HW-8

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Problem 1. *Is $\mathbb{R}^n$ a polyhedron?*

Proof. Let $n \in \mathbb{N}_0$. To show $\mathbb{R}^n$ is a polyhedron. Then we must show there exists a collection of halfspaces, $H = \{H_i : i \in I\}$, such that $|I| \in \mathbb{N}_0$ and $\mathbb{R}^n = \bigcap H = \bigcap_{i \in I} H_i$. Take $H = \emptyset$ i.e. $I = \emptyset$. Then

$$\bigcap_{i \in I} H_i = \bigcap H = \bigcap \emptyset = U$$

where U is the universal set for the universe of our proof. Our proof is regarding halfspaces of $\mathbb{R}^n$ and as such $U = \mathbb{R}^n$. $\emptyset$ is finite and is (vacuously) a collection of halfspaces. Therefore $\mathbb{R}^n$ is a polyhedron. $\square$

Problem 2. *Describe how one would need to modify the proof for Theorem* 10.4 *to prove the following. Let $P, \tilde{P}$ be disjoint polyhedra in $\mathbb{R}^n$. Then there exists two disjoint generalized half-spaces, $H, \tilde{H}$, such that $P \subseteq H$ and $\tilde{P} \subseteq \tilde{H}$.*

Proof. The given statement and Theorem 10.4 differ only in that the given statement does not require non-emptiness. Thus, to achieve the given statement, add the following case to the proof of Theorem 10.4. Without loss of generality, suppose $\tilde{P} = \emptyset$. Then $\tilde{H} = \emptyset$. Thus $\tilde{H}$ is disjoint (vacuously) with every set. Thus take H to be any half-space such that $P \subseteq H$ which, in the case that $P \neq \emptyset$, there is at least one of because P is defined as a intersection of half-spaces. $\square$

Problem 3. *Find a strictly complementary solution to the following linear programming problem (primal and dual).*

$$\begin{array}{lccccc} \textit{maximize} & 2x_1 & + & x_2 & & \\ \textit{subject to} & 4x_1 & + & 2x_2 & \leq & 6 \\ & 0x_1 & + & x_2 & \leq & 1 \\ & 2x_1 & + & x_2 & \leq & 3 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lccccccc} \textit{minimize} & 6y_1 & + & y_2 & + & 3y_3 & & \\ \textit{subject to} & 4y_1 & + & 0y_2 & + & 2y_3 & \geq & 2 \\ & 2y_1 & + & y_2 & + & y_3 & \geq & 1 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

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Proof. Consider the optimal primal solution and dual solutions respectively.

$$(\mathbf{x}^*, \mathbf{w}^*, \zeta^*) = \left(\begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, 3 \right), (\mathbf{y}^*, \mathbf{z}^*, \xi^*) = \left(\begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 3 \right)$$

Notice

$$\mathbf{x} + \mathbf{z} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \end{bmatrix}, \mathbf{y} + \mathbf{w} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

are both non-zero for all of their terms.

□

Problem 4. *Complete the proof of Theorem* 10.7*.*

Proof. As mentioned we consider the following cases.
Case ($t > 0$): Suppose the optimal t is strictly positive. Then, the x_j^* we are currently analyzing is non-null. Thus $x_j^* + z_j^* > 0$.
Case ($t = 0$): Suppose the optimal t vanishes. Then, the x_j^* we are currently analyzing is null. Thus z_j^* is non-null. Thus $x_j^* + z_j^* > 0$.
We see the same logic holds for $\mathbf{y}, \mathbf{w}$. Therefore, if an optimal solution exists for a given linear programming problem then there must exist optimal solutions to the primal and dual such that $x_j^* + z_j^* > 0$ and $y_i^* + w_i^* > 0$ for all i, j.

□

Problem 5. *Consider the following game. Players A and B each hide a nickel or a dime. If the hidden coins match, player A gets both, otherwise player B gets both. Answer the following.*

(1) *What are the optimal strategies?*

(2) *Who has the advantage?*

(3) *What are the optimal strategies for arbitrary coins, a and b?*

(1) *Proof.* This game can be represented with the following payout matrix where A is the minimizer (rows) and B is the maximizer (columns).

$$P = \begin{bmatrix} -20 & 15 \\ 15 & -10 \end{bmatrix}$$

By the symmetry of this matrix we know that both players have the same optimal strategy of $(\frac{5}{12}, \frac{7}{12})$ with an optimal expected value of $\frac{5}{12}$.

$\square$

(2) As shown above, the expected value of the optimal strategies is positive so player A has the advantage.

(3) In a generalized form, we can represent this game as the following matrix.

$$\begin{bmatrix} -2a & a+b \\ a+b & -2b \end{bmatrix}$$

Observe the derivation of the optimal strategies.

$$\begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2a & a+b \\ a+b & -2b \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = -2b - 4(a+b)pq + (a+3b)p + (a+3b)q$$

Thus the optimal strategy for both players is

$$\left(\frac{a+3b}{4(a+b)}, 1 - \frac{a+3b}{4(a+b)} \right)$$

Problem 6. *Let $A \in M_{m \times n}$ be a matrix with $\mathbf{r}_i$ as row vectors and $\mathbf{s}_j$ as column vectors for $i \in \mathbb{N} \cap [1, m]$ and $j \in \mathbb{N} \cap [1, n]$. Prove the following.*

(1) *If $\mathbf{r}_i$ dominates $\mathbf{r}_k$ for some $i, k \in \mathbb{N} \cap [1, m]$ where $i \neq k$, then there exists an optimal strategy for the row player, $\mathbf{y}^*$, such that $y_i^* = 0$.*

(2) *If $\mathbf{s}_j$ is dominated by $\mathbf{s}_l$ for some $j, l \in \mathbb{N} \cap [1, n]$ where $j \neq l$, then there exists an optimal strategy for the column player, $\mathbf{x}^*$, such that $x_j^* = 0$.*

Use the consequences of these proofs to reduce the following payout matrix.

$$\begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & -3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix}$$

(1) *Proof.* Assume the given premise and supppose we have a feasible strategy for the row player, $\mathbf{y}$, such that $\mathbf{y}$ is optimal other than, possibly, for y_i, y_k. Because $\mathbf{r}_i$ dominates $\mathbf{r}_k$, we know we can choose, at least equivalently optimal, $y_i' = y_i + \varepsilon, y_k' = y_k - \varepsilon$ where $\varepsilon \in [-y_i, y_k]$ with guaranteed consequences to the expected payout value of our game. Choosing $\varepsilon > 0$ will raise the expected payout while choosing $\varepsilon < 0$ will lower the expected payout. Since the row player wishes to minimize this expected value, they will optimally choose $\varepsilon = -y_i$. Thus, updating $\mathbf{y}$ with y_i', y_k' will produce an optimal solution for which $y_i' = 0, y_k' = y_i + y_k$.

□

(2) *Proof.* This proof is a consequence of applying the previous proof to the transpose of a given payout matrix. Regardless we provide the following proof. Assume the given premise and supppose we have a feasible strategy for the column player, $\mathbf{x}$, such that $\mathbf{x}$ is optimal other than, possibly, for y_j, y_l. Because $\mathbf{s}_j$ is dominated by $\mathbf{s}_l$, we know we can choose, at least equivalently optimal, $x_j' = x_j - \delta, x_l' = x_l + \delta$ where $\delta \in [-x_l, x_j]$ with guaranteed consequences to the expected payout value of our game. Choosing $\delta > 0$ will raise the expected payout while choosing $\delta < 0$ will lower the expected payout. Since the column player wishes to maximize this expected value, they will optimally choose $\delta = x_j$. Thus, updating $\mathbf{x}$ with x_j', x_l' will produce an optimal solution for which $x_j' = 0, x_l' = x_j + x_l$.

□

(3) *Proof.* Observe the reduction of the given payout matrix.

$$P = \begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & -3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix} \xrightarrow{P/\mathbf{r}_4} \begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & -3 & -3 & -8 & -2 \end{bmatrix} \xrightarrow{P/\mathbf{s}_4}$$

$$\begin{bmatrix} -6 & 2 & -4 & -5 \\ 0 & 4 & -2 & -1 \\ -7 & -3 & -3 & -2 \end{bmatrix} \xrightarrow{P/\mathbf{r}_2} \begin{bmatrix} -6 & 2 & -4 & -5 \\ -7 & -3 & -3 & -2 \end{bmatrix} \xrightarrow{P/\mathbf{s}_1} \begin{bmatrix} 2 & -4 & -5 \\ -3 & -3 & -2 \end{bmatrix} \xrightarrow{P/\mathbf{s}_2} \begin{bmatrix} 2 & -5 \\ -3 & -2 \end{bmatrix}$$

□

Problem 7. *Consider the following game. Two players simultaneously raise* 1 *or* 2 *fingers and guesses the* 2*-parity of the sum. If a player guesses correctly whilst the opposing player does not, then that player wins points equal to the sum while the opposing player loses that many points. Do the following.*

(1) *List the pure strategies of this game.*

(2) *Write down the payout matrix for this game.*

(3) *Formulate the row player's strategy as a linear programming problem.*

(4) *Find the optimal expected value of this game.*

(5) *Find the optimal strategies to this game.*

Before doing the above, we create the following framework. The choices of play in this game are

$$C = \{(a, b) : a \in \{1, 2\}, b \in \{T, F\}\}$$

where a represents the number of fingers raised and b represents the predicted truth value of the statement, "the sum is even". We choose to model C using binary numbers i.e.

$$a'b' = \begin{cases} a = 1 \wedge b = F & 00 \\ a = 1 \wedge b = T & 01 \\ a = 2 \wedge b = F & 10 \\ a = 2 \wedge b = T & 11 \end{cases}$$

(1) The pure strategies of this game are the elementary basis of $\mathbb{F}^4$ i.e.

$$\{\mathbf{e}_k \in \mathbb{F}^4 : k \in \mathbb{N} \cap [1, 4]\}$$

(2) The following is this game's payout matrix, P (with $P_{(i,j)}$ representing the row-column choices with index of the binary representation $+1$ i.e. $i = a'b' + 1$ for some $a', b' \in \{0, 1\}$ and similarly with j).

$$P = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

(3) The optimal row player strategy can be formulated as the following linear programming problem.

$$\begin{array}{rrcl} \text{minimize} & \mathbf{y}^T P\mathbf{x} & & \\ \text{subject to} & \mathbf{e}^T\mathbf{y} & = & 1 \\ & \forall_{j=1}^4 y_j & \geq & 0 \end{array}$$

Where P is the payout matrix, $\mathbf{y}$ is our collection of decision variables for the row player, and $\mathbf{x}$ is a given collection of decision variables for the column player.

(4) P is a skew matrix ($P^T = P$). As such, its optimal expected value is 0 (because every move by one player can be exactly undone by the other).

(5) The optimal strategies for row-player and column-player respectively

$$\mathbf{y} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$