HW-7

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Problem 1. *Consider the following Linear Programming Problem.*

$$\begin{array}{lccccccccc} \textit{maximize} & x_1 & + & 2x_2 & + & x_3 & + & x_4 & & \\ \textit{subject to} & 2x_1 & + & x_2 & + & 5x_3 & + & x_4 & \leq & 8 \\ & 2x_1 & + & 2x_2 & + & 0x_3 & + & 4x_4 & \leq & 12 \\ & 3x_1 & + & x_2 & + & 2x_3 & + & 0x_4 & \leq & 18 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

This is its final dictionary where x_5, x_6, x_7 are slack variables.

$$\begin{array}{|ccccccccccc|} \hline \zeta & = & \frac{62}{5} & - & \frac{6}{5}x_1 & - & \frac{1}{5}x_5 & - & \frac{9}{10}x_6 & - & \frac{14}{5}x_4 \\ \hline x_2 & = & 6 & - & x_1 & + & 0x_5 & - & \frac{1}{2}x_6 & - & 2x_4 \\ x_3 & = & \frac{2}{5} & - & \frac{1}{5}x_1 & - & \frac{1}{5}x_5 & + & \frac{1}{10}x_6 & + & \frac{1}{5}x_4 \\ x_7 & = & \frac{56}{5} & - & \frac{8}{5}x_1 & + & \frac{2}{5}x_5 & + & \frac{3}{10}x_6 & + & \frac{8}{5}x_4 \\ \hline \end{array}$$

What is the optimal solution for each of the modified problems?

(1) *The objective function is $3x_1 + 2x_2 + x_3 + x_4$.*

(2) *The objective function is $x_1 + 2x_2 + \frac{1}{2}x_3 + x_4$.*

(3) *The second constraint is $2x_1 + 2x_2 + 0x_3 + 4x_4 \leq 26$.*

(1) The optimal solution is $\mathbf{x} = \begin{bmatrix} 2 & 4 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}^T$ with $\zeta = 14$.

(2) The optimal solution is $\mathbf{x} = \begin{bmatrix} 0 & 6 & \frac{2}{5} & 0 & 0 & 0 & \frac{56}{5} \end{bmatrix}^T$ with $\zeta = \frac{61}{5}$.

(3) The optimal solution is $\mathbf{x} = \begin{bmatrix} 0 & 8 & 0 & 0 & 0 & 10 & 10 \end{bmatrix}^T$ with $\zeta = 16$.

Problem 2. *In reference to the previous problem, find the range over the objective coefficients for which the final dictionary remains optimal.*

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Proof. In other words, we wish to find $\mathbf{c}$ such that $\langle \mathbf{c}, \mathbf{x}_{\mathscr{N}} \rangle$ is optimal where $\mathbf{x}_{\mathscr{N}} = \begin{bmatrix} 0 \\ 6 \\ \frac{2}{5} \\ 0 \end{bmatrix}$. Notice in the final dictionary above, x_1, x_4 are non-basic with objective coefficients of $-\frac{6}{5}, -\frac{14}{5}$ respectively. Thus c_1, c_4 must be at most $\frac{6}{5}, \frac{14}{5}$ respectively. Furthermore, c_2, c_3 must be chosen such that the resulting objective coefficients attached to x_5, x_6 are at most $\frac{1}{5}, \frac{9}{10}$ respectively. This yields $c_2 \in [-\frac{6}{5}, \infty)$ and $c_3 \in [-1, 9]$. Thus

$$\mathbf{c} \in \left(-\infty, \frac{6}{5} \right] \times \left[-\frac{6}{5}, \infty \right) \times \left[-1, 9 \right] \times \left(-\infty, \frac{14}{5} \right]$$

□

Problem 3. *Consider the following dictionary.*

$$\begin{array}{|ccccccc|}
\hline
\zeta & = & -3 & - & (11+5\mu)x_1 & - & (-2+2\mu)x_4 \\
\hline
x_3 & = & (-2+\mu) & - & 5x_1 & + & x_4 \\
x_2 & = & (3-\mu) & + & x_1 & + & x_4 \\
x_5 & = & (1+2\mu) & + & 3x_1 & + & x_4 \\
\hline
\end{array}$$

For what values of μ is this dictionary optimal?

Proof. For this dictionary to be optimal, the following must be true.

$$11 + 5\mu \geq 0, -2 + 2\mu \geq 0, -2 + \mu \geq 0, 3 - \mu \geq 0, 1 + 2\mu \geq 0$$

i.e.

$$\mu \geq -\frac{11}{5}, \mu \geq 1, \mu \geq 2, \mu \leq 3, \mu \geq \frac{1}{2}$$

Thus

$$\mu \in \left[-\frac{11}{5}, \infty \right) \cap [1, \infty) \cap [2, \infty) \cap (\infty, 3] \cap \left[\frac{1}{2}, \infty \right) = [2, 3]$$

□

Problem 4. *Let $A \in M_{m \times n}(\mathbb{F})$ and $\mathbf{c} \in \mathbb{F}^n$ for some $m, n \in \mathbb{N}_0$. Let $\xi^* : \mathbb{F}^m \to \mathbb{F}$ be a function such that for each $\mathbf{b} \in \mathbb{F}^m, \xi^*(\mathbf{b})$ is the optimal objective function value for the following linear programming problem.*

$$
\begin{array}{lrcl}
\textit{maximize} & \mathbf{c}^T\mathbf{x} & & \\
\textit{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\
& \forall_{i=1}^n x_i & \geq & 0
\end{array}
$$

Suppose $\xi^*(\mathbf{b}) < \infty$ *for every* $\mathbf{b} \in \mathbb{F}^m$*. Prove* ξ^* *is a concave function.*

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathbb{F}^m$ and $t \in (0, 1)$. We wish to prove

$$\xi^*(t\mathbf{u} + (1-t)\mathbf{v}) \geq t\xi^*(\mathbf{u}) + (1-t)\xi^*(\mathbf{v})$$

Without loss of generality, assume $\mathbf{v}\mathbf{x}_u \geq \mathbf{v}\mathbf{x}_v$. Observe.

$$t\xi^*(\mathbf{u}) + (1-t)\xi^*\mathbf{v} = t\mathbf{u}^T\mathbf{x}_u + (1-t)\mathbf{v}^T\mathbf{x}_v \leq t\mathbf{u}^T\mathbf{x}_u + (1-t)\mathbf{v}^T\mathbf{x}_u =$$

$$(t\mathbf{u} + (1-t)\mathbf{v})^T\mathbf{x}_u \leq \xi^*(t\mathbf{u} + (1-t)\mathbf{v})$$

$\square$