THINKING LIKE A COMPUTER SCIENTIST

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Definition

A square root of a number x is a number y such that

$$y^2 = x$$

In other words, it is a number y such that the result of multiplying y by itself is x:

$$y \cdot y = x$$

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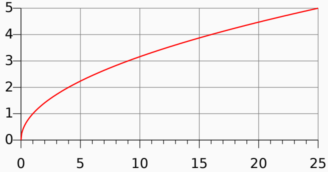

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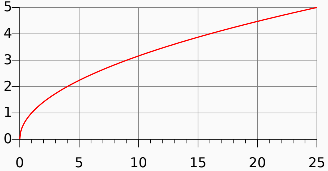

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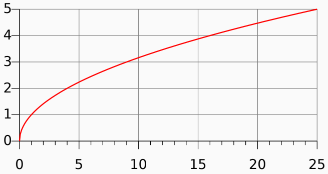

- Some algebraic properties include: $\sqrt{xy} = \sqrt{x}\sqrt{y}$, $\sqrt{x} = x^{1/2}$, $\sqrt{x^2} = |x|$ and so on...
- Square roots are mentioned in ancient Babylonian clay tablets from 1800 BC.

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- We call it declarative knowledge.

“How to” knowledge

- How to do things.
- Recipes, procedures, instructions, methods, algorithms.
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Do you think this will work?

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$$\dots = \frac{x - g^2}{2g + \cancel{\epsilon}} \approx \frac{x - g^2}{2g}$$

Since we have established that the error

$$\epsilon \approx \frac{x - g^2}{2g}$$

we can replace our guess with an *improved* guess (by compensating for this error) as

$$g_{\mathrm{new}} = g + \epsilon \approx g + \frac{x - g^2}{2g} = \frac{2g^2 + x - g^2}{2g} = \frac{g^2 + x}{2g} = \frac{g + \frac{x}{g}}{2}$$

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Indeed, suppose g is an overestimation, that is $g > y$.

Then x/g must be an underestimation:

$$\frac{1}{g} < \frac{1}{y} \xRightarrow{x>0} \frac{x}{g} < \frac{x}{y} \xRightarrow{x=y^2} \frac{x}{g} < \frac{y^2}{y} \implies \frac{x}{g} < y$$

(And vice versa.)

Improving the guess

Another way of thinking about our improved guess

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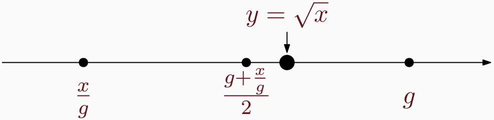
(And vice versa.) Thus, for our new guess g_{new} we take the average between an overestimation and an underestimation.



Putting it all together

So now we have the complete recipe for finding square roots:

procedure SquareRoot(x)
    Choose desired accuracy, *e.g.* $A \leftarrow 0.01$
    Make an initial guess $g \leftarrow x$
    while the error $|g^2 - x| > A$ **repeat the following:**
        replace g with an improved guess: $g \leftarrow \frac{g+x/g}{2}$
    Finally, g is the result, *i.e.* $g \approx \sqrt{x}$
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    Choose desired accuracy, e.g. A ← 0.01
    Make an initial guess g ← x
    while the error |g^2 − x| > A repeat the following:
        replace g with an improved guess: g ← (g+x/g)/2
    Finally, g is the result, i.e. g ≈ √x
end procedure
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Let's try finding some square roots with this recipe!

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Find the square root of 20 *with the error no greater than* 0.001

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Current guess	*Error*	*Good enough?*	*Improved guess*

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$$g = 20$$

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20.0000	380.0000		

$|g \cdot g - x|$

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Solution

Current guess	*Error*	*Good enough?*	*Improved guess*
20.0000	380.0000	*No*	

Is $|g \cdot g - x| < 0.001$ *?*

Problem

Find the square root of 20 *with the error no greater than* 0.001

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Current guess	*Error*	*Good enough?*	*Improved guess*
20.0000	380.0000	*No*	10.5000

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Therefore $\sqrt{20} \approx 4.4721$

A computer scientist will further study the problem to answer questions like

- Does this method work correctly?
- Does it *always* work? For all numbers?
  If not, when does it not work?
- How fast does it work? How many repetitions do we need to make to reach the desired accuracy?
- Are there faster methods to find square roots?
- Will this method always stop?
- How much of other resources (*e.g.* memory) are needed?
- Can this method be used to solve other problems apart from square roots?

More on this in CM1103, CM1208, CM2104, CM2207, CM2208, CM2307...

Questions and exercises

- What does the symbol ☡ mean in my lectures?
- What will happen if you use our recipe to find the square root of 20 with the initial guess $g = 0$?
- What will happen if you try computing the square root of a negative number?
- Try computing the square root of 10 to two decimal places.
- ☡ Can you come up with a different recipe for finding square roots?
- ☡ How would you modify our method to find *cube* roots?

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- Alexander I is a son of Paul I

And so on… Alice has written down all the known facts about who is a child of whom.

Can you help Alice devise procedures to answer questions like:

- Who are the parents of Alexander I?
- Is it true that Alexander II is a grandson of Paul I?
- How many children and grandchildren did Peter I "The Great" have?
- Who was the last Russian emperor?
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