HW 1

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Problem 1. *Let S be the set of $2x2$ matrices.*

(1) *Verify that S is a vector space over $\mathbb{R}$ under matrix addition and scalar multiplication.*
(2) *What is the dimension of this S? Justify your answer with a basis.*

Proof. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in S$. Then $\mathbf{A} = \begin{bmatrix} a_{(0,0)} & a_{(0,1)} \\ a_{(1,0)} & a_{(1,1)} \end{bmatrix}$ where $a_{(0,0)}, a_{(0,1)}, a_{(1,0)}, a_{(1,1)} \in \mathbb{R}$ and similarly for $\mathbf{B}$ and $\mathbf{C}$. Additionally, let $k_0, k_1 \in \mathbb{R}$ and $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We will now prove S has the properties of a vector space.

(1) Additive Associativity:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{bmatrix} a_{(0,0)} & a_{(0,1)} \\ a_{(1,0)} & a_{(1,1)} \end{bmatrix} + \begin{bmatrix} b_{(0,0)} + c_{(0,0)} & b_{(0,1)} + c_{(0,1)} \\ b_{(1,0)} + c_{(1,0)} & b_{(1,1)} + c_{(1,1)} \end{bmatrix} =$$

$$\begin{bmatrix} a_{(0,0)} + b_{(0,0)} + c_{(0,0)} & a_{(0,1)} + b_{(0,1)} + c_{(0,1)} \\ a_{(1,0)} + b_{(1,0)} + c_{(1,0)} & a_{(1,1)} + b_{(1,1)} + c_{(1,1)} \end{bmatrix} =$$

$$\begin{bmatrix} a_{(0,0)} + b_{(0,0)} & a_{(0,1)} + b_{(0,1)} \\ a_{(1,0)} + b_{(1,0)} & a_{(1,1)} + b_{(1,1)} \end{bmatrix} + \begin{bmatrix} c_{(0,0)} & c_{(0,1)} \\ c_{(1,0)} & c_{(1,1)} \end{bmatrix} = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

(2) Additive Commutativity:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{(0,0)} + b_{(0,0)} & a_{(0,1)} + b_{(0,1)} \\ a_{(1,0)} + b_{(1,0)} & a_{(1,1)} + b_{(1,1)} \end{bmatrix} = \begin{bmatrix} b_{(0,0)} + a_{(0,0)} & b_{(0,1)} + a_{(0,1)} \\ b_{(1,0)} + a_{(1,0)} & b_{(1,1)} + a_{(1,1)} \end{bmatrix} = \mathbf{B} + \mathbf{A}$$

(3) Additive Identity:

$$\mathbf{A} + \mathbf{O} = \begin{bmatrix} a_{(0,0)} + 0 & a_{(0,1)} + 0 \\ a_{(1,0)} + 0 & a_{(1,1)} + 0 \end{bmatrix} = \begin{bmatrix} a_{(0,0)} & a_{(0,1)} \\ a_{(1,0)} & a_{(1,1)} \end{bmatrix} = \mathbf{A}$$

(4) Additive Inverse:

$$\mathbf{A} + (-A) = \begin{bmatrix} a_{(0,0)} - a_{(0,0)} & a_{(0,1)} - a_{(0,1)} \\ a_{(1,0)} - a_{(1,0)} & a_{(1,1)} - a_{(1,1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$$

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(5) Scalar-Multiplicative Associativity:

$$k_0(k_1\mathbf{A}) = k_0 \begin{bmatrix} k_1 a_{(0,0)} & k_1 a_{(0,1)} \\ k_1 a_{(1,0)} & k_1 a_{(1,1)} \end{bmatrix} = \begin{bmatrix} (k_0k_1) a_{(0,0)} & (k_0k_1) a_{(0,1)} \\ (k_0k_1) a_{(1,0)} & (k_0k_1) a_{(1,1)} \end{bmatrix} = (k_0k_1)\mathbf{A}$$

(6) Scalar-Multiplicative Identity:

$$1\mathbf{A} = \begin{bmatrix} 1a_{(0,0)} & 1a_{(0,1)} \\ 1a_{(1,0)} & 1a_{(1,1)} \end{bmatrix} = \begin{bmatrix} a_{(0,0)} & a_{(0,1)} \\ a_{(1,0)} & a_{(1,1)} \end{bmatrix} = \mathbf{A}$$

(7) First Law of Scalar-Multiplicative Distributivity:

$$k_0(\mathbf{A}+\mathbf{B}) = \begin{bmatrix} k_0(a_{(0,0)} + b_{(0,0)}) & k_0(a_{(0,1)} + b_{(0,1)}) \\ k_0(a_{(1,0)} + b_{(1,0)}) & k_0(a_{(1,1)} + b_{(1,1)}) \end{bmatrix} =$$

$$\begin{bmatrix} k_0 a_{(0,0)} + k_0 b_{(0,0)} & k_0 a_{(0,1)} + k_0 b_{(0,1)} \\ k_0 a_{(1,0)} + k_0 b_{(1,0)} & k_0 a_{(1,1)} + k_0 b_{(1,1)} \end{bmatrix} = k_0\mathbf{A} + k_0\mathbf{B}$$

(8) Second Law of Scalar-Multiplicative Distributivity:

$$k_0\mathbf{A} + k_1\mathbf{A} = \begin{bmatrix} k_0 a_{(0,0)} + k_1 a_{(0,0)} & k_0 a_{(0,1)} + k_1 a_{(0,1)} \\ k_0 a_{(1,0)} + k_1 a_{(1,0)} & k_0 a_{(1,1)} + k_1 a_{(1,1)} \end{bmatrix} =$$

$$\begin{bmatrix} (k_0+k_1) a_{(0,0)} & (k_0+k_1) a_{(0,1)} \\ (k_0+k_1) a_{(1,0)} & (k_0+k_1) a_{(1,1)} \end{bmatrix} = (k_0+k_1)\mathbf{A}$$

□

$\dim(S) = 4$. For justification, $S_{\mathscr{B}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of S.

Proof. We will prove $S_{\mathscr{B}}$ is a basis of S. Let $\mathbf{A} \in S$.

$$\mathbf{A} = a_{(0,0)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{(0,1)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{(1,0)} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{(1,1)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Problem 2. *Show that no bounded subset of* $\mathbb{R}^n$ *is a non-trivial subspace.*

Proof. Let $S \subseteq \mathbb{R}^n$ such that S is bounded. Then, for every neighborhood about $\mathbf{0}, E$, there exists $k_0 \in \mathbb{R}_{>0}$, such that for every $k \in \mathbb{R}_{\geq |k_0|}, S \subseteq kE$. If $S = \emptyset$ then $\mathbf{0} \notin S$ so it would not be a subset thus we assume S is non-empty moving forward. By way of contradiction, suppose S is a non-trivial subspace of $\mathbb{R}^n$. Then for every $\mathbf{v} \in S$ and every $c \in \mathbb{R}, c\mathbf{v} \in S$ and there exists $\mathbf{v} \in S$ such that $\mathbf{v} \neq \mathbf{0}$. Let E be a neighborhood about $\mathbf{0}$ and $k_0 \in \mathbb{R}$ such that for every $k > |k_0|, S \subseteq kE$. Then $c\mathbf{v} \in kE$. Without loss of generality, we may restrict E to being an $n-$sphere about $\mathbf{0}$ with a radius of k_0 i.e. $E = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}|_2 \leq k_0\}$. Suppose $\mathbf{v} \in S$ and $\mathbf{v} \neq \mathbf{0}$. Then for any $c \in \mathbb{R}$, $c\mathbf{v} \in k_0E$. Therefore, $\forall c \in \mathbb{R}, k_0 > c$. $\mathbb{R}$ is an ordered field and is therefore unbounded; however, k_0 is a bound on $\mathbb{R}$. This is a contradiction. □

Problem 3. *Consider the vector space of real-valued functions defined on* $[0,1]$.

(1) *Show that the set of upto 3rd-order polynomials,* $P_{3[0,1]}$, *is a subspace. What is the dimension of* $P_{3[0,1]}$*?*
(2) *Show* $F = \{\mathbf{f_0} = f_0(t) = 1, \mathbf{f_1} = f_1(t) = t, \mathbf{f_2} = f_2(t) = t^2, \mathbf{f_3} = f_3(t) = t^3\}$ *is a basis of* $P_{3[0,1]}$.

Proof. The zero function, $\mathbf{0}$, is a zero-degree polynomial so $\mathbf{0} \in F$. Let $\mathbf{g}, \mathbf{h} \in P_{3[0,1]}$. Then $\mathbf{g} = g(t) = g_0 + g_1 t + g_2 t^2 + g_3 t^3$ for some $g_0, g_1, g_2, g_3 \in \mathbb{R}$ and similarly with $\mathbf{h}$. $\mathbf{g} + \mathbf{h} = (g_0 + h_0) + (g_1 t + h_1 t) + (g_2 t^2 + h_2 t^2) + (g_3 t^3 + h_3 t_3) = (g_0 + h_0) + (g_1 + h_1)t + (g_2 + h_2)t^2 + (g_3 + h_3)t^3 \in P_{3[0,1]}$. Let $c \in \mathbb{R}$. $c\mathbf{g} = (cg_0) + (cg_1)t + (cg_2)t^2 + (cg_3)t^3 \in P_{3[0,1]}$. $\square$

Problem 4. *Formulate the problem in exercise 1.2 as a linear programming problem.*

Let Y, B, M be the classes described in the problem and let a subscript of $0, 1, 2$ represent Ithica-Newark, Newark-Boston, and Ithaca-Boston respectively. The problem becomes the following.
Maximize

$$(300Y_0 + 220B_0 + 100M_0) + (160Y_1 + 130B_1 + 80M_1) + (360Y_2 + 280B_2 + 140M_2)$$

subject to

$$\begin{gathered} Y_0 \leq 4, B_0 \leq 8, M_0 \leq 22 \\ Y_1 \leq 8, B_1 \leq 13, M_1 \leq 20 \\ Y_2 \leq 3, B_2 \leq 10, M_2 \leq 18 \\ (Y_0 + B_0 + M_0) + (Y_2 + B_2 + M_2) \leq 30 \\ (Y_1 + B_1 + M_1) + (Y_2 + B_2 + M_2) \leq 30 \end{gathered}$$

Problem 5. *Formulate the problem in exercise 1.3 as a linear programming problem.*

Maximize

$$\sum_{j=1}^{n} p_j x_j$$

subject to

$$\begin{gathered} \sum_{j=1}^{n} q_j x_j \leq \beta \\ \sum_{j=1}^{n} p_j = 1 \\ \sum_{j=1}^{n} q_j = 1 \\ \forall n \in \mathbb{N} \cap [1, n], x_j, p_j, q_j \in [0, 1] \end{gathered}$$