QUADRATIC EQUATIONS

ASSUMED KNOWLEDGE

- Facility with solving linear equations
- All of the content of the module, *Factorisation.*
- Facility with arithmetic of positive and negative numbers

MOTIVATION

In the module, *Linear equations* we saw how to solve various types of linear equations. Such equations arise very naturally when solving elementary everyday problems.

A **linear equation** involves the unknown quantity occurring to the first power, thus, for example,

$$2x - 7 = 9$$

$$3(x + 2) - 5(x - 8) = 16$$

$$\frac{5x}{3} = 8$$

are all examples of linear equations.

Roughly speaking, quadratic equations involve the square of the unknown. Thus, for example, $2x^2 - 3 = 9$, $x^2 - 5x + 6 = 0$, and $\frac{6x^2}{5} - 4x = 2x - 1$ are all examples of quadratic equations. The equation $\frac{x+1}{x+5} = \frac{2x+5}{3x+7}$ is also a quadratic equation.

The essential idea for solving a linear equation is to isolate the unknown. We keep rearranging the equation so that all the terms involving the unknown are on one side of the equation and all the other terms to the other side. The rearrangements we used for linear equations are helpful but they are not sufficient to solve a quadratic equation. In this module we will develop a number of methods of dealing with these important types of equations.

While quadratic equations do not arise so obviously in everyday life, they are equally important and will frequently turn up in many areas of mathematics when more sophisticated problems are encountered. Both in senior mathematics and in tertiary and engineering mathematics, students will need to be able to solve quadratic equations with confidence and speed. Surprisingly, when mathematics is employed to solve complicated and important real world problems, quadratic equations very often make an appearance as part of the overall solution.

The history of quadratics will be further explored in the History section, but we note here that these types of equations were solved by both the Babylonians and Egyptians at a very early stage of world history. The techniques of solution were further refined by the Greeks, the Arabs and Indians, and finally a complete and coherent treatment was completed once the notion of complex numbers was understood. Thus quadratic equations have been central to the history and applications of mathematics for a very long time.

CONTENT

QUADRATIC EQUATIONS

A **quadratic** is an expression of the form $ax^2 + bx + c$, where a, b and c are given numbers and $a \neq 0$.

The **standard form** of a **quadratic equation** is an equation of the form

$ax^2 + bx + c = 0$, where a, b and c are given numbers and $a \neq 0$.

We seek to find the value(s) of which make the statement true, or to show that there are no such values.

Thus, for example, the values $x = 3$ and $x = 2$ satisfy the equation, $x^2 - 5x + 6 = 0$. This is easily checked by substitution.

These values are called the **solutions** of the equation. Linear equations that are written in the standard form , $ax + b = 0$, $a \neq 0$, have one solution. Quadratic equations may have no solutions, one solution, or, as in the above example, two solutions.

There are two special types of quadratic equations, that are best dealt with separately.

Quadratic equations with no term in x

When there is no term in x we can move the constant to the other side.

> **EXAMPLE**
>
> Solve $x^2 - 9 = 0$.
>
> **SOLUTION**
>
> $x^2 - 9 = 0$
>
> $x^2 = 9$
>
> $x = 3$ or $x = -3$.

(Note that this equation can also be solved by factoring using the difference of squares identity. While this is a valid approach, it makes a simple problem appear complicated, which is, in general, not a good way to do mathematics.)

Quadratic equations with no constant term

> **EXAMPLE**
>
> Solve $x^2 - 9x = 0$.
>
> **SOLUTION**
>
> In this case, we can write
>
> $x^2 - 9 = 0$
>
> $x(x - 9) = 0$
>
> Since the product of the two factors is 0, one or both of the factors is zero, $x(x) - 9 = 0$.
>
> so $x = 0$ or $x - 9 = 0$
>
> Hence the two solutions are or $x = 0$, or $x = 9$.

These two methods work just as well when the coefficient of x^2 is not one.

The two previous examples were relatively easy since in the first case it was easy to isolate the unknown while in the second, a common factor enabled the left-hand side to be easily factored.

SOLVING QUADRATIC EQUATIONS WITH THREE TERMS

We will now deal with the equation $ax^2 + bx + c = 0$ in which neither a nor b nor c are zero.

There are three basic methods of solving such quadratic equations:

- by factoring
- by completing the square
- by the quadratic formula

Each method is important and needs to be mastered. Different situations will require different approaches, and while the last two methods always work, the method of factoring is very quick and accurate, provided the equation has rational solutions.

Solving quadratic equations by factoring

The method of solving quadratic equations by factoring rests on the simple fact, used in example (2) above, that if we obtain zero as the product of two numbers then at least one of the numbers must be zero.

That is, if $AB = 0$ then $A = 0$ or $B = 0$

In the module, *Factorisation,* we first saw how to factor monic quadratics, then we learnt how to factorise non-monic quadratics.

To factor $x^2 + bx + c$ we try to find two numbers whose sum is b and whose product is c. We now apply this idea to solving quadratic equations.

EXAMPLE

Solve $x^2 - 7x + 12 = 0$.

SOLUTION

We factor the left-hand side by finding two numbers whose product is 12 and whose sum is -7. Clearly, -4, -3 are the desired numbers. We can then factor as:

$$x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

Since the product of the two factors is zero, one of the factors is zero.

Thus $\quad x - 4 = 0$, or $x - 3 = 0$

so $\quad x = 4$, or $x = 3$

The same method can also be applied to non-monic quadratic equations. A **non-monic quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where and are given numbers, and $a \neq 1$ or 0. This is the general case.

Thus $2x^2 + 5x + 3 = 0$ is an example of a non-monic quadratic equation.

EXAMPLE

Solve the equation $2x^2 + 5x + 3 = 0$.

SOLUTION

Using the factoring method from the module *Factorisation* , we multiply 2 and 3 to give 6 and find two numbers that multiply to give 6 and add to give 5. The desired numbers are 2 and 3. We use these numbers to split the middle term and factor in pairs.

$$2x^2 + 5x + 3 = 0$$

$$2x^2 + 2x + 3x + 3 = 0 \qquad \text{(split the middle term)}$$

$$2x(x + 1) + 3(x + 1) = 0$$

$$(x + 1)(2x + 3) = 0$$

We can now equate each factor to zero and obtain

$$x + 1 = 0, \text{ or } 2x + 3 = 0$$

$$x = -1, \text{ or } x = -\tfrac{3}{2}.$$

As was pointed out in the module, *Factorisation,* the order in which the middle terms are written does not affect the final factorisation, and hence does not effect the solutions of the quadratic.

EXERCISE 1

Solve the equations.

a $4x^2 - 20 = 0$ **b** $x^2 - x - 12 = 0$ **c** $3x^2 + 2x - 8 = 0$

Note: While the values of which satisfy $2x^2 + 5x + 3 = 0$ are $x = -1$ or $x = -\tfrac{3}{2}$, we often say that the solution of $2x^2 + 5x + 3 = 0$ are $x = -1$ and $x = -\tfrac{3}{2}$.

Common simplifications of quadratics

It is often convenient to simplify a quadratic equation before any method of solution is applied.

- If the coefficient of x^2 is negative multiply through by -1.

  $-x^2 + 5x - 6 = 0$ becomes $x^2 - 5x + 6 = 0$

- 'multiply out fractions'

  $\frac{x^2}{2} - \frac{5x}{2} + 3$ becomes $x^2 - 5x + 6 = 0$

- If there is a common factor divide through by it.

  $3x^2 - 15x + 18 = 0$ becomes $x^2 - 5x + 6 = 0$

Equations that can be rearranged to be a quadratic equation in standard form

The **standard form** for a quadratic equation is $ax^2 + bx + c = 0, a \neq 0$.

We may however, be given a quadratic equation that is not in this form and so our first step is to re-write the equation into this standard form.

EXAMPLE

Solve $\frac{x-2}{3} = \frac{5}{x}$.

SOLUTION

$$\frac{x-2}{3} = \frac{5}{x}$$

$$x(x-2) = 3 \times 5 \qquad \text{(cross-multiplication)}$$

$$x^2 - 2x = 15$$

$$x^2 - 2x - 15 = 0 \qquad \text{(Rearrange)}$$

$$(x+3)(x-5) = 0$$

$$x + 3 = 0 \text{ or } x - 5 = 0$$

$$x = -3 \text{ or } x = 5$$

EXERCISE 2

Solve $\frac{1}{3-x} - \frac{4}{5} = \frac{1}{9-2x}$.

Applications

EXAMPLE

A rectangle has one side 3cm longer than the other. The rectangle has area 28cm^2. What is the length of the shorter side?

SOLUTION

Let x cm be the length of the shorter side. The other side has length $(x + 3)$cm.

Area $= x(x + 3) = 28\text{cm}^2$

$$x^2 + 3x - 28 = 0$$

$$(x-4)(x+7) = 0$$

$$x - 4 = 0 \text{ or } x + 7 = 0$$

$$x = 4 \text{ or } x = -7$$

Since length must be positive, the solution to the problem is $x = 4$. The shorter side has length 4cm.

EXERCISE 3

Each number in the sequence 5, 9, 13, 17, ... is obtained by adding 4 to the previous number. The sum S of the first n numbers in the sequence is given by $S = 2n^2 + 3n$.

How many numbers must be added to make the sum equal to 152?

Completing the square

The quadratic equations encountered so far, had one or two solutions that were rational. There are many quadratics that have irrational solutions, or in some cases no real solutions at all.

For example, it is not easy at all to see how to factor the quadratic $x^2 - 5x - 3 = 0$. Indeed it has no rational solutions. We will see shortly that the solutions are $x = \frac{5+\sqrt{37}}{2}$ and $x = \frac{5-\sqrt{37}}{2}$.

To deal with more general quadratics, we employ a technique known as *completing the square*. Historically, this was the most commonly used method of solution.

The technique of completing the square is used not only for solving quadratic equations, but also in further mathematics for such things as:

- finding the centre and radius of a circle – given its algebraic equation,
- finding the maximum or minimum of a quadratic function,
- finding the axis of symmetry of a parabola,
- putting integrals into standard form in calculus.

This is an important technique that will appear in other settings and so is a basic skill that students who intend to proceed to senior mathematics need to master.

In the early stages, students will need to be told when to apply which method.With experience, they will use completing the square whenever they cannot see how to apply the factor method shown above. The method of completing the square works in every case, including the situation in which the factor method applies.

In earlier modules we have seen the two identities referred to as **perfect squares:**

$$a^2 + 2ab + b^2 = (a+b)^2 \text{ and } a^2 - 2ab + b^2 = (a-b)^2.$$

Thus, for example, $x^2 + 6x + 9 = (x+3)^2$ and $x^2 - 4x + 4 = (x-2)^2$.

Notice that in the quadratics above, the constant term in each case is the square of **half** the coefficient of x. The method of completing the square simply involves adding in a number make a given quadratic expression into a perfect square.

Completing the square in an expression

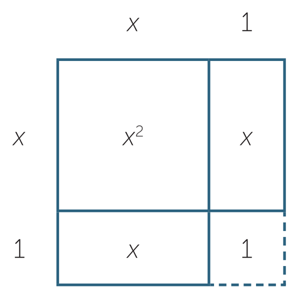
We begin by completing the square on the quadratic expression $x^2 + 2x - 6$.

We focus on $x^2 + 2x$ and ask: What number must be added to $x^2 + 2x$ to make the expression into a perfect square?

The key step is to take half the coefficient of and square it. The same rule applies when the coefficient of is x negative.

In this case, the answer is 1, since $x^2 + 2x + 1 = (x + 1)^2$.

This can be seen diagrammatically, where a square is added to 'complete the square'.



We can then write:

$$x^2 + 2x - 6 = (x^2 + 2x + 1) - 1 - 6 \qquad \text{(add and subtract 1)}$$

$$= (x + 1)^2 - 7.$$

In the case when the coefficient of x is odd, we will need to use fractions. For example, to complete the square on $x^2 - 3x + 1$, we note that half of -3 is $-\frac{3}{2}$ and $\frac{9}{4}$. Hence we have $x^2 - 3x + 1 = \left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$.

Solving quadratic equations by completing the square

We can now apply the method of completing the square to solve quadratic equations. To complete the square for an equation, we will add in a factor on each side to produce a square.

EXAMPLE

Solve $x^2 + 2x - 6 = 0$.

SOLUTION

It is easiest to move the constant term onto the other side first and then complete the square.

$$x^2 + 2x - 6 = 0$$

$$x^2 + 2x = 6$$

$$x^2 + 2x + 1 = 7 \qquad \text{(Add 1 to both sides to produce a square)}$$

$$(x + 1)^2 = 7$$

We can now take the positive and negative square roots to obtain

$$x + 1 = \sqrt{7} \text{ or } x + 1 = -\sqrt{7},$$

so, $\quad x = -1 + \sqrt{7}$ or $x = -1 - \sqrt{7}$.

Notice that the solutions are irrational, and so this equation could not be easily solved using the factoring method.

EXAMPLE

Solve $x^2 - 6x - 2 = 0$.

SOLUTION

$$x^2 - 6x - 2 = 0$$

$$x^2 - 6x + 9 - 9 - 2 = 0 \qquad \text{(Complete the square.)}$$

$$(x - 3)^2 = 11$$

$$x - 3 = \sqrt{11} \text{ or } x - 3 = -\sqrt{11}$$

Hence $\qquad x = 3 + \sqrt{11}$ or $x = 3 - \sqrt{11}$.

There are, of course, quadratic equations which cannot be solved using real numbers. For example, if we apply the method to the equation $x^2 - 6x + 12 = 0$, we obtain $(x - 3)^2 = -3$ and $(x - 3)^2 \geq -3$ since the equation cannot be solved.

EXERCISE 4

Solve $x^2 - 5x - 3 = 0$ by completing the square and also show that $x^2 - 5x + 7 = 0$ has no solutions.

Non-monic quadratics

You will have noticed that we have not solved any non-monic quadratics by completing the square. This is generally rather tricky for students to complete and non-monic quadratics that cannot be solved by factoring can always be solved using the quadratic formula.

To solve a non-monic quadratic by completing the square, it is easiest to divide the equation by the leading coefficient and so make the quadratic monic. This will lead to fractions as the following example shows.

EXAMPLE

Solve $3x^2 - 5x + 1 = 0$.

SOLUTION

Divide the equation by 3 and shift the constant term to the other side.

$$3x^2 - 5x + 1 = 0$$

$$x^2 - \frac{5}{3}x = -\frac{1}{3}$$

$$x^2 - \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} - \frac{1}{3}$$

$$\left(x - \frac{5}{3}\right)^2 = \frac{13}{36}$$

So, $\qquad x = \frac{5 + \sqrt{13}}{6}$ or $x = \frac{5 - \sqrt{13}}{6}$.

As mentioned above, these solutions can also be found using the quadratic formula.

The quadratic formula

The method of completing the square always works. By applying it to the general quadratic equation $ax^2 + bx + c = 0$ we obtain the well-known **quadratic formula.**

To derive the formula, we will begin by multiplying the equation through by $4a$, which although not the usual first step in completing the square, will make the algebra much easier.

$$ax^2 + bx + c = 0$$

$$4a^2x^2 + 4abx + 4ac = 0$$

We now note that $(2ax + b)^2 = 4a^2x^2 + 4abx + b^2$ so adding b^2 will produce a square.

$$4a^2x^2 + 4abx = -4ac$$

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$(2ax + b)^2 = b^2 - 4ac.$$

We pause at this stage to note that if $b^2 - 4ac$ is negative, then there is no solution. If $b^2 - 4ac$ is positive, we then proceed to take the positive and negative square roots to solve for x. If $b^2 - 4ac$ is equal to 0, then there will only be 1 solution. We suppose then that $b^2 - 4ac$ is positive and proceed to find the solutions.

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \sqrt{b^2 - 4ac} \text{ or } 2ax + b = -\sqrt{b^2 - 4ac}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

This last formula is called the quadratic formula, sometimes written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If the quantity $b^2 - 4ac = 0$ then there will only be one solution, $x = -\frac{b}{2a}$.

In this case, the quadratic will be a perfect square. The quantity $b^2 - 4ac$ plays an important role in the theory of quadratic equations and is called the *discriminant*.

Thus, in summary, when solving $ax^2 + bx + c = 0$, first calculate the discriminant $b^2 - 4ac$. Then,

- if $b^2 - 4ac$ is negative, then there is no solution.
- if $b^2 - 4ac$ is positive, then the solutions are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.
- if $b^2 - 4ac$ is zero, then there is only one solution $x = -\frac{b}{2a}$.

While students do not need to learn the derivation of the formula, they do need to remember the formula itself.

Note: If $b^2 - 4ac$ is zero, then the quadratic is a perfect square.

EXAMPLE

Solve $x^2 - 10x - 3 = 0$ by

a using the formula. **b** completing the square.

SOLUTION

a Here $a = 1$, $b = -10$, $c = -3$,

$$\begin{aligned} \text{so } b^2 - 4ac &= 100 + 12 \\ &= 112. \end{aligned}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{10 + \sqrt{112}}{2} \quad \text{or} \quad x = \frac{10 - \sqrt{112}}{2}$$

$$x = \frac{10 + 4\sqrt{7}}{2} \quad \text{or} \quad x = \frac{10 - 4\sqrt{7}}{2} \qquad \text{(Simplify the surd.)}$$

$$x = \frac{2(5 + 2\sqrt{7})}{2} \quad \text{or} \quad x = \frac{2(5 - 2\sqrt{7})}{2}$$

$$x = 5 + 2\sqrt{7} \quad \text{or} \quad x = 5 - 2\sqrt{7}$$

b

$$\begin{aligned} x^2 - 10x - 3 &= 0 \\ x^2 - 10x + 25 - 25 - 3 &= 0 \\ (x - 5)^2 &= 28 \\ x - 5 &= \sqrt{28} \quad \text{or} \quad x - 5 = -\sqrt{28} \\ x &= 5 + 2\sqrt{7} \quad \text{or} \quad x = 5 - 2\sqrt{7} \end{aligned}$$

EXERCISE 5

Re-solve the quadratic equation $3x^2 - 5x + 1 = 0$ using the quadratic formula.

A further application

One very interesting application involves a number known to the Greeks as the **golden ratio**.

A **golden rectangle** is a rectangle such as *ACDF* drawn below, with sides of length 1 and x, and with the property that if a 1×1 square (*BCDE*)is removed, the resulting rectangle (*ABEF*) is similar to the original one. That is, *ACDF* is an enlargement of *ABEF*.