EULER-POISSON INTEGRAL PROOF

In this article I will explain how derive formula for Euler-Poisson integral (or more commonly known as Gaussian integral). The proof itself is based on radial symmetry of Gaussian function, even in context of multivariable system.

Euler-Poisson integral (from now will be called Gaussian) has following structure:

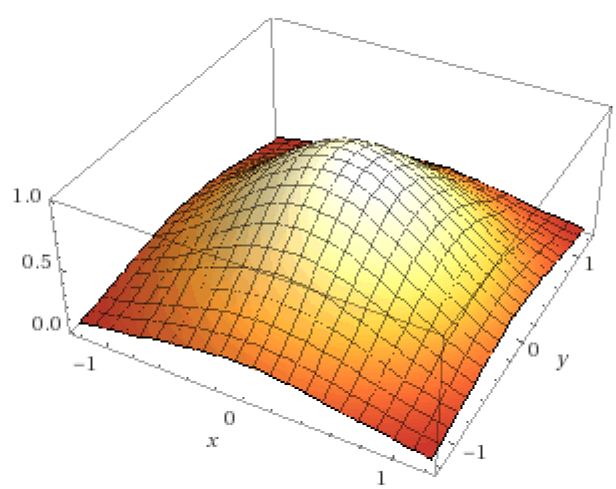
$$I = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Assuming that

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \int_{-\infty}^{+\infty} e^{-y^2}\, dy$$

then it's obvious that multiplying both integrals can transform Gaussian function into 3D space and open new ways for advancement.

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx \int_{-\infty}^{+\infty} e^{-y^2}\, dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy = I^2$$



Exponent $-(x^2 + y^2)$ is very simillar to Carthesian system circle equation with origin (0,0). This fact allows us to replace it with r^2 and treat the integral in circular ways. We need to, however, adjust the integral for Polar coordinate system instead of Carthesian, since it offers more efficient approaches to solve the integral.

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-r^2} \, dx \, dy$$

The idea behind solving this integral is such that we are stacking infinite amount of hollow cylinders under the function in order to find it's volume. Because it is radially symmetric, after using enough cylinders we can get close enough value of the volume.

One can imagine such cylinder as a volume of arc length of vector rotated under angle $d\varphi$ with infinitely small thickness and height equal to functional value. So if we assume standard radian definition

$$\varphi = \frac{arc\ length}{radius}$$

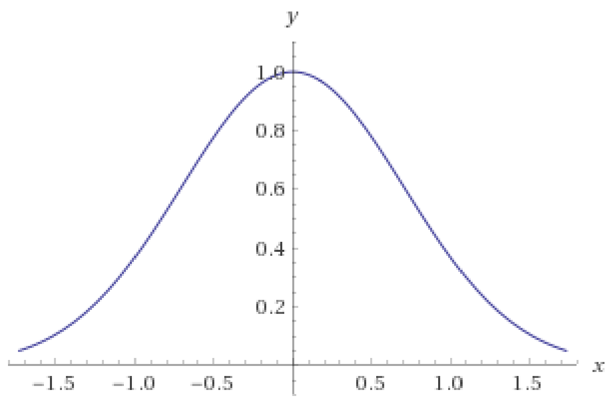
we can easily conclude that $arc\ length = d\varphi * r$ and since we want to turn it into cylinder, we need to apply infinitely small thickness which is equal to radius differential dr. With such approach we derived formula for content of bottom side of cylinder.

$$s = \ r * dr * d\varphi$$

To make it 3D hollow cylinder, we also need to give it a height and therfore multiply with our functional value.

$$c = e^{-r^2} * r * dr * d\varphi$$

Now we need to adjust limits of integration. In Polar system, cylinder is common circle, and therefore it's a vector rotated under the angle of 2π. Limits of the other integral will scale from 0 to ∞, because radius of our pseudo-circles can be expanding forever.



As functional value approaches zero, radiuses of imaginary cylinders keep expanding.

For the reasons explained above, our integral will look like this

$$\int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, r\, dr\, d\varphi$$

and we can easily evaluate it using substitution rule.

$u = r^2$

$du = -2r\, dr$

$\int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, r\, dr = \int_0^{2\pi} d\varphi \int_0^{\infty} e^u - \frac{du}{2} = 2\pi\left(-\frac{1}{2}\right)\int_0^{\infty} e^u\, du$

$e^{-\infty} - e^0 = -1$

$2\pi\left(-\frac{1}{2}\right)(-1) = 2\pi\,\frac{1}{2} = \pi$

$I = \sqrt{I^2} = \sqrt{\pi}$