HW-5

DREW MORRIS

Problem 1. *Consider the following linear programming problem.*

$$\begin{array}{lrcrcrcrcl} \textit{maximize} & 2x_0 & + & 8x_1 & - & x_2 & - & 2x_3 & & \\ \textit{subject to} & 2x_0 & + & 3x_1 & + & 0x_2 & + & 6x_3 & \leq & 6 \\ & -2x_0 & + & 4x_1 & + & 3x_2 & + & 0x_3 & \leq & \frac{3}{2} \\ & 3x_0 & + & 2x_1 & - & 2x_2 & - & 4x_3 & \leq & 4 \\ & x_0 & , & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

Do the dual primal/dual solutions from the previous homework satisfy the complementary slackness theorem.

Proof. Observe the initial primal and dual initial dictionaries.

$$\begin{array}{|ccccccccccc|} \hline \zeta & = & 0 & + & 2x_0 & + & 8x_1 & - & x_2 & - & 2x_3 \\ \hline u_0 & = & 6 & - & 2x_0 & - & 3x_1 & + & 0x_2 & - & 6x_3 \\ u_1 & = & \frac{3}{2} & + & 2x_0 & - & 4x_1 & - & 3x_2 & + & 0x_3 \\ u_2 & = & 4 & - & 3x_0 & - & 2x_1 & + & 2x_2 & + & 4x_3 \\ \hline \end{array} \qquad \begin{array}{|ccccccccc|} \hline \xi & = & 0 & + & 6y_0 & + & \frac{3}{2}y_1 & + & 4y_2 \\ \hline v_0 & = & 2 & - & 2y_0 & + & 2y_1 & - & 3y_2 \\ v_1 & = & 8 & - & 3y_0 & - & 4y_1 & - & 2y_2 \\ v_2 & = & -1 & + & 0y_0 & - & 3y_1 & + & 2y_2 \\ v_3 & = & -2 & - & 6y_0 & + & 0y_1 & + & 4y_2 \\ \hline \end{array}$$

Recall that our primal solution was $\mathbf{x} = \begin{bmatrix} \frac{13}{16} & \frac{3}{8} & 0 & \frac{65}{304} \end{bmatrix}^T$. If this solution satisfies the complementary slackness theorem then $\mathbf{v} = \begin{bmatrix} 0 & 0 & v_2 & 0 \end{bmatrix}$ where $v_2 \in \mathbb{F}$. Looking at the dual problem we see the dual solution must satisfy the following matrix solution.

$$\left[\begin{array}{cccc|c} 2 & -2 & 3 & 0 & 2 \\ 3 & 4 & 2 & 0 & 8 \\ 0 & 3 & -2 & v_2 & -1 \\ 6 & 0 & -4 & 0 & -2 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 2 & -2 & 3 & 0 & 2 \\ 3 & 4 & 2 & 0 & 8 \\ 6 & 0 & -4 & 0 & -2 \\ 0 & 3 & -2 & v_2 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 3 & 4 & 2 & 0 & 8 \\ 6 & 0 & -4 & 0 & -2 \\ 0 & 3 & -2 & v_2 & -1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \cdots \xrightarrow{R_4 - 6R_1}$$

Date: October 9th, 2023.

$$\left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 0 & 7 & -\frac{5}{2} & 0 & 5 \\ 0 & 6 & -13 & 0 & -8 \\ 0 & 3 & -2 & v_2 & -1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 0 & 1 & -\frac{5}{14} & 0 & \frac{5}{7} \\ 0 & 6 & -13 & 0 & -8 \\ 0 & 3 & -2 & v_2 & -1 \end{array}\right] \xrightarrow{R_3-6R_2} \cdots \xrightarrow{R_4-3R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 0 & 1 & -\frac{5}{14} & 0 & \frac{5}{7} \\ 0 & 0 & -\frac{76}{7} & 0 & -\frac{26}{7} \\ 0 & 0 & -\frac{13}{14} & v_2 & -\frac{22}{7} \end{array}\right] \xrightarrow{-\frac{7}{76}R_3} \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 0 & 1 & -\frac{5}{14} & 0 & \frac{5}{7} \\ 0 & 0 & 1 & 0 & \frac{13}{38} \\ 0 & 0 & -\frac{13}{14} & v_2 & -\frac{22}{7} \end{array}\right] \xrightarrow{R_4+\frac{13}{14}R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & 0 & 1 \\ 0 & 1 & -\frac{5}{14} & 0 & \frac{5}{7} \\ 0 & 0 & 1 & 0 & \frac{13}{38} \\ 0 & 0 & 0 & v_2 & -\frac{1503}{532} \end{array}\right] \xrightarrow{R_1-\frac{3}{2}R_3} \cdots \xrightarrow{R_2+\frac{5}{14}R_3} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & \frac{37}{76} \\ 0 & 1 & 0 & 0 & \frac{445}{532} \\ 0 & 0 & 1 & 0 & \frac{13}{38} \\ 0 & 0 & 0 & v_2 & \frac{1503}{532} \end{array}\right] \xrightarrow{R_1+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{176}{133} \\ 0 & 1 & 0 & 0 & \frac{445}{532} \\ 0 & 0 & 1 & 0 & \frac{13}{38} \\ 0 & 0 & 0 & v_2 & \frac{1503}{532} \end{array}\right]$$

Thus we see $v_2 \neq 0 \implies x_2 = 0$. Additionally, $\mathbf{y} = \begin{bmatrix} \frac{176}{133} & \frac{445}{532} & \frac{13}{38} & \frac{1503}{532} \end{bmatrix}^T$ so $\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Thus

$$6 - 2\left(\frac{13}{16}\right) - 3\left(\frac{3}{8}\right) - 6\left(\frac{65}{304}\right) = 0$$

i.e.

$$\frac{299}{152} = 0$$

which is false. Therefore, our solutions are non-optimal.

$\square$

Problem 2. *Consider the following process.*
Take a linear programming problem in standard form. Form its dual problem. Replace the minimization in the dual problem with maximization. By strong duality, the optimal solution to this new problem must be equal to the original problem's optimal solution. By the following inequalities.

$$\zeta^*(a_{(i,j)}, b_i, c_j) \leq \zeta^*(-a_{(j,i)}, -c_j, b_i) \leq \zeta^*(a_{(i,j)}, -b_i, -c_j) \leq \zeta^*(-a_{(j,i)}, c_j, -b_i) \leq \zeta^*(a_{(i,j)}, b_i, c_j)$$

While this logic may sometimes work, it is not always true. What is the flaw in this logic? Can you a state a correct (non-trivial) theorem along the same kind of reasoning? Can you give an example where these above inequalities are equalities?

Proof. Swapping min with max without negating the objective function does not create an equivalent dual problem and as such, strong duality cannot be applied. If we make this adjustment, we get a rephrasing of the Strong Duality Theorem. A (albeit trivial) example that does use the non-negated maximization version of the dual problem that satisfies making all the listed inequalities into equality would be a case where the objective function is the zero function. □

Problem 3. *Let m be the number of nutrients found to be important to a subjects diet and let $b_i, i \in \mathbb{N} \cap [1, m]$ denote the minimum daily requirements for each nutrient. Now let n be a list of foods such that $c_j, j \in \mathbb{N} \cap [1, n]$ denotes the prices (per unit) of each of these foods. Finally, let $a_{(i,j)}$ be the amount of nutrient i contained in food j. The following linear programming problem will minimizes cost while still satisfying the minimum required nutrients.*

$$\begin{array}{rlcl} \textit{minimize} & \sum_{j=1}^{n} c_j x_j & & \\ \textit{subject to} & \forall_{i=1}^{m} \sum_{j=1}^{n} a_{(i,j)} x_j & \geq & b_i \\ & \forall_{j=1}^{n} x_j & \geq & 0 \end{array}$$

Formulate the dual problem. Can you introduce another person into the given story whose problem would naturally be to solve the dual problem?

Proof. The dual problem would be the following.

$$\begin{array}{rlcl} \text{maximize} & \sum_{i=1}^{m} b_i y_i & & \\ \text{subject to} & \forall_{j=1}^{n} \sum_{i=1}^{m} a_{(j,i)} y_i & \leq & c_j \\ & \forall_{i=1}^{m} y_i & \leq & 0 \end{array}$$

In words, maximize the nutrients of a diet subject to budget constraints. A character whose natural problem would be to solve this would be a nurse caring for nutrient-deficient patients. This nurse is alloted a budget for their patient and, with that, wishes to maximize their patient's nutrient intake. □