HW 2

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Problem 1. *Solve the following constrained maximization problem using simplex.*

$$\begin{array}{|ccccccccc|}\hline 6x_0 & + & 8x_1 & + & 5x_2 & + & 9x_3 & = & \xi \\ \hline 2x_0 & + & x_1 & + & x_2 & + & 3x_3 & \leq & 5 \\ x_0 & + & 3x_1 & + & x_2 & + & 2x_3 & \leq & 3 \\ x_0 & , & x_1 & , & x_2 & , & x_3 & \geq & 0 \\ \hline \end{array}$$

Observe.

$$\begin{array}{|ccccccccccc|}\hline \xi & = & 0 & + & 6x_0 & + & 8x_1 & + & 5x_2 & + & 9x_3 \\ \hline y_0 & = & 5 & - & 2x_0 & - & x_1 & - & x_2 & - & 3x_3 \\ y_1 & = & 3 & - & x_0 & - & 3x_1 & - & x_2 & - & 2x_3 \\ \hline \end{array} \quad \to \quad x_3 = \tfrac{3}{2} \quad \to$$

$$\begin{array}{|ccccccccccc|}\hline \xi & = & \frac{27}{2} & + & \frac{3}{2}x_0 & - & \frac{11}{2}x_1 & + & \frac{1}{2}x_2 & - & \frac{9}{2}y_1 \\ \hline y_0 & = & \frac{1}{2} & - & \frac{1}{2}x_0 & + & \frac{7}{2}x_1 & + & \frac{1}{2}x_2 & + & \frac{3}{2}y_1 \\ x_3 & = & \frac{3}{2} & - & \frac{1}{2}x_0 & - & \frac{3}{2}x_1 & - & \frac{1}{2}x_2 & - & \frac{1}{2}y_1 \\ \hline \end{array} \quad \to \quad x_0 = 1 \quad \to$$

$$\begin{array}{|ccccccccccc|}\hline \xi & = & 15 & - & 3y_0 & + & 5x_1 & + & 2x_2 & + & 0y_1 \\ \hline x_0 & = & 1 & - & 2y_0 & + & 7x_1 & + & x_2 & + & 3y_1 \\ x_3 & = & 1 & + & y_0 & - & 5x_1 & - & x_2 & - & 2y_1 \\ \hline \end{array} \quad \to \quad x_1 = \tfrac{1}{5} \quad \to$$

$$\begin{array}{|ccccccccccc|}\hline \xi & = & 16 & - & 2y_0 & - & x_3 & + & x_2 & - & 2y_1 \\ \hline x_0 & = & \frac{12}{5} & - & \frac{3}{5}y_0 & - & \frac{7}{5}x_3 & - & \frac{2}{5}x_2 & + & \frac{1}{5}y_1 \\ x_1 & = & \frac{1}{5} & + & \frac{1}{5}y_0 & - & \frac{1}{5}x_3 & - & \frac{1}{5}x_2 & - & \frac{2}{5}y_1 \\ \hline \end{array} \quad \to \quad x_2 = 1 \quad \to$$

$$\begin{array}{|ccccccccccc|}\hline \xi & = & 17 & - & y_0 & - & 2x_3 & - & 5x_1 & - & 4y_1 \\ \hline x_0 & = & 2 & - & y_0 & - & x_3 & + & 2x_1 & + & y_1 \\ x_2 & = & 1 & + & y_0 & - & x_3 & - & 5x_1 & - & 2y_1 \\ \hline \end{array} \quad \to \quad \xi = 17$$

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Problem 2. *Solve the following constrained maximization problem using simplex.*

$$
\begin{array}{|ccccc|}
\hline
3x_0 & + & 2x_1 & = & \xi \\
\hline
x_0 & - & 2x_1 & \leq & 1 \\
x_0 & - & x_1 & \leq & 2 \\
2x_0 & - & x_1 & \leq & 6 \\
x_0 & + & 0x_1 & \leq & 5 \\
2x_0 & + & x_1 & \leq & 16 \\
x_0 & + & x_1 & \leq & 12 \\
x_0 & + & 2x_1 & \leq & 21 \\
0x_0 & + & x_2 & \leq & 10 \\
x_0 & , & x_1 & \geq & 0 \\
\hline
\end{array}
$$

Observe.

$$
\begin{array}{|ccccccc|l}
\cline{1-7}
\xi & = & 0 & + & 3x_0 & + & 2x_1 & \\
\cline{1-7}
y_0 & = & 1 & - & x_0 & + & 2x_1 & \\
y_1 & = & 2 & - & x_0 & + & x_1 & \\
y_2 & = & 6 & - & 2x_0 & + & x_1 & \\
y_3 & = & 5 & - & x_0 & + & 0x_1 & \quad \to \quad x_0 = 1 \quad \to \\
y_4 & = & 16 & - & 2x_0 & - & x_1 & \\
y_5 & = & 12 & - & x_0 & - & x_1 & \\
y_6 & = & 21 & - & x_0 & - & 2x_1 & \\
y_7 & = & 10 & + & 0x_0 & - & x_1 & \\
\cline{1-7}
\xi & = & 3 & - & 3y_0 & + & 8x_1 & \\
\cline{1-7}
x_0 & = & 1 & - & y_0 & + & 2x_1 & \\
y_1 & = & 1 & + & y_0 & - & x_1 & \\
y_2 & = & 4 & + & 2y_0 & - & 3x_1 & \\
y_3 & = & 4 & + & y_0 & - & 2x_1 & \quad \to \quad x_1 = 1 \quad \to \\
y_4 & = & 14 & + & 2y_0 & - & 5x_1 & \\
y_5 & = & 11 & + & y_0 & - & 3x_1 & \\
y_6 & = & 20 & + & y_0 & - & 4x_1 & \\
y_7 & = & 10 & + & 0y_0 & - & x_1 & \\
\cline{1-7}
\end{array}
$$

$$
\begin{array}{|ccccccc|l}
\hline
\xi & = & 11 & + & 5y_0 & - & 8y_1 & \\
\hline
x_0 & = & 3 & + & y_0 & - & 2y_1 & \\
x_1 & = & 1 & + & y_0 & - & y_1 & \\
y_2 & = & 1 & - & y_0 & + & 3y_1 & \\
y_3 & = & 2 & - & y_0 & + & 2y_1 & \to \quad y_0 = 1 \quad \to \\
y_4 & = & 9 & - & 3y_0 & + & 5y_1 & \\
y_5 & = & 8 & - & 2y_0 & + & 3y_1 & \\
y_6 & = & 16 & - & 3y_0 & + & 4y_1 & \\
y_7 & = & 9 & - & y_0 & + & y_1 & \\
\hline
\xi & = & 16 & - & 5y_2 & + & 7y_1 & \\
\hline
x_0 & = & 4 & - & y_2 & + & y_1 & \\
x_1 & = & 2 & - & y_2 & + & 2y_1 & \\
y_0 & = & 1 & - & y_2 & + & 3y_1 & \\
y_3 & = & 1 & + & y_2 & - & y_1 & \to \quad y_1 = 1 \quad \to \\
y_4 & = & 6 & + & 3y_2 & - & 4y_1 & \\
y_5 & = & 6 & + & 2y_2 & - & 3y_1 & \\
y_6 & = & 13 & + & 3y_2 & - & 5y_1 & \\
y_7 & = & 8 & + & y_2 & - & 2y_1 & \\
\hline
\xi & = & 23 & + & 2y_2 & - & 7y_3 & \\
\hline
x_0 & = & 5 & + & 0y_2 & - & y_3 & \\
x_1 & = & 4 & + & y_2 & - & 2y_3 & \\
y_0 & = & 4 & + & 2y_2 & - & 3y_3 & \\
y_1 & = & 1 & + & y_2 & - & y_3 & \to \quad y_2 = 2 \quad \to \\
y_4 & = & 2 & - & y_2 & + & 4y_3 & \\
y_5 & = & 3 & - & y_2 & + & 3y_3 & \\
y_6 & = & 8 & - & 2y_2 & + & 5y_3 & \\
y_7 & = & 6 & - & y_2 & + & 2y_3 & \\
\hline
\end{array}
$$

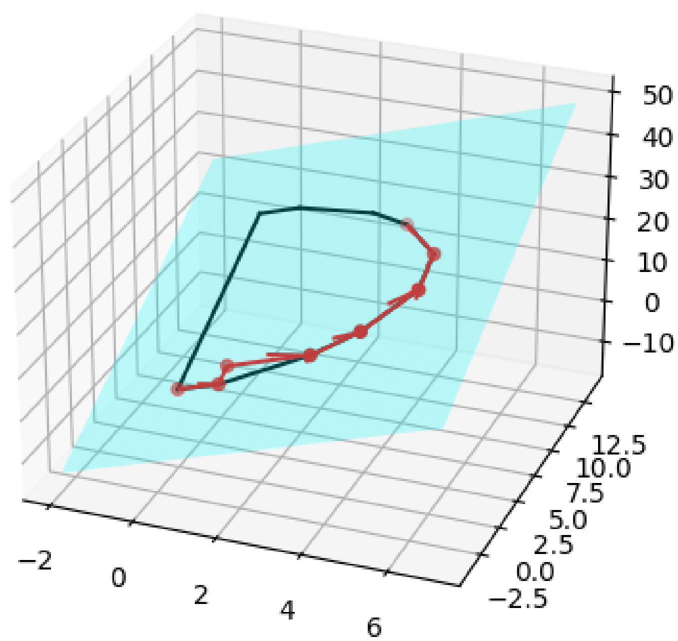
$$
\begin{array}{|ccccccc|c}
\hline
\xi & = & 27 & - & 2y_4 & + & y_3 & \\
\hline
x_0 & = & 5 & + & 0y_4 & - & y_3 & \\
x_1 & = & 6 & - & y_4 & + & 2y_3 & \\
y_0 & = & 8 & - & 2y_4 & + & 5y_3 & \\
y_1 & = & 3 & - & y_4 & + & 3y_3 & \quad \rightarrow \quad y_3 = 1 \quad \rightarrow \\
y_2 & = & 2 & - & y_4 & + & 4y_3 & \\
y_5 & = & 1 & + & y_4 & - & y_3 & \\
y_6 & = & 4 & + & 2y_4 & - & 3y_3 & \\
y_7 & = & 4 & + & y_4 & - & 2y_3 & \\
\hline
\xi & = & 28 & - & y_4 & - & y_5 & \\
\hline
x_0 & = & 4 & - & y_4 & + & y_5 & \\
x_1 & = & 8 & + & y_4 & - & 2y_5 & \\
y_0 & = & 13 & + & 3y_4 & - & 5y_5 & \\
y_1 & = & 6 & + & 2y_4 & - & 3y_5 & \quad \rightarrow \quad \xi = 28 \\
y_2 & = & 6 & + & 3y_4 & - & 4y_5 & \\
y_3 & = & 1 & + & y_4 & - & y_5 & \\
y_6 & = & 1 & - & y_4 & + & 3y_5 & \\
y_7 & = & 2 & - & y_4 & + & 2y_5 & \\
\hline
\end{array}
$$

Problem 3. *Show that the following dictionary cannot be the optimal dictionary for any linear programming problem in which y_0, y_1 are the initial slack variables.*

$$\begin{array}{|ccccccc|}\hline \xi & = & 4 & - & y_0 & - & 2x_1 \\ \hline x_0 & = & 3 & + & 0y_0 & - & 2x_1 \\ y_1 & = & 1 & + & y_0 & - & x_1 \\ \hline \end{array}$$

If y_0 were an initial slack variable then we'd see $0ky_0 = k3 - kx_0 - k2x_1$ where $k \in \mathbb{R}$ is some number such that $0k = 1$. No such number exists.

Problem 4. *Graph the region of feasible solutions for* 2.8 *and show the sequence of dictionary solutions.*



Problem 5. *Give an example showing that the variable that becomes basic in one iteration of the simplex method can become nonbasic in the next iteration.*

Consider the following problem.

$$\begin{array}{|ccccc|}\hline 3x_0 & + & 4x_1 & = & \xi \\ \hline x_0 & + & 0x_1 & \leq & 4 \\ 2x_0 & - & x_1 & \leq & 6 \\ x_0 & + & 2x_1 & \leq & 2 \\ x_0 & , & x_1 & \geq & 0 \\ \hline \end{array}$$

Observe.

$$\begin{array}{|ccccccc|c}\hline \xi & = & 0 & + & 3x_0 & + & 4x_1 & \\ \hline y_0 & = & 4 & - & x_0 & + & 0x_1 & \\ y_1 & = & 6 & - & 2x_0 & + & x_1 & \to \quad x_1 = 1 \quad \to \\ y_2 & = & 2 & - & x_0 & - & 2x_1 & \\ \hline \xi & = & 4 & + & x_0 & - & 2y_2 & \\ \hline y_0 & = & 4 & - & x_0 & + & 0y_2 & \\ y_1 & = & 7 & - & \frac{5}{2}x_0 & - & \frac{1}{2}y_2 & \to \quad x_0 = 2 \quad \to \\ x_1 & = & 1 & - & \frac{1}{2}x_0 & - & \frac{1}{2}y_2 & \\ \hline \xi & = & 6 & - & 2x_1 & - & 3y_2 & \\ \hline y_0 & = & 2 & + & 2x_1 & + & y_2 & \\ y_1 & = & 2 & + & 5x_1 & + & 2y_2 & \to \qquad \xi = 6 \\ x_0 & = & 2 & - & 2x_1 & - & y_2 & \\ \hline \end{array}$$

After the first iteration, x_1 became a basic variable. Then after the second iteration, x_1 became a nonbasic variable.

Problem 6. *Show that the variable that becomes nonbasic in one iteration of the simplex method cannot become basic in the next iteration.*

As show in the example for the previous problem, we can see whenever a variable becomes nonbasic, its leading coefficient in the equation that we are attempting to maximize will be negative, thus it will not be a candidate for our next entering variable and as such cannot become a basic variable during the next iteration.

Problem 7. *Suppose that a linear programming problem has the following property. Its initial dictionary is not degenerate and when solved by the simplex method there is never a tie for the choice of the leaving variable.*

(1) *Can such a problem have degenerate dictionaries? Explain.*
(2) *Can such a problem cycle? Explain.*

(1) Such a problem cannot yield a degenerate dictionary because there are no ties in selection which means a degenerate dictionary could only follow from a pervious dictionary, of which there is none because the initial dictionary is non-degenerate.
(2) Such a problem cannot cycle because of what is mentioned above. To cycle, a problem would have to contain a series of iterations through which you yield the dictionary that started with at the beginning of the cycle. Because we cannot create any additional degeneracies by the answer above, there is no first degeneracy to begin a cycle of degeneracies.