HW-9

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Problem 1. *Consider the set of resource allocation games of* 2 *agents. Do the following and justify each answer.*

(1) *Find the price of anarchy for all such games.*

(2) *Use your solution and a Venn Diagram to fill out the values of* v_S *for all* S.

(3) *Determine whether or not* $a^{eq} = \{a_1^{eq}, a_2^{eq}\}$ *is an equillibrium.*

(4) *Determine whether or not* $a^{opt}\{a_1^{opt}, a_2^{opt}\}$ *is optimal.*

(5) *Determine the value of the price of anarchy.*

(1) Solving the linear programming problem from class yields a price of anarchy of 1.

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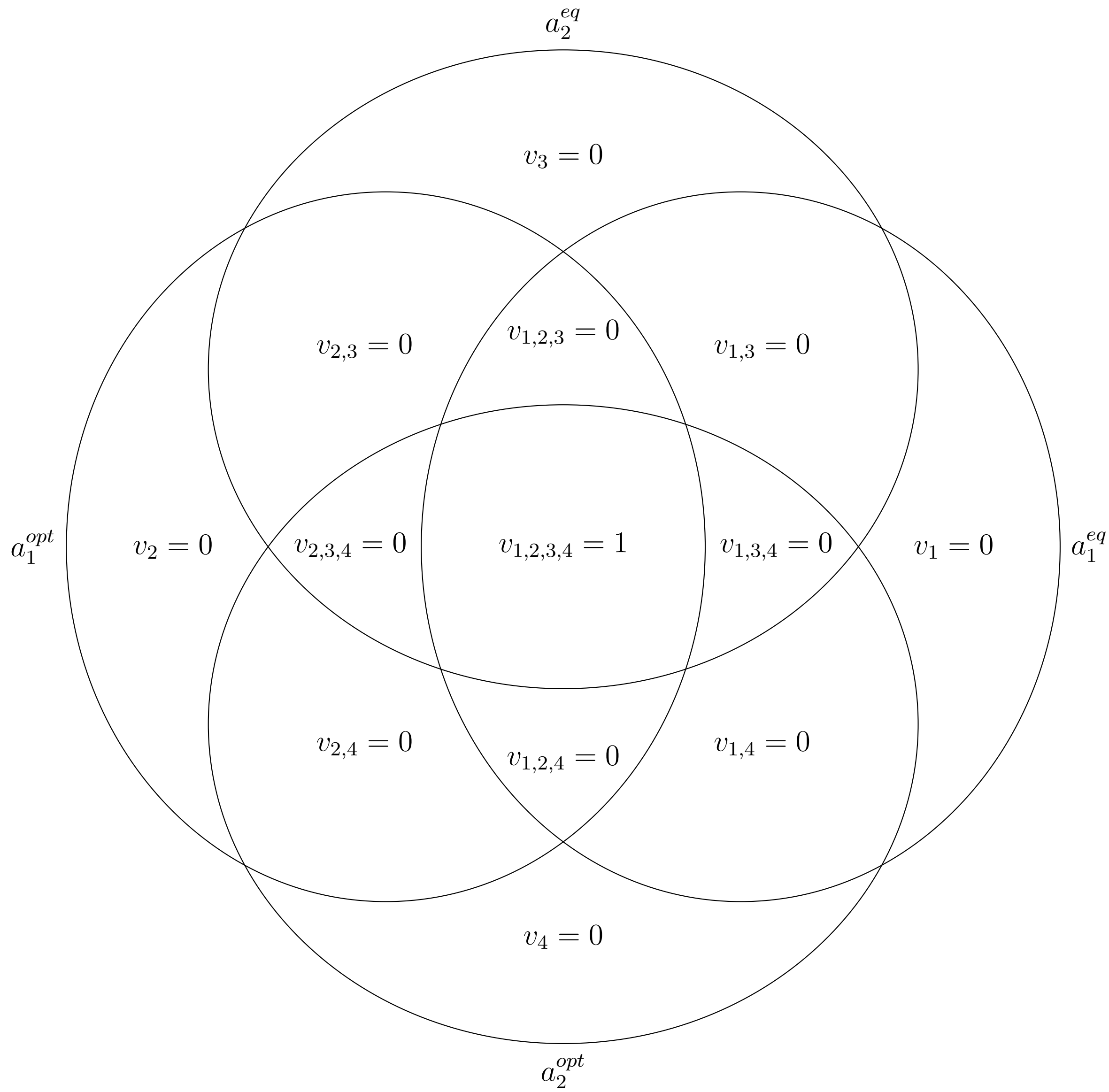
(2) Below is the requested diagram.



(3) a^{eq} is an equillibrium by assumption.

(4) a^{opt} is an optimal set of actions by assumption.

(5) The value of the price of anarchy is 1, meaning, in the case of two agents, the only equillibria are optima.

Problem 2. *Find the L^2-regression for the data in Figure 12.8.*

Proof. Given the data,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \in M_{4\times 2}(\mathbb{R}), \mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^4$$

we wish to find

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2, \mathbf{n} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} \in \mathbb{R}^4$$

such that

$$A\mathbf{x} = \mathbf{y} - \mathbf{n}$$

and

$$\mathbf{n} = \min_{\mathbf{k}\in\mathbb{R}^4} ||\mathbf{k}||_2$$

Notice

$$A^H = \begin{bmatrix} 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$A^H A = \begin{bmatrix} 21 & 7 \\ 7 & 4 \end{bmatrix}$$

and

$$A^H \mathbf{y} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

Thus, finding $\mathbf{x}$, is equivalent to solving

$$A^H A\mathbf{x} = A^H \mathbf{y}$$

i.e.

$$\mathbf{x} = (A^H A)^{-1} A^H \mathbf{y}$$

Notice

$$(A^H A)^{-1} = \frac{1}{35} \begin{bmatrix} 4 & -7 \\ -7 & 21 \end{bmatrix}$$

Thus

$$\mathbf{x} = \frac{1}{35} \begin{bmatrix} 4 & -7 \\ -7 & 21 \end{bmatrix} \begin{bmatrix} 13 \\ 6 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 10 \\ 35 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$$

Therefore, the L^2-regression is

$$y = \frac{2}{7} x + 1$$

$\square$

Problem 3. *Find the L^1-regression for the data in Figure* 12.8.

Proof. Given the data,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \in M_{4\times 2}(\mathbb{R}), \mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} \in \mathbb{R}^4$$

we wish to find

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2, \mathbf{n} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} \in \mathbb{R}^4$$

such that

$$A\mathbf{x} = \mathbf{y} - \mathbf{n}$$

and

$$\mathbf{n} = \min_{\mathbf{k} \in \mathbb{R}^4} ||\mathbf{k}||_1$$

Notice

$$\mathbf{n} = \mathbf{y} - A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0a + b \\ a + b \\ 2a + b \\ 4a + b \end{bmatrix} = \begin{bmatrix} -b \\ 3 - a - b \\ 1 - 2a - b \\ 2 - 4a - b \end{bmatrix}$$

Thus we wish to solve

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ such that } \min_{a,b \in \mathbb{R}} |-b| + |3 - a - b| + |1 - 2a - b| + |2 - 4a - b|$$

Which yields

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

□

Problem 4. *Given a sorted set,* $(b_i)_{i=1}^m \subseteq \mathbb{R}$*, show the midrange of* $(b_i)_{i=1}^m$*,*

$$\tilde{x} = \frac{b_1 + b_m}{2}$$

satisfies

$$\tilde{x} = \text{argmin}_{x \in \mathbb{R}} \max_{i \in \mathbb{N} \cap [1,m]} |x - b_i|$$

Proof. Since $(b_i)_{i=1}^m$ is sorted, we know for any given $x \in \mathbb{R}$, $(b_i - x)_{i=1}^m$ is sorted. Thus $\tilde{x}$ needs to be the median of the range spanned by $(b_i)_{i=1}^m$ i.e. $\tilde{x}$ is the median of $[b_1, b_m]$. Therefore

$$\tilde{x} = \frac{b_1 + b_m}{2}$$

□

Problem 5. *Given a sorted set, $(b_i)_{i=1}^m \subseteq \mathbb{R}^2$, show the midrange of $(b_i)_{i=1}^m$,*

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} b_i$$

satisfies

$$\bar{x} = \min_{x \in \mathbb{R}^2} \sum_{i=1}^{m} ||x - b_i||_2^2$$

Proof. Due to the lack of time my reasoning is not as thorough as usual. We know that $\bar{x}$ needs minimize its difference between the given points on both axes with respect to Euclidean distance (2-norm). Thus we take the mean across the 2-norms distances of the difference between x and the given points. $\square$