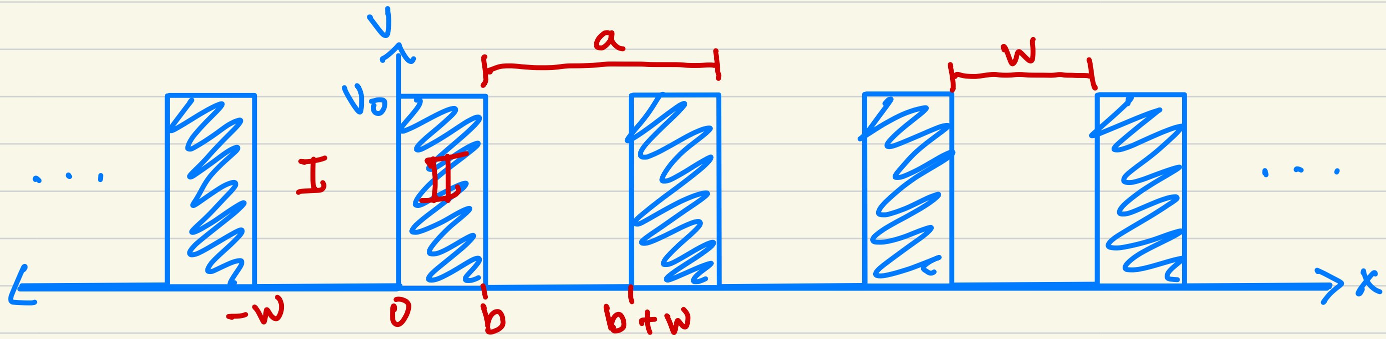
Kronig - Penney model

- Now lets consider a specific $V(x)$: periodic array of square wells



- "Lattice constant" $a = w + b$
- in "unit cell" $-w < x < b$, I: $-w < x < 0$ is well
  II: $0 < x < b$ is barrier
- Recall S.E. in finite well, $0 < E < V_0$

$$\psi_I(x) = A e^{iqx} + B e^{-iqx}, \quad q = \sqrt{2mE/\hbar^2}$$

$$\psi_{II}(x) = C e^{\beta x} + D e^{-\beta x}, \quad \beta = \sqrt{2m(V_0 - E)/\hbar^2}$$

Constants A-D from boundary conditions:

$$\psi_I(0) = \psi_{II}(0), \qquad \left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$$

periodicity

$$\psi_{II}(b) = e^{ika}\,\psi_I(-w), \qquad \left.\frac{d\psi_{II}}{dx}\right|_{x=b} = e^{ika} \left.\frac{d\psi_I}{dx}\right|_{x=-w}$$

periodicity

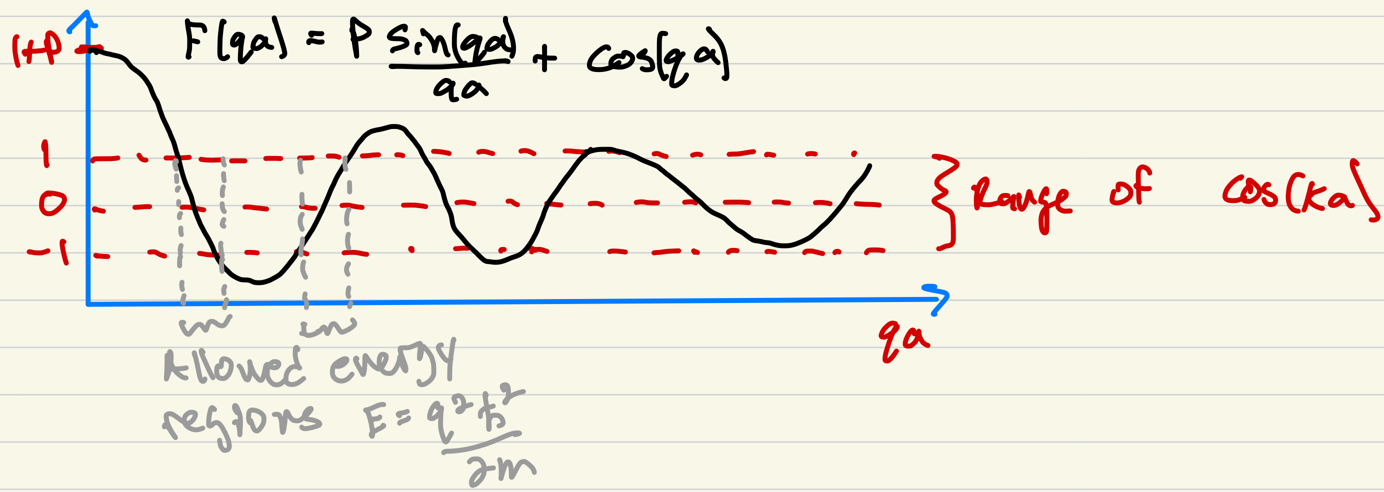
- Solving set of linear equations:

$$\frac{\beta^2 - q^2}{2q\beta} \sinh(\beta b)\sin(qw) + \cosh(\beta b)\cos(qw) = \cos(ka)$$

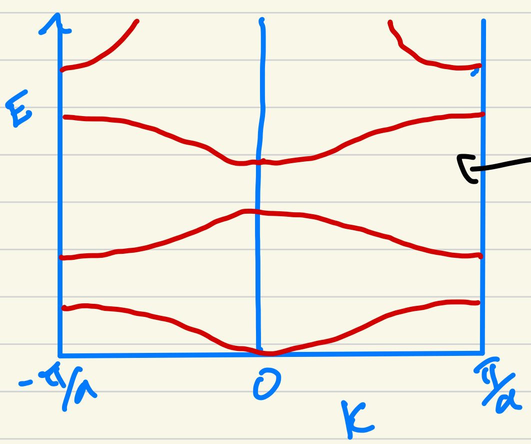
- Additional simplification: $b \to 0$, $V_0 \to \infty$ s.t. $V_0 b = \text{Const.}$
  results in δ-like potential barriers

  gives: $$\underbrace{\frac{m V_0 b a}{\hbar^2}}_{\text{unitless} \equiv P} \frac{\sin(qa)}{qa} + \cos(qa) = \underbrace{\cos(ka)}_{\text{ranges from } -1 \text{ to } 1}$$

- Solve graphically:



- "Band structure", $E(k)$ vs. k in first BZ



looks like free electron w/ gaps at $k = 0, \pm \pi/a$