HW-12

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Problem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x, y) = x^2 - 2xy + y^2$$

Prove f is convex.

Proof. Notice

$$\nabla f(x, y) = \begin{bmatrix} 2x - 2y & 2y - 2x \end{bmatrix}^T$$

and

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Furthermore,

$$\det(\nabla^2 f(x, y) - \lambda I) = \begin{vmatrix} 2 - \lambda & -2 \\ -2 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

Thus, the eigenvalues of $\nabla^2 f(x, y)$ are $4, 0$, which are both non-negative. Therefore, $\nabla^2 f(x, y)$ is positive semi-definite which means $f(x, y)$ is convex.

$\square$

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