Assume that stock prices are a stochastic process:

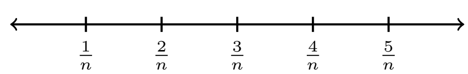
$$\{S(t)\}_{t \geq 0}$$

The price of t at any time is random. Lets try to answer the question: What is the distribution of these random variables?

1 Assumptions:

A. Efficient Market Hypothesis: roughly speaking states that the stock price of say t = 10 shouldn't depend on the stock price at time 0, 1, 2, etc. only the instantaneous future. So the stock price at t = 10.0001 should only depend on the stock price at time 10. In other words. This is a Markov assumption that the stock price only depends on the immediate past. No one queries the price of a stock in 1932 to figure out the stock price tomorrow - the stock could be bankrupt or could be thriving... we don't know.

B. Breaking it down into n-steps follows a Normal Distribution:
If the one year returns is $N(0, 1)$, then $\frac{1}{n}$ time steps are $N(0, \frac{1}{n})$ as shown by the number line below:



If you break up the year into n equal pieces, all of those pieces must have the same distribution$N(0, \frac{1}{n})$ and be independent of each other. This means that $\sigma \approx \frac{1}{\sqrt{n}}$

C. The expected return and volatility do not depend on the price but rather the percentages do (relative price): If you have 100 shares in the stock and you expect a 10% return if you have 10,000 shares instead you should reasonably expect the same return of 10%. Same for volatility. If you own 100 dollars worth of a position that shouldn't change if you own 1000 dollars worth of a position. The volatility shouldn't change based on the scale it's a relative ratio. We can define this mathematically as:

$$\frac{\Delta s}{s}$$

2 Formulating a Model

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

The change in a stock price can be defined as a constant drift (expected return) μ over time Δt and the volatility σ times ΔW where delta W represents the increments of randomness in stock price movements.

$$\Delta W = \sqrt{\Delta t}$$

We know that ΔW follows a Normal Distribution. According to the Central Limit Theorem, if $\Delta W = W_{t+\Delta t} - W_t \sim N(0, \Delta t)$ the standard deviation is $\sqrt{\Delta t}$.

In continuous time we rewrite the equation as:

$$dS = \mu S dt + \sigma S dW$$

In this context the dW is the differential Brownian Motion. This is our Lognormal model of stock growth.

3 Why it is Lognormal

If we consider f(S) = ln(S) and we use Itô's formula:

$$df = f_t + \mu S f_s + \frac{1}{2}\sigma^2 S^2 f_{ss} + \sigma S f_s dW$$

Where $f_t + \mu S f_s + \frac{1}{2}\sigma^2 S^2 f_{ss}$ is the deterministic component and $\sigma S f_s dW$ is the random component
Now the derivative of f with respect to S $f_s = \frac{1}{S}$ and $f_{ss} = \frac{-1}{S^2}$ so df satisfies the following Itô:

$$df = f_t + \mu S f_s + \frac{1}{2}\sigma^2 S^2 f_{ss}$$

f_t can be ignored because there is no t dependence

$$\mu S f_s = \mu S \frac{1}{S} = \mu$$

$$\frac{1}{2}\sigma^2 S^2 f_{ss} = \frac{1}{2}\sigma^2 S^2 \frac{-1}{S^2} = -\frac{1}{2}\sigma^2$$

$$\sigma S f_s dW = \sigma S \frac{1}{S} dW = \sigma dW$$

$$\text{So } df = [\mu - \frac{1}{2}\sigma^2]dt + \sigma dW$$

Now we know what f is if we integrate the stochastic equation we will get:

$$ln(\frac{S(t)}{S(0)}) = [\mu - \frac{1}{2}\sigma^2]t + \sigma W(t) \text{ Therefore: } S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma)t + \sigma W(t)}$$

It is an exponential with a normally distributed variable which is why it is called a lognormal distribution.