

CM1103 PROBLEM SOLVING WITH PYTHON

# LOGIC

AN INTRODUCTION

---

Dr Jing Wu

School of Computer Science & Informatics

Cardiff University

# Propositional logic

## Definition

A proposition is a sentence declaring a fact that is either ***True*** or ***False***.

## Example

Stuart Allen is the module leader of CM1103.	
Cardiff is the capital of France.	
$8 + 2 = 10$	
$8 + 2 = 15$	
Is it Tuesday?	
Read this.	
$n < 5$	
If $n$ is 4, then $n < 5$	

# Combining propositions

## Definition

Compound propositions are formed from other propositions using logical operators.

## Logical operators

Negation	$\neg$
Conjunction	$\wedge$
Disjunction	$\vee$
Exclusive or	$\oplus$
Conditional / Implication	$\Rightarrow$
Biconditional	$\Leftrightarrow$

# Negation $\neg$

## Definition

Let  $p$  be a proposition. The *negation* of  $p$ , denoted  $\neg p$ , is the statement “*It is not the case that  $p$* ”.

## Example

$p$	$\neg p$
Cardiff is the capital of France.	
$8 + 2 = 10$	
My hard drive has at least 500 GB.	

# Conjunction $\wedge$ and disjunction $\vee$

## Definition

Let  $p$  and  $q$  be propositions. The *conjunction* of  $p$  and  $q$ , denoted  $p \wedge q$ , is the proposition “ $p$  and  $q$ ”. It is true if both  $p$  and  $q$  are true, and false otherwise.

## Definition

Let  $p$  and  $q$  be propositions. The *disjunction* of  $p$  and  $q$ , denoted  $p \vee q$ , is the proposition “ $p$  or  $q$ ”. It is true if either  $p$  or  $q$  (or both) are true, and false otherwise.

## Example

Let  $p$  be the proposition “*Cardiff is the capital of France*”, and let  $q$  be the proposition  $2 + 2 = 4$ .

$p$	$q$	$p \wedge q$	$p \vee q$	$\neg p \wedge q$

# Truth tables

A truth table gives the truth value of a compound proposition for all possible combinations of truth-values of the individual propositions.

Each row shows one possible combination.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

VS

	$p$	
$\wedge$	T	F
$q$ T	T	F
F	F	F



$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

# Truth tables

$p$	$\neg p$
T	F
F	T

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

# Truth tables

Add operands;

Compare different compound propositions.

$p$	$q$	$r$	$p \wedge q$	$(p \wedge q) \wedge r$	$(p \wedge q) \vee r$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	F	T
T	F	F	F	F	F
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	F	F	T
F	F	F	F	F	F



# Truth tables

Construct truth tables for complex propositions, e.g.,

$$(p \wedge q) \vee \neg r$$

# Short circuit evaluation

For some Boolean operators in Python (and in some other programming languages as well), the second argument is executed or evaluated only if the first argument does not suffice to determine the value of the expression:

- When the first argument of the AND function evaluates to *false*, the overall value must be *false*;
- When the first argument of the OR function evaluates to *true*, the overall value must be *true*.

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

# Exclusive or $\oplus$

## Definition

Let  $p$  and  $q$  be propositions. The *exclusive or* of  $p$  and  $q$ , denoted  $p \oplus q$ , is the proposition that is true if exactly one of  $p$  and  $q$  are true, and false otherwise.

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

## Inclusive vs. Exclusive Or

I'd like some tea **or** coffee.

I'd like some milk **or** sugar with my tea.

# Translate English into Logic

Example:

Let  $p$  be: Today is hot.

Let  $q$  be: Today is sunny.

Translate:

Today it is hot, but not sunny.

Today it is neither hot nor sunny.

## Knights & Knaves example

A very special island is inhabited only by knights and knaves.  
Knights always tell the truth, and knaves always lie.

You meet two inhabitants: Zoey and Mel. Zoey tells you that Mel is a knave.  
Mel says, “Neither Zoey nor I are knaves.”

Can you determine who is a knight and who is a knave?

## More Knights and Knaves puzzles

<http://philosophy.hku.hk/think/logic/knights.php>

The key is to translate their statements into logic.

Logic operators may not be obvious.

Always ask yourself what's the statement *'in other words'*

# Implication

## Definition

Let  $p$  and  $q$  be propositions. The *implication* or *conditional statement*  $p \Rightarrow q$ , is the proposition “if  $p$  then  $q$ ”. We call  $p$  the hypothesis,  $q$  the conclusion. It is false if  $p$  is true and  $q$  is false, and true otherwise. It has the truth table:

$p$	$q$	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Think it as a promise.

When the promise is broken.

## Example

If you tidy your room, you will get a candy.

$p \Rightarrow q$

### Examples: Are these implications true or false?

If  $4 \times 4 = 16$ , then  $(-1) \times (-1) = 1$ .

If  $2 + 2 = 4$ , then all dogs are black.

If  $2 + 2 = 5$ , then all dogs are black.

If it rains today, then 2 is a prime number.



# Contrapositive

## Definition (Contrapositive)

The *contrapositive* of the implication  $p \Rightarrow q$  is  $\neg q \Rightarrow \neg p$ .

## Example

$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
If you tidy your room, you will get a candy.	
If you get full marks on CM1103 exam, you will pass the module.	

$p$	$q$	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T			
T	F	F			
F	T	T			
F	F	T			

# Converse

## Definition (Converse)

The *converse* of the implication  $p \Rightarrow q$  is  $q \Rightarrow p$ .

## Example

$$p \Rightarrow q$$

If you get full marks on CM1103 exam, you will pass the module.

$$q \Rightarrow p$$

$p$	$q$	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	
T	F	F	
F	T	T	
F	F	T	

# Biconditional

## Definition

Let  $p$  and  $q$  be propositions. The *biconditional of  $p$  and  $q$*  is the proposition “ $p$  if, and only if,  $q$ ” and is denoted  $p \Leftrightarrow q$ . It is true if both  $p$  and  $q$  have the same truth values and is false if  $p$  and  $q$  have opposite truth values.

$p$	$q$	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T			
T	F	F			
F	T	F			
F	F	T			

# Tautologies and contradictions

## Definition (Tautology)

A *tautology* is a compound proposition that is always true.

## Definition (Contradiction)

A *contradiction* is a compound proposition that is always false.

## Example

Tautology	Contradiction

# Logical equivalence $\equiv$

## Definition

Compound propositions  $p$  and  $q$  are *logically equivalent* if, and only if,  $p$  and  $q$  have the same truth table. The logical equivalence of  $p$  and  $q$  is denoted by  $p \equiv q$ .

## Logical equivalence: example

Prove that propositions  $p \Rightarrow q$  and  $\neg p \vee q$  are logically equivalent.

# De Morgan's laws

## De Morgan's laws

Let  $p$  and  $q$  be propositions. Then:

$\neg(p \vee q)$  is logically equivalent to  $\neg p \wedge \neg q$

$\neg(p \wedge q)$  is logically equivalent to  $\neg p \vee \neg q$

## Example

Given a day  $d$  and month  $m$ , when is it not April 23<sup>rd</sup>?

# Logical equivalences

<b>Commutative</b>	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
<b>Associative</b>	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
<b>Distributive</b>	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
<b>Identity</b>	$p \wedge T \equiv p$	$p \vee F \equiv p$
<b>Negation</b>	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
<b>Idempotent</b>	$p \wedge p \equiv p$	$p \vee p \equiv p$
<b>De Morgan</b>	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
<b>Double negation</b>	$\neg(\neg p) \equiv p$	



# Using equivalences

## Example

Show  $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

### Example

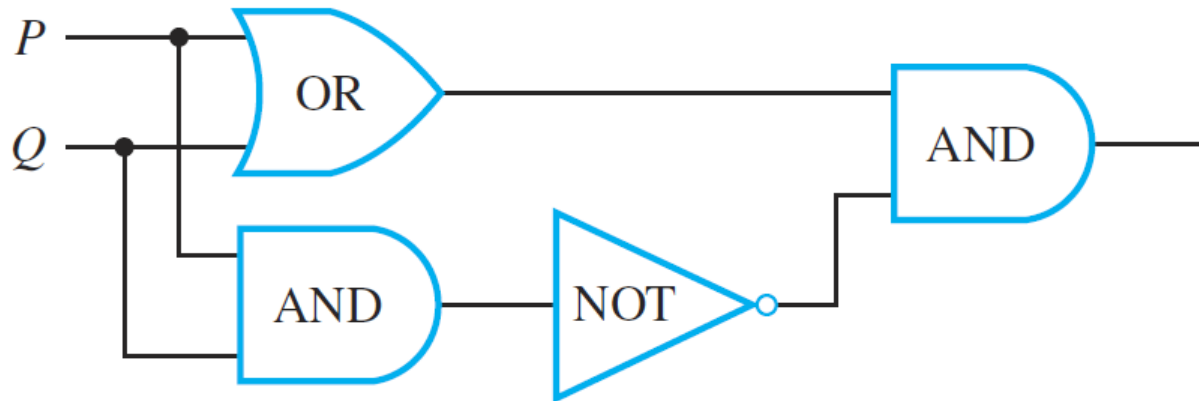
Show  $\neg(p \Rightarrow q)$  and  $p \wedge \neg q$  are logically equivalent.

### Example

Simplify proposition  $\neg(p \wedge q) \vee q$ .

# Application: Digital Logic Circuits

Finding a Boolean expression for a Circuit



Constructing Circuits for Boolean Expressions

$$(\neg P \wedge Q) \vee \neg Q$$

# Summary

You should be able to:

- Describe basics of propositional logic
- Combine propositions to express English sentences
- Derive truth tables of compound propositions
- Show logical equivalence using truth tables
- Show logical equivalence using known logical equivalence

# Definitions covered

- Proposition and compound proposition
- $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\oplus$ ,  $\Rightarrow$ ,  $\Leftrightarrow$  with terms (negation, conjunction, etc.)
- Contrapositive, converse, tautology, and contradiction
- Truth table
- Logical equivalence