Rotation Solutions Notes

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1 Introduction

The purpose of these notes is to describe solutions of the SVK model for rotating solids.

2 Governing equations

Let $\mathbf{a} = (a, b)$ measure position in the lab (Eulerian) frame and let $\mathbf{x} = (x, y)$ measure position in the material reference (Lagrangian) frame. Both $\mathbf{a}$ and $\mathbf{x}$ are Cartesian coordinates. The momentum equation in the reference frame is

$$\frac{\partial}{\partial t} v_j = \frac{\partial}{\partial x} P_{1j} + \frac{\partial}{\partial y} P_{2j}, \qquad j = 1\,,\,2 \tag{1}$$

where $\mathbf{v} = (v_1, v_2)$ is velocity, P_{ij} are components of the nominal stress tensor and the density has been set to one for convenience. The nominal stress is given by

$$P = SF^T, \qquad S = \lambda(\operatorname{tr} E)\, I + 2\mu E, \qquad E = \frac{1}{2}\left(F^T F - I\right), \qquad F = \frac{\partial \mathbf{a}}{\partial \mathbf{x}}, \tag{2}$$

where S is the PK2 stress tensor, E is the Green strain tensor and F is the deformation gradient tensor. The components of velocity are given by

$$v_1 = \frac{\partial a}{\partial t}, \qquad v_2 = \frac{\partial b}{\partial t} \tag{3}$$

3 Polar coordinates

The first task is to write the governing equations in terms of the polar coordinates (r, θ) in the reference frame. Let us begin with the momentum equation in (1). The chain rule gives

$$\frac{\partial}{\partial t} v_j = \left[r_x (P_{1j})_r + \theta_x (P_{1j})_\theta\right] + \left[r_y (P_{2j})_r + \theta_y (P_{2j})_\theta\right].$$

Use

$$r_x = \frac{1}{J} y_\theta, \qquad \theta_x = -\frac{1}{J} y_r, \qquad r_y = -\frac{1}{J} x_\theta, \qquad \theta_y = \frac{1}{J} x_r, \qquad J = x_r y_\theta - x_\theta y_r,$$

to give

$$\begin{aligned}
\frac{\partial}{\partial t} v_j &= \frac{1}{J}\left[y_\theta (P_{1j})_r - y_r (P_{1j})_\theta\right] + \frac{1}{J}\left[-x_\theta (P_{2j})_r + x_r (P_{2j})_\theta\right], \\
&= \frac{1}{J}\left[y_\theta P_{1j} - x_\theta P_{2j}\right]_r + \frac{1}{J}\left[-y_r P_{1j} + x_r P_{2j}\right]_\theta,
\end{aligned}$$

Use $x = r\cos\theta$ and $y = r\sin\theta$ to give

$$\frac{\partial}{\partial t} v_j = \frac{1}{r}\left[r\cos\theta (P_{1j}) + r\sin\theta (P_{2j})\right]_r + \frac{1}{r}\left[-\sin\theta (P_{1j}) + \cos\theta (P_{2j})\right]_\theta \tag{4}$$

Define

$$\tilde{P} = RP, \qquad R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{5}$$

so that (4) becomes

$$\frac{\partial}{\partial t} v_j = \frac{1}{r}(r\tilde{P}_{1j})_r + \frac{1}{r}(\tilde{P}_{2j})_\theta, \qquad j = 1\,,\,2. \tag{6}$$

Now let us consider the stress-strain formulas in (2). Using the chain rule, the deformation gradient tensor becomes

$$F = \begin{bmatrix} a_r & a_\theta \\ b_r & b_\theta \end{bmatrix} \begin{bmatrix} r_x & r_y \\ \theta_x & \theta_r \end{bmatrix} = \begin{bmatrix} a_r & a_\theta/r \\ b_r & b_\theta/r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Define

$$\tilde{F} = \begin{bmatrix} a_r & a_\theta/r \\ b_r & b_\theta/r \end{bmatrix} \tag{7}$$

so that

$$F = \tilde{F}R$$

Also, define

$$\tilde{S} = RSR^T, \qquad \tilde{E} = RER^T,$$

so that (2) becomes

$$\tilde{P} = \tilde{S}\tilde{F}^T, \qquad \tilde{S} = \lambda(\operatorname{tr}\tilde{E})\, I + 2\mu\tilde{E}, \qquad \tilde{E} = \frac{1}{2}\left(\tilde{F}^T\tilde{F} - I\right), \tag{8}$$

where $\tilde{F}$ is given in (7). Note that we have used

$$\operatorname{tr}\tilde{E} = \operatorname{tr}(RER^T) = \operatorname{tr}(RR^TE) = \operatorname{tr}E.$$

The equations in (3) for the components of velocity are unaffected by the change to polar coordinates.

4 Equations governing rotation solutions

We now consider rotation solutions of the form

$$a = (r+u)\cos(\theta+\phi), \qquad b = (r+u)\sin(\theta+\phi), \tag{9}$$

where $u = u(r,\theta,t)$ and $\phi = \phi(r,\theta,t)$ are radial and angular displacements, respectively. Let us consider various special cases of the form given in (9).

4.1 Constant rotation

Let us assume radial and angular displacements of the form

$$u = u(r,t), \qquad \phi = \omega_0 t, \tag{10}$$

where ω_0 is a constant rate of rotation. For this choice,

$$a_r = (1+u_r)\cos\bar{\theta}, \qquad b_r = (1+u_r)\sin\bar{\theta}, \qquad a_\theta = -(r+u)\sin\bar{\theta}, \qquad b_\theta = (r+u)\cos\bar{\theta},$$

where $\bar{\theta} = \theta + \phi$. The deformation gradient tensor in (7) becomes

$$\tilde{F} = \begin{bmatrix} (1+u_r)\cos\bar{\theta} & -(1+u/r)\sin\bar{\theta} \\ (1+u_r)\sin\bar{\theta} & (1+u/r)\cos\bar{\theta} \end{bmatrix} = \bar{R}^T\bar{F},$$

where

$$\bar{R} = \begin{bmatrix} \cos\bar{\theta} & \sin\bar{\theta} \\ -\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix}, \qquad \bar{F} = \begin{bmatrix} 1+u_r & 0 \\ 0 & 1+u/r \end{bmatrix}.$$

Using the stress-strain formulas in (8) we find

$$\tilde{E} = \begin{bmatrix} u_r + (u_r)^2/2 & 0 \\ 0 & u/r + (u/r)^2/2 \end{bmatrix}, \qquad \tilde{S} = \begin{bmatrix} (\lambda+2\mu)\tilde{E}_{11} + \lambda\tilde{E}_{22} & 0 \\ 0 & \lambda\tilde{E}_{11} + (\lambda+2\mu)\tilde{E}_{22} \end{bmatrix},$$

and

$$\tilde{P} = \tilde{S}\tilde{F}^T = \tilde{S}\left(\bar{R}^T\bar{F}\right)^T = \tilde{S}\bar{F}^T\bar{R} = \begin{bmatrix} (1+u_r)\tilde{S}_{11} & 0 \\ 0 & (1+u/r)\tilde{S}_{22} \end{bmatrix}\bar{R} = \begin{bmatrix} \bar{P}_{11} & 0 \\ 0 & \bar{P}_{22} \end{bmatrix}\bar{R}. \tag{11}$$

We now use the forms for the position (a,b) in (9) with (10) to work out the left-hand side of the momentum equation (6), and equate that to the right-hand side using the components of the mapped nominal stress given by (11). This results in the two equations

$$\left[u_{tt} - \omega_0^2(r+u)\right]\cos\bar{\theta} - 2\omega_0 u_t \sin\bar{\theta} = \frac{1}{r}\left[\left(r\bar{P}_{11}\right)_r - \bar{P}_{22}\right]\cos\bar{\theta},$$

$$\left[u_{tt} - \omega_0^2(r+u)\right]\sin\bar{\theta} + 2\omega_0 u_t \cos\bar{\theta} = \frac{1}{r}\left[\left(r\bar{P}_{11}\right)_r - \bar{P}_{22}\right]\sin\bar{\theta}.$$

Multiplying the first equation by $-\sin\bar{\theta}$ and adding it to $\cos\bar{\theta}$ times the second gives

$$2\omega_0 u_t = 0,$$

which implies $\omega_0 = 0$ or $u_t = 0$. Assuming $\omega_0 \neq 0$, we have $u_t = u_{tt} = 0$ so that $\cos\bar{\theta}$ times the first equation plus $\sin\bar{\theta}$ times the second gives

$$-\omega_0^2(r+u) = \frac{1}{r}\left[\left(r\bar{P}_{11}\right)_r - \bar{P}_{22}\right]. \tag{12}$$

This is a second-order nonlinear ODE which could be solved for $r \in [r_0, r_1]$, for an annulus say, subject to boundary conditions on $r = r_0$ and $r = r_1$, given by zero-traction BCs, say. The solution has the form $u = u(r)$ which defines a steady solution rotating with constant angular velocity ω_0.

4.2 Non-constant rotation

A more general solution with non-constant rotation rate has radial and angular displacements of the form

$$u = u(r,t), \qquad \phi = \phi(r,t). \tag{13}$$

For this choice, the deformation gradient tensor in (7) becomes

$$\tilde{F} = \begin{bmatrix} (1+u_r)\cos\bar{\theta} - (1+u/r)\,r\phi_r\sin\bar{\theta} & -(1+u/r)\sin\bar{\theta} \\ (1+u_r)\sin\bar{\theta} + (1+u/r)\,r\phi_r\cos\bar{\theta} & (1+u/r)\cos\bar{\theta} \end{bmatrix} = \bar{R}^T\begin{bmatrix} 1+u_r & 0 \\ (1+u/r)\,r\phi_r & 1+u/r \end{bmatrix}$$

where $\bar{\theta} = \theta + \phi$ as before and $\bar{R}$ is the rotation matrix defined above. Using the stress-strain formulas in (8) we find

$$\tilde{E} = \frac{1}{2}\begin{bmatrix} (1+u_r)^2 + (1+u/r)^2(r\phi_r)^2 - 1 & (1+u/r)^2 r\phi_r \\ (1+u/r)^2 r\phi_r & (1+u/r)^2 - 1 \end{bmatrix}, \tag{14}$$

$$\tilde{S} = \begin{bmatrix} (\lambda+2\mu)\tilde{E}_{11} + \lambda\tilde{E}_{22} & 2\mu\tilde{E}_{12} \\ 2\mu\tilde{E}_{21} & \lambda\tilde{E}_{11} + (\lambda+2\mu)\tilde{E}_{22} \end{bmatrix}, \tag{15}$$

and

$$\tilde{P} = \tilde{S}\tilde{F}^T = \begin{bmatrix} (1+u_r)\tilde{S}_{11} & (1+u/r)(r\phi_r\tilde{S}_{11} + \tilde{S}_{12}) \\ (1+u_r)\tilde{S}_{21} & (1+u/r)(r\phi_r\tilde{S}_{21} + \tilde{S}_{22}) \end{bmatrix}\bar{R} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}\bar{R}. \tag{16}$$

As before, we use the forms for the position (a,b) in (9) but now with (13) to work out the left-hand side of the momentum equation (6). We then equate this to the right-hand side using the components of the mapped nominal stress given by (16). This results in the two equations

$$
\begin{aligned}
&\left[u_{tt}-(r+u)(\phi_t)^2\right]\cos\bar{\theta}-\left[(r+u)\phi_{tt}+2u_t\phi_t\right]\sin\bar{\theta} \\
&\qquad=\frac{1}{r}\Big[\big(r\bar{P}_{11}\big)_r-r\phi_r\bar{P}_{12}-\bar{P}_{22}\Big]\cos\bar{\theta}-\frac{1}{r}\Big[\big(r\bar{P}_{12}\big)_r+r\phi_r\bar{P}_{11}+\bar{P}_{21}\Big]\sin\bar{\theta}, \\
&\left[u_{tt}-(r+u)(\phi_t)^2\right]\sin\bar{\theta}+\left[(r+u)\phi_{tt}+2u_t\phi_t\right]\cos\bar{\theta} \\
&\qquad=\frac{1}{r}\Big[\big(r\bar{P}_{11}\big)_r-r\phi_r\bar{P}_{12}-\bar{P}_{22}\Big]\sin\bar{\theta}+\frac{1}{r}\Big[\big(r\bar{P}_{12}\big)_r+r\phi_r\bar{P}_{11}+\bar{P}_{21}\Big]\cos\bar{\theta}.
\end{aligned}
$$

These imply the two evolution equations

$$
\begin{aligned}
u_{tt}-(r+u)(\phi_t)^2&=\frac{1}{r}\Big[\big(r\bar{P}_{11}\big)_r-r\phi_r\bar{P}_{12}-\bar{P}_{22}\Big], \\
(r+u)\phi_{tt}+2u_t\phi_t&=\frac{1}{r}\Big[\big(r\bar{P}_{12}\big)_r+r\phi_r\bar{P}_{11}+\bar{P}_{21}\Big],
\end{aligned}
\tag{17}
$$

which are coupled wave equations, presumably, for $u(r,t)$ and $\phi(r,t)$. Note that the second of the two equations in (17) gives $2\omega_0 u_t=0$ for the special case when $\phi=\omega_0 t$, while the first equation reduces to the steady equation in (12).

5 Initial conditions and boundary conditions

Let us consider a rotating annulus defined for $r\in[r_0,r_1]$ in the reference polar coordinates. At $t=0$, we assume that the displacement is zero, and therefore the stress is zero, and that the velocity is given by

$$
v_1=-\omega_0 y,\qquad v_2=\omega_0 x,
$$

where ω_0 is a given initial rotation rate and (x,y) are Cartesian reference coordinates. In terms of the form of the solution given in (9), the condition of zero displacement implies

$$
u=\phi=0,\qquad \text{at }\ t=0\,.
$$

Using (9), the velocity at $t=0$ is given by

$$
a_t=u_t\cos\theta-r\phi_t\sin\theta,\qquad b_t=u_t\sin\theta+r\phi_t\cos\theta
$$

Equating these velocities to (v_1,v_2) given above implies the second initial condition

$$
u_t=0,\qquad \phi_t=\omega_0,\qquad \text{at }\ t=0\,.
$$

For boundary conditions, we assume that the traction force on the boundaries of the rotating annulus is zero. The unit outward normals to the boundaries $r=r_0$ and $r=r_1$ in the reference coordinates are $\mathbf{n}_0=-(\cos\theta,\sin\theta)$ and $\mathbf{n}_1=-\mathbf{n}_0$, respectively. Since the traction force is taken to be zero, the plus/minus signs are irrelevant so that the conditions on both boundaries are

$$
P_{11}\cos\theta+P_{21}\sin\theta=0,\qquad P_{12}\cos\theta+P_{22}\sin\theta=0.
$$

In terms of the components of the mapped stress tensor $\tilde{P}$ defined in (5), we have

$$
\tilde{P}_{11}=\tilde{P}_{12}=0,\qquad \text{on }\ r=r_0\ \text{ and }\ r=r_1\,.
$$

The boundary conditions for the components of $\tilde{P}$ imply conditions on the components of the rotated stress tensor $\bar{P}$ defined by $\tilde{P}=\bar{P}\bar{R}$ in (11) or (16). These conditions are

$$
\tilde{P}_{11}=\bar{P}_{11}\cos\bar{\theta}-\bar{P}_{12}\sin\bar{\theta}=0,\qquad \tilde{P}_{12}=\bar{P}_{11}\sin\bar{\theta}+\bar{P}_{12}\cos\bar{\theta}=0,
$$

which imply the boundary conditions

$$\bar{P}_{11} = \bar{P}_{12} = 0, \qquad \text{on } r = r_0 \text{ and } r = r_1 .$$

Note that if $\bar{P}_{11} = 0$, then $\tilde{S}_{11} = 0$ from (16), and if $\bar{P}_{12} = 0$ and $\tilde{S}_{11} = 0$, then $\tilde{S}_{12} = 0$. The conditions $\tilde{S}_{11} = \tilde{S}_{12} = 0$ imply

$$(\lambda + 2\mu)(2u_r + (u_r)^2) + \lambda(2u/r + (u/r)^2) = 0, \qquad \phi_r = 0, \qquad \text{on } r = r_0 \text{ and } r = r_1 , \tag{18}$$

from (14) and (15).

6 Linearized equations

In order to get a feel for the structure of the evolution equations in (17), we now consider a linearization for small displacements. We begin with a simplification of the terms on the right-hand sides of the two equations in (17) by assuming

$$|u|/r \ll 1, \qquad |u_r| \ll 1, \qquad r|\phi_r| \ll 1.$$

With these approximations at hand, the rotated Green strain tensor in (14) becomes

$$\tilde{E} = \frac{1}{2} \begin{bmatrix} 2u_r & r\phi_r \\ r\phi_r & 2u/r \end{bmatrix},$$

so that the components of the rotated PK2 stress tensor in (15) become

$$\tilde{S}_{11} = (\lambda + 2\mu)u_r + \lambda u/r, \qquad \tilde{S}_{12} = \tilde{S}_{21} = \mu r \phi_r, \qquad \tilde{S}_{22} = \lambda u_r + (\lambda + 2\mu)u/r.$$

Also, we note that $\bar{P} = \tilde{S}$ from (16), to leading order, so that the right-hand sides of the two equations in (17) are

$$\frac{\lambda + 2\mu}{r} \left[\left(r u_r\right)_r - \frac{u}{r} \right], \qquad \frac{\mu}{r} \left[\left(r^2 \phi_r\right)_r + r\phi_r \right].$$

It is convenient to set

$$\phi(r, t) = \omega_0 t + w/r,$$

where $w(r, t)$ measures circumferential displacement relative to that given by the initial rotation $\omega_0 t$. In terms of $w(r, t)$, the right-hand side of the second equation in (17) becomes

$$\frac{\mu}{r} \left[\left(r w_r\right)_r - \frac{w}{r} \right],$$

which has a similar form to the right-hand side of the first equation. The left-hand sides of (17) also simply assuming $|w_t| \ll r\omega_0$ to give the coupled wave equations

$$u_{tt} = \frac{c_p^2}{r} \left[\left(r u_r\right)_r - \frac{u}{r} \right] + r\omega_0^2, \qquad w_{tt} = \frac{c_s^2}{r} \left[\left(r w_r\right)_r - \frac{w}{r} \right] - 2\omega_0 u_t, \tag{19}$$

where $c_p^2 = \lambda + 2\mu$ and $c_s^2 = \mu$. (Recall that the density was set to one at the beginning.) The initial conditions are

$$u = w = 0, \qquad u_t = w_t = 0, \qquad \text{at } t = 0 ,$$

and the boundary conditions in (18), when linearized, are

$$(\lambda + 2\mu)u_r + \frac{\lambda u}{r} = 0, \qquad w_r - \frac{w}{r} = 0, \qquad \text{on } r = r_0 \text{ and } r = r_1 .$$

We can make some general observations about the coupled system in (19). The radial displacement $u(r, t)$ is forced by the centrifugal acceleration given by $r\omega_0^2$. The circumferential displacement $w(r, t)$, in turn, is forced by the acceleration $-2\omega_0 u_t$ associated with the radial velocity. Thus, given that ω_0 must be small for the linearization to be valid, the size of the solution for u is $O(\omega_0^2)$, i.e. $(\text{small})^2$, while the size of w is $O(\omega_0 u_t) = O(\omega_0^3)$, i.e. $(\text{small})^3$. Also, homogeneous solutions of both equations in (19) have the form

$$\sum_\kappa Q_\kappa(r) e^{i\kappa ct}$$

where c is a wave speed (either c_p or c_s) and $Q_\kappa(r)$ is an eigenfunction belonging to the eigenvalue κ. The eigenvalue problem for $Q(r)$ is

$$r^2 Q'' + rQ' + (\kappa r^2 - 1)Q = 0, \qquad r_0 < r < r_1$$

with

$$Q'(r_0) + zQ(r_0) = Q'(r_0) + zQ(r_0) = 0,$$

where $z = \lambda/(\lambda + 2\mu)$ or $z = -1$. Solutions can be worked out in terms of Bessel functions of the first kind. For the case of $z = \lambda/(\lambda + 2\mu)$ (associated with the problem for u), the smallest eigenvalue is

$$\kappa \approx 1.311353019$$

according to Maple. For the case of $z = -1$ (associated with the problem for w), the smallest eigenvalue is zero. The next smallest eigenvalue is

$$\kappa \approx 6.813842853$$

This suggests that the natural response of the perturbations in u are at a much lower frequency than that for w. So, the equation for w is forced at a much lower frequency than its natural response. (This should be checked.)

7 Singularity at $r = 0$

For the case of a disk, $0 < r < r_1$ ($r_0 = 0$), the singularity in the evolution equations in (17) at $r = 0$ needs to be removed. At the origin, the radial displacement is zero and the angular displacement is bounded so that

$$u(r,t) \to 0, \qquad \phi(r,t) \to \phi_0(t), \qquad \text{as } r \to 0.$$

Thus, for the first of the two equations in (17), we require that

$$\left[\left(r\bar{P}_{11}\right)_r - r\phi_r \bar{P}_{12} - \bar{P}_{22}\right]\Big|_{r=0} = \frac{\partial}{\partial r}\left[\left(r\bar{P}_{11}\right)_r - r\phi_r \bar{P}_{12} - \bar{P}_{22}\right]\Big|_{r=0} = 0,$$

so that $u_{tt} = 0$ at $r = 0$. These two constraints imply

$$\bar{P}_{11} - \bar{P}_{22} = 2\bar{P}_{11,r} - \phi_r \bar{P}_{12} - \bar{P}_{22,r} = 0. \tag{20}$$

For the second of the two equations in (17), we require that

$$\left[\left(r\bar{P}_{12}\right)_r + r\phi_r \bar{P}_{11} + \bar{P}_{21}\right]\Big|_{r=0} = \frac{\partial}{\partial r}\left[\left(r\bar{P}_{12}\right)_r + r\phi_r \bar{P}_{11} + \bar{P}_{21}\right]\Big|_{r=0} = 0,$$

so that ϕ_{tt} is bounded at $r = 0$. These two constraints imply

$$\bar{P}_{12} + \bar{P}_{21} = 2\bar{P}_{12,r} + \phi_r \bar{P}_{11} + \bar{P}_{21,r} = 0. \tag{21}$$

It is noted that the equations in (17) are unchanged under the transformations

$$r \to -r, \qquad u \to -u, \qquad \phi \to \phi,$$

which suggests that $u(r,t)$ is an odd function and $\phi(r,t)$ is an even function with respect to r. With this behavior in mind, we have

$$\tilde{E} = \frac{1}{2}\begin{bmatrix} (1+u_r)^2 - 1 & 0 \\ 0 & (1+u_r)^2 - 1 \end{bmatrix} + O(r^2) = \frac{1}{2}\left[(1+u_r)^2 - 1\right] I + O(r^2),$$

so that

$$\tilde{S} = (\lambda+\mu)\left[(1+u_r)^2-1\right] I + O(r^2), \qquad \bar{P} = (1+u_r)\tilde{S} + O(r^2), \tag{22}$$

as $r \to 0$. From the behaviors in (22) and given that ϕ is an even function, it is evident that the constraints in (20) and (21) hold.

In order to compute solutions of the equation in (17) for the case when $r_0 = 0$, we need to determine the value for ϕ_{tt} at $r = 0$. This value is obtained by evaluating the removable singularity in the second equation in (17). We have

$$\phi_{tt} = -\frac{2u_t\phi_t}{r+u} + \frac{1}{r(r+u)}\Big[\big(r\bar{P}_{12}\big)_r + r\phi_r\bar{P}_{11} + \bar{P}_{21}\Big]. \tag{23}$$

The first term on the right-hand side of (23) is a $0/0$ indeterminate form as $r \to 0$, while the second term has the form $0^2/0^2$. Using L'Hospital's rule, the first term becomes

$$-\frac{2u_{rt}\phi_t}{1+u_r}\bigg|_{r=0}.$$

The second term requires more work. Note that

$$\begin{aligned}
\bar{P}_{12} &= (1+u_r)\left\{(\lambda+\mu)\left[(1+u_r)^2-1\right] + 2\mu(1+u_r)^2\right\} r\phi_r + O(r^4),\\
\bar{P}_{11} &= (\lambda+\mu)(1+u_r)\left[(1+u_r)^2-1\right] + O(r^2),\\
\bar{P}_{21} &= 2\mu(1+u_r)^3 r\phi_r + O(r^4),
\end{aligned}$$

and thus

$$\begin{aligned}
(r\bar{P}_{12})_r &= 3(1+u_r)\left\{(\lambda+\mu)\left[(1+u_r)^2-1\right] + 2\mu(1+u_r)^2\right\} r^2\phi_{rr} + O(r^4),\\
r\phi_r\bar{P}_{11} &= (\lambda+\mu)(1+u_r)\left[(1+u_r)^2-1\right] r^2\phi_{rr} + O(r^4),\\
\bar{P}_{21} &= 2\mu(1+u_r)^3 r^2\phi_{rr} + O(r^4).
\end{aligned}$$

The second term becomes

$$4\big\{(\lambda+\mu)\left[(1+u_r)^2-1\right] + 2\mu(1+u_r)^2\big\}\phi_{rr}\Big|_{r=0}.$$

Putting both terms together, we have

$$\phi_{tt} = -\frac{2u_{rt}\phi_t}{1+u_r} + 4\big\{(\lambda+\mu)\left[(1+u_r)^2-1\right] + 2\mu(1+u_r)^2\big\}\phi_{rr},$$

at $r = 0$.

8 Numerical approach

The coupled wave equations in (17) for $u(r,t)$ and $\phi(r,t)$ may be solved numerically. The equations have the form

$$u_{tt} - (r+u)\phi_t^2 = \mathcal{N}(u,\phi), \qquad (r+u)\phi_{tt} + 2u_t\phi_t = \mathcal{M}(u,\phi), \tag{24}$$

where $\mathcal{N}$ and $\mathcal{M}$ are the second-order differential operators (in r) on the right-hand sides of the equations in (17). Let u_j^n and ϕ_j^n denote approximations for u and ϕ on a uniform grid (r_j, t_n). A two-level time-marching scheme for (24) is

$$\begin{aligned}
u_j^{n+1} &= 2u_j^n - u_j^{n-1} + \Delta t^2\mathcal{N}_h(u_j^n,\phi_j^n) + \frac{1}{4}(r_j+u_j^n)(\phi_j^{n+1}-\phi_j^{n-1})^2,\\
\phi_j^{n+1} &= 2\phi_j^n - \phi_j^{n-1} + \frac{1}{r_j+u_j^n}\Big[\Delta t^2\mathcal{M}_h(u_j^n,\phi_j^n) - \frac{1}{2}(u_j^{n+1}-u_j^{n-1})(\phi_j^{n+1}-\phi_j^{n-1})\Big],
\end{aligned}$$

Grid	$\mathcal{E}_u$	rate	$\mathcal{E}_v$	rate	$\mathcal{E}_P$	rate
annulus40	1.72e−6		6.16e−5		3.01e−5	
annulus80	5.00e−7	1.79	1.49e−5	2.05	9.07e−6	1.73
annulus160	1.36e−7	1.88	3.47e−6	2.11	2.56e−6	1.82
sicFixede4.order2	1.56e−4		1.52e−3		1.52e−3	
sicFixede8.order2	3.33e−5	2.23	3.91e−4	1.96	3.64e−4	1.93
sicFixede16.order2	7.52e−6	2.15	9.71e−5	2.01	8.82e−5	2.02

Table 1: Maximum errors and rates for an annulus (Grid = annulus) and a disk (Grid = sic) at $t = 0.5$.

where $\mathcal{N}_h$ and $\mathcal{M}_h$ are second-order centered difference operators corresponding to $\mathcal{N}$ and $\mathcal{M}$, respectively. The time-marching scheme is implicit and Newton's method is used to obtain u_j^{n+1} and ϕ_j^{n+1} at each step.

We consider the initial conditions

$$u = \phi = 0, \qquad u_t = 0, \qquad \phi_t = \omega(r), \qquad \text{at } t = 0\,,$$

where $\omega(r)$ is a given initial angular rotation rate. It is helpful to consider an initial rotation rate that depends on r . If $\omega(r) \to 0$ as r tends to r_0 or r_1 , then the resulting solution of the problem with zero traction on the boundary is smoother than if $\omega(r) = \text{constant}$. The initial conditions are implemented numerically by setting

$$u_j^0 = \phi_j^0 = 0, \qquad u_j^1 = \frac{\Delta t^2}{2} r_j \omega(r_j)^2, \qquad \phi_j^1 = \Delta t \omega(r_j).$$

The boundary conditions in (18) are approximated using second-order centered differences.

9 Some numerical results

Results are obtained for the initial rotation rate given by

$$\omega(r) = \omega_{\max} \left[\frac{4(r - r_0)(r_1 - r)}{(r_0 + r_1)^2} \right]^2$$

so that $\omega(r)$ tends to zero quadratically as r tends to r_0 or r_1 . Also,

$$\max_{r \in [r_0, r_1]} \omega(r) = \omega\big((r_0 + r_1)/2\big) = \omega_{\max}.$$

Solutions of the reduced equations are compared with full solutions obtained from `cgsm`. Grid functions for the components of the Cartesian displacement (u_1, u_2) , components of the velocity (v_1, v_2) and components of the nominal stress $(P_{11}, P_{12}, P_{21}, P_{22})$ are constructed using a numerical solution of the reduced equations and these are compared with the corresponding grid functions obtained by `cgsm`. The numerical solution of the reduced equations are performed on a very fine grid so that grid functions obtained from them are considered to be exact. Table 1 gives the maximum errors in the displacement, velocity and stress at $t = 0.5$ for an annulus with $r_0 = 0.5$ and $r_1 = 1.0$ and for a disk with $r_0 = 0$ and $r_1 = 1.0$. The maximum rotation rate is taken to be $\omega_{\max} = 0.5$ for both cases. The errors suggest second-order accuracy.