HW-4

DREW MORRIS

Problem 1. *Illustrate Theorem* 5.2 *on the problem from Exercise* 2.1.

Proof. Here is the primal problem from Exercise 2.1.

$$\begin{array}{|ccccccccccc|}
\hline
\zeta & = & 0 & + & 6x_0 & + & 8x_1 & + & 5x_2 & + & 9x_3 \\
\hline
u_0 & = & 5 & - & 2x_0 & - & x_1 & - & x_2 & - & 3x_3 \\
u_1 & = & 3 & - & x_0 & - & 3x_1 & - & x_2 & - & 2x_3 \\
\hline
\end{array}$$

Now we will analyze the dual problem.

$$\begin{array}{|ccccccc|}
\hline
-\xi & = & 0 & - & 5y_0 & - & 3y_1 \\
\hline
v_0 & = & -6 & + & 2y_0 & + & y_1 \\
v_1 & = & -8 & + & y_0 & + & 3y_1 \\
v_2 & = & -5 & + & y_0 & + & y_1 \\
v_3 & = & -9 & + & 3y_0 & + & 2y_1 \\
\hline
\end{array} \quad \rightarrow \quad y_0 = 3 \quad \rightarrow$$

$$\begin{array}{|ccccccc|}
\hline
-\xi & = & -15 & + & \frac{5}{2}v_0 & - & \frac{1}{2}y_1 \\
\hline
y_0 & = & 3 & - & \frac{1}{2}v_0 & - & \frac{1}{2}y_1 \\
v_1 & = & -5 & - & \frac{1}{2}v_0 & + & \frac{5}{2}y_1 \\
v_2 & = & -2 & - & \frac{1}{2}v_0 & + & \frac{1}{2}y_1 \\
v_3 & = & 0 & - & \frac{3}{2}v_0 & + & \frac{1}{2}y_1 \\
\hline
\end{array} \quad \rightarrow \quad y_1 = 0 \quad \rightarrow$$

$$\begin{array}{|ccccccc|}
\hline
-\xi & = & -15 & + & v_0 & + & v_3 \\
\hline
y_0 & = & 3 & - & 2v_0 & + & v_3 \\
v_1 & = & -5 & + & 7v_0 & - & 5v_3 \\
v_2 & = & -2 & + & v_0 & - & v_3 \\
y_1 & = & 0 & + & 3v_0 & + & 2v_3 \\
\hline
\end{array} \quad \rightarrow \quad \xi = 15$$

This is an optimal solution of the dual problem, thus $\zeta = 15$ is also an optimal solution for the primal problem. ∎

Date: October 2nd 2023.

Problem 2. *Consider the following linear programming problem.*

$$\begin{array}{|ccccccccccc|}\hline \zeta & = & 0 & + & 2x_0 & + & 8x_1 & - & x_2 & - & 2x_3 \\ \hline u_0 & = & 6 & - & 2x_0 & - & 3x_1 & + & 0x_2 & - & 6x_3 \\ u_1 & = & \frac{3}{2} & + & x_0 & - & 4x_1 & - & 3x_2 & + & 0x_3 \\ u_2 & = & 4 & - & 3x_0 & - & 2x_1 & + & 2x_2 & + & 4x_3 \\ \hline \end{array}$$

Suppose in solving this problem you arrive at the following dictionary.

$$\begin{array}{|ccccccccccc|}\hline \zeta & = & \frac{7}{2} & - & \frac{1}{4}u_0 & + & \frac{25}{4}x_1 & - & \frac{1}{2}u_2 & - & \frac{3}{2}x_3 \\ \hline x_0 & = & 3 & - & \frac{1}{2}u_0 & - & \frac{3}{2}x_1 & + & 0u_2 & - & 3x_3 \\ u_1 & = & 0 & + & \frac{5}{4}u_0 & - & \frac{13}{4}x_1 & - & \frac{3}{2}u_2 & + & \frac{27}{2}x_3 \\ u_2 & = & \frac{5}{2} & - & \frac{3}{4}u_0 & - & \frac{5}{4}x_1 & + & \frac{1}{2}u_2 & - & \frac{13}{2}x_3 \\ \hline \end{array}$$

Do the following.

(1) *Write the dual problem.*
(2) *Which variables are basic/non-basic in the given dictionary?*
(3) *Is the primal solution of the given dictionary optimal/degenerate?*
(4) *Write down the corresponding dual dictionary.*
(5) *Is the dual solution feasible?*
(6) *Is the current primal solution optimal?*
(7) *For the next primal pivot, which variable will enter/leave under the largest-coefficient rule and will the pivot be degenerate?*

(1) Here is the dual problem.

$$\begin{array}{|ccccccccc|}\hline -\xi & = & 0 & - & 6y_0 & - & \frac{3}{2}y_1 & - & 4y_2 \\ \hline v_0 & = & -2 & + & 2y_0 & - & y_1 & + & 3y_2 \\ v_1 & = & -8 & + & 3y_0 & + & 4y_1 & + & 2y_2 \\ v_2 & = & 1 & + & 0y_0 & + & 3y_1 & - & 2y_2 \\ v_3 & = & 2 & + & 6y_0 & + & 0y_1 & - & 4y_2 \\ \hline \end{array}$$

(2) x_0, x_1, x_2, x_3 are non-basic and u_0, u_1, u_2 are basic.

(3) The solution is $\zeta = \frac{7}{2}$ which is feasible but degenerate.

(4) Here is the corresponding dual dictionary.

$-\xi$	$=$	$-\frac{7}{2}$	$-$	$3y_0$	$+$	$0v_1$	$-$	$\frac{5}{2}v_2$
v_0	$=$	$\frac{1}{4}$	$+$	$\frac{1}{2}y_0$	$-$	$\frac{5}{4}v_1$	$+$	$\frac{3}{4}v_2$
y_1	$=$	$-\frac{25}{4}$	$+$	$\frac{3}{2}y_0$	$+$	$\frac{13}{4}v_1$	$+$	$\frac{5}{4}v_2$
v_2	$=$	$\frac{1}{2}$	$+$	$0y_0$	$+$	$\frac{3}{2}v_1$	$-$	$\frac{1}{2}v_2$
y_3	$=$	$\frac{3}{2}$	$+$	$3y_0$	$-$	$\frac{27}{2}v_1$	$+$	$\frac{13}{2}v_2$

(5) The solution is $-\xi = \frac{7}{2}$ and it is infeasible.

(6) The solution of $\zeta = \frac{7}{2}$ is sub-optimal.

(7) The next primal pivot would yield x_1 as the entering variable and u_1 as the exiting variable. This is a degenerate pivot.

Problem 3. *Solve the linear programming problem from Exercise* 2.4 *using the dual-primal two-phase algorithm.*

Proof. The initial primal dictionary is this.

ζ	$=$	0	$-$	x_0	$-$	$3x_1$	$-$	x_2
u_0	$=$	-5	$-$	$2x_0$	$+$	$5x_1$	$-$	x_2
u_1	$=$	4	$-$	$2x_0$	$+$	x_1	$-$	$2x_2$

The initial dual dictionary is this (feasible) with the following solution.

$-\xi$	$=$	0	$+$	$5y_0$	$-$	$4y_1$	
v_0	$=$	1	$+$	$2y_0$	$+$	$2y_1$	
v_1	$=$	3	$-$	$5y_0$	$-$	y_1	$\rightarrow \quad y_0 = \frac{3}{5} \quad \rightarrow$
v_2	$=$	1	$+$	y_0	$+$	$2y_1$	

$-\xi$	$=$	3	$-$	v_1	$-$	$5y_1$	
v_0	$=$	$\frac{11}{5}$	$-$	$\frac{2}{5}v_1$	$+$	$\frac{8}{5}y_1$	
y_0	$=$	$\frac{3}{5}$	$-$	$\frac{1}{5}v_1$	$-$	$\frac{1}{5}y_1$	$\rightarrow \quad -\xi = 3 \quad \rightarrow$
v_2	$=$	$\frac{8}{5}$	$-$	$\frac{1}{5}v_1$	$+$	$\frac{9}{5}y_1$	

Therefore, $\zeta = 3$.

□

Problem 4. *Solve the linear programming problem from Exercise* 2.6 *using the dual-primal two-phase algorithm.*

Proof. The initial primal dictionary (infeasible) is this.

$$\begin{array}{|ccccccc|}\hline \zeta & = & 0 & + & x_0 & + & 3x_1 \\ \hline u_0 & = & -3 & + & x_0 & + & x_1 \\ u_1 & = & -1 & + & x_0 & - & x_1 \\ u_2 & = & 2 & - & x_0 & - & 2x_1 \\ \hline \end{array}$$

The initial dual dictionary (infeasible) is this.

$$\begin{array}{|ccccccccc|}\hline -\xi & = & 0 & + & 3y_0 & + & y_1 & - & 2y_2 \\ \hline v_0 & = & -1 & - & y_0 & - & y_1 & + & y_2 \\ v_1 & = & -3 & - & y_0 & + & y_1 & + & 2y_2 \\ \hline \end{array}$$

The auxiliary dual dictionary is this with the following solution.

$$\begin{array}{|ccccccccc|}\hline -\xi' & = & 0 & + & 3y_0 & + & y_1 & - & 2y_2 \\ \hline v_0 & = & 1 & - & y_0 & - & y_1 & + & y_2 \\ v_1 & = & 1 & - & y_0 & + & y_1 & + & 2y_2 \\ \hline -\xi' & = & 3 & - & 3v_0 & - & 2y_1 & + & y_2 \\ \hline y_0 & = & 1 & - & v_0 & - & y_1 & + & y_2 \\ v_1 & = & 0 & + & v_0 & + & 2y_1 & + & y_2 \\ \hline \end{array} \quad \begin{array}{l} \rightarrow \; y_0 = 1 \; \rightarrow \\ \\ \\ \rightarrow \; -\xi' = \infty \end{array}$$

The auxiliary dual problem is unbounded, thus the initial is infeasible. $\square$