Numerical Simulations Homework # 2

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February 1, 2017

1 Problem 1

1.1 Part a

The given problem is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < 1 \quad 0 < y < 1 \tag{1}$$

We are given that $T(0, y) = T(1, y) = T(x, 1) = 0$ and $T(x, 0) = T_o$, i.e. a unit square with a value of 1 along the x-axis and 0 along the other sides. Let us presume $T(x, y) = X(x)Y(y)$. Substitute into Eq. 1 and divide through by $T(x, y)$:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

This is satisfied by letting $X(x) = A\sin(n\pi x)$ with n as yet being undetermined. Meanwhile, $Y(y)$ has the form of a exponential decay; however no individual expression of the form $\sinh y$ will satisfy both conditions. Instead, we use a Fourier series of the form:

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\sinh[n\pi\,(1 - y)]$$

and A_n is determined by

$$A_n = \frac{2T_o}{\sinh(n\pi)}\int_0^1 \sin(n\pi x)dx$$

$$A_n = \frac{4T_o}{n\pi \sinh(n\pi)}$$

From the shape of the graph, only the odd A_n terms contribute.

Now, we plot.

1.2 Part c

The given problem is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < 1 \quad 0 < y < 1 \tag{2}$$

We are given that $T(0, y) = T(1, y) = 0$ and $T(x, 0) = T(x, 1) = T_o$, i.e. a unit square with a value of 1 along bothsides parallel to the x-axis and 0 along the sides parallel to the y-axis. The solution follows precisely above, except we will use a $cosh(y)$ expression shifted to center on $y = 1/2$, to give us the symmetric "valley" shape.

I.e.

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \cosh[n\pi\,(y - 1/2)]$$

with B_n being

$$B_n = \frac{4T_o}{n\pi \cosh(n\pi/2)}$$

Again, we only retain odd B_n terms.

2 Problem 2

2.1 Part a

The given equation is $\partial^2 u/\partial x \partial t = 0$ and we wish to know what type of second-order PDE it is. The type is determined by the discriminant of the coefficients of the second-order derivatives of our function $u(x, y)$:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D(x, y) = 0$$

If $B^2 - 4AC > 0$, our equation is hyperbolic.

If $B^2 - 4AC = 0$, our equation is parabolic.

If $B^2 - 4AC < 0$, our equation is elliptic.

$B^2 - 4AC$ in our case will be $1^2 - 4 \cdot 0 \cdot 0 = 1 > 0$ so our equation is hyperbolic.

2.2 Part b

The given system of equations is $\partial v/\partial x = 0$ and $\partial u/\partial t - v = 0$.

If we take a spatial derivative of the second equation, we achieve:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - v\right) = 0$$

$$\frac{\partial^2 u}{\partial x \partial t} - \cancelto{0}{\frac{\partial v}{\partial x}} = 0$$

$$\frac{\partial^2 u}{\partial x \partial t} = 0$$

as above, hence this is a hyperbolic equation.

For finding characteristics, we wish to find total time derivatives.

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\frac{dx}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial x}{\partial x} = 0$$

We have lost our x-dependence, thus we know we have no information about our equation based on the domain. Thus, only our coordinates, and thus their axes, give us characteristics.