DSZOB
Digitálne spracovanie zvuku, obrazu a biosignálov

Fourier Transform

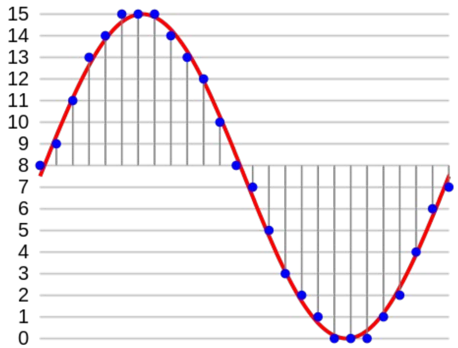
PCM, DPCM, Delta.. modulation

Pulse Code Modulation (PCM)

Transmission of the signal information

Simple use the digitalised samples of analog signals

-> PCM corresponds to the digital signal derived by sampling and quantization



Differential Pulse Code Modulation (DPCM)

Differences of subsequent digital samples

$d_t = x_t - x_{t-1}$

Due to the correlation of samples differences tend to be smaller values

-> we can use only e.g. 4 bits instead of original 16 bits

DPCM Example

Changes between adjacent samples are typical small changes -> use fewer bits

Example:

E.g., 220, 218, 221, 219, 220, 221, 222, 218,.. (all values between 218 and 222)

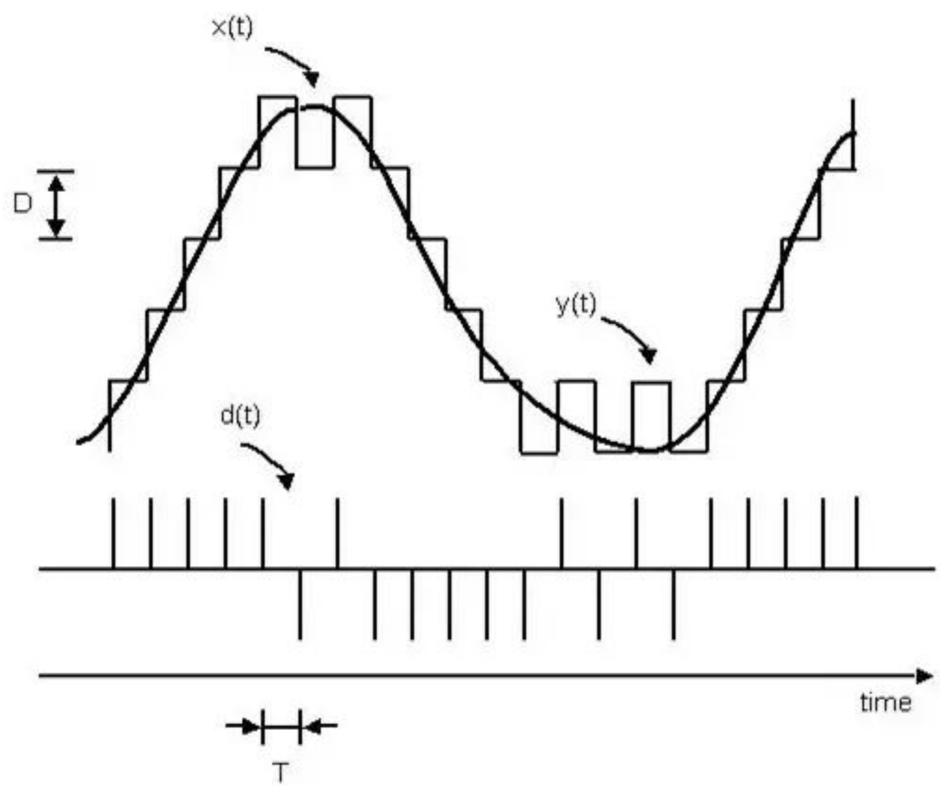
Difference sequence sent: 220, -2, +3, -2, +1, +1, +1, -4, ..

Result: originally for encoding sequence 0..255 numbers need 8 bits;

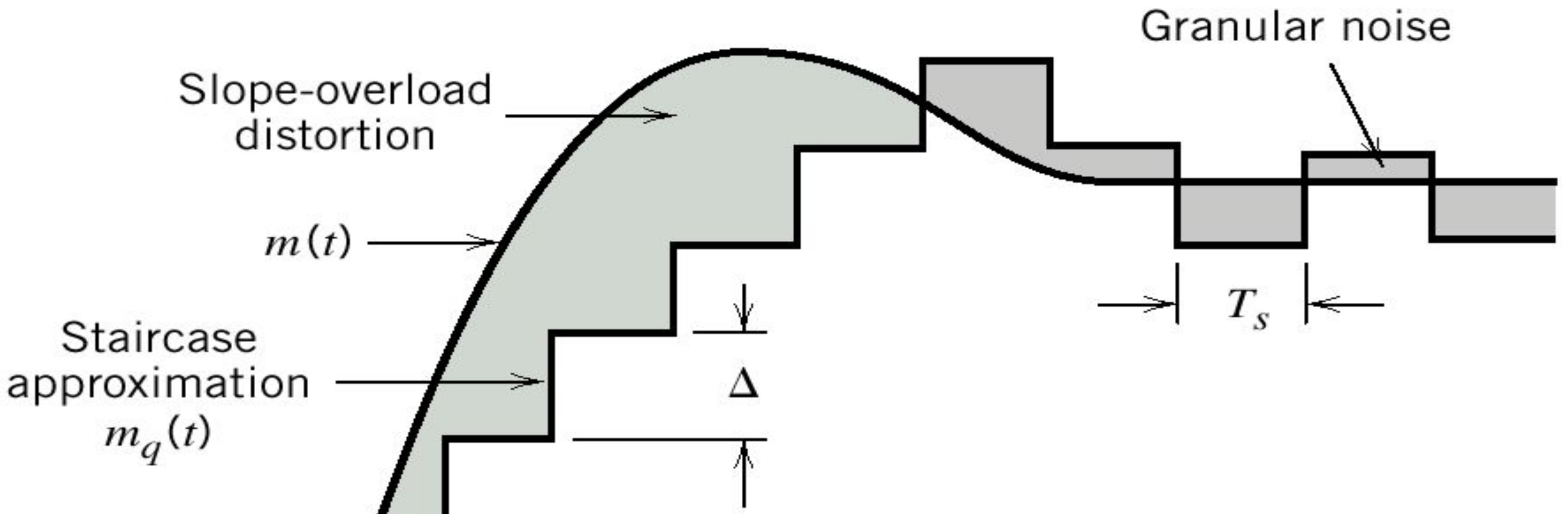
Difference coding: need only 3 bits

Delta modulation

- simplest form of differential pulse-code modulation (DPCM)
-the transmitted data is reduced to a 1-bit data stream.



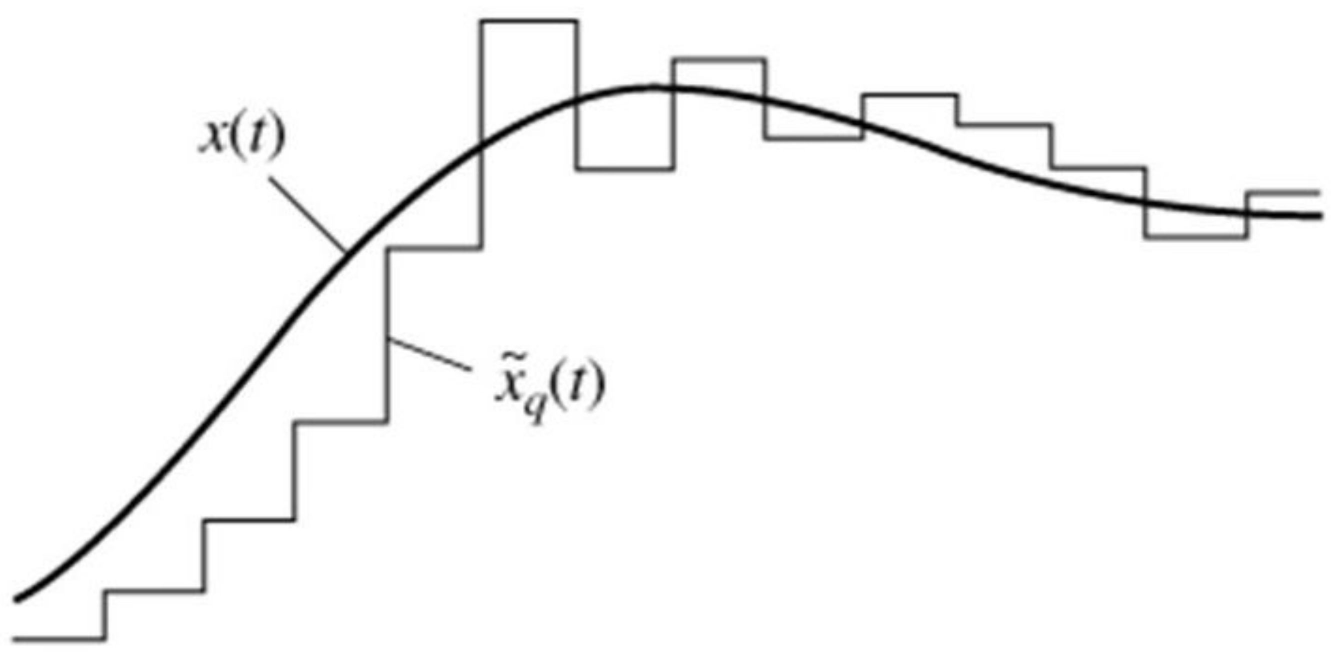
Delta modulation - noise



Adaptive Delta Modulation

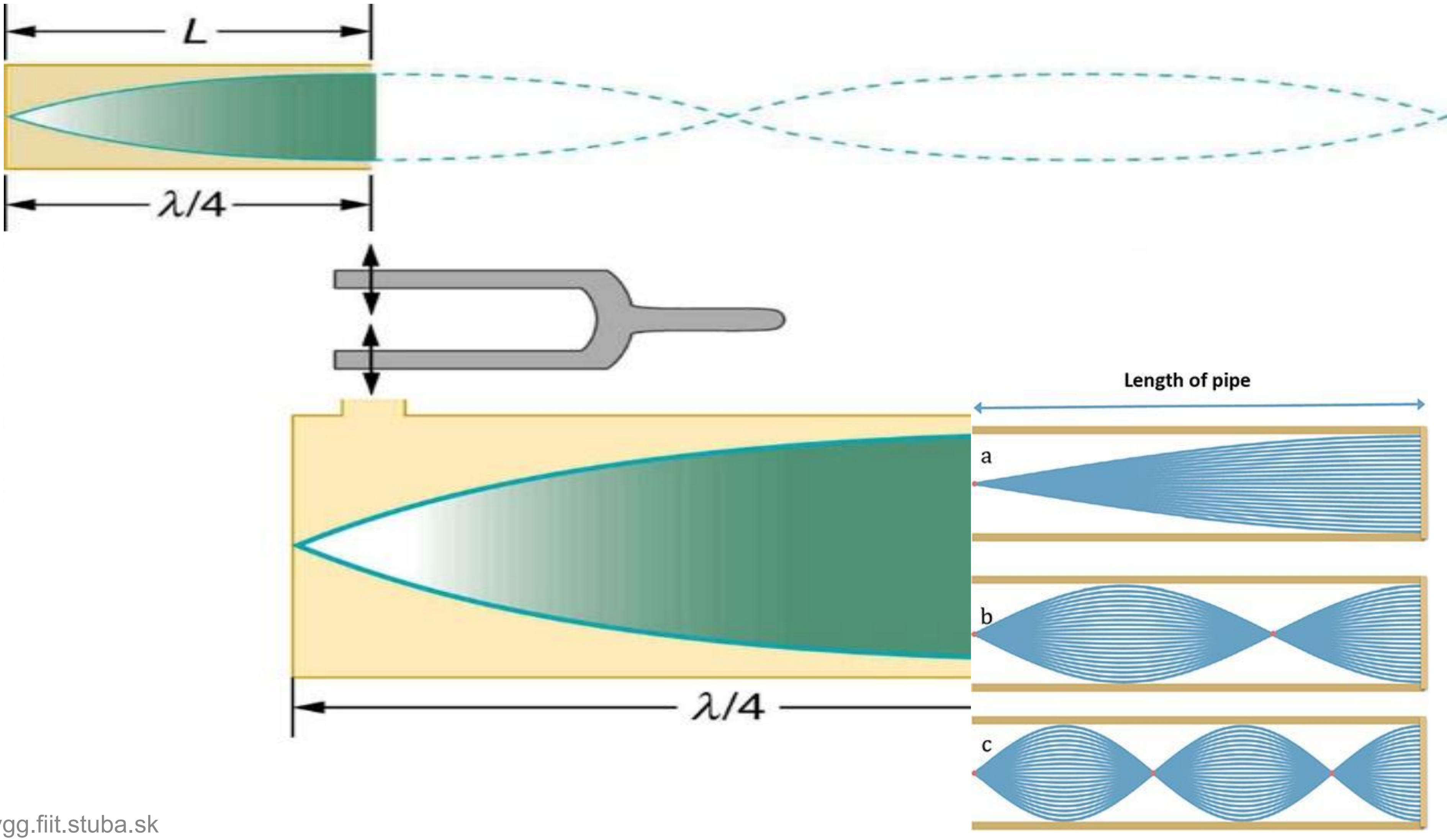
Modification of basic delta modulation

The size of the quantization step is variable - adaptive



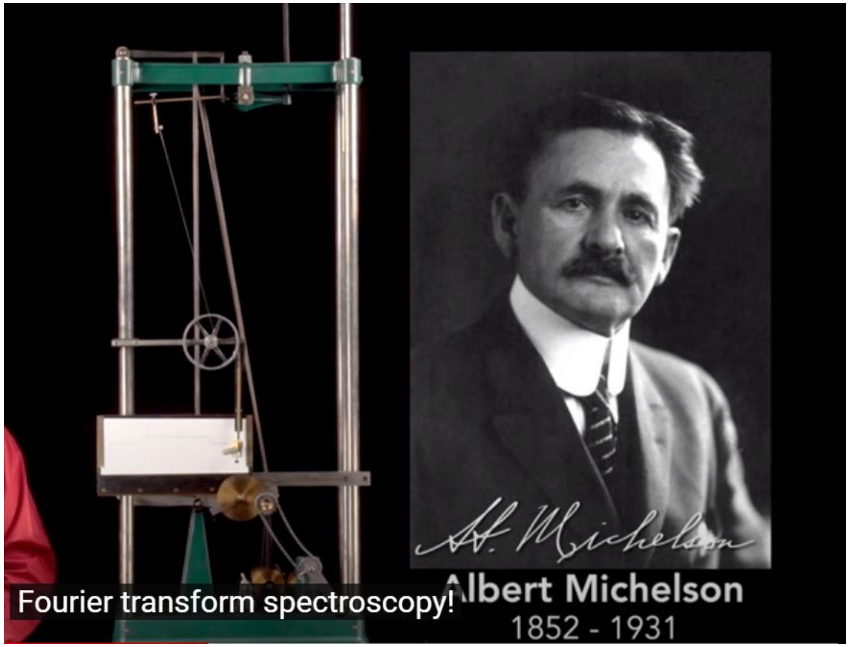
Harmonic analysis /Spectral Analysis

Sound generation by musical instruments



Mechanical spectral analysis

100-year-old mechanical computer

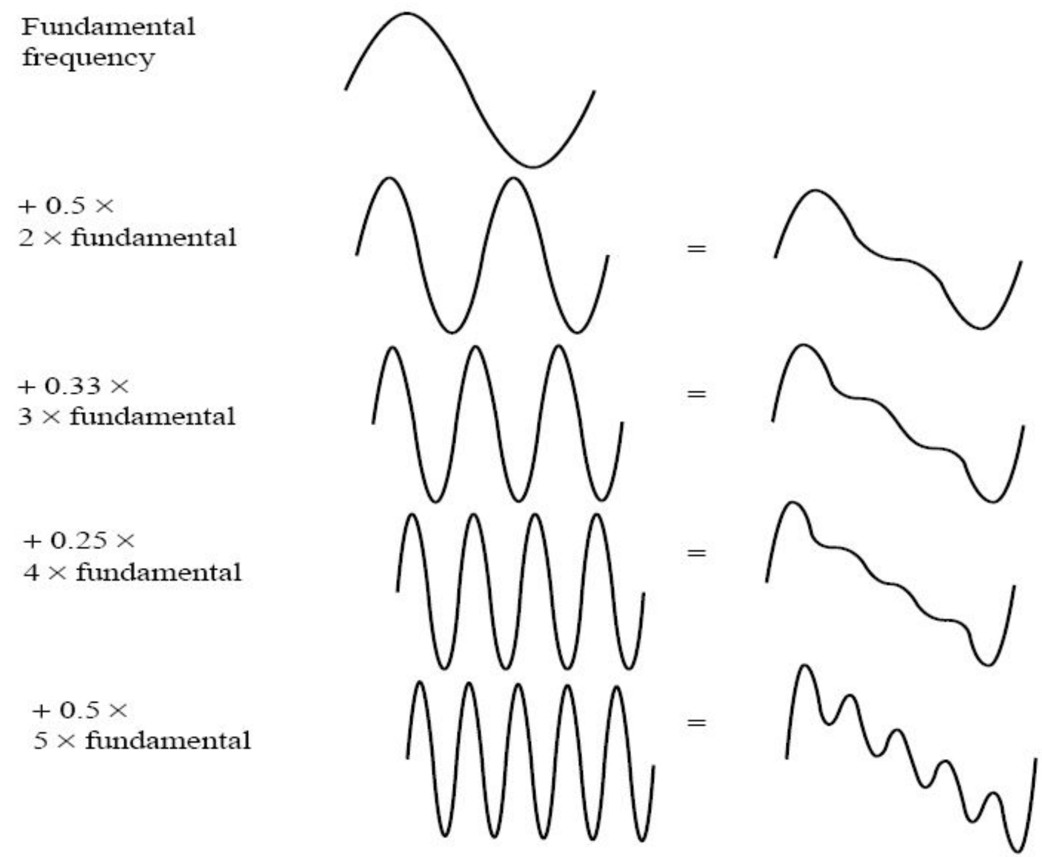


Fourier Spectral Analysis

Understanding the Spectral Analysis

Signals can be decomposed to sine and cosine functions with various parameters.

...sum of weighted sinus can compose quite complex signals...

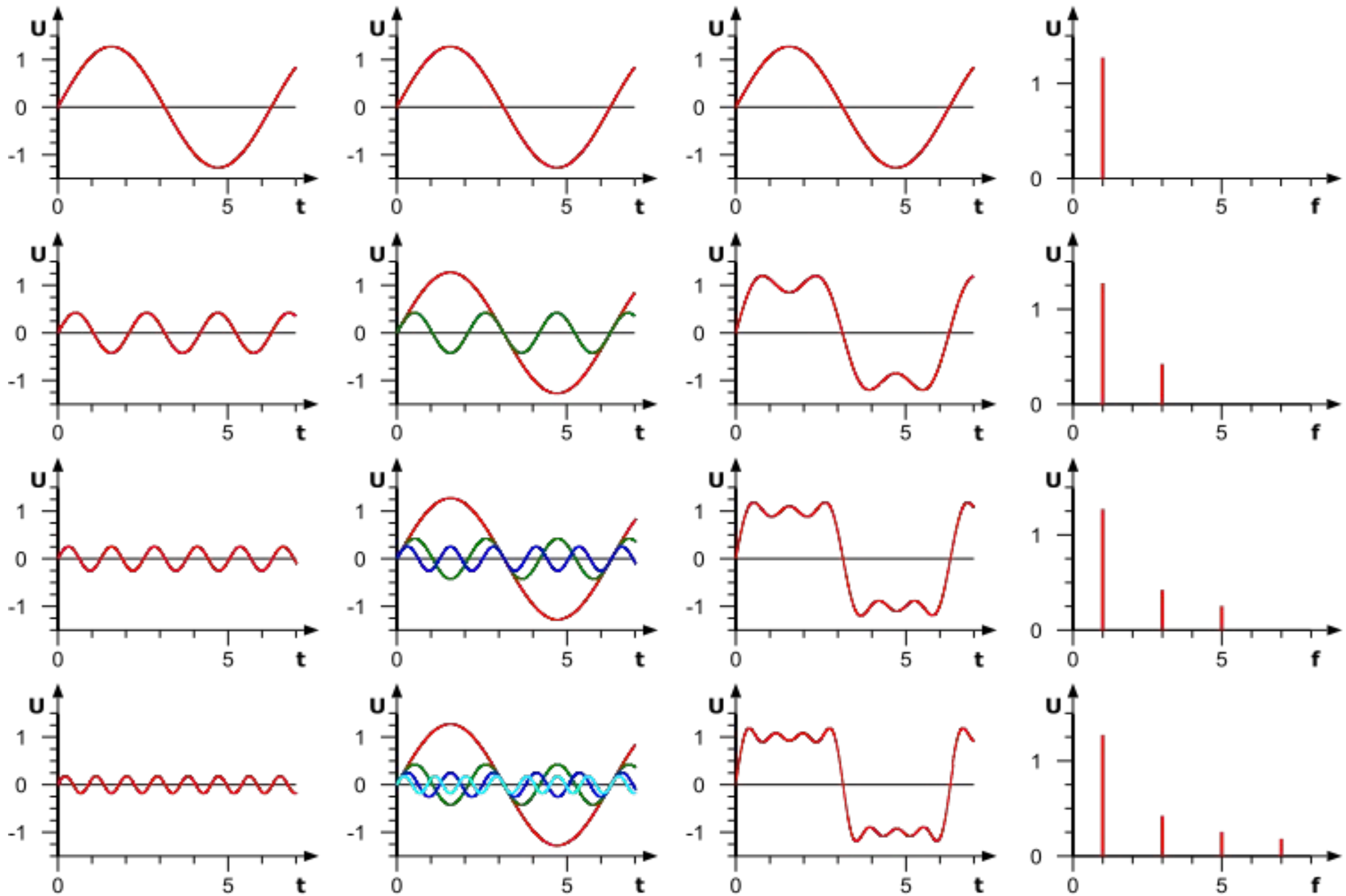


Fourier series is for periodic signals

periodic functions and signals may be expanded into a series of sine and cosine functions

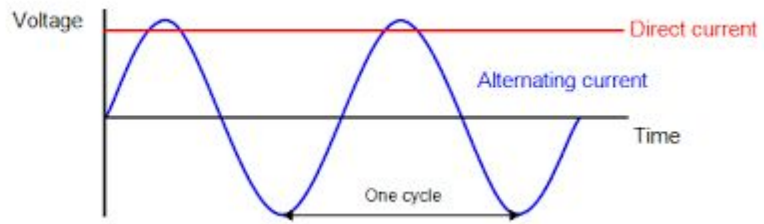
Interaktivny nastroj:
http://www.falstad.com/fourier/

Basics of Fourier analysis / synthesis

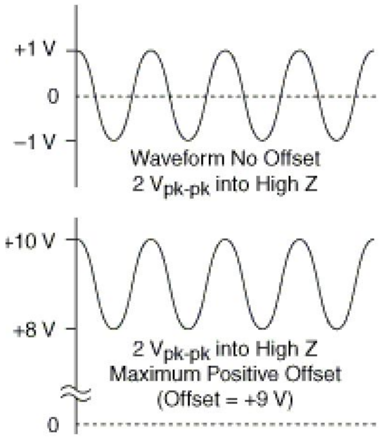


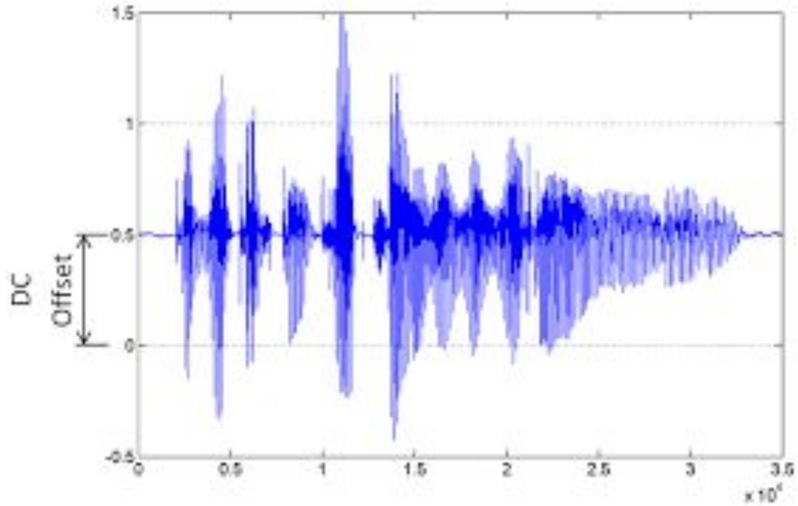
DC component in spectral analysis

DC – Direct current



DC offset positions a waveform around an arbitrary DC value.





Additional informations...on web

http://madebyevan.com/dft/

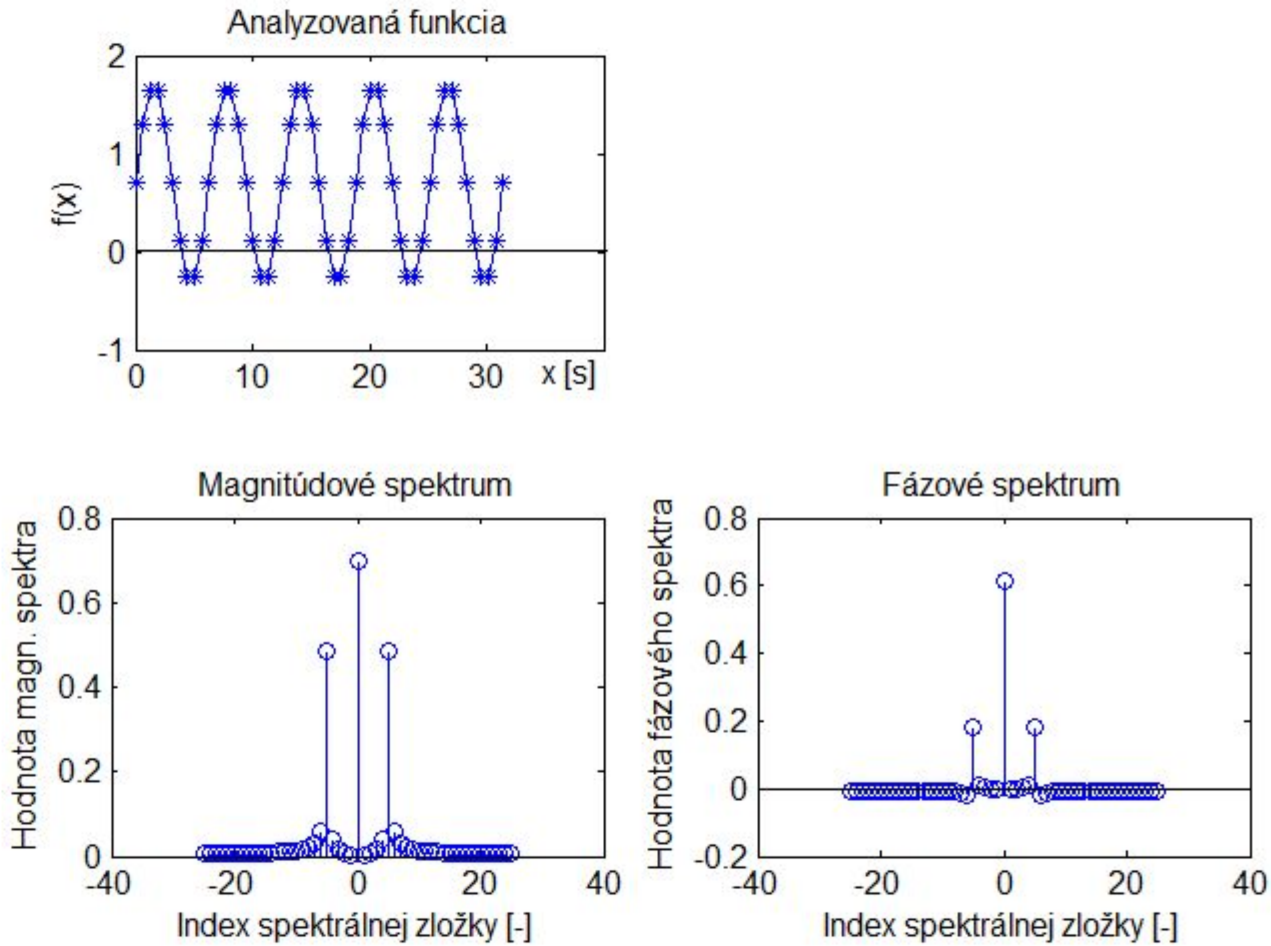
Fourier transform explanation:
https://www.youtube.com/watch?v=mkGsMWi_j4Q

The Fourier Transforms and its Applications

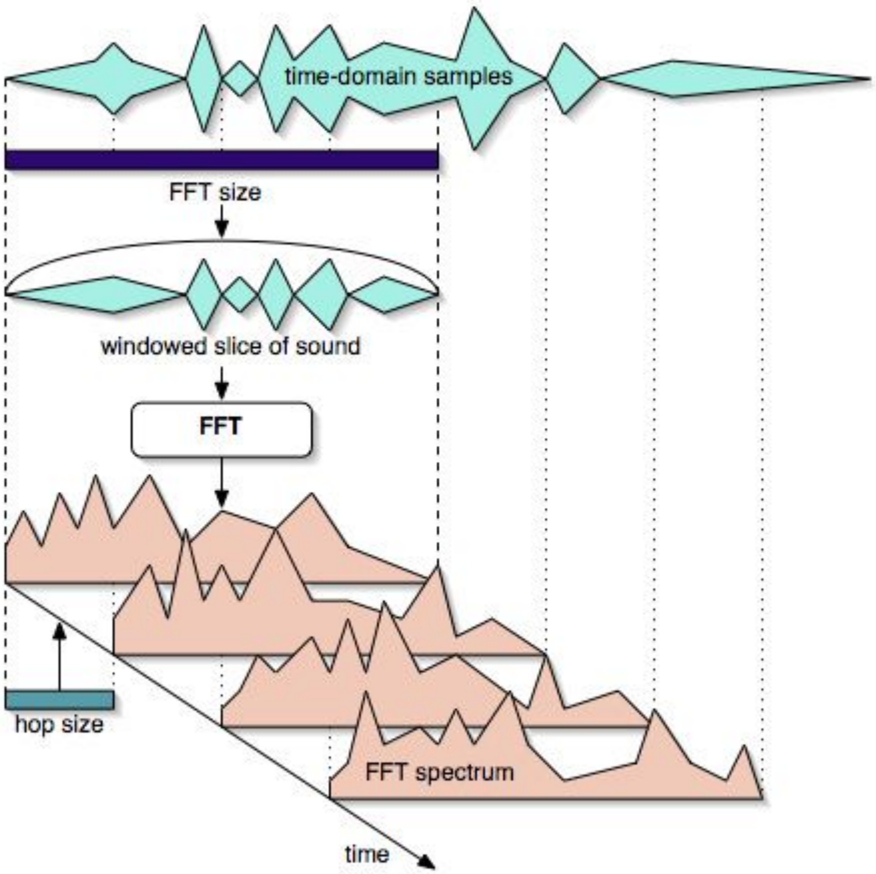
Lecture by Professor Brad Osgood

https://www.youtube.com/watch?v=gZNm7L96pfY&ab_channel=Stanford

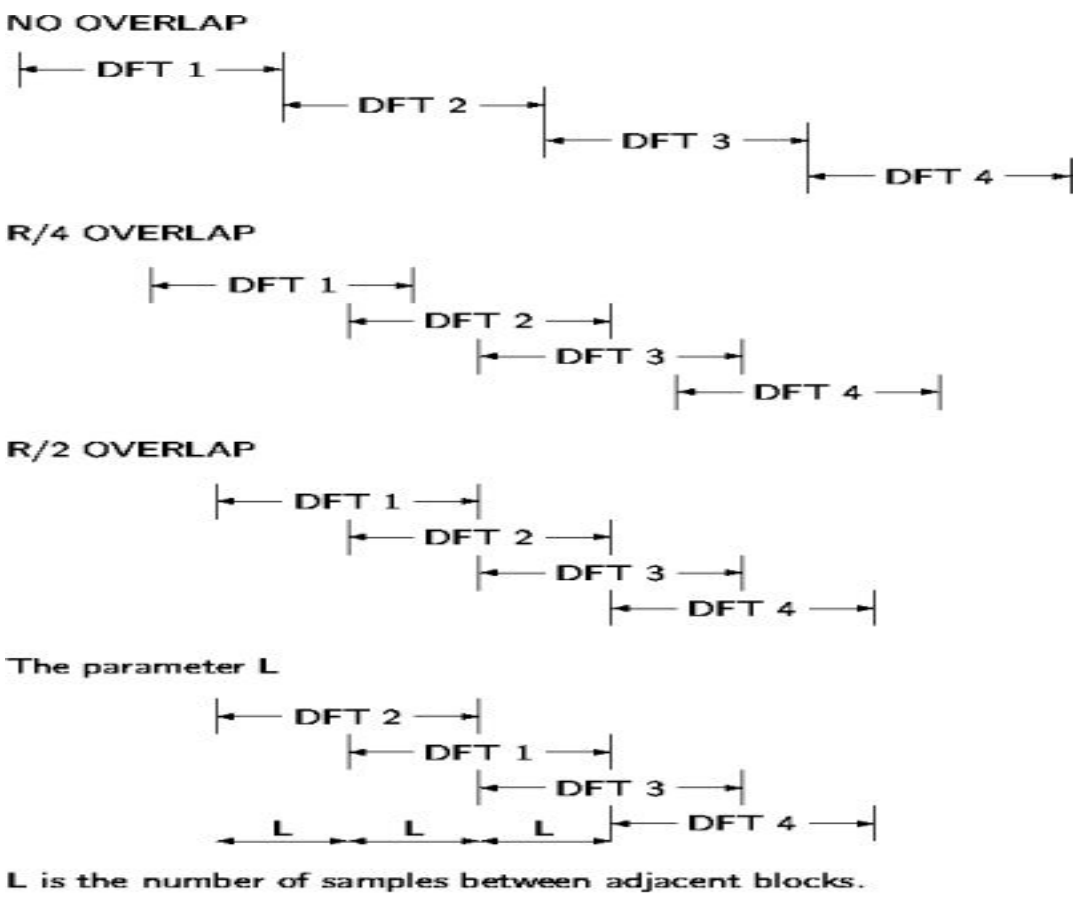
Example of Fourier spectral analysis



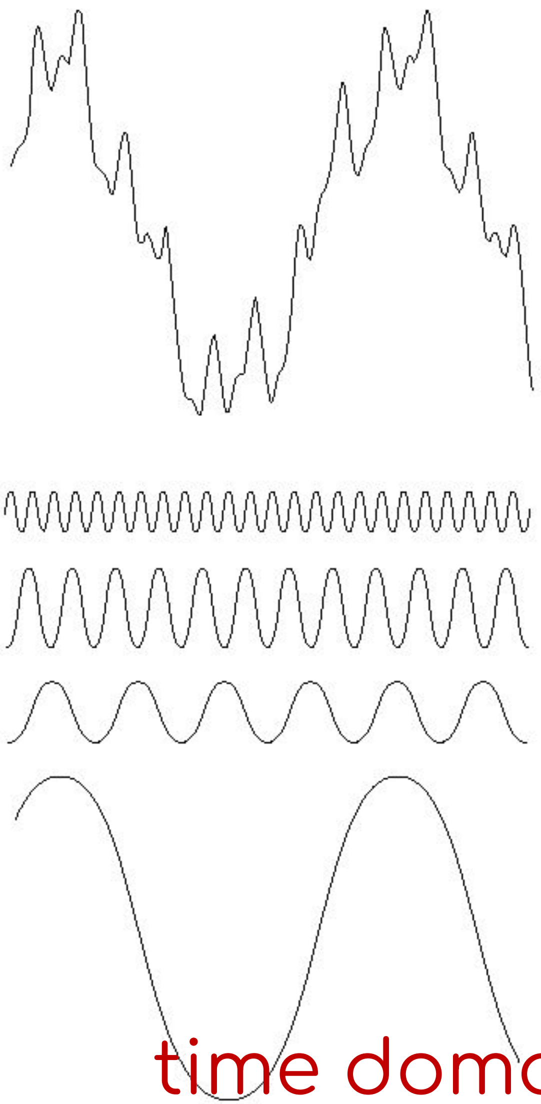
Short-Time Fourier analysis



Short-Time Fourier Analysis
Overlap / no overlap



Fourier series (Fourierov rad)



Fourier series:

Všetky periodické signály môžeme považovať za váženú sumu sínusových signálov s rôznymi frekvenciami

time domain ⇔ frequency domain

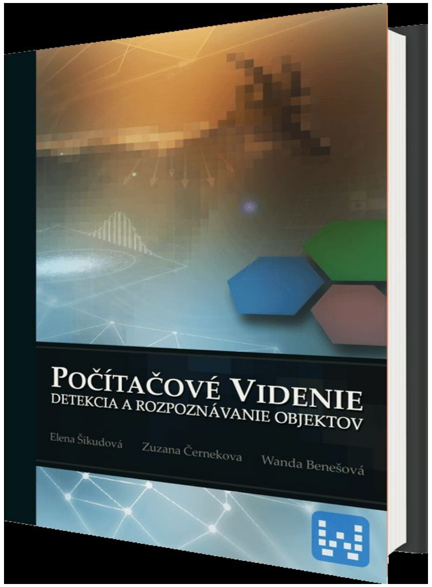
časová doména ⇔ frekvenčná doména

Fourier transform

Fourier transform in this book -> in Slovak Language

Fourierova transformácia

https://vgg.fiit.stuba.sk/kniha/



Fourierov rad (Fourier series)

Fourierov rad (*Fourier series*)

Máme periodický signál, ktorý matematicky môžeme zapísať ako funkciu f (x) s periódou T > 0.

Túto periodickú funkciu môžeme interpretovať ako jej rozvoj na nekonečný rad sínusových a kosínusových funkcií.

We have a periodic signal that can be written mathematically as a function f (x) with a period T> 0.

This periodic function can be interpreted as an expansion into an infinite series of sine and cosine functions.

Fourierov rad je teda rozvoj periodickej funkcie f (x) na nekonečnú sumu sínusov a kosínusov.

A Fourier series is an expansion of a periodic_function f(x) in terms of an infinite sum of sines and cosines.

A sum of sines and cosines

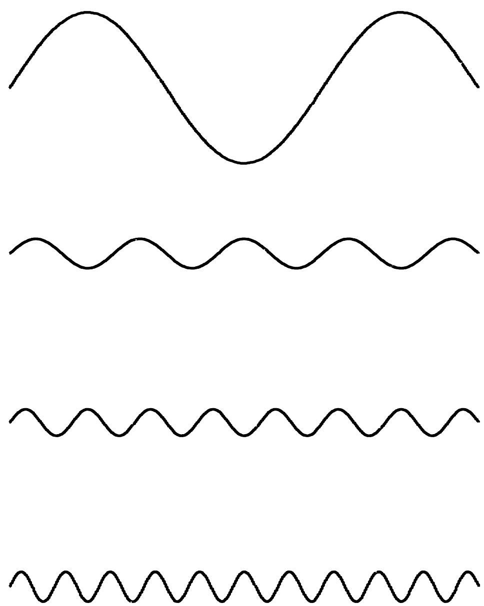
=

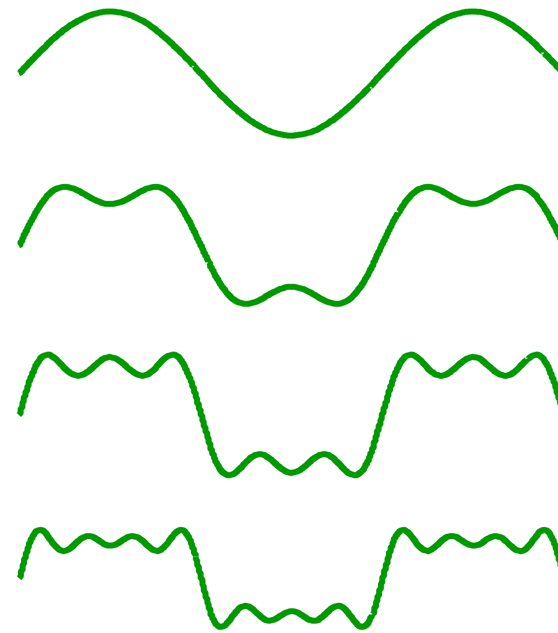
$3 \sin(x)$ A

$+ 1 \sin(3x)$ B

$+ 0.8 \sin(5x)$ C

$+ 0.4 \sin(7x)$ D





A+B

A+B+C

A+B+C+D

Fourier series (Fourierov rad)

Na intervale [−π,π] je Fourierov rad funkcie f (x) definovaný ako:

over interval [−π,π] , the Fourier series of a function is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Fourierov rad môžeme teda rozpísať na jednotlivé členy: $f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \ldots$

We can therefore break down the Fourier series into individual members:

Intuitívne si môžeme predstaviť, že akýkoľvek spojitý periodický signál aproximujeme váhovaným súčtom spojitých sínusových a kosínusových funkcií s rôznou uhlovou frekvenciou.

Intuitively, we can imagine that any continuous periodic signal is approximated by a weighted sum of continuous sine and cosine functions with different angular frequency.

Fourier series (Fourierov rad) pre párne a nepárne funkcie

Ak by sme použili iba kosínusové funkcie, tak by sme mohli aproximovať iba párne funkcie.

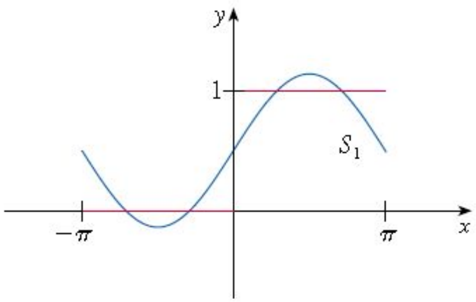
(Párne funkcie sú funkcie, pre ktoré platíf (−x) = f (x) na intervale [−π, π], teda sú symetrické okolo y-novej osi podobne ako funkcia kosínus).

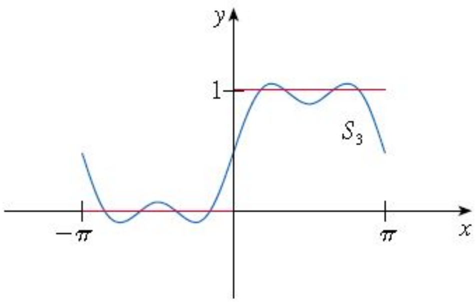
Ak by sme použili iba sínusové funkcie, tak by sme mohli aproximovať iba nepárne funkcie.

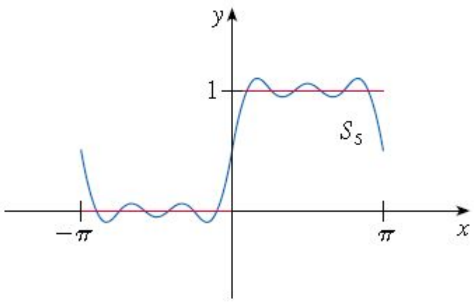
(Nepárne funkcie sú funkcie, pre ktoré platí, že: f (−x) = −f (x) na intervale [−π,π], teda sú symetrické okolo nuly podobne ako sínus.)

Pre aproximáciu ľubovoľného periodického signálu Fourierovym radom teda potrebujeme kombináciu sínusových aj kosínusových funkcií.

Harmonic analysis (harmonická analýza)







Harmonická analýza.

Rozklad periodického signálu na Fourierov rad sa často nazýva aj harmonická analýza.

Takéto označenie sa často používa v technickej praxi.

The decomposition of a periodic signal into a Fourier series is often called harmonic analysis.

Such a term is often used in technical practice.

Harmonic analysis (harmonická analýza)

Terminológia, ktorú potom používame je nasledovná:

Nultý koeficiet nazývame „nultou harmonickou." Je to priemerná hodnota signálu, označujeme ju tiež z angličtiny ako DC (direct current) komponent.

Prvý koeficient je „základná frekvencia ." Ostatné koeficienty sú tzv. „vyššie harmonické."

Ďalšie koeficienty prislúchajú násobným frekvenciám základnej frekvencie a nazývame ich **druhá harmonická, tretia harmonická** . . .

$$f(x) = \underbrace{a_0}_{\text{DC}} + \underbrace{a_1 \cos(x) + b_1 \sin(x)}_{\text{základná harmonická}} + \underbrace{a_2 \cos(2x) + b_2 \sin(2x)}_{\text{2. harmonická}} +$$
$$+ \underbrace{a_3 \cos(3x) + b_3 \sin(3x)}_{\text{3. harmonická}} \ldots$$

The terminology we then use is as follows:

We call the zero coefficient "zero harmonic." It is the average value of the signal, we also call it from English as a DC (direct current) component. The first coefficient is the "fundamental frequency." The other coefficients are the so-called "Higher harmonics" are integer multiples of the fundamental frequency

Continuous Fourier Transform

Spojitá Fourierova transformácia (*Continuous Fourier transform*)

Fourierovou transformáciou spojitej funkcie *f* (*x*) jednej premennej nazveme funkciu *F*(*u*)

Fourier transform of continuous function f (x) is the function F (u)

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \mathrm{d}x,$$

inverzná spojitá Fourierova transformácia je potom definovaná:

The inverse continuous Fourier transform is then defined:

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} \mathrm{d}u$$

$$j = \sqrt{-1},$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$2\pi f = \omega,$$

ω je uhlová frekvencia v radiánoch [rad] a f je frekvencia v [Hz]

ω is the angular frequency (radial frequency) in radians [rad] and f is the frequency in [Hz]..

Fourierova transformácia - v obore komplexných čísiel

$$j = \sqrt{-1},$$

F(u) je z oboru komplexných čísiel

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = |F(u)|\, e^{i\varphi(u)}$$

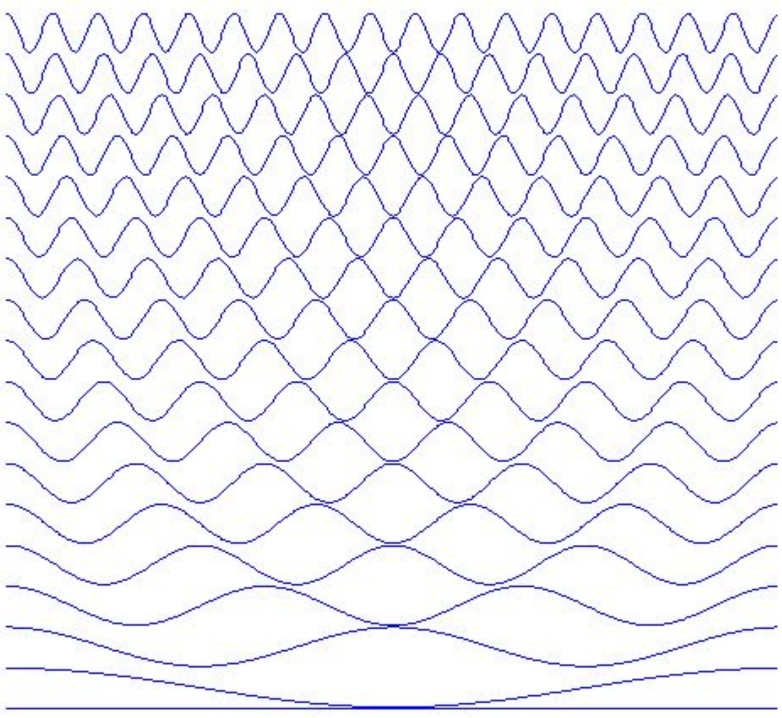
magnitúdové spectrum
amplitude spectrum

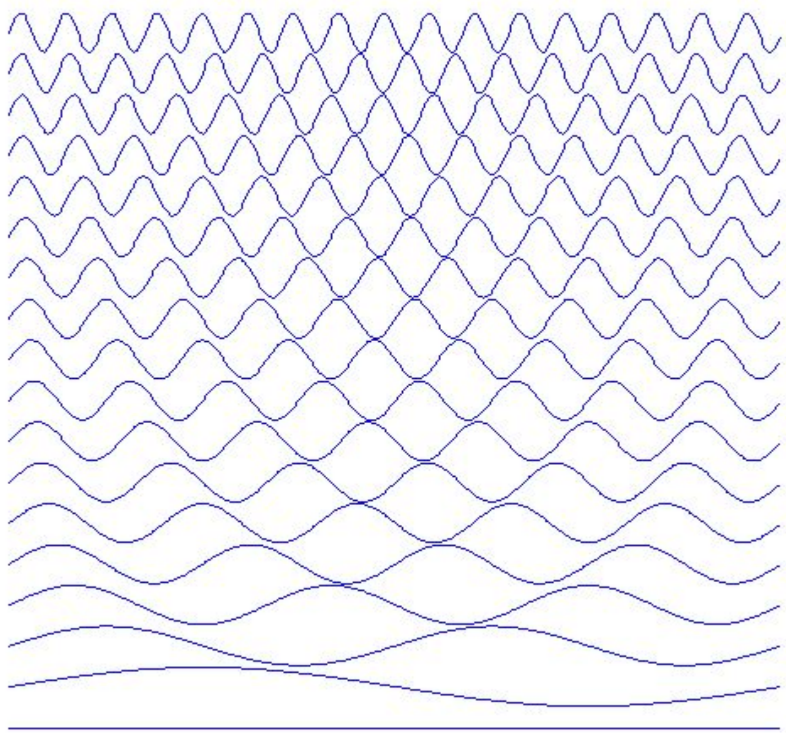
$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{\frac{1}{2}},$$

fázové spectrum
phase spectrum

$$\varphi(u) = \arctan\left[\frac{I(u)}{R(u)}\right]$$

Spojité bázové funkcie Fourier Transf.
(Continuous basis functions FT)





Ukážka kosínusových C0-C17 a sínusových funkcií S0-S17 s rôznou uhlovou frekvenciou.

Example of cosine C0-C17 and sine functions S0-S17 with different angular frequency.

Diskrétna Fourierova Transformácia (DFT)
Discrete Fourier Transform (DFT)

Ekvivalentom spojitej Fourierovej transformácie pre spojité signály je

Diskrétna Fourierova transformácia (DFT) pre diskrétne signály.

An equivalent of a continuous Fourier transform for continuous signals is Discrete Fourier Transform (DFT) for discrete signals.

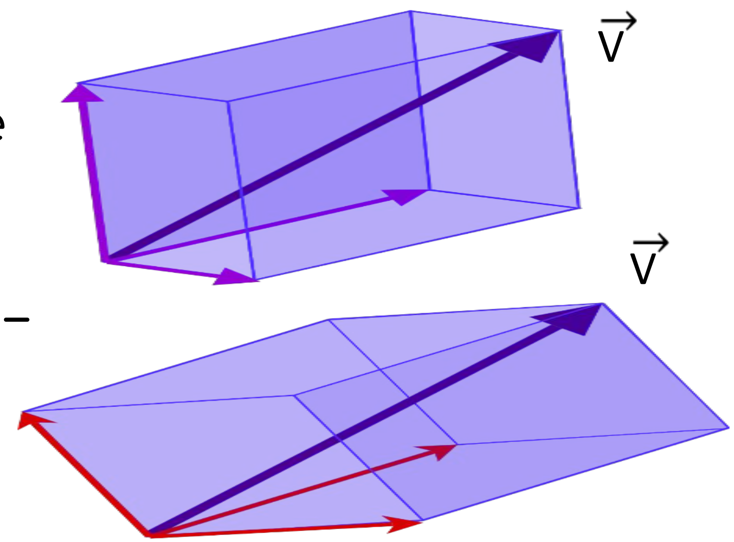
Linear transformation

Linear transformation – intuitive approach - example

Vector $\vec{v}$ -> linear combination of basis set of vectors

basis set of vectors (in example – purple vectors)

new basis set definition (in the example – red vectors)



Basis B:
- complete orthogonal set of vectors
- vectors are linearly independent

Vector $\vec{v}$ represented by two different bases (purple and red arrows).

Linear transformation - change of basis

Linear transformation – intuitive approach

a set vectors B in a vector space V is called a basis, if

- every element of V may be written in a unique way as a (finite) linear combination of elements of B.

- The coefficients of this linear combination are referred to as components or coordinates on B of the vector.

- The elements of a basis are called basis vectors.

Diskrétna lineárna transformácia (*The discrete linear transformation*)

Diskrétna lineárna transformácia v prípade 1-dimenzionálneho signálu priraďuje postupnosti hodnôt číslicového signálu (súradniciam v pôvodnom lineárnom priestore) postupnosť spektrálnych koeficientov (súradnice v novom lineárnom priestore).

Transformácia z jedného lineárneho priestoru => do iného lineárneho priestoru.

Lineárny priestor je definovaný bázou – bázovými funkciami.

The discrete linear transformation in the case of a 1-dimensional signal assigns a sequence of spectral coefficients (coordinates in the new linear space) to the sequence of digital signal values (coordinates in the original linear space).

Transformation from one linear space => to another linear space.

Linear space is defined by a basis - basis functions.

Diskrétna Fourierova Transformácia (DFT) - špeciálny prípad lineárnej transformácie

- Diskrétny signál $F_s(n)$ pre $n = 0, 1 \ldots N - 1$, získaný digitalizáciou 1D signálu na konečnom intervale s počtom vzoriek N, môžeme reprezentovať ako jeden vektor v N-rozmernom komplexnom priestore C_N s ortonormálnou bázou Φ_0
- Pomocou lineárnej transformácie (napr. Fourierovej) môžeme súradnice každého takéhoto vektora transformovať do iných súradníc v N-rozmernom priestore V2 s novou bázou Φ.
- Spätne môžeme vyjadriť pôvodný číslicový signál ako sumu N vážených bázových funkcií bázy Φ. (až na násobnú konštantu). Tieto váhové koeficienty sú komlexné spektrálne koeficienty získané výpočtom danej lineárnej transformácie.

- *The discrete signal $F_s(n)$ for $n = 0, 1 \ldots N - 1$, obtained by digitizing the 1D signal on a finite interval with the number of samples N, can be represented as a single vector in the N-dimensional complex space C_N with orthonormal basis Φ_0.*
- *Using a linear transformation (eg. Fourier), we can transform the coordinates of each such vector into other coordinates in the N-dimensional space V2 with a new base Φ.*
- *Then, we can also express the original digital signal as the sum of N weighted base functions of the base Φ(up to multiple constant). These weighting coefficients are complex spectral coefficients obtained by calculating this given linear transformation.*

Linear transformations as matrix vector products

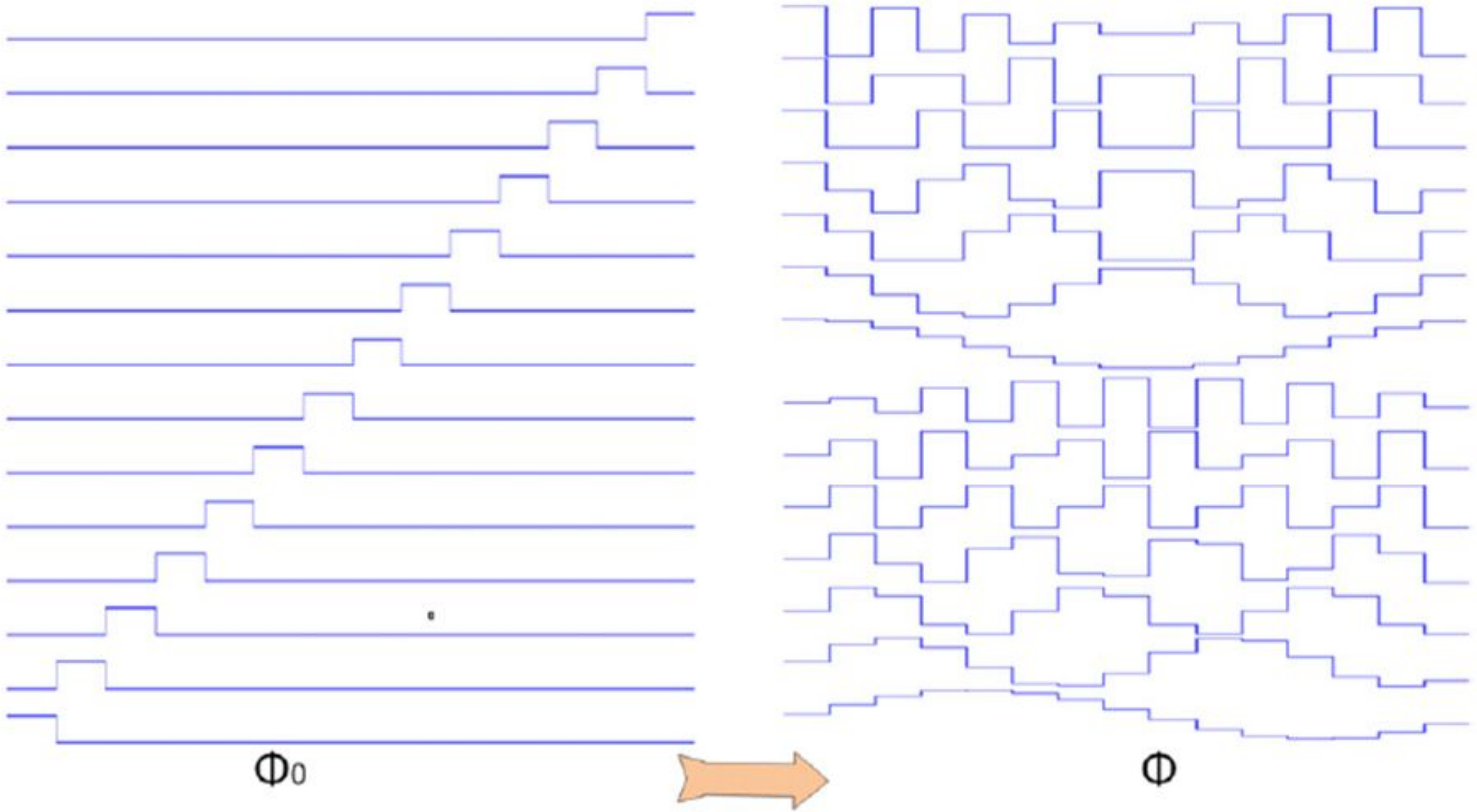
In linear algebra, linear transformations can be represented by matrices.

If T is a linear transformation mapping $\mathbb{R}^n to \mathbb{R}^m$
and x is a column vector with n entries, then

$$T(x) = A.x$$

for some *m x n* matrix A, called the transformation matrix of T.

Discrete Base functions (Diskrétne bázové funkcie)- príklady



Ilustratívny príklad bázových funkcií Φ_0 a Φ

Diskrétna Fourierova Transformácia (DFT) - Definition

Diskrétna Fourierova Transformácia (DFT)

1-D, Discrete case

Fourier Tr.:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u = 0,\ldots,M\text{-}1$$

Inv. Fourier Tr.:

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \qquad x = 0,\ldots,M\text{-}1$$

Diskrétna Fourierova Transformácia (DFT)

Euler's formula links the trigonometric functions to the complex exponential function:

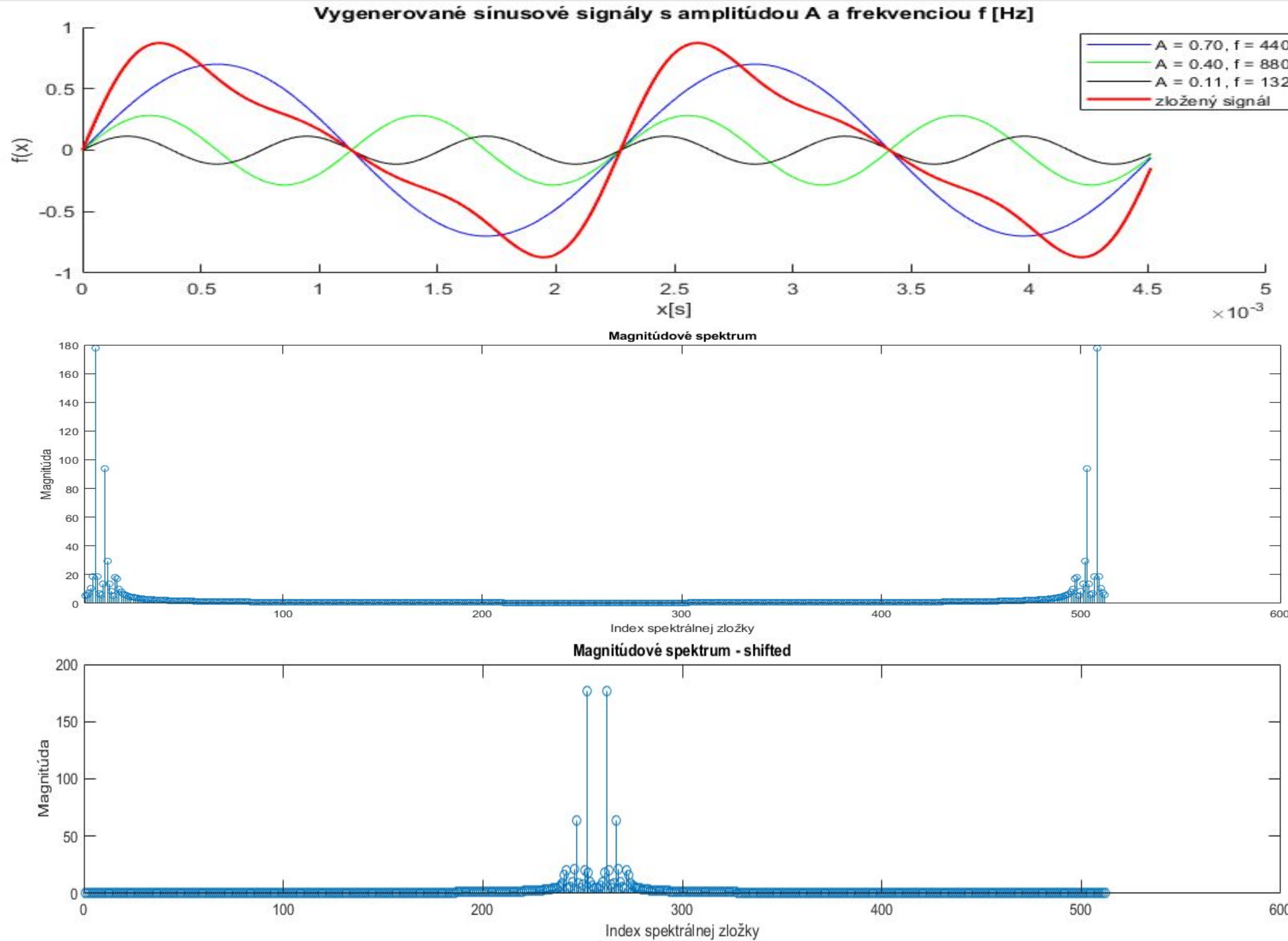
$e^{-jx}=cos(x)-j.sin(x)$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[cos(\tfrac{2\pi}{M}ux) - j.sin(\tfrac{2\pi}{M}ux)\right]$$

u = 0,...,M-1

M	number of samples
x	current sample
u	current spectral coefficient
f(x)	input function
F(u)	Fourier spectrum

DFT Example



Vygenerované sínusové signály s amplitúdou A a frekvenciou f [Hz]
A = 0.70, f = 440
A = 0.40, f = 880
A = 0.11, f = 1320
zložený signál
f(x)
x[s]
×10^-3
Magnitúdové spektrum
Magnitúda
Index spektrálnej zložky
Magnitúdové spektrum - shifted
Magnitúda
Index spektrálnej zložky

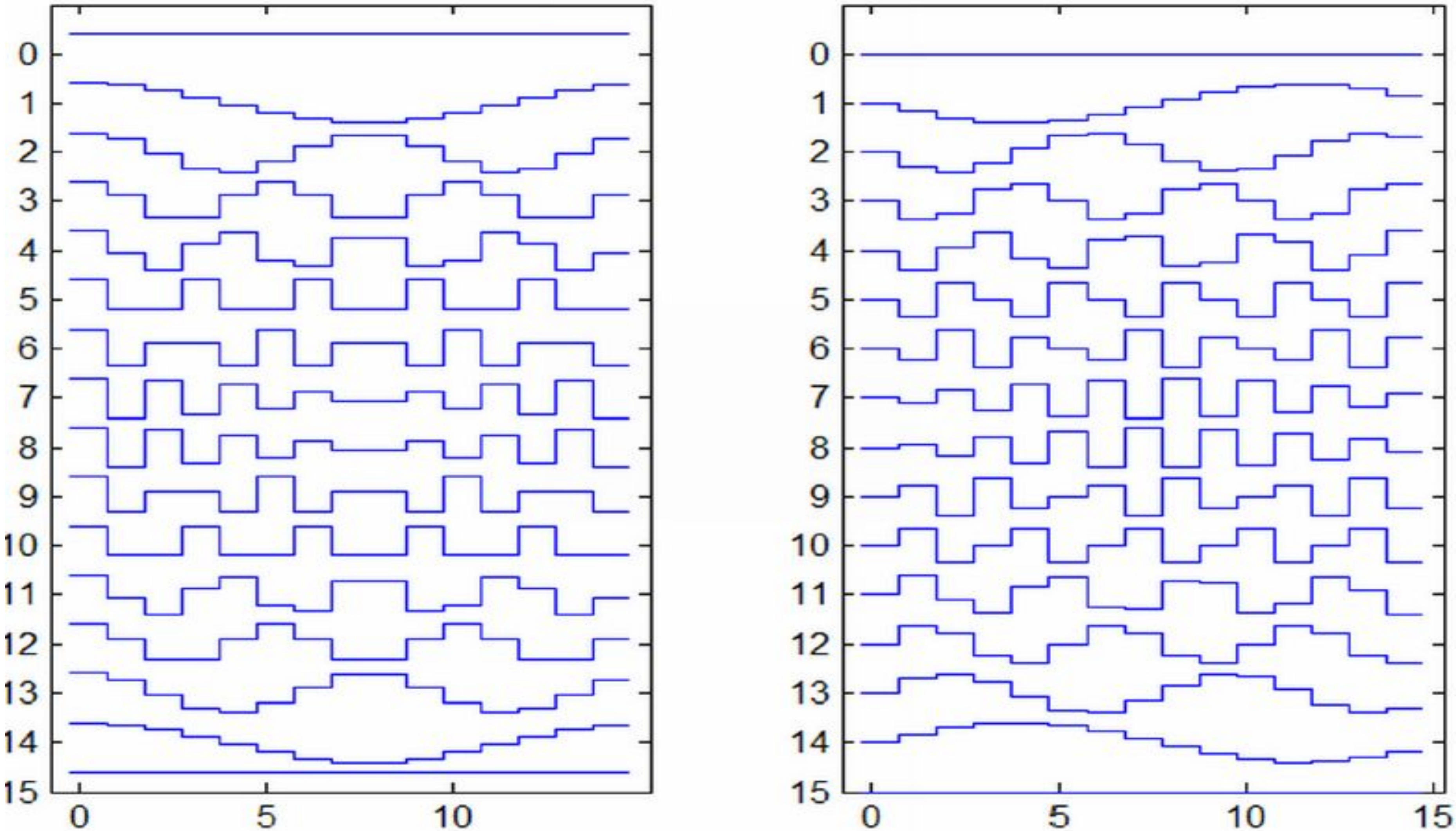
Diskrétna Fourierova Transformácia (DFT) magnitúda a fáza

Fourierova transformácia je definovaná nad oborom komplexných čísiel a spektrálne koeficienty sú komplexné čísla, pozostávajú z reálnej a imaginárnej zložky.

V praxi sa častejšie používa prepočet reálnej a imaginárnej zložky na magnitúdu a fázu komplexného čísla, teda máme spektrum magnitúdové a fázové.

The Fourier transform is defined over the domain of complex numbers and the spectral coefficients are complex numbers, consisting of a real and an imaginary component.
In practice, the conversion of real and imaginary is more often used components for magnitude and phase of a complex number, so we have a magnitude and phase spectrum.

DFT base functions - example



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