# CptS 442/542 (Computer Graphics)
# Unit 9: Extrusion

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# Motivation

- We already have a track design, but it's not 3-dimensional.
- When we go to 1st person mode, a polyline track won't be very impressive.
- Rollercoasters run on tracks with ties to keep rails from separating.

# Background: Parametric Lines

- We've already talked about the parametric form of a line: $\mathbf{P}(u) = (1-u)\mathbf{P}_0 + u\mathbf{P}_1$
- Recall that there are no restrictions on dimensionality for this.
- A more general parametric line form (in 3D) is:

$$\mathbf{P}(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix}$$

where $x(u)$, $y(u)$, and $z(u)$ are arbitrary (scalar) functions of $u$.

# Try These Parameterizations

What lines are given by:

$$\mathbf{P}(u) = \begin{bmatrix} cos(2\pi u) \\ sin(2\pi u) \end{bmatrix} \qquad \mathbf{P}(u) = \begin{bmatrix} u \\ u^2 \end{bmatrix}$$

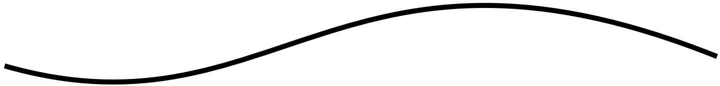
$$\mathbf{P}(u) = \begin{bmatrix} cos(2\pi u) \\ sin(2\pi u) \\ u \end{bmatrix} \qquad \mathbf{P}(u) = \begin{bmatrix} cos(4\pi u) \\ sin(2\pi u) \\ u \end{bmatrix}$$

In 2D: https://graphsketch.com/parametric.php

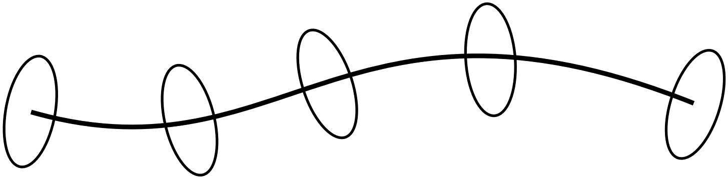
In 3D: http://www.math.uri.edu/~bkaskosz/flashmo/as3/motion3d/motion3d.html

# The Problem I

Suppose we're given a parametric curve $\mathbf{P}(u)$ in 3D:

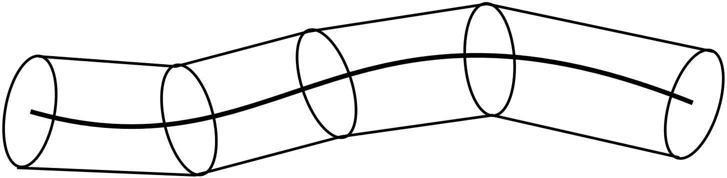


with the goal of "extruding" a circle along it...



# The Problem II

...to form a polyhedral “tube”:



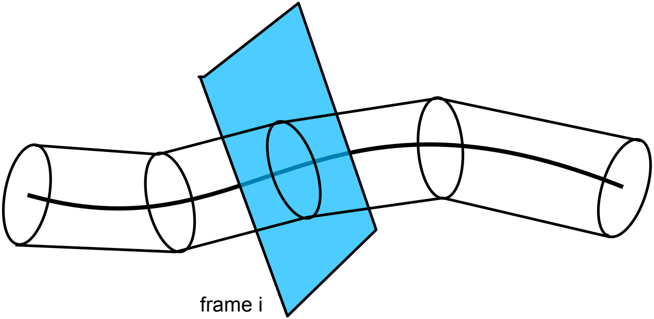
Implementing this in OpenGL will mean that:

- The circles are polygons.
- The sides of the tube are triangles.

How shall we approach this problem? Piece-by-piece.

# Frames

We know how to draw a polygon in a plane, so come up with a plane for each circle. Call it a “frame”:




How shall we represent this plane?

# Parametric Form

We have

$$\mathbf{Q} = \mathbf{P} + U\widehat{\mathbf{U}} + V\widehat{\mathbf{V}}$$

- $\mathbf{P}$ is easy: It's just $\mathbf{P}(u)$, the point along our curve.
- We get $u$ and $v$ from our planar circle formula:

$$\begin{aligned} U &= R\cos\theta \\ V &= R\sin\theta \end{aligned}$$

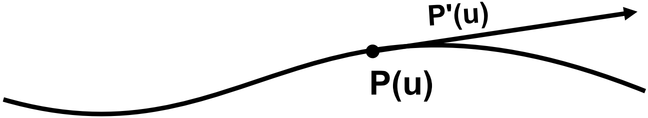
- $R$ is the (given) radius of the cross section.
- $0 \leq \theta \leq 2\pi$, sampled (`nI`, as we've done all along)

Q: But how can we find **U** and **V**?

A: Frenet coordinate frames

# The Tangent Vector

Given any curve $\mathbf{P}(u)$, we can always compute a tangent to the curve at any point by computing the derivative with respect to u, $\mathbf{P}'(u)$. This is a vector (why?).




- $|\mathbf{P}'(u)|$ is not important (yet), but the direction of $\mathbf{P}'(u)$ is. The tangent is:
$$\widehat{\mathbf{T}}(u) = \widehat{\mathbf{P}}'(u)$$
- We will make $\widehat{\mathbf{T}}(u)$ the normal (call it $\widehat{\mathbf{W}}$ of our frame)
- **u** and **v** must be perpendicular to it.

# Computing The Tangent Vector

There are two ways to compute $\widehat{\mathbf{T}}(u)$:

- Analytically:
  If we have $\mathbf{P}(u)$, use calculus to provide $\mathbf{P}'(\mathrm{u})$ as a separate function. Normalize the result to get $\widehat{\mathbf{T}}(u)$
- Numerically:
  (This is cheating, but ...)

$$\widehat{\mathbf{T}}(u) \approx \widehat{\mathbf{P}(u+h) - \mathbf{P}(u-h)}$$

for $h$ “small, but not too small”.

# Finding $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$

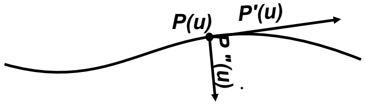
There are an infinite number of vectors perpendicular to T.

- Frenet's idea:
  - Find some vector $\mathbf{S}$ which is not parallel to $\mathbf{T}$.
  - $\widehat{\mathbf{W}} = \widehat{\mathbf{T}}$
  - $\widehat{\mathbf{U}} = \widehat{\mathbf{S} \times \mathbf{T}}$
  - $\widehat{\mathbf{V}} = \widehat{\mathbf{W}} \times \widehat{\mathbf{U}}$
    - No normalization required: Why?
- Candidates for $\mathbf{S}$:
  - any coordinate axis
    This can lead to problems.
  - the "acceleration" vector
  - a user-specified "never parallel" vector

$\mathbf{P}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{V}}$, and $\widehat{\mathbf{W}}$ – all of which depend on $u$ – form a parameterized *coordinate frame*.

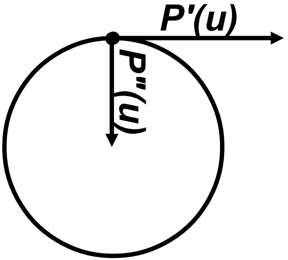
# The "Acceleration" Vector

(Repeat: This is *not* what we will use for PA06.)
$\mathbf{P}''(u)$ is the second derivative of $\mathbf{P}(u)$:




If $\mathbf{P}(u)$ is smooth (enough), so is $\mathbf{P}''(u)$.

Recall physics: If $u$ were time (which it isn't here), what is the second derivative of position related to?

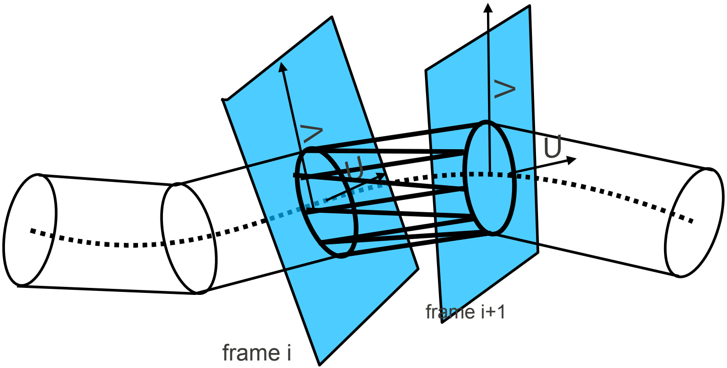


# The "Never Parallel" Vector

- Specified by the user, this is a vector that is guaranteed to never be parallel to the tangent.
- What happens to $\widehat{\mathbf{U}}$ if it is?
- Usually set when the curve is instantiated.

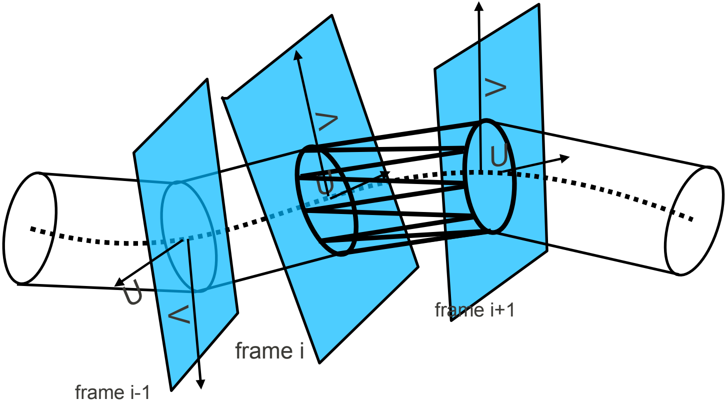
# Tube Tesselation

We can now draw tessellate these “tubes” into (2D) Meshes



# ...or Not Tube

A problem arises with changes in acceleration...



# Using Arbitrary 3D Curves

We can buld coordinate frames with *any* 3D curve, as long as:

- The curve is differentiable for any $u$.
- The magnitude of the first derivative doesn't vanish.
- If we're using an acceleration vector, the magnitude of the second derivative doesn't vanish.

# Example: A Toroidal Spiral

Here's a toroidal spiral:

$$\mathbf{P}(u) = \begin{bmatrix} (a\sin(2\pi ct) + b)\cos(2\pi t) \\ (a\sin(2\pi ct) + b)\sin(2\pi t) \\ a\cos(2\pi ct) \end{bmatrix}$$

What is its derivative?