CptS 442/542 (Computer Graphics)
Unit 6: Vectors in Computer Graphics

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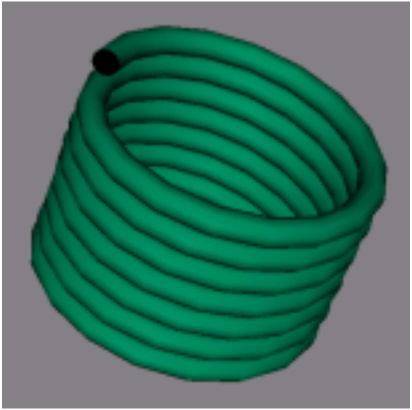
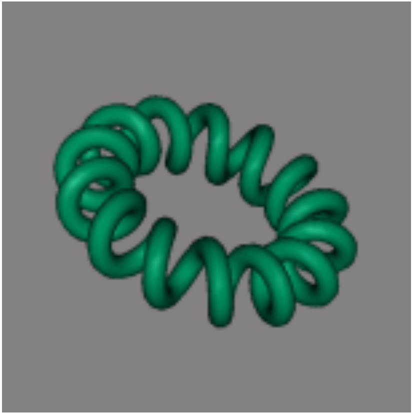
Motivation



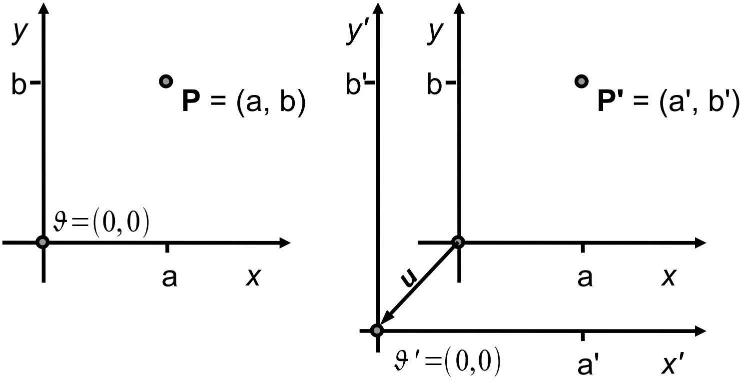


We Already Know These:

- scalars
- points
  - "n-tuples"
  - depend on origin of coordinate system
- lines
  - parametric vs. implicit vs. slope-intercept
- segments
- rays

Points

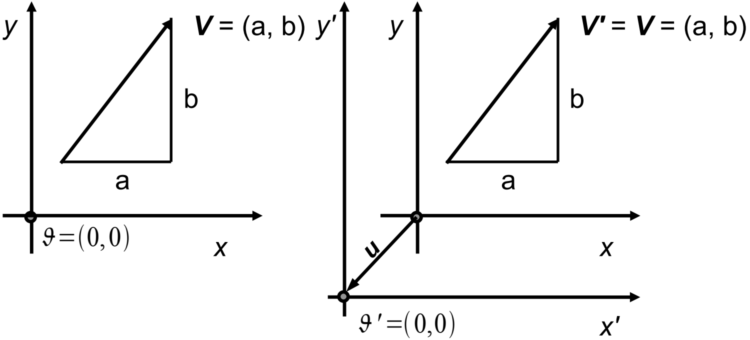
Points vary under an axis translation:



a' = a + u_x   b' = b + u_y

Vectors

Like points, they're an n-tuple, but they represent a displacement, not a position. Vectors and points may look alike, but they're not the same thing. They are invariant under translation.



Vector Representation

There are several ways to represent vectors:

- list or tuple: (a, b)
- basis unit vectors: $a\mathbf{i} + b\mathbf{j}$ or $a\widehat{x} + b\widehat{y}$
- row vector: $\begin{bmatrix} a & b \end{bmatrix}$
- column vector: $\begin{bmatrix} a \\ b \end{bmatrix}$

Many (classic, even) pre-1985 graphics publications use row vectors. Most now use column vectors.

Vector Dimensionality

- 2D: $\begin{bmatrix} a \\ b \end{bmatrix}$
- 3D: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- 4D: $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

These are the dimensionalities we'll use in this class.

Homogeneous Notation

It is convenient to distinguish points from vectors by adding an additional dimension, which we call "w"...

type	point	vector
2DH	$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$	$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$
3DH	$\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$	$\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$

"w" = 0 or 1

OpenGL uses 3DH (float) coordinates internally for vertices.

Vector Operations (2D)

- vector addition:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

- vector subtraction:

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$$

- vector-scalar multiplication:

$$s \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$$

- vector magnitude:

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

The extensions to 3D are obvious.

Vector Normalization

The normalization of a vector $\mathbf{v}$ is:

$$\widehat{\mathbf{v}} \equiv \frac{\mathbf{v}}{|\mathbf{v}|}$$

- We use the "$\widehat{\ }$" to as both operator (as above) and as part of the vector name (e.g. $\widehat{\mathbf{x}}$).
- What is the magnitude of $\frac{\mathbf{v}}{|\mathbf{v}|}$?
- Are there any special conditions to watch for?

This is called "normalization," not to be confused with the term "normal" (perpendicular to a surface) in geometry. Normal vectors are *usually* normalized, but not all normalized vectors are normal vectors. (Ugh!)

Dot Products

In 2D:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} \equiv u_x v_x + u_y v_y$$

In N dimensions:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{N} u_i v_i$$

In matrix notation:

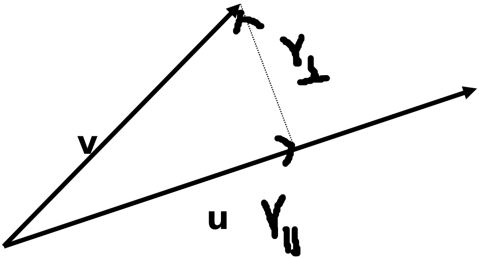
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t \mathbf{v}$$

This is also known as an *inner product*.

Properties of the Dot Product

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	The dot product is *commutative*.
$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$	The dot product is *distributive*.
$(s\mathbf{u}) \cdot \mathbf{v} = s\,(\mathbf{u} \cdot \mathbf{v})$	The dot product is *associative*.
$\lvert\mathbf{u}\rvert^2 = \mathbf{u} \cdot \mathbf{u}$	The dot product is related to a vector's magnitude.
$\mathbf{u} \cdot \mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert \cos\theta$	The dot product has geometric meaning. (We'll use this a lot.)

Application: Projection

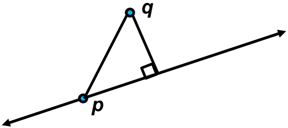


It is often convenient to express a vector **v** as

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

(read as "v par plus v perp") wrt another vector u.
Given **v** and **u**, how can we compute these?
(Hint: Find $\mathbf{v}_{\parallel}$ first.)

Application: Distance from a Line to a Point I



Given a line of the (implicit) form:

$$ax + by + c = 0$$

Its normal $\widehat{\mathbf{n}}$ is

$$\widehat{\mathbf{n}} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}$$

And the projection of a vector $\mathbf{q} - \mathbf{p}$ for *any* point $\mathbf{p}$ on the line gives us the answer:

$$D = (\mathbf{q} - \mathbf{p}) \cdot \widehat{\mathbf{n}}$$

So, how can we get (any) $\mathbf{p}$?

Application: Distance from a Line to a Point II

Answer: Remember that

$$P = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

is on the line, so

$$aP_x + bP_y + c = 0$$

but this is just

$$\left(\sqrt{a^2 + b^2}\,\widehat{\mathbf{n}}\right) \cdot \mathbf{P} + c = 0$$

so we can solve for

$$\widehat{\mathbf{n}} \cdot \mathbf{P} = -\frac{c}{\sqrt{a^2 + b^2}}$$

Hence,

$$\begin{aligned} D(q) &= (\mathbf{q} - \mathbf{p}) \cdot \widehat{\mathbf{n}} \\ &= \mathbf{q} \cdot \widehat{\mathbf{n}} - \mathbf{p} \cdot \widehat{\mathbf{n}} \\ &= \mathbf{q} \cdot \widehat{\mathbf{n}} + \frac{c}{\sqrt{a^2 + b^2}} \end{aligned}$$

Planes (Review, I Hope)

- Implicit form:

$$ax + by + cz + d = 0$$

- Parametric form:

$$\mathbf{P} = \mathbf{C} + s\widehat{\mathbf{s}} + t\widehat{\mathbf{t}}$$

- Point-Normal form:

$$\mathbf{N} \cdot (\mathbf{P} - \mathbf{P}_0) = 0$$

Example: Using Linear Algebra

Suppose you're given the equations of three planes:

$$
\begin{aligned}
a_0 x + b_0 y + c_0 z + d_0 &= 0 \\
a_1 x + b_1 y + c_1 z + d_1 &= 0 \\
a_2 x + b_2 y + c_2 z + d_2 &= 0
\end{aligned}
$$

- How can they intersect?
- How would you find their intersection?
- What can go wrong?

(This was a real problem I had to solve once!)

Cross Products

For 3D vectors (only), the cross product is defined as:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

and best memorized with the "pseudodeterminant":

$$\begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

Q: Is there a matrix form for this?

A: No, but there's a *tensor* one.

Properties of Cross Products

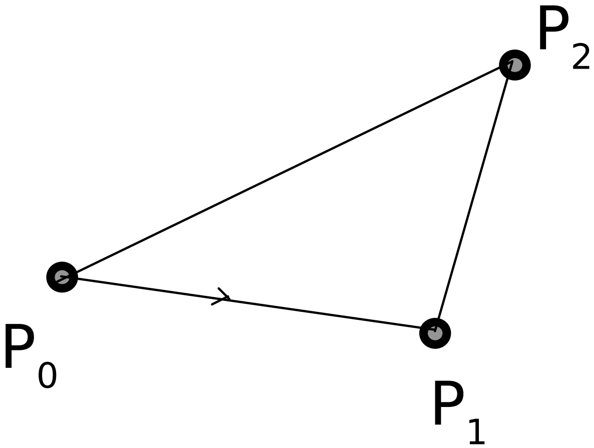
$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$	The cross product is *anti-commutative*.
$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$	The cross product is *distributive*.
$(s\mathbf{u}) \times \mathbf{v} = s(\mathbf{u} \times \mathbf{v})$	The cross product is *associative*.

And the following geometric properties:

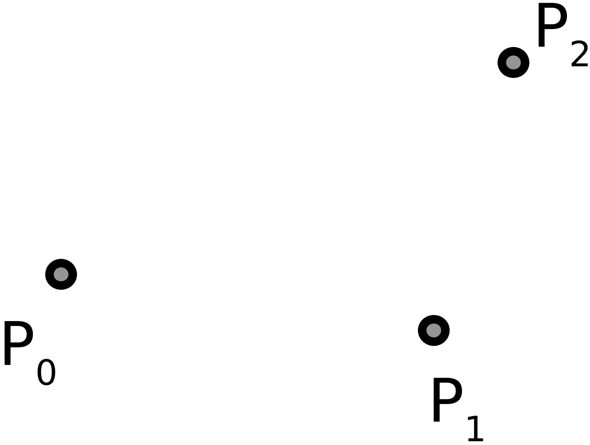
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \times \mathbf{v}$ follows the *right-hand rule*.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where $0 \leq \theta \leq \pi$

~~Can we extend this to a convex polygon? An arbitrary polygon?~~

Application: Finding the Normal of a Triangle



Application: Finding the Equation of a Plane



Application: Finding the Area of a Triangle