$

$$\begin{aligned}\mathbf{J}^{\mathrm{p}} &= \sum_{\nu=1}^{M_f}\mathbf{J}_\nu^{\mathrm{p}} \in \mathbb{R}^{1\times 6}, \\ &= \sum_{\nu=1}^{M_f}\left(\frac{2}{N_\nu}\mathbf{u}_{\nu l}^T\mathbf{S_P}\left(\mathbf{T}_{\mathrm{p}} - \frac{1}{N_\nu}\mathbf{C}_\nu\mathbf{F}\right)\mathbf{C}_{f_{\nu\mathrm{p}}}\mathbf{T}_{\mathrm{p}}^T\mathbf{V}_{\nu l}^T\right).\end{aligned} \tag{150}$$

Next, we calculate the partial derivative $\frac{\partial(\mathbf{J}^{\mathrm{p}})^T}{\partial \mathbf{c}_{f_{ij}}}$. Note that in the summation of (150), only the i-th summation term (i.e., $\nu = i$) is related to $\mathbf{C}_{f_{ij}}$ (hence $\mathbf{c}_{f_{ij}}$), hence,

$$\frac{\partial\left(\mathbf{J}^{\mathrm{p}}\right)^T}{\partial \mathbf{c}_{f_{ij}}} = \frac{\partial}{\partial \mathbf{c}_{f_{ij}}}\left(\frac{2}{N_i}\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}\right). \tag{151}$$

Since $\mathbf{C}_i = \sum_{\nu=1}^{M_p}\mathbf{T}_\nu\mathbf{C}_{f_{i\nu}}\mathbf{T}_\nu^T$, $\mathbf{u}_{il}$ is the eigenvector associated to the l-th largest eigenvalue of matrix $\mathbf{A}(\mathbf{C}_i)$, and $\mathbf{V}_{il} = \begin{bmatrix} -\lfloor \mathbf{u}_{il} \rfloor & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 3} & \mathbf{u}_{il}\end{bmatrix}$, the derivative with respect to $\mathbf{c}_{f_{ij}}$ on the right hand side of (151) consists of four terms respectively from $\mathbf{V}_{il}$, $\mathbf{C}_{f_{i\mathrm{p}}}$ (only when $\mathrm{p} = j$), $\mathbf{C}_i$, and $\mathbf{u}_{il}$. Combining with Lemma 3 and Lemma 4, we have

$$\mathbf{L}_{ij}^{\mathrm{p}} \triangleq \frac{\partial\left(\mathbf{J}^{\mathrm{p}}\right)^T}{\partial \mathbf{c}_{f_{ij}}} \in \mathbb{R}^{6\times 9} \tag{152}$$

$$\begin{aligned} &= \frac{2}{N_i}\Big(\mathbf{g}_2(\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}) \\ &+ \mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\Big)\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} \\ &- \frac{2}{N_i^2}\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}\mathbf{F}\mathbf{T}_j\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &+ \frac{2}{N_i}\mathbb{1}_{\mathrm{p}=j}\cdot\left(\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{g}_1\Big((\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}\Big)\right) \end{aligned} \tag{153}$$

Only the term $\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}}$ is unknown in this formula. To compute it, we apply the second conclusion in Lemma 1 (i.e., (2)) to all components of $\mathbf{c}_{f_{ij}}$ and use the notation trick similar to (85):

$$\frac{\partial \mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} = \sum_{k=1,k\neq l}^{3}\frac{1}{\lambda_{il} - \lambda_{ik}}\mathbf{u}_{ik}\frac{\partial\left(\mathbf{u}_{ik}^T\mathbf{A}(\mathbf{C}_i)\mathbf{u}_{il}\right)}{\partial \mathbf{c}_{f_{ij}}} \tag{154}$$

From (30) of Lemma 2,

$$\mathbf{u}_{ik}^T\mathbf{A}(\mathbf{C}_i)\mathbf{u}_{il} = \frac{1}{N_i}\mathbf{u}_{ik}^T\mathbf{S_P}$$
$$\left(\sum_{\mu=1}^{M_p}\mathbf{T}_\mu\mathbf{C}_{f_{i\mu}}\mathbf{T}_\mu^T - \frac{1}{N_i}\sum_{\mu=1}^{M_p}\sum_{\nu=1}^{M_p}\mathbf{T}_\mu\mathbf{C}_{f_{i\mu}}\mathbf{F}\mathbf{C}_{f_{i\nu}}\mathbf{T}_\nu^T\right)\mathbf{S_P}^T\mathbf{u}_{il}$$

Denote

$$\begin{aligned}\mathbf{G}_{kl}^{ij} &\triangleq \frac{\partial \mathbf{u}_{ik}^T\mathbf{A}(\mathbf{C}_{f_i})\mathbf{u}_{il}}{\partial \mathbf{c}_{f_{ij}}} = \frac{1}{N_i}\mathbf{u}_{ik}^T\mathbf{S_P}\mathbf{T}_j\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &- \frac{1}{N_i^2}\mathbf{u}_{ik}^T\mathbf{S_P}\sum_{\mu=1}^{M_p}\mathbf{T}_\mu\mathbf{C}_{f_{i\mu}}\mathbf{F}\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &- \frac{1}{N_i^2}\mathbf{u}_{ik}^T\mathbf{S_P}\mathbf{T}_j\mathbf{g}_1\Big(\sum_{\nu=1}^{M_p}\mathbf{F}\mathbf{C}_{f_{i\nu}}\mathbf{T}_\nu^T\mathbf{S_P}^T\mathbf{u}_{il}\Big) \in \mathbb{R}^{1\times 9} \\ &= \frac{1}{N_i}\mathbf{u}_{ik}^T\mathbf{S_P}\Big(\mathbf{T}_j\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) - \frac{1}{N_i}\mathbf{C}_i\mathbf{F}\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &\qquad - \frac{1}{N_i}\mathbf{T}_j\mathbf{g}_1(\mathbf{F}\mathbf{C}_i\mathbf{S_P}^T\mathbf{u}_{il})\Big)\end{aligned} \tag{155}$$

Substitute it into $\mathbf{L}_{ij}^{\mathrm{p}}$:

$$\begin{aligned}\mathbf{L}_{ij}^{\mathrm{p}} &= \frac{2}{N_i}\Big(\mathbf{g}_2(\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}) + \\ &\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}(\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\Big)\left(\sum_{k=1,k\neq l}^{3}\frac{\mathbf{u}_{ik}\mathbf{G}_{kl}^{ij}}{\lambda_{il} - \lambda_{ik}}\right) \\ &- \frac{2}{N_i^2}\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{C}_{f_{i\mathrm{p}}}\mathbf{F}\mathbf{T}_j\mathbf{g}_1(\mathbf{T}_j^T\mathbf{S_P}^T\mathbf{u}_{il}) \\ &+ \frac{2}{N_i}\mathbb{1}_{\mathrm{p}=j}\cdot\left(\mathbf{V}_{il}\mathbf{T}_{\mathrm{p}}\mathbf{g}_1\Big((\mathbf{T}_{\mathrm{p}}^T - \frac{1}{N_i}\mathbf{F}\mathbf{C}_i)\mathbf{S_P}^T\mathbf{u}_{il}\Big)\right).\end{aligned} \tag{156}$$

Since $\mathbf{J} = \begin{bmatrix} \cdots & \mathbf{J}^{\mathrm{p}} & \cdots \end{bmatrix}$ and (145), we obtain

$$\frac{\partial \mathbf{J}^T}{\partial \mathbf{c}_{f_{ij}}} = \begin{bmatrix} \vdots \\ \frac{\partial(\mathbf{J}^{\mathrm{p}})^T}{\partial \mathbf{c}_{f_{ij}}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \mathbf{L}_{ij}^{\mathrm{p}} \\ \vdots \end{bmatrix} \triangleq \mathbf{L}_{ij} \in \mathbb{R}^{6M_p\times 9}, \tag{157}$$

$$\frac{\partial \mathbf{J}^T\left(\mathbf{T}^\star, \mathbf{C}_f\right)}{\partial \mathbf{C}_f}\delta\mathbf{C}_f = \sum_{i=1}^{M_f}\sum_{j=1}^{M_p}\mathbf{L}_{ij}\delta\mathbf{c}_{f_{ij}}. \tag{158}$$

Finally, from (52) of the main texts, we have

$$\delta\mathbf{T}^\star = \mathbf{H}^{-1}\left(\sum_{i=1}^{M_f}\sum_{j=1}^{M_p}\mathbf{L}_{ij}\delta\mathbf{c}_{f_{ij}}\right) \in \mathbb{R}^{6M_p}, \tag{159}$$

$$\mathbf{\Sigma}_{\delta\mathbf{T}^\star} = \mathbf{H}^{-1}\left(\sum_{i=1}^{M_f}\sum_{j=1}^{M_p}\mathbf{L}_{ij}\mathbf{\Sigma}_{\mathbf{c}_{f_{ij}}}\mathbf{L}_{ij}^T\right)\mathbf{H}^{-1}, \tag{160}$$

where

$$\mathbf{\Sigma}_{\mathbf{c}_{f_{ij}}} = \sum_{k=1}^{N_{ij}} \mathbf{B}_{f_{ijk}} \mathbf{\Sigma}_{\mathbf{p}_{f_{ijk}}} \mathbf{B}_{f_{ijk}}^T, \tag{161}$$

which is obtained from (143) and can be computed beforehand without enumerating each raw point in the run time.

Algorithm 1: LM optimization

Input: Initial poses $\mathbf{T}$;
Point cluster in the local frame $\mathbf{C}_{f_{ij}}$;

1 $\mu = 0.01,\ \nu = 2,\ j = 0$;
2 **repeat**
3 $\quad j = j + 1$;
4 $\quad \mathbf{J} = \mathbf{0}_{1\times 6M_p}, \mathbf{H} = \mathbf{0}_{6M_p \times 6M_p}$;
5 $\quad$ **foreach** $i \in \{1, \cdots, M_f\}$ **do**
6 $\quad\quad$ Compute $\mathbf{J}_i$ and $\mathbf{H}_i$ from (30) and (31);
7 $\quad\quad \mathbf{J} = \mathbf{J} + \mathbf{J}_i;\ \mathbf{H} = \mathbf{H} + \mathbf{H}_i$
8 $\quad$ **end**
9 $\quad$ Solve $(\mathbf{H} + \mu \mathbf{I})\Delta\mathbf{T} = -\mathbf{J}^T$;
10 $\quad \mathbf{T}' = \mathbf{T} \boxplus \Delta\mathbf{T}$;
11 $\quad$ Compute current cost $c = c(\mathbf{T})$ and the new cost $c' = c(\mathbf{T}')$ from (22);
12 $\quad \rho = (c - c')/(\frac{1}{2}\Delta\mathbf{T} \cdot (\mu\Delta\mathbf{T} - \mathbf{J}^T))$;
13 $\quad$ **if** $\rho > 0$ **then**
14 $\quad\quad \mathbf{T} = \mathbf{T}'$;
15 $\quad\quad \mu = \mu * \max\left(\frac{1}{3}, 1 - (2\rho - 1)^3\right);\ \nu = 2$;
16 $\quad$ **else**
17 $\quad\quad \mu = \mu * \nu;\ \nu = 2 * \nu$;
18 $\quad$ **end**
19 **until** $\| \Delta\mathbf{T} \| < \epsilon$ **or** $j \geq j_{max}$;

Output: Final optimized states $\mathbf{T}$;

III. Time Complexity Analysis

In this section, we analyze the time complexity of the proposed BA solver in Algorithm 1 and compare it with other similar BA methods, including BALM [32] and Plane Adjustment [34]. The computation time of Algorithm 1 mainly consists of the evaluation of the Jacobian $\mathbf{J}$ and Hessian matrix $\mathbf{H}$ on Line 6, and the solving of the linear equation on Line 9. For the evaluation of the Jacobian $\mathbf{J}$, according to (42), it consists of M_f items, each item $\mathbf{J}_i$ requires to evaluate M_p block elements, and each block $\mathbf{g}_{ll}^j$ requires a constant computation time according to (34). Therefore, the time complexity for the evaluation of $\mathbf{J}$ is $O\left(M_f M_p\right)$. Similarly, for the Hessian matrix, according to (42), it consists of M_f items, each item $\mathbf{H}_i$ requires to evaluate M_p^2 block elements, and each block $\mathbf{H}^{ij}$ requires a constant computation time according to (35). Therefore, the time complexity for the evaluation of $\mathbf{H}$ is $O\left(M_f M_p^2\right)$. On Line 9, the linear equation has a dimension of $6M_p$, solving the lienar equation requires an inversion of the Hessian matrix which contributes a time complexity of $O(M_p^3)$. As a result, the overall time complexity of Algorithm 1 is $O\left(M_f M_p + M_f M_p^2 + M_p^3\right)$.

Our previous work BALM [32] also eliminated the feature parameters from the optimization, leading to an optimization similar to (9) whose dimension of Jacobian and Hessian are also $6M_p$. To solve the cost function, BALM [32] adopted a second order solver similar to Algorithm 1, where the linear solver on Line 9 has the same time complexity $O(M_p^3)$. However, when deriving the Jacobian and Hessian matrix of the cost function, the chain rule was used where the Jacobian and Hessian is the multiplication of the cost w.r.t. each point of a feature and the derivative of the point w.r.t. scan poses (see Section III. C of [32]). As a consequence, for each feature, the evaluation of Jacobian has complexity of $O(NM_p)$ and the Hessian has $O(N^2 M_p^2)$, where N is the average number of points on each feature. Therefore, the time complexity including the evaluation of all features' Jacobian and Hessian matrices and the linear solver are $O(NM_f M_p + N^2 M_f M_p^2 + M_p^3)$.

The plane adjustment method in [34] is a direct mimic of the visual bundle adjustment, which does not eliminate the feature parameters but optimizes them along with the poses in each iteration. The resultant linear equation at each iteration is in the form of $\mathbf{J}^T\mathbf{J}\delta\mathbf{x} = \mathbf{b}$, where $\mathbf{J}$ is the Jacobian of the cost function w.r.t. both the pose and feature parameters. Due to a reduced residual and Jacobian technique similar to our point cluster method, the evaluation of $\mathbf{J}$ does not need to enumerate each point of a feature, so the time complexity of computing $\mathbf{J}$ for all M_f features is $O(M_f M_p + M_f)$ by noticing the inherent sparsity. Let $\mathbf{H} = \mathbf{J}^T\mathbf{J} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}^T & \mathbf{H}_{22} \end{bmatrix}, \delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{T} \\ \delta\boldsymbol{\pi} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\delta\mathbf{T} \in \mathbb{R}^{6M_p}$ is the pose component and $\delta\boldsymbol{\pi} \in \mathbb{R}^{3M_f}$ is feature component. The plane adjustment in [34] further used a Schur complement technique similar to visual bundle adjustment, leading to the optimization of the pose vector only: $\left(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\right)\delta\mathbf{T} = \mathbf{b}_1 - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{b}_2$. Since $\mathbf{H}_{22}$ is block diagonal, its inverse has a time complexity of $O(M_f)$. As a consequence, constructing this linear equation would require computing $\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}$ and $\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{b}_2$, which have time complexity of $O(M_p^2 M_f)$, and solving this linear equation has a complexity of $O(M_p^3)$. As a consequence, the overall time complexity is $O(M_f + M_f M_p + M_f M_p^2 + M_p^3)$.

In summary, BALM [32] has a complexity $O(NM_f M_p + N^2 M_f M_p^2 + M_p^3)$, which is linear to M_f, the number of feature, quadratic to N, the number of point, and cubic to M_p, the number of pose. The plane adjustment [34] has a complexity $O(M_f + M_f M_p + M_f M_p^2 + M_p^3)$, which is linear to M_f, irrelevant to N, and cubic to M_p. Our proposed method has a complexity of $O\left(M_f M_p + M_f M_p^2 + M_p^3\right)$, which is similar to [34] (i.e., linear to M_f, irrelevant to N, and cubic to M_p) but has less operations.

IV. Additional Results

Three figures are provided in this section. Fig. 16 is the plots of point-to-plane versus optimization time in real-world datasets including Hilti, VIRAL and UrbanLoco, totally 19 sequences. Fig. 17 shows the plots of point-to-plane versus optimization time in KITTI dataset, from 00 to 10, totally 11 sequences. In Fig. 18, we display the comparison of map consistency between CT-ICP and our method in KITTI sequence 02, 08, 09.

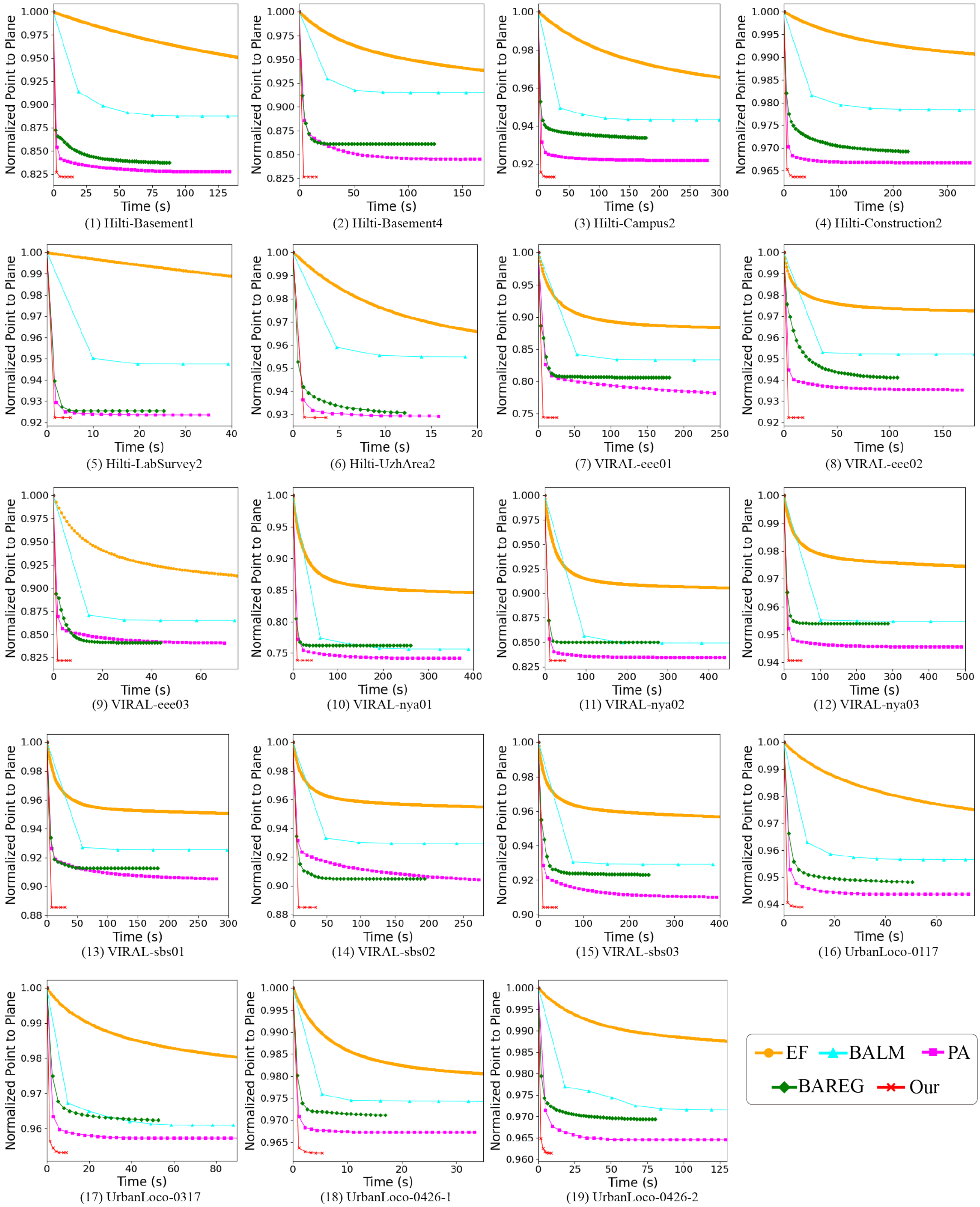


Fig. 16. Point-to-plane distance versus optimization time in real-world datasets including Hilti, VIRAL, and UrbanLoco. All methods have the same initial pose (hence the same initial point-to-plane distance) and have their point-to-plane distance all normalized by the initial values.

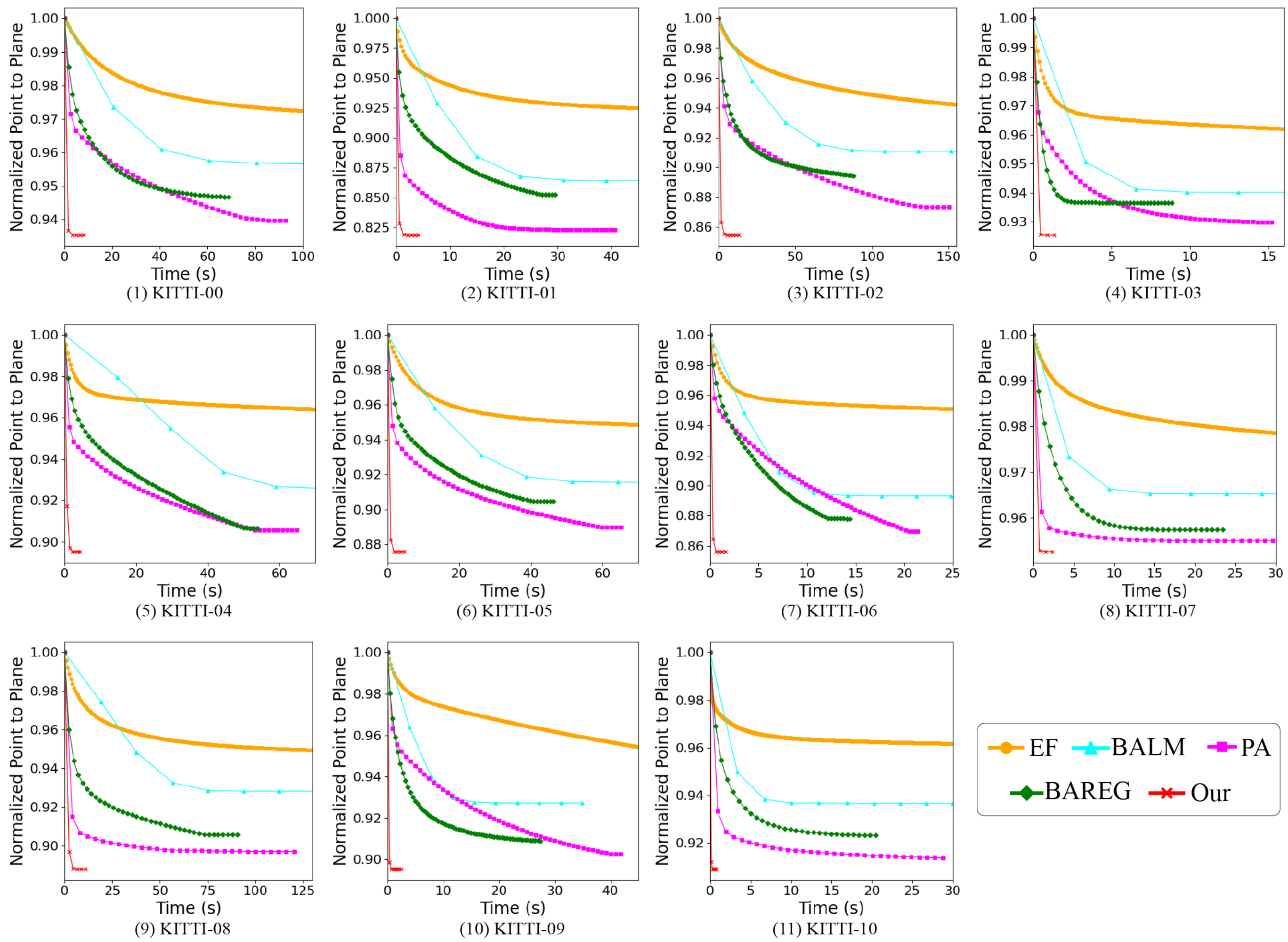
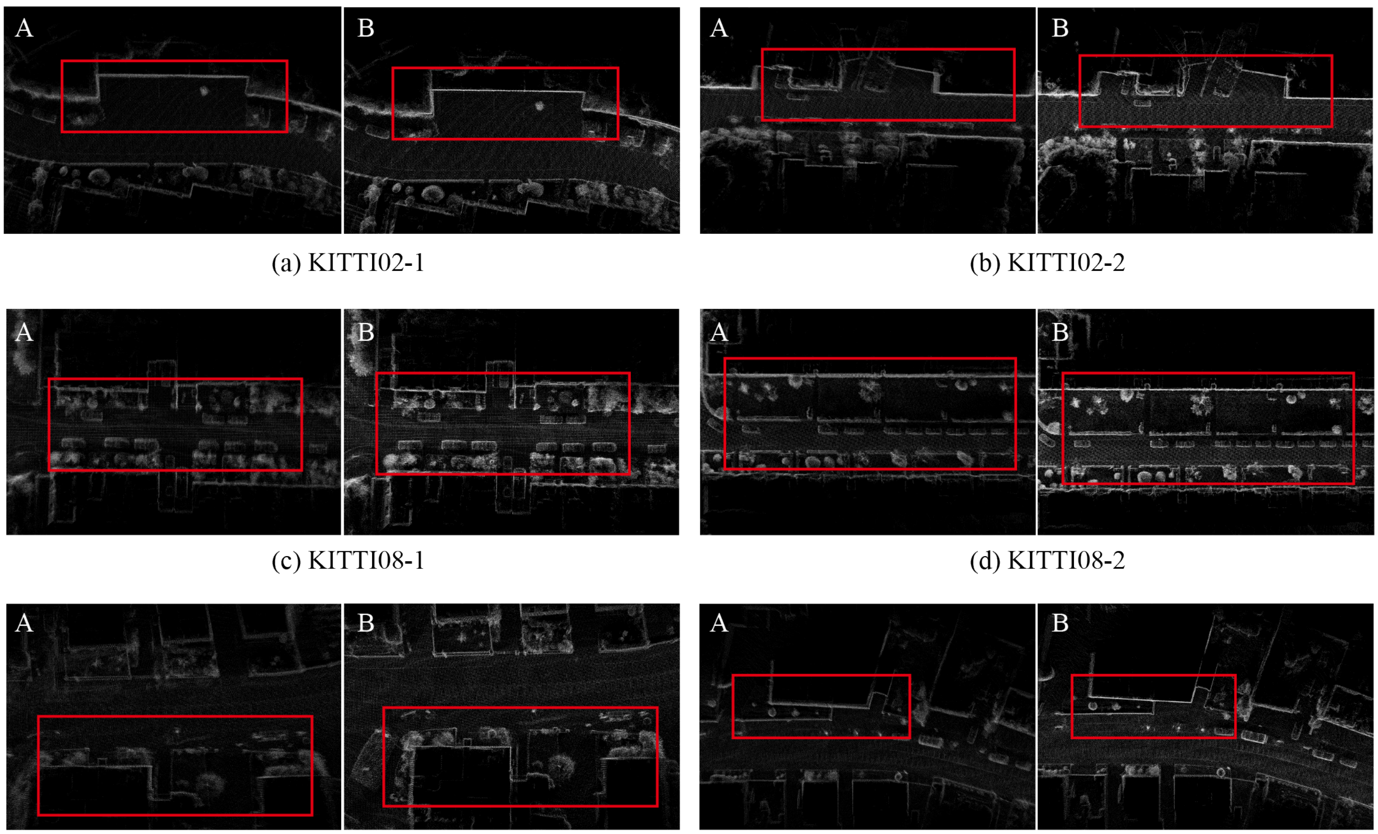


Fig. 17. Point-to-plane distance versus optimization time in KITTI dataset. All methods have the same initial pose (hence the same initial point-to-plane distance) and have their point-to-plane distance all normalized by the initial values.



(a) KITTI02-1 (b) KITTI02-2

(c) KITTI08-1 (d) KITTI08-2

(e) KITTI09-1 (f) KITTI09-2

Fig. 18. Comparison of map consistency between CT-ICP(A) and our method(B) in KITTI sequence 02, 08 and 09.