Numerical Simulations Homework # 3

Drew Coffin

February 8, 2017

1 Problem 1

1.1 Part a

Our exact function is $\widetilde{T}$, which has the general Taylor series:

$$\frac{\partial \widetilde{T}}{\partial x} = \widetilde{T}(x_0) + \Delta x \frac{\partial \widetilde{T}}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \widetilde{T}}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 \widetilde{T}}{\partial x^3} + ...$$

We are approximating this derivative via three pieces: a spatial step "back," the current location, and a spatial step forward:

$$\left[\frac{\partial \widetilde{T}}{\partial x}\right]_j^n = a\widetilde{T}_{j-1}^n + b\widetilde{T}_j^n + c\widetilde{T}_{j+1}^n + O(\Delta x^m)$$

and either of the two spatial steps can be expressed as Taylor series. The $\widetilde{T}_{j-1}^n$ term is with a spatial step of $-\Delta x$, meaning its Taylor series, with a coefficient of a, will *alternate* sign. Meanwhile, the $\widetilde{T}_{j+1}^n$ term is with a spatial step of $+\Delta x$. Our current position, $\widetilde{T}_j^n$, has no associated Δx, giving us only the $b\widetilde{T}_j^n$ term. Combine these two series and our $b\widetilde{T}_j^n$ term and we have

$$\left[\frac{\partial \widetilde{T}}{\partial x}\right]_j^n = (a+b+c)\widetilde{T}_j^n + (-a+c)\Delta x\left[\frac{\partial \widetilde{T}}{\partial x}\right]_j^n + (a+c)\frac{\Delta x^2}{2}\left[\frac{\partial^2 \widetilde{T}}{\partial x^2}\right]_j^n + (-a+c)\frac{\Delta x^3}{6}\left[\frac{\partial^3 \widetilde{T}}{\partial x^3}\right]_j^n + ...$$

1.2 Part b

Since we have

$$\left[\frac{\partial \widetilde{T}}{\partial x}\right]_j^n = a\widetilde{T}_{j-1}^n + b\widetilde{T}_j^n + c\widetilde{T}_{j+1}^n + O(\Delta x^m)$$

we know that our coefficients of $\widetilde{T}_j^n$ must vanish, i.e. $a+b+c=0$. Likewise, we want $(-a+c)\Delta x = 1$ to return our first order derivative. Thus, $a = c - 1/\Delta x$ and $b = -2c + 1/\Delta x$. Note that since a and c depend on $1/\Delta x$, the term we wish to eliminate in our Taylor approximation is the $(a + c)\Delta x^2$ term. Thus, $a = -c = -1/2\Delta x$ and $b = 0$. This gives us a final approximation of

$$\left[\frac{\partial \widetilde{T}}{\partial x}\right]_j^n = \frac{\widetilde{T}_{j+1}^n - \widetilde{T}_{j-1}^n}{2\Delta x} + O(\Delta x^2)$$

1.3 Part c

Working with the second derivative, now we want $(a + c)\Delta x^2/2 = 1$, so $a = -c + 2/\Delta x^2$, and we also know $(-a + c)\Delta x^3/6 = 0$ to zero out what will become our Δx term. So $a = c = 1/\Delta x^2$, and from our first term, $a + b + c = 0$ so $b = -2/\Delta x^2$. Thus we obtain

$$\left[\frac{\partial^2 \widetilde{T}}{\partial x^2}\right]_j^n = \frac{\widetilde{T}_{j+1}^n - 2\widetilde{T}_j^n + \widetilde{T}_{j-1}^n}{\Delta x^2} + O(\Delta x^2)$$

2 Problem 2

2.1 Part a

For $y = \sin \pi x/2$, we are evaluating dy/dx at $x = 0.5$. Explicitly, $dy/dx = (\pi/2)\cos \pi x/2$.

Three point symmetric error:

$$\frac{dy}{dx} \approx \frac{y_{j+1} - y_{j-1}}{2\Delta x} = \frac{\sin 0.3\pi - \sin 0.2\pi}{2 \times 0.1} \approx 1.10616$$

Our error is $\pi/2 \cos \pi x/2$ minus the above, which is -0.0045620.

We compare this to

$$\Delta x^2 \frac{f_{xxx}}{6} = (0.1^2)\frac{(-\pi/2)^3 \cos 0.25\pi}{6} \approx -0.0045677$$

which differs by 5 parts in one million.
Forward difference:

$$\frac{dy}{dx} \approx \frac{y_{j+1} - y_j}{\Delta x} = \frac{\sin 0.3\pi - \sin 0.25\pi}{0.1} \approx 1.019102$$

Our error is $\pi/2 \cos \pi x/2$ minus the above, which is 0.091618.

We compare this to

$$\Delta x \frac{f_{xx}}{2} = 0.1 \frac{(\pi/2)^2(-\sin 0.25\pi)}{2} \approx 0.01935900$$

which differs by eight parts in one hundred.
Five point symmetric error:

$$\frac{dy}{dx} \approx \frac{y_{j+2} - 8y_{j+1} + 8y_{j-1} - y_{j-2}}{12\Delta x} \approx 1.1106982$$

Our error is $\pi/2 \cos \pi x/2$ minus the above, which is 2.2474×10^{-5}.
We compare this to

$$\Delta x^4 \frac{f_{xxxxx}}{30} = 10^{-5} \frac{(\pi/2)^5(\cos 0.25\pi)}{30} \approx 3.177879 \times 10^{-6}$$

3 Problem 3

3.1 Part a

Our equation is

$$\frac{\partial \widetilde{T}}{\partial t} - \alpha \frac{\partial^2 \widetilde{T}}{\partial x^2} = 0$$

We insert our first and second derivative expanisons from Problem 1 and include only first-order terms. Note we only have a forward step in time, not a centered derivative like in the spatial terms.

$$\frac{\widetilde{T}_j^{n+1} - \widetilde{T}_j^n}{\Delta t} - \alpha \frac{\widetilde{T}_{j+1}^n - 2\widetilde{T}_j^n + \widetilde{T}_{j-1}^n}{\Delta x^2} = 0$$

$$\widetilde{T}_j^{n+1} - \widetilde{T}_j^n = \frac{\alpha \Delta t}{\Delta x^2} \left(\widetilde{T}_{j+1}^n - 2\widetilde{T}_j^n + \widetilde{T}_{j-1}^n \right)$$

Let $\alpha \Delta t / \Delta x^2 = s$, so we have

$$\widetilde{T}_j^{n+1} = \widetilde{T}_j^n + s \left(\widetilde{T}_{j+1}^n - 2\widetilde{T}_j^n + \widetilde{T}_{j-1}^n \right)$$

$$\widetilde{T}_j^{n+1} = s\widetilde{T}_{j+1}^n + (1 - 2s)\widetilde{T}_j^n + s\widetilde{T}_{j-1}^n$$

3.2 Part c

The five-point symmetric scheme should converge much more rapidly.