

SETS AND VENN DIAGRAMS

ASSUMED KNOWLEDGE

- Addition and subtraction of whole numbers.
- Familiarity with the English words

'and', 'or', 'not', 'all', 'if...then'.

MOTIVATION

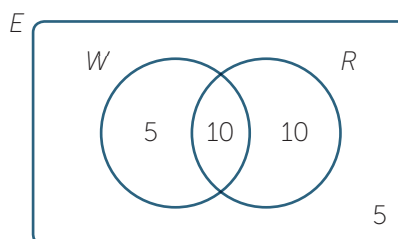
In all sorts of situations we classify objects into sets of similar objects and count them. This procedure is the most basic motivation for learning the whole numbers and learning how to add and subtract them.

Such counting quickly throws up situations that may at first seem contradictory.

'Last June, there were 15 windy days and 20 rainy days, yet 5 days were neither windy nor rainy.'

How can this be, when June only has 30 days? A Venn diagram, and the language of sets, easily sorts this out.

Let W be the set of windy days,
and R be the set of rainy days.
Let E be the set of days in June.
Then W and R together have size 25,
so the overlap between W and R is 10.;
The Venn diagram opposite displays
the whole situation.



The purpose of this module is to introduce language for talking about sets, and some notation for setting out calculations, so that counting problems such as this can be sorted out. The Venn diagram makes the situation easy to visualise.

CONTENT

DESCRIBING AND NAMING SETS

A set is just a collection of objects, but we need some new words and symbols and diagrams to be able to talk sensibly about sets.

In our ordinary language, we try to make sense of the world we live in by classifying collections of things. English has many words for such collections. For example, we speak of 'a flock of birds', 'a herd of cattle', 'a swarm of bees' and 'a colony of ants'.

We do a similar thing in mathematics, and classify numbers, geometrical figures and other things into collections that we call sets. The objects in these **sets** are called the **elements** of the set.

Describing a set

A set can be described by **listing** all of its elements. For example,

$$S = \{ 1, 3, 5, 7, 9 \},$$

which we read as 'S is the set whose elements are 1, 3, 5, 7 and 9'. The five elements of the set are separated by commas, and the list is enclosed between curly brackets.

A set can also be described by **writing a description** of its elements between curly brackets. Thus the set S above can also be written as

$$S = \{ \text{odd whole numbers less than } 10 \},$$

which we read as 'S is the set of odd whole numbers less than 10'.

A set must be **well defined**. This means that our description of the elements of a set is clear and unambiguous. For example, { tall people } is not a set, because people tend to disagree about what 'tall' means. An example of a well-defined set is

$$T = \{ \text{letters in the English alphabet} \}.$$

Equal sets

Two sets are called **equal** if they have exactly the same elements. Thus following the usual convention that 'y' is not a vowel,

$$\{\text{vowels in the English alphabet}\} = \{a, e, i, o, u\}$$

On the other hand, the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ are not equal, because they have different elements. This is written as

$$\{1, 3, 5\} \neq \{1, 2, 3\}.$$

The order in which the elements are written between the curly brackets does not matter at all. For example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\}.$$

If an element is listed more than once, it is only counted once. For example,

$$\{a, a, b\} = \{a, b\}.$$

The set $\{a, a, b\}$ has only the two elements a and b . The second mention of a is an unnecessary repetition and can be ignored. It is normally considered poor notation to list an element more than once.

The symbols \in and \notin

The phrases 'is an element of' and 'is not an element of' occur so often in discussing sets that the special symbols \in and \notin are used for them. For example, if $A = \{3, 4, 5, 6\}$, then

$$3 \in A \text{ (Read this as '3 is an element of the set A'.)}$$

$$8 \notin A \text{ (Read this as '8 is not an element of the set A'.)}$$

Describing and naming sets

- A **set** is a collection of objects, called the **elements** of the set.
- A set must be **well defined**, meaning that its elements can be described and listed without ambiguity. For example:

$$\{1, 3, 5\} \text{ and } \{\text{letters of the English alphabet}\}.$$

- Two sets are called **equal** if they have exactly the same elements.
 - The order is irrelevant.
 - Any repetition of an element is ignored.
- If a is an element of a set S , we write $a \in S$.
- If b is not an element of a set S , we write $b \notin S$.

EXERCISE 1

- a Specify the set A by listing its elements, where
 $A = \{ \text{whole numbers less than 100 divisible by 16} \}.$
- b Specify the set B by giving a written description of its elements, where
 $B = \{ 0, 1, 4, 9, 16, 25 \}.$
- c Does the following sentence specify a set?
 $C = \{ \text{whole numbers close to 50} \}.$

Finite and infinite sets

All the sets we have seen so far have been finite sets, meaning that we can list all their elements. Here are two more examples:

$$\{ \text{whole numbers between 2000 and 2005} \} = \{ 2001, 2002, 2003, 2004 \}$$

$$\{ \text{whole numbers between 2000 and 3000} \} = \{ 2001, 2002, 2003, \dots, 2999 \}$$

The three dots ' \dots ' in the second example stand for the other 995 numbers in the set. We could have listed them all, but to save space we have used dots instead. This notation can only be used if it is completely clear what it means, as in this situation.

A set can also be **infinite** – all that matters is that it is well defined. Here are two examples of infinite sets:

$$\{ \text{even whole numbers} \} = \{ 0, 2, 4, 6, 8, 10, \dots \}$$

$$\{ \text{whole numbers greater than 2000} \} = \{ 2001, 2002, 2003, 2004, \dots \}$$

Both these sets are infinite because no matter how many elements we list, there are always more elements in the set that are not on our list. This time the dots ' \dots ' have a slightly different meaning, because they stand for infinitely many elements that we could not possibly list, no matter how long we tried.

The numbers of elements of a set

If S is a finite set, the symbol $|S|$ stands for the **number of elements** of S . For example:

$$\text{If } S = \{ 1, 3, 5, 7, 9 \}, \text{ then } |S| = 5.$$

$$\text{If } A = \{ 1001, 1002, 1003, \dots, 3000 \}, \text{ then } |A| = 2000.$$

$$\text{If } T = \{ \text{letters in the English alphabet} \}, \text{ then } |T| = 26.$$

The set $S = \{ 5 \}$ is a **one-element set** because $|S| = 1$. It is important to distinguish between the number 5 and the set $S = \{ 5 \}$:

$$5 \in S \text{ but } 5 \neq S.$$

The empty set

The symbol \emptyset represents the empty set, which is the set that has no elements at all. Nothing in the whole universe is an element of \emptyset :

$|\emptyset| = 0$ and $x \notin \emptyset$, no matter what x may be.

There is only one empty set, because any two empty sets have exactly the same elements, so they must be equal to one another.

Finite and Infinite sets

- A set is called **finite** if we can list all of its elements.
- An **infinite** set has the property that no matter how many elements we list, there are always more elements in the set that are not on our list.
- If S is a finite set, the symbol $|S|$ stands for the number of elements of S .
- The set with no elements is called the empty set, and is written as \emptyset . Thus $|\emptyset| = 0$.
- A one-element set is a set such as $S = \{5\}$ with $|S| = 1$.

EXERCISE 2

- a** Use dots to help list each set, and state whether it is finite or infinite.
- $A = \{\text{even numbers between 10 000 and 20 000}\}$
 - $B = \{\text{whole numbers that are multiples of 3}\}$
- b** If the set S in each part is finite, write down $|S|$.
- $S = \{\text{primes}\}$
 - $S = \{\text{even primes}\}$
 - $S = \{\text{even primes greater than 5}\}$
 - $S = \{\text{whole numbers less than 100}\}$
- c** Let F be the set of fractions in simplest form between 0 and 1 that can be written with a single-digit denominator. Find F and $|F|$.

SUBSETS AND VENN DIAGRAMS

Subsets of a set

Sets of things are often further subdivided. For example, owls are a particular type of bird, so every owl is also a bird. We express this in the language of sets by saying that the set of owls is a subset of the set of birds.

A set S is called a subset of another set T if every element of S is an element of T . This is written as

$$S \subseteq T \text{ (Read this as 'S is a subset of T'.)}$$

The new symbol \subseteq means 'is a subset of'. Thus $\{\text{owls}\} \subseteq \{\text{birds}\}$ because every owl is a bird. Similarly,

$$\text{if } A = \{2, 4, 6\} \text{ and } B = \{0, 1, 2, 3, 4, 5, 6\}, \text{ then } A \subseteq B,$$

because every element of A is an element of B .

The sentence ' S is not a subset of T ' is written as

$$S \not\subseteq T.$$

This means that at least one element of S is *not an element of* T . For example,

$$\{\text{birds}\} \not\subseteq \{\text{flying creatures}\}$$

because an ostrich is a bird, but it does not fly. Similarly,

$$\text{if } A = \{0, 1, 2, 3, 4\} \text{ and } B = \{2, 3, 4, 5, 6\}, \text{ then } A \not\subseteq B,$$

because $0 \in A$, but $0 \notin B$.

The set itself and the empty set are always subsets

Any set S is a subset of itself, because every element of S is an element of S . For example:

$$\{\text{birds}\} \subseteq \{\text{birds}\} \text{ and } \{1, 2, 3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

Furthermore, the empty set \emptyset is a subset of every set S , because every element of the empty set is an element of S , there being no elements in \emptyset at all. For example:

$$\emptyset \subseteq \{\text{birds}\} \text{ and } \emptyset \subseteq \{1, 2, 3, 4, 5, 6\}.$$

Every element of the empty set is a bird, and every element of the empty set is one of the numbers 1, 2, 3, 4, 5 or 6.

Subsets and the words 'all' and 'if ... then'

A statement about subsets can be rewritten as a sentence using the word 'all'. For example,

$$\{\text{owls}\} \subseteq \{\text{birds}\} \quad \text{means} \quad \text{'All owls are birds.'}$$

$$\{\text{multiples of 4}\} \subseteq \{\text{even numbers}\} \quad \text{means} \quad \text{'All multiples of 4 are even.'}$$

$$\{\text{rectangles}\} \subseteq \{\text{rhombuses}\} \quad \text{means} \quad \text{'Not all rectangles are rhombuses.'}$$

They can also be rewritten using the words 'if ... then'. For example,

$\{\text{owls}\} \subseteq \{\text{birds}\}$ means 'If a creature is an owl, then it is a bird.'

$\{\text{multiples of 4}\} \subseteq \{\text{even numbers}\}$ means 'If a number is a multiple of 4, then it is even':

$\{\text{rectangles}\} \subseteq \{\text{rhombuses}\}$ means 'If a figure is a rectangle, then it may not be a square.'

Venn diagrams

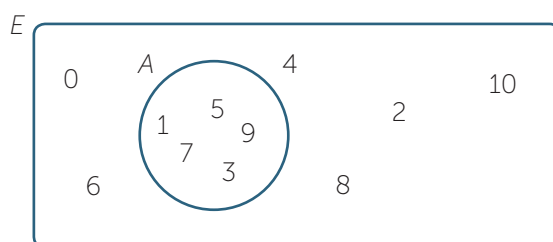
Diagrams make mathematics easier because they help us to see the whole situation at a glance. The English mathematician John Venn (1834–1923) began using diagrams to represent sets. His diagrams are now called **Venn diagrams**.

In most problems involving sets, it is convenient to choose a larger set that contains all of the elements in all of the sets being considered. This larger set is called the **universal set**, and is usually given the symbol E . In a Venn diagram, the universal set is generally drawn as a large rectangle, and then other sets are represented by circles within this rectangle.



For example, if $V = \{\text{vowels}\}$, we could choose the universal set as $E = \{\text{letters of the alphabet}\}$ and all the letters of the alphabet would then need to be placed somewhere within the rectangle, as shown below.

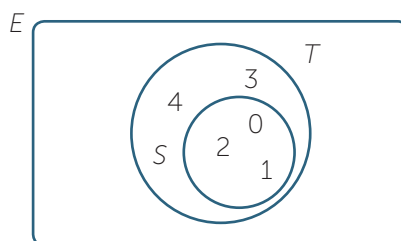
In the Venn diagram below, the universal set is $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and each of these numbers has been placed somewhere within the rectangle.



The region inside the circle represents the set A of odd whole numbers between 0 and 10. Thus we place the numbers 1, 3, 5, 7 and 9 inside the circle, because $A = \{1, 3, 5, 7, 9\}$. Outside the circle we place the other numbers 0, 2, 4, 6, 8 and 10 that are in E but not in A .

Representing subsets on a Venn diagram

When we know that S is a subset of T , we place the circle representing S inside the circle representing T . For example, let $S = \{0, 1, 2\}$, and $T = \{0, 1, 2, 3, 4\}$. Then S is a subset of T , as illustrated in the Venn diagram below.



Make sure that 5, 6, 7, 8, 9 and 10 are placed outside both circles>

Subsets and the number line

The whole numbers are the numbers 0, 1, 2, 3,... These are often called the 'counting numbers', because they are the numbers we use when counting things. In particular, we have been using these numbers to count the number of elements of finite sets. The number zero is the number of elements of the empty set.

The set of all whole numbers can be represented by dots on the number line.



Any finite subset of set of whole numbers can be represented on the number line. For example, here is the set $\{0, 1, 4\}$.



Subsets of a st

- If all the elements of a set S are elements of another set T , then S is called a subset of T . This is written as $S \subseteq T$.
- If at least one element of S is not an element of T , then S is not a subset of T . This is written as $S \not\subseteq T$.
- If S is any set, then $\emptyset \subseteq S$ and $S \subseteq S$.
- A statement about a subset can be rewritten using the words 'all' or 'if ... then'.
- Subsets can be represented using a **Venn diagram**.
- The set $\{0, 1, 2, 3, 4, \dots\}$ of whole numbers is infinite.
- The set of whole numbers, and any finite subset of them, can be represented on the number line.

EXERCISE 3

- a Rewrite in set notation:
- All squares are rectangles.
 - Not all rectangles are rhombuses.
- b Rewrite in an English sentence using the words 'all' or 'not all':
- $\{ \text{whole number multiples of } 6 \} \subseteq \{ \text{even whole numbers} \}$.
 - $\{ \text{square whole numbers} \} \subseteq \{ \text{even whole numbers} \}$.
- c Rewrite the statements in part (b) in an English sentence using the words 'if ..., then'.
- d Given the sets $A = \{ 0, 1, 4, 5 \}$ and $B = \{ 1, 4 \}$:
- Draw a Venn diagram of A and B using the universal set $U = \{ 0, 1, 2, \dots, 8 \}$.
 - Graph A on the number line.

COMPLEMENTS, INTERSECTIONS AND UNIONS

The complement of a set

Suppose that a suitable universal set E has been chosen. The **complement** of a set S is the set of all elements of E that are not in S . The complement of S is written as S^c . For example,

If $E = \{ \text{letters} \}$ and $V = \{ \text{vowels} \}$, then $V^c = \{ \text{consonants} \}$

If $E = \{ \text{whole numbers} \}$ and $O = \{ \text{odd whole numbers} \}$,
then $O^c = \{ \text{even whole numbers} \}$.

Complement and the word 'not'

The word 'not' corresponds to the complement of a set. For example, in the two examples above,

$V^c = \{ \text{letters that are not vowels} \} = \{ \text{consonants} \}$

$O^c = \{ \text{whole numbers that are not odd} \} = \{ \text{even whole numbers} \}$

The set V^c in the first example can be represented on a Venn diagram as follows.



The intersection of two sets

The intersection of two sets A and B consists of all elements belonging to A and to B . This is written as $A \cap B$. For example, some musicians are singers and some play an instrument.

If $A = \{ \text{singers} \}$ and $B = \{ \text{instrumentalists} \}$, then

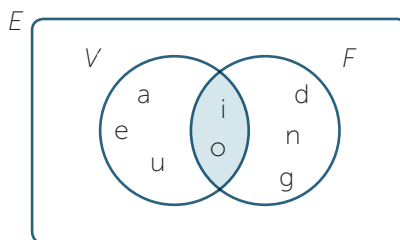
$$A \cap B = \{ \text{singers who play an instrument} \}.$$

Here is an example using letters.

If $V = \{ \text{vowels} \}$ and $F = \{ \text{letters in 'dingo'} \}$, then

$$V \cap F = \{ i, o \}.$$

This last example can be represented on a Venn diagram as follows.



Intersection and the word 'and'

The word 'and' tells us that there is an intersection of two sets. For example:

$$\{ \text{singers} \} \cap \{ \text{instrumentalists} \} = \{ \text{people who sing and play an instrument} \}$$

$$\{ \text{vowels} \} \cap \{ \text{letters of 'dingo'} \} = \{ \text{letters that are vowels and are in 'dingo'} \}$$

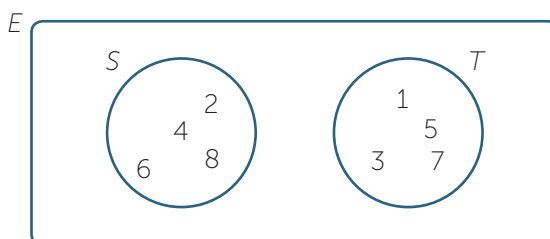
The union of two sets

The union of two sets A and B consists of all elements belonging to A or to B . This is written as $A \cup B$. Elements belonging to both set belong to the union. Continuing with the example of singers and instrumentalists:

If $A = \{ \text{singers} \}$ and $B = \{ \text{instrumentalists} \}$, then $A \cup B = \{ \text{musical performers} \}$.

In the case of the sets of letters:

If $V = \{ \text{vowels} \}$ and $F = \{ \text{letters in 'dingo'} \}$, then $V \cup F = \{ a, e, i, o, u, d, n, g \}$.



Union and the word 'or'

The word 'or' tells us that there is a union of two sets. For example:

$$\{\text{singers}\} \cup \{\text{instrumentalists}\} = \{\text{people who sing or play an instrument}\}$$

$$\{\text{vowels}\} \cup \{\text{letters in 'dingo'}\} = \{\text{letters that are vowels or are in 'dingo'}\}$$

The word 'or' in mathematics always means 'and/or', so there is no need to add 'or both' to these descriptions of the unions. For example,

If $A = \{0, 2, 4, 6, 8, 10, 12, 14\}$ and $B = \{0, 3, 6, 9, 12\}$, then

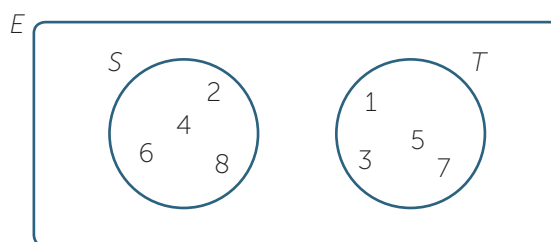
$$A \cup B = \{0, 2, 3, 4, 6, 8, 9, 10, 12, 14\}.$$

Here the elements 6 and 12 are in both sets A and B .

Disjoint sets

Two sets are called disjoint if they have no elements in common. For example:

The sets $S = \{2, 4, 6, 8\}$ and $T = \{1, 3, 5, 7\}$ are disjoint.



Another way to define disjoint sets is to say that their intersection is the empty set,

Two sets A and B are disjoint if $A \cap B = \emptyset$.

In the example above,

$S \cap T = \emptyset$ because no number lies in both sets.

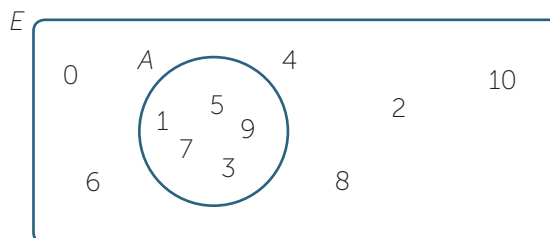
Complement, intersection and union

Let A and B be subsets of a suitable universal set E .

- The complement A^c is the set of all elements of E that are not in A .
- The intersection $A \cap B$ is the set of all elements belonging to A and to B .
- The union $A \cup B$ is the set of all elements belonging to A or to B .
- In mathematics, the word 'or' always means 'and/or', so all the elements that are in both sets are in the union.
- The sets A and B are called disjoint if they have no elements in common, that is, if $A \cap B = \emptyset$.

Representing the complement on a Venn diagram

Let $A = \{1, 3, 5, 7, 9\}$ be the set of odd whole numbers less than 10, and take the universal set as $E = \{0, 1, 2, \dots, 10\}$. Here is the Venn diagram of the situation.



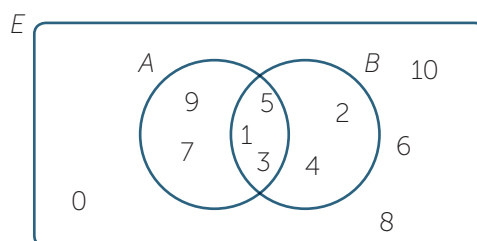
The region inside the circle represents the set A , so we place the numbers 1, 3, 5, 7 and 9 inside the circle. Outside the circle, we place the other numbers 0, 2, 4, 6, 8 and 10 that are not in A . Thus the region outside the circle represents the complement $A^c = \{0, 2, 4, 6, 8, 10\}$.

Representing the intersection and union on a Venn diagram

The Venn diagram below shows the two sets

$$A = \{1, 3, 5, 7, 9\} \quad \text{and} \quad B = \{1, 2, 3, 4, 5\}.$$

- The numbers 1, 3 and 5 lie in both sets, so we place them in the overlapping region of the two circles.
- The remaining numbers in A are 7 and 9. These are placed inside A , but outside B .
- The remaining numbers in B are 2 and 4. These are placed inside B , but outside A .



Thus the overlapping region represents the intersection $A \cap B = \{1, 3, 5\}$, and the two circles together represent the union $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$.

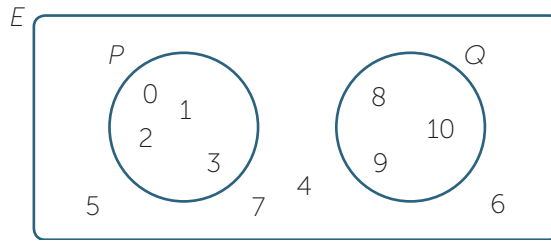
The four remaining numbers 0, 6, 8 and 10 are placed outside both circles.

Representing disjoint sets on a Venn diagram

When we know that two sets are disjoint, we represent them by circles that do not intersect. For example, let

$$P = \{0, 1, 2, 3\} \quad \text{and} \quad Q = \{8, 9, 10\}$$

Then P and Q are disjoint, as illustrated in the Venn diagram below.



Venn diagrams with complements, unions and intersections

- Sets are represented in a Venn diagram by circles drawn inside a rectangle representing the universal set.
- The region outside the circle represents the complement of the set.
- The overlapping region of two circles represents the intersection of the two sets.
- Two circles together represent the union of the two sets.
- When two sets are disjoint, we can draw the two circles without any overlap.
- When one set is a subset of another, we can draw its circle inside the circle of the other set.

EXERCISE 4

Let the universal set be $E = \{\text{whole numbers less than } 20\}$, and let

$$A = \{\text{squares less than } 20\}$$

$$B = \{\text{even numbers less than } 20\}$$

$$C = \{\text{odd squares less than } 20\}$$

- Draw A and C on a Venn diagram, and place the numbers in the correct regions.
- Draw B and C on a Venn diagram, and place the numbers in the correct regions.
- Shade $A \cap B$ on a Venn diagram, and place the numbers in the correct regions.
- Shade $A \cup B$ on a Venn diagram, and place the numbers in the correct regions.

SOLVING PROBLEMS USING A VENN DIAGRAM

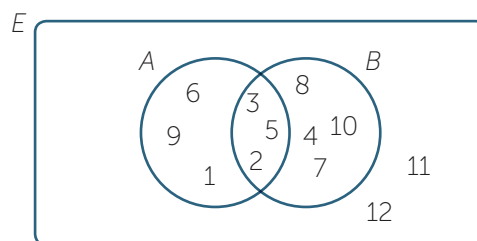
Keeping count of elements of sets

Before solving problems with Venn diagrams, we need to work out how to keep count of the elements of overlapping sets.

The upper diagram to the right shows two sets A and B inside a universal set E , where

$$|A| = 6 \quad \text{and} \quad |B| = 7,$$

with 3 elements in the intersection $A \cap B$.



The lower diagram to the right shows only the number of elements in each of the four regions.

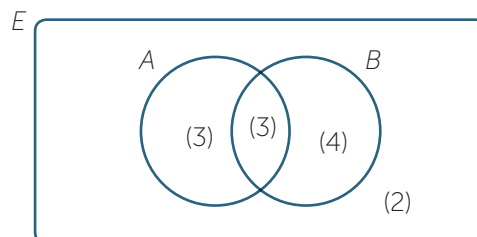
These numbers are placed inside round brackets so that they don't look like elements.

You can see from the diagrams that

$$|A| = 6 \text{ and } |B| = 7, \text{ but } |A \cup B| \neq 6 + 7.$$

The reason for this is that the elements inside the overlapping region $A \cap B$ should only be counted once, not twice. When we subtract the three elements of $A \cap B$ from the total, the calculation is then correct.

$$|A \cup B| = 6 + 7 - 3 = 10.$$

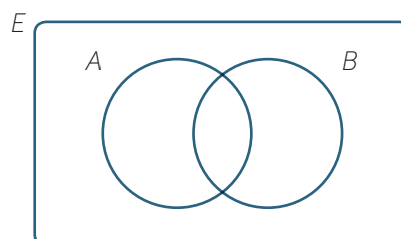


EXAMPLE

In the diagram to the right,

$$|A| = 15, |B| = 25, |A \cap B| = 5 \text{ and } |E| = 50.$$

- Insert the number of elements into each of the four regions.
- Hence find $|A \cup B|$ and $|A \cap B^c|$



SOLUTION

- We begin at the intersection and work outwards.

The intersection $A \cap B$ has 5 elements.

Hence the region of A outside $A \cap B$ has 10 elements, and the region of B outside $A \cap B$ has 20 elements.

This makes 35 elements so far, so the outer region has 15 elements.

- From the diagram,

$$|A \cup B| = 35 \text{ and } |A \cap B^c| = 10.$$

EXERCISE 5

- Draw a Venn diagram of two sets S and T
- Given that $|S| = 15$, $|T| = 20$, $|S \cup T| = 25$ and $|E| = 50$, insert the number of elements into each of the four regions.
- Hence find $|S \cap T|$ and $|S \cap T^c|$.

Number of elements in the regions of a Venn diagram

- The numbers of elements in the regions of a Venn diagram can be done by working systematically around the diagram.
- The number of elements in the union of two sets A and B is
 - Number of elements in $A \cup B$ = number of elements in A
 - + number of elements in B
 - number of elements in $A \cap B$.
- Writing this formula in symbols, $|A \cup B| = |A| + |B| - |A \cap B|$.

Solving problems by drawing a Venn diagram

Many counting problems can be solved by identifying the sets involved, then drawing up a Venn diagram to keep track of the numbers in the different regions of the diagram.

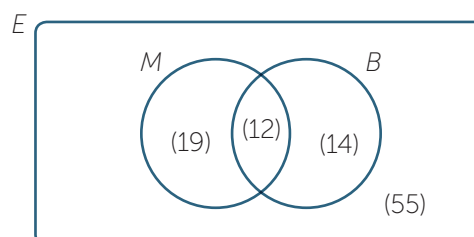
EXAMPLE

A travel agent surveyed 100 people to find out how many of them had visited the cities of Melbourne and Brisbane. Thirty-one people had visited Melbourne, 26 people had been to Brisbane, and 12 people had visited both cities. Draw a Venn diagram to find the number of people who had visited:

- Melbourne or Brisbane
- Brisbane but not Melbourne
- only one of the two cities
- neither city.

SOLUTION

Let M be the set of people who had visited Melbourne, and let B be the set of people who had visited Brisbane. Let the universal set E be the set of people surveyed.



The information given in the question can now be rewritten as

$$|M| = 31, \quad |B| = 26, \quad |M \cap B| = 12 \quad \text{and} \quad |E| = 100.$$

Hence number in M only = $31 - 12$

$$= 19$$

and number in B only = $26 - 12$

$$= 14.$$

- a** Number visiting Melbourne or Brisbane = $19 + 14 + 12 = 45$.
- b** Number visiting Brisbane only = 14.
- c** Number visiting only one city = $19 + 14 = 33$.
- d** Number visiting neither city = $100 - 45 = 55$.

Problem solving using Venn diagrams

- First identify the sets involved.
- Then construct a Venn diagram to keep track of the numbers in the different regions of the diagram.

EXERCISE 6

Twenty-four people go on holidays. If 15 go swimming, 12 go fishing, and 6 do neither, how many go swimming and fishing? Draw a Venn diagram and fill in the number of people in all four regions.

EXERCISE 7

In a certain school, there are 180 pupils in Year 7. One hundred and ten pupils study French, 88 study German and 65 study Indonesian. Forty pupils study both French and German, 38 study German and Indonesian only. Find the number of pupils who study:

- a** all three languages **b** Indonesian only
- c** none of the languages **d** at least one language
- e** either one or two of the three languages.