HW-6

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Problem 1. *Consider the following linear programming problem.*

$$\begin{array}{lccccccccccc} \textit{maximize} & x_0 & + & 2x_1 & + & 4x_2 & + & 8x_3 & + & 16x_4 & & \\ \textit{subject to} & x_0 & + & 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & \leq & 2 \\ & 7x_0 & + & 5x_1 & - & 3x_2 & - & 2x_3 & + & 0x_4 & \leq & 0 \\ & x_0 & , & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

Consider the situation where x_2, x_4 are basic and the rest are non-basic. Find the following.

(1) $\mathscr{B}$

(2) $\mathscr{N}$

(3) B

(4) N

(5) $\mathbf{b}$

(6) $\mathbf{c}_{\mathscr{B}}$

(7) $\mathbf{c}_{\mathscr{N}}$

(8) $B^{-1}N$

(9) $\mathbf{x}^*_{\mathscr{B}} = B^{-1}\mathbf{b}$

(10) $\zeta^* = \mathbf{c}^T_{\mathscr{B}} B^{-1}\mathbf{b}$

(11) $\mathbf{z}^*_{\mathscr{N}} = (B^{-1}N)^T \mathbf{c}_{\mathscr{B}} - \mathbf{c}_{\mathscr{N}}$

(12) *The dictionary representation of this basis.*

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Proof. Observe.

(1) $\mathscr{B} = \{2, 4\}$

(2) $\mathscr{N} = \{0, 1, 3, 5, 6\}$

(3) $B = \Big[\mathbf{v}_i \in A : i \in \mathscr{B}\Big] = \begin{bmatrix} 3 & 5 \\ -3 & 0 \end{bmatrix}$

(4) $N = \Big[\mathbf{v}_i \in A : i \in \mathscr{N}\Big] = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix}$

(5) $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(6) $\mathbf{c}_{\mathscr{B}} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$

(7) $\mathbf{c}_{\mathscr{N}} = \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix}$

(8) $B^{-1}N = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 7 & 5 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} & -\frac{5}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{8}{5} & \frac{7}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$

(9) $\mathbf{x}^*_{\mathscr{B}} = B^{-1}\mathbf{b} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{5} \end{bmatrix}$

(10) $\zeta^* = \mathbf{c}^T_{\mathscr{B}} B^{-1}\mathbf{b} = \mathbf{c}^T_{\mathscr{B}} \mathbf{x}^*_{\mathscr{B}} = \begin{bmatrix} 4 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{2}{5} \end{bmatrix} = \frac{32}{5}$

(11) $$\mathbf{z}^*_{\mathscr{N}} = (B^{-1}N)^T\mathbf{c}_{\mathscr{B}} - \mathbf{c}_{\mathscr{N}} = \begin{bmatrix} -\frac{7}{3} & \frac{8}{5} \\ -\frac{5}{3} & \frac{7}{5} \\ \frac{2}{3} & \frac{2}{5} \\ 0 & \frac{1}{5} \\ -\frac{1}{3} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 244 \\ 236 \\ 136 \\ 48 \\ 28 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 229 \\ 206 \\ 16 \\ 48 \\ 28 \end{bmatrix}$$

(12) The dictionary representation of this basis is the following.

$$\begin{array}{|ccccccccccccc|} \hline \zeta & = & \frac{32}{5} & - & \frac{56}{5}x_0 & - & \frac{104}{5}x_1 & - & \frac{8}{5}u_1 & + & \frac{16}{5}x_3 & - & \frac{16}{5}u_0 \\ \hline x_2 & = & 0 & + & \frac{7}{3}x_0 & - & \frac{5}{3}x_1 & + & \frac{1}{3}x_2 & - & \frac{2}{3}x_3 & + & 0u_0 \\ x_4 & = & \frac{2}{5} & - & \frac{28}{15}x_0 & - & \frac{32}{15}x_1 & - & \frac{4}{15}u_1 & + & \frac{8}{15}x_3 & - & \frac{1}{5}u_0 \\ \hline \end{array}$$

□

Problem 2. *Solve Exercise* 2.1 *using the matrix form of the primal-simplex method.*

Proof. Recall the initial primal dictionary for Exercise 2.1 is the following.

$$\begin{array}{|ccccccccccc|} \hline \zeta & = & 0 & + & 6x_0 & + & 8x_1 & + & 5x_2 & + & 9x_3 \\ \hline u_0 & = & 5 & - & 2x_0 & - & x_1 & - & x_2 & - & 3x_3 \\ u_1 & = & 3 & - & x_0 & - & 3x_1 & - & x_2 & - & 2x_3 \\ \hline \end{array}$$

Thus we have the following initial basis.

$$\mathscr{B} = \{4,5\}, \mathscr{N} = \{0,1,2,3\}, \zeta^* = 0$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \end{bmatrix}, B^{-1}N = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \mathbf{c}_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{c}_{\mathscr{N}} = \begin{bmatrix} 6 \\ 8 \\ 5 \\ 9 \end{bmatrix}, \mathbf{x}^*_{\mathscr{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \mathbf{z}^*_{\mathscr{N}} = \begin{bmatrix} -6 \\ -8 \\ -5 \\ -9 \end{bmatrix}$$

To reach our next basis we choose the following.

$$j = 3, t = \frac{3}{5}, i = 0, s = 3$$

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \Delta\mathbf{x}_{\mathscr{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{e}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Delta\mathbf{z}_{\mathscr{N}} = \begin{bmatrix} -2 \\ -1 \\ -1 \\ -3 \end{bmatrix}$$

This yields our second basis.

$$\mathscr{B} = \{3, 5\}, \mathscr{N} = \{0, 1, 2, 4\}$$

$$B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, N = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}, B^{-1}N = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{x}^*_{\mathscr{B}} = \begin{bmatrix} \frac{16}{5} \\ \frac{9}{5} \end{bmatrix}, \mathbf{z}^*_{\mathscr{N}} = \begin{bmatrix} 0 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

To reach our next basis we choose the following.

$$j = 1, t = \frac{5}{48}, i = 0, s = 15$$

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \Delta\mathbf{x}^*_{\mathscr{B}} = \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \end{bmatrix}, \mathbf{e}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Delta\mathbf{z}^*_{\mathscr{N}} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

This yields our third (and final) basis.

$$\mathscr{B} = \{3, 0\}, \mathscr{N} = \{5, 1, 2, 4\}$$

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}, B^{-1}N = \begin{bmatrix} 2 & 5 & 1 & -1 \\ -3 & -7 & -1 & 2 \end{bmatrix}$$

$$\mathbf{x}^*_{\mathscr{B}} = \begin{bmatrix} \frac{2279}{720} \\ \frac{1121}{720} \end{bmatrix}, \mathbf{z}^*_{\mathscr{N}} = \begin{bmatrix} 10 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Thus $\zeta^* = 6x_0 + 8x_1 + 5x_2 + 9x + 3 = 6(2) + 8(0) + 5(1) + 9(0) = 17.$
(I honestly don't know where I went wrong with $\mathbf{x}^*_{\mathscr{B}}$.)

$\square$