Notes on Solving the Equations of Solid Mechanics

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1 Nomenclature

ρ	density	(1)
u_i	displacement vector	(2)
ϵ_{ij}	strain tensor	(3)
τ_{ij}	stress tensor	(4)
λ	shear modulus, Lamé constant	(5)
μ	Lamé constant	(6)

2 Introduction and Governing Equations

The equations of linear elasticity for a homogeneous isotropic material are governed by

$$\rho \partial_t^2 u_i = \partial_{x_j} \tau_{ij} + \rho f_i \tag{7}$$
$$\tau_{ij} = \lambda \partial_{x_k} u_k \delta_{ij} + 2\mu \epsilon_{ij} \tag{8}$$
$$\epsilon_{ij} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j) \tag{9}$$

In two-dimensions,

$$\rho \partial_t^2 u = \partial_x(\lambda(\partial_x u + \partial_y v) + 2\mu \partial_x u) + \partial_y(\mu(\partial_x v + \partial_y u)) \tag{10}$$
$$\rho \partial_t^2 v = \partial_x(\mu(\partial_x v + \partial_y u)) + \partial_y(\lambda(\partial_x u + \partial_y v) + 2\mu \partial_y v) \tag{11}$$

or

$$\rho \partial_t^2 u = \partial_x((\lambda + 2\mu)\partial_x u) + \partial_x(\lambda \partial_y v) + \partial_y(\mu \partial_x v) + \partial_y(\mu \partial_y u)) \tag{12}$$
$$\rho \partial_t^2 v = \partial_x(\mu \partial_x v) + \partial_x(\mu \partial_y u)) + \partial_y(\lambda \partial_x u) + \partial_y((\lambda + 2\mu)\partial_y v) \tag{13}$$

For constant μ and λ the equations become:

$$\rho \partial_t^2 u_i = (\lambda + \mu)\partial_{x_i} \partial_{x_k} u_k + \mu \partial_{x_k}^2 u_i + \rho f_i \tag{14}$$
$$\rho \mathbf{u}_{tt} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u} + \rho \mathbf{f} \tag{15}$$

If the dilatation and curl are denoted by

$$\delta = \nabla \cdot \mathbf{u}, \tag{16}$$
$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{17}$$

then

$$\rho \delta_{tt} = (\lambda + 2\mu)\Delta \delta + \rho \nabla \cdot \mathbf{f} \tag{18}$$
$$\rho \omega_{tt} = \mu \Delta \boldsymbol{\omega} + \rho \nabla \times \mathbf{f} \tag{19}$$

2.1 Boundary conditions

2.2 Displacement boundary condition

A *displacement* boundary condition is one where the displacements are specified

$$\mathbf{u} = \mathbf{g}(\mathbf{x}, t) \qquad \mathbf{x} \in \partial\Omega \tag{20}$$

If $\mathbf{g}(\mathbf{x}, t) = 0$, the boundary is said to be *clamped*.

2.3 Traction boundary condition

Stress or *traction* boundary conditions are

$$\mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{g}(\mathbf{x}, t) \qquad \mathbf{x} \in \partial\Omega \tag{21}$$

or

$$\lambda(\nabla \cdot \mathbf{u})n_i + \mu n_j [u_{j,i} + u_{i,j}] = g_i \tag{22}$$

or

$$
\begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \lambda\nabla\cdot\mathbf{u}+2\mu u_x & \mu(u_y+v_x) & \mu(u_z+w_x) \\ \mu(u_y+v_x) & \lambda\nabla\cdot\mathbf{u}+2\mu v_y & \mu(v_z+w_y) \\ \mu(u_z+w_x) & \mu(v_z+w_y) & \lambda\nabla\cdot\mathbf{u}+2\mu w_z \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{23}
$$

For a straight wall at $x = 0$ this becomes

$$
\begin{aligned}
u_x &= -\frac{\lambda}{\lambda+2\mu}(v_y+w_z) + \frac{n_1}{\lambda+2\mu}g_1 && (24)\\
v_x &= -u_y + \frac{n_1}{\mu}g_2 && (25)\\
w_x &= -u_z + \frac{n_1}{\mu}g_3 && (26)
\end{aligned}
$$

For a straight wall at $y = 0$,

$$
\begin{aligned}
u_y &= -v_x + \frac{n_2}{\mu}g_1 && (27)\\
v_y &= -\frac{\lambda}{\lambda+2\mu}(u_x+w_z) + \frac{n_2}{\lambda+2\mu}g_2 && (28)\\
w_y &= -v_z + \frac{n_2}{\mu}g_3 && (29)
\end{aligned}
$$

For a straight wall at $z = 0$,

$$
\begin{aligned}
u_z &= -w_x + \frac{n_1}{\mu}g_1 && (30)\\
v_z &= -w_y + \frac{n_2}{\mu}g_2 && (31)\\
w_z &= -\frac{\lambda}{\lambda+2\mu}(u_x+v_y) + \frac{n_3}{\lambda+2\mu}g_3 && (32)
\end{aligned}
$$

2.4 Slip-wall boundary condition

The slip-wall boundary condition imposes the normal-component of the displacement to be zero and the tangential component of the traction to be zero:

$$
\begin{aligned}
\mathbf{n}\cdot\mathbf{u} &= 0, && (33)\\
\mathbf{n}\cdot\boldsymbol{\tau}\cdot\hat{\mathbf{t}}_m &= 0\ . && (34)
\end{aligned}
$$

Here $\hat{\mathbf{t}}_m$, $m = 1, 2$ are the tangent vectors. This can also be written as the vector equations (*check this*)

$$
(\mathbf{n}\cdot\mathbf{u})\ \mathbf{n} + \mathbf{n}\cdot\boldsymbol{\tau}\times\mathbf{n} = 0 \tag{35}
$$

Note that taking the dot product of $\mathbf{n}$ with (35) gives (33) and taking cross-product of $\mathbf{n}$ with (35) gives (34). To see this recall $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{b}(\mathbf{a}\cdot\mathbf{b})$ so that taking $\mathbf{n}\times$ equation (35) gives

$$
\mathbf{n}\times(\mathbf{g}\times\mathbf{n}) = \mathbf{g} - (\mathbf{n}\cdot\mathbf{g})\mathbf{n}
$$

where $\mathbf{g} = \mathbf{n}\cdot\boldsymbol{\tau}$ is the traction vector. This may be a convenient form since there is no need to explicitly form the tangent vectors.

2.5 Corner compatibility conditions

Corner compatibility conditions are used to derive discrete boundary conditions for the ghost points near the corners.

2.5.1 Traction-Traction corner

Consider a 2D rectangular domain with a traction-traction corner at $\mathbf{x} = 0$,

$$u_x(0, y) = -\alpha v_y(0, y) \tag{36}$$
$$v_x(0, y) = -u_y(0, y) \tag{37}$$
$$u_y(x, 0) = -v_x(x, 0) \tag{38}$$
$$v_y(x, 0) = -\alpha u_x(x, 0) \tag{39}$$
$$u_{tt}(x, y) = (\lambda + \mu)(u_{xx} + v_{xy}) + \mu(u_{xx} + u_{yy}) \tag{40}$$
$$v_{tt}(x, y) = (\lambda + \mu)(u_{xy} + v_{yy}) + \mu(v_{xx} + v_{yy}) \tag{41}$$

By using the above expressions and their derivatives, it follows that at the corner

$$u_x(0, 0) = -\alpha v_y(0, 0) \tag{42}$$
$$v_y(0, 0) = -\alpha u_x(0, 0) \tag{43}$$
$$u_{yy}(0, 0) = -v_{xy}(0, 0) = \alpha u_{xx}(0, 0) \tag{44}$$
$$v_{xx}(0, 0) = -u_{xy}(0, 0) = \alpha v_{yy}(0, 0) \tag{45}$$

which implies

$$u_x(0, 0) = 0 \tag{46}$$
$$v_y(0, 0) = 0 \tag{47}$$
$$u_{yy}(0, 0) = \alpha u_{xx}(0, 0) \tag{48}$$
$$v_{xx}(0, 0) = \alpha v_{yy}(0, 0) \tag{49}$$

From $u_x(0, 0, t) = 0$, $v_y(0, 0, t) = 0$, (40) and (41) it follows that

$$u_{xtt}(0, 0) = (\lambda + \mu)(u_{xxx} + v_{xxy}) + \mu(u_{xxx} + u_{xyy}) = 0 \tag{50}$$
$$v_{ytt}(0, 0) = (\lambda + \mu)(u_{xyy} + v_{yyy}) + \mu(v_{xxy} + v_{yyy}) = 0 \tag{51}$$

or

$$\begin{aligned} (\lambda + 2\mu)u_{xxx}(0, 0) &= -(\lambda + \mu)v_{xxy} - \mu u_{xyy} && (52)\\ &= \alpha(\lambda + \mu)u_{xxx} + \alpha\mu v_{yyy} && (53)\\ (\lambda + 2\mu)v_{yyy}(0, 0) &= -(\lambda + \mu)u_{xyy} + \mu v_{xxy} && (54)\\ &= \alpha(\lambda + \mu)v_{yyy} + \alpha\mu u_{xxx} && (55) \end{aligned}$$

which implies

$$u_{xxx}(0, 0) = 0 \tag{56}$$
$$v_{yyy}(0, 0) = 0 \tag{57}$$

From $u_x(0, 0, t) = 0$, $v_y(0, 0, t) = 0$, (40) and (41) it follows that

$$u_{xtt}(0, 0) = (\lambda + \mu)(u_{xxx} + v_{xxy}) + \mu(u_{xxx} + u_{xyy}) = 0 \tag{58}$$
$$v_{ytt}(0, 0) = (\lambda + \mu)(u_{xyy} + v_{yyy}) + \mu(v_{xxy} + v_{yyy}) = 0 \tag{59}$$

From $u_{yy}(0, 0, t) = \alpha u_{xx}(0, 0)$, $v_{xx}(0, 0, t) = \alpha v_{yy}(0, 0, t)$, (40) and (41) it follows that

$$\begin{aligned} u_{yytt}(x, y) - \alpha u_{xx} &= (\lambda + \mu)(u_{xxyy} + v_{xyyy}) + \mu(u_{xxyy} + u_{yyyy}) && (60)\\ &\quad - \alpha\Big((\lambda + \mu)(u_{xxxx} + v_{xxxy}) + \mu(u_{xxxx} + u_{xxyy})\Big) && (61) \end{aligned}$$

2.5.2 Slip-wall traction corner

Consider a 2D rectangular domain with a vertical slip-wall ($x = 0$) next to a horizontal traction wall ($y = 0$) and a corner at $\mathbf{x} = 0$. The equations in 2D are:

$$\rho u_{tt} = \partial_x \sigma_{11} + \partial_y \sigma_{12}, \tag{62}$$

$$\rho v_{tt} = \partial_x \sigma_{21} + \partial_y \sigma_{22}, \tag{63}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \alpha u_x + \lambda v_y & \mu(u_y + v_x) \\ \mu(u_y + v_x) & \alpha v_y + \lambda u_x \end{bmatrix} \tag{64}$$

where $\alpha = \lambda + 2\mu$. We will assume that $\sigma_{12} = \sigma_{21}$. The boundary conditions are

$$u(0, y) = 0, \qquad \text{(slip wall)} \tag{65}$$

$$v_x(0, y) = 0, \qquad \text{(slip wall, } \sigma_{12} = 0) \tag{66}$$

$$u_y(x, 0) = -v_x(x, 0), \qquad \text{(traction, } \sigma_{21} = 0) \tag{67}$$

$$\alpha v_y(x, 0) = -\lambda u_x(x, 0), \qquad \text{(traction, } \sigma_{22} = 0) \tag{68}$$

These equations provide relations between the first derivatives at the corner. From these equations we can determine relations between the second derivatives at the corner

$$u_{yy}(0, 0) = 0, \qquad \text{(from } \partial_y^2 \text{ of (65))} \tag{69}$$

$$u_{xy}(0, 0) = -v_{xx}(0, 0), \qquad \text{(from } \partial_x \text{ of (67))} \tag{70}$$

$$v_{xy}(0, 0) = 0, \qquad \text{(from } \partial_y \text{ of (66))} \tag{71}$$

$$u_{xx}(0, 0) = 0, \qquad \text{(from } \partial_x \text{ of (68) and } v_{xy}(0, 0) = 0) \tag{72}$$

Suppose we have discretized the problem and we need to determine values the solution at the ghost points $\mathbf{u}(-\Delta x, 0)$, $\mathbf{u}(0, -\Delta y)$ and $\mathbf{u}(-\Delta x, -\Delta y)$ (what we write will also apply to $\mathbf{u}(-2\Delta x, 0)$, etc.) We can first determine the ghost values on the extended boundaries:

$$u(-\Delta x, 0) = 2u(0, 0) - u(\Delta x, 0), \qquad \text{(from } u_{xx}(0, 0) = 0) \tag{73}$$

$$v(-\Delta x, 0) = v(\Delta x, 0), \qquad \text{(from } v_x(0, 0) = 0) \tag{74}$$

$$u(0, -\Delta y) = 0, \qquad \text{(from } u(0, y) = 0) \tag{75}$$

$$v(0, -\Delta y) = v(0, \Delta y) + (2\Delta y)(\lambda/\alpha) u_x(0, 0), \qquad \text{(from } \alpha v_y(0, 0) + \lambda u_x(0, 0) = 0) \tag{76}$$

where for the last equation we use $u_x(0, 0) = (u(\Delta x, 0) - u(-\Delta x, 0))/(2\Delta x)$, using the value for $u(-\Delta x, 0)$ that we have just computed from (73) . For the corner ghost points $\mathbf{u}(-\Delta x, -\Delta y)$ we use Taylor series,

$$\mathbf{u}(-\Delta x, -\Delta y) = \mathbf{u}(0, 0) - \Delta x\ \mathbf{u}_x(0, 0) - \Delta y\ \mathbf{u}_y(0, 0) + \tag{77}$$

$$(\Delta x^2/2)\ \mathbf{u}_{xx}(0, 0) + \Delta x \Delta y\ \mathbf{u}_{xy}(0, 0) + (\Delta y^2/2)\ \mathbf{u}_{yy}(0, 0) + O(\Delta x^3 + \Delta y^3) \tag{78}$$

and thus we can evaluate the corner ghost points from

$$u(-\Delta x, -\Delta y) \approx u(0, 0) - \Delta x\ u_x(0, 0) - \Delta x \Delta y\ v_{xx}(0, 0), \tag{79}$$

$$v(-\Delta x, -\Delta y) \approx v(0, 0) - \Delta y\ v_y(0, 0) + (\Delta x^2/2) v_{xx}(0, 0) + (\Delta y^2/2) v_{yy}(0, 0), \tag{80}$$

where in these last expressions we discretize the terms on the RHS using central differences such as $u_x(0, 0) = (u(\Delta x, 0) - u(-\Delta x, 0))/(2\Delta x)$, etc. and we use the values computed from (73)-(76). Note that we can use the Taylor series expression to also compute $\mathbf{u}(-2\Delta x, -\Delta y)$, $\mathbf{u}(-\Delta x, -2\Delta y)$ etc.

Now consider computation of the stress on the ghost points. Given the values for (u, v) that we have determined above we can then compute numerical approximations to $\boldsymbol{\sigma}(0,0)$, $\partial_x\boldsymbol{\sigma}(0,0)$ and $\partial_y\boldsymbol{\sigma}(0,0)$ from:

$$\partial_x\sigma_{11}(0,0) = \alpha u_{xx} + \lambda v_{xy} = 0 \tag{81}$$
$$\partial_y\sigma_{11}(0,0) = \alpha u_{xy} + \lambda v_{yy} = -\alpha v_{xx} + \lambda v_{yy} \tag{82}$$
$$\partial_x\sigma_{22}(0,0) = \alpha v_{xy} + \lambda u_{xx} = 0 \tag{83}$$
$$\partial_y\sigma_{22}(0,0) = \alpha v_{yy} + \lambda u_{xy} = \alpha v_{yy} - \lambda v_{xx} \tag{84}$$
$$\partial_x\sigma_{12}(0,0) = \mu(u_{xy} + v_{xx}) = 0 \tag{85}$$
$$\partial_y\sigma_{12}(0,0) = \mu(u_{yy} + v_{xy}) = 0 \tag{86}$$

From these above expressions we can compute $\boldsymbol{\sigma}(-\Delta x, 0)$ and $\boldsymbol{\sigma}(0, -\Delta y)$. For example

$$\sigma_{11}(-\Delta x, 0) = \sigma_{11}(\Delta x, 0), \tag{87}$$
$$\sigma_{22}(0, -\Delta y) = \sigma_{22}(0, \Delta y) - (2\Delta y)(\alpha v_{yy}(0,0) - \lambda v_{xx}(0,0)). \tag{88}$$

We can also say something about the mixed second derivatives of $\boldsymbol{\sigma}$:

$$\partial_x\partial_y\sigma_{11}(0,0) = 0, \qquad \text{(from } \partial_y \text{ of (62))} \tag{89}$$
$$\partial_x\partial_y\sigma_{22}(0,0) = 0, \qquad \text{(from } \partial_x \text{ of (63))} \tag{90}$$
$$(\lambda+\alpha)\partial_x\partial_y\sigma_{12}(0,0) = -\lambda\partial_x^2\sigma_{11}(0,0) - \alpha\partial_y^2\sigma_{22}(0,0), \qquad \text{(from } \partial_t^2 \text{ of } \alpha v_y + \lambda u_x = 0). \tag{91}$$

From Taylor series,

$$\boldsymbol{\sigma}(-\Delta x, -\Delta y) \approx \boldsymbol{\sigma}(0,0) - \Delta x\ \boldsymbol{\sigma}_x(0,0) - \Delta y\ \boldsymbol{\sigma}_y(0,0) + \tag{92}$$
$$(\Delta x^2/2)\ \boldsymbol{\sigma}_{xx}(0,0) + \Delta x\Delta y\ \boldsymbol{\sigma}_{xy}(0,0) + (\Delta y^2/2)\ \boldsymbol{\sigma}_{yy}(0,0), \tag{93}$$

and we can use this expression to compute $\boldsymbol{\sigma}(-\Delta x, -\Delta y)$ since we can compute approximations to all the second derivatives of $\boldsymbol{\sigma}$.

3 Invariants of the Motion

The equations of linear elasticity have certain invariants of the motion. These are special solutions that satisfy homogeneous boundary conditions and $\partial_t^2\mathbf{u} = 0$.

For the case of constant λ and μ we see that the function

$$\mathbf{u} = (\mathbf{a}x + \mathbf{b}y + \mathbf{c})(\mathbf{d} + t\mathbf{e})$$

with constant $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$, will be a solution to the interior equations with $\partial_t^2\mathbf{u} = 0$.

Of course there are many other solutions to the homogeneous equations but these cannot be made to satisfy the boundary conditions (why is this true?)

Claim: for a given domain with traction boundary conditions it follows that

$$\partial_y u_1 + \partial_x u_2 = 0 \tag{94}$$
$$\partial_z u_2 + \partial_y u_3 = 0 \tag{95}$$
$$\partial_x u_3 + \partial_z u_1 = 0 \tag{96}$$
$$\partial_x u_1 = \partial_y u_2 = \partial_z u_3 = 0 \tag{97}$$

Note that the restricted solution that satisfies these conditions will also satisfy the equations with variable λ and μ.

To see why these conditions are true consider different points on the boundary. The traction boundary condition is given by (23). As the normal varies over the boundary we see that $\lambda\nabla\cdot\mathbf{u} + 2\mu\partial_x u_1 = 0$ where

$\mathbf{n} = [1, 0, 0]$, and $\lambda \nabla \cdot \mathbf{u} + 2\mu \partial_y u_2 = 0$ where $\mathbf{n} = [0, 1, 0]$ and $\lambda \nabla \cdot \mathbf{u} + 2\mu \partial_z u_3 = 0$ where $\mathbf{n} = [0, 0, 1]$. Note that the stress components are constant in space. We thus have 3 equations for the three unknowns $\partial_x u_1$, $\partial_y u_2$, and $\partial_z u_3$ and it follows that these must all be zero (assuming $\lambda > 0$ and $\mu > 0$).

The general form of the the invariant solution for homogeneous traction boundaries in two-dimensions is then

$$
\begin{aligned}
u_1 &= (a + cy)\ (d + et), \\
u_2 &= (b - cx)\ (d + et).
\end{aligned}
$$

These motions consists of translations $c = 0$ and the *scale-rotate* mode $a = b = 0$, $c \neq 0$:

$$
\begin{aligned}
u_1 &= y\ (d + et), \\
u_2 &= -x\ (d + et).
\end{aligned}
$$

The *scale-rotate* consists of a partial rotation and scaling.

Note that a rotation from the reference cooridinates is given by

$$
\widetilde{\mathbf{x}} = \mathbf{x} + \mathbf{u} = R\mathbf{x}
$$

where $\widetilde{\mathbf{x}}$ are the physical space coordinates and R is a rotation matrix. This gives the form

$$
\begin{aligned}
u_1 &= (\cos(\theta) - 1)x + \sin(\theta)y, \\
u_2 &= \sin(\theta)x + (\cos(\theta) - 1)y
\end{aligned}
$$

and this is not an invariant when $\cos(\theta) \neq 0$.

4 Traveling wave solutions

Consider the equations of linear elasticity,

$$\rho \mathbf{u}_{tt} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u},$$

which can be written as

$$\mathbf{u}_{tt} = A\mathbf{u}_{xx} + B\mathbf{u}_{xy} + C\mathbf{u}_{yy}. \tag{98}$$

Let us look for traveling wave solutions of the form

$$\begin{aligned} &\mathbf{u} = \mathbf{f}(\boldsymbol{\kappa} \cdot \mathbf{x} - ct) = \mathbf{f}(\xi), \\ &\xi = \boldsymbol{\kappa} \cdot \mathbf{x} - ct, \end{aligned}$$

where $\kappa = |\boldsymbol{\kappa}| = 1$. Subsituting the traveling wave solution into (98) gives the eigenvalue problem

$$\Big[- c^2\mathbf{I} + \kappa_1^2 A + \kappa_1\kappa_2 B + \kappa_2^2 B \Big]\mathbf{f}'' = 0$$

The eigenvalues are

$$c_p^2 = \frac{(\lambda + 2\mu)}{\rho}, \quad c_s^2 = \frac{\mu}{\rho},$$

with corresponding eigenfunctions

$$\begin{aligned} &\mathbf{f}_m = \phi_m(\xi)\mathbf{e}^m, \; m = p, s, \\ &\mathbf{e}^p = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}, \quad \mathbf{e}^s = \begin{bmatrix} \kappa_2 \\ -\kappa_1 \end{bmatrix}, \end{aligned}$$

where $\phi_m = \phi_m(\xi)$ are arbitrary functions of ξ. (Note that we also have the solution with $\mathbf{f}'' = 0$, i.e. $\mathbf{u}(x,t) = \mathbf{a} + \mathbf{b}\xi$ for any c.) We thus have traveling wave solutions for p-waves and -s-waves

$$\begin{aligned} \mathbf{u}_p(x,t) &= \mathbf{e}^p\phi_p(\xi), \quad \xi = \boldsymbol{\kappa} \cdot \mathbf{x} - c_p t, \quad c_p = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}, \\ \mathbf{u}_s(x,t) &= \mathbf{e}^s\phi_s(\xi), \quad \xi = \boldsymbol{\kappa} \cdot \mathbf{x} - c_s t, \quad c_s = \sqrt{\frac{\mu}{\rho}}. \end{aligned}$$

The velocity and stress tensor are given by

$$\begin{aligned} \mathbf{v}_m &= \partial_t \mathbf{u}_m = -c_m\phi'(\xi)\ \mathbf{e}^m, \\ \partial_{x_i}\mathbf{u}_m &= \kappa_i\phi'(\xi)\ \mathbf{e}^m, \\ \sigma_{ij}^m &= \lambda\nabla \cdot \mathbf{u}_m\delta_{ij} + \mu(\partial_{x_j}u_i + \partial_{x_i}u_j) \\ &= \Big\{\lambda(\boldsymbol{\kappa} \cdot \mathbf{e}^m)\delta_{ij} + \mu(\kappa_j e_i^m + \kappa_i e_j^m)\Big\}\phi_m' \end{aligned}$$

Let us now consider particular traveling wave solutions where the velocity, $\mathbf{v}_m = \partial_t\mathbf{u}_m$ is piecewise constant. One such solution is given by

$$\begin{aligned} \mathbf{u}_m(\xi) &= \begin{cases} -\xi\ v_0/c_m\ \mathbf{e}^m & \text{for } \xi < 0, \\ 0 & \text{for } \xi > 0 \end{cases}, \\ \mathbf{v}_m(\xi) &= \begin{cases} v_0\mathbf{e}^m & \text{for } \xi < 0, \\ 0 & \text{for } \xi > 0, \end{cases} \end{aligned}$$

Define

$$\widetilde{H}(\xi) = \begin{cases} 1 & \text{for } \xi < 0, \\ 0 & \text{for } \xi > 0 \end{cases},$$
$$G(\xi) = \begin{cases} \xi & \text{for } \xi < 0, \\ 0 & \text{for } \xi > 0 \end{cases},$$

Here $\widetilde{H}(\xi) = 1 - H(\xi)$ where $H(\xi)$ is the Heaviside function and $G(\xi) = -\int_\xi^\infty \widetilde{H}(x)\, dx$. Then

$$\begin{aligned}
\phi_m(\xi) &= -v_0/c_m \; G(\xi), \\
\phi_m'(\xi) &= -v_0/c_m \; \widetilde{H}(\xi), \\
\mathbf{u}_m(\xi) &= -v_0/c_m \; \mathbf{e}^m \; G(\xi), \\
\mathbf{v}_m(\xi) &= v_0 \mathbf{e}^m \widetilde{H}(\xi),
\end{aligned}$$

and the stress tensor is

$$\begin{aligned}
\sigma_{ij}^p &= \left\{ \lambda\delta_{ij} + 2\mu\kappa_i\kappa_j \right\} \; \phi_p', \\
\sigma_{ij}^s &= \left\{ \mu(\kappa_j e_i^s + \kappa_i e_j^s) \right\} \; \phi_s',
\end{aligned}$$

or explicitly in two-dimensions

$$\begin{aligned}
\sigma_{ij}^p &= \begin{bmatrix} \lambda + 2\mu\kappa_1^2 & 2\mu\kappa_1\kappa_2 \\ 2\mu\kappa_1\kappa_2 & \lambda + 2\mu\kappa_2^2 \end{bmatrix} \; \phi_p', \\
\sigma_{ij}^s &= \begin{bmatrix} 2\mu\kappa_1\kappa_2 & \mu(-\kappa_1^2 + \kappa_2^2) \\ \mu(-\kappa_1^2 + \kappa_2^2) & -2\mu\kappa_1\kappa_2 \end{bmatrix} \; \phi_s'.
\end{aligned}$$

4.1 Traveling Wave Convergence Results

Here are some numerical results when solving for a traveling wave solution. The solution consisted of a p-wave and s-wave. These tables computed using `conv.p traveling.conv [args]` in the `cg/sm/conv` dir. The errors and rates are to first order bascially the same for all the methods.

1. Table 1 : SOS, non-conservative, max-norm : rate($\mathbf{u}$) = 0.83.
2. Table 4 : SOS, non-conservative, L_1-norm : rate($\mathbf{u}$) = 1.57.
3. Table 2 : SOS, conservative, max-norm : rate($\mathbf{u}$) = 0.62.
4. Table 5 : SOS, conservative, L_1-norm : rate($\mathbf{u}$) = 1.36.
5. Table 3 : FOS, Godunov, max-norm : rate($\mathbf{u}$) = 0.75, rate($\mathbf{v}$) = 0, rate($\boldsymbol{\sigma}$) = 0.
6. Table 6 : FOS, Godunov, L_1-norm : rate($\mathbf{u}$) = 1.37, rate($\mathbf{v}$) = 0.63, rate($\boldsymbol{\sigma}$) = 0.64.

Max-norm results for the traveling waves:

grid	N	u_1	u_2
sq32	1	3.7×10^{-3}	4.6×10^{-3}
sq64	2	1.8×10^{-3}	2.5×10^{-3}
sq128	4	1.0×10^{-3}	1.3×10^{-3}
sq256	8	6.1×10^{-4}	7.0×10^{-4}
sq512	16	3.6×10^{-4}	4.8×10^{-4}
rate		0.83	0.84

Table 1: SM, traveling.nc.max, $t = 0.2$, pv=nc diss=10, norm=max, traveling-wave, Sun Apr 26 10:28:55 2009

grid	N	u_1	u_2
sq32	1	3.1×10^{-3}	2.8×10^{-3}
sq64	2	1.9×10^{-3}	1.8×10^{-3}
sq128	4	1.1×10^{-3}	1.1×10^{-3}
sq256	8	6.7×10^{-4}	7.4×10^{-4}
sq512	16	3.9×10^{-4}	5.2×10^{-4}
rate		0.75	0.62

Table 2: SM, traveling.c.max, $t = 0.2$, pv=c diss=10, norm=max, traveling-wave, Sun Apr 26 10:43:21 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
sq32	1	2.1×10^{-1}	2.2×10^{-1}	2.7×10^{-1}	1.9×10^{-1}	1.9×10^{-1}	3.7×10^{-1}	2.5×10^{-3}	2.9×10^{-3}
sq64	2	2.1×10^{-1}	2.3×10^{-1}	2.6×10^{-1}	1.9×10^{-1}	1.9×10^{-1}	3.9×10^{-1}	1.4×10^{-3}	1.4×10^{-3}
sq128	4	2.2×10^{-1}	2.2×10^{-1}	2.8×10^{-1}	2.1×10^{-1}	2.1×10^{-1}	3.7×10^{-1}	8.5×10^{-4}	7.6×10^{-4}
sq256	8	2.2×10^{-1}	2.3×10^{-1}	2.7×10^{-1}	2.0×10^{-1}	2.0×10^{-1}	3.8×10^{-1}	5.2×10^{-4}	4.8×10^{-4}
rate		-0.03	-0.01	-0.01	-0.04	-0.04	-0.01	0.75	0.87

Table 3: SM, traveling.g.max, $t = 0.2$, pv=g diss=10, norm=max, traveling-wave, Sun Apr 26 10:36:26 2009

L_1-norm results for the traveling waves:

grid	N	u_1	u_2
sq32	1	3.2×10^{-4}	4.2×10^{-4}
sq64	2	1.1×10^{-4}	1.3×10^{-4}
sq128	4	3.4×10^{-5}	3.9×10^{-5}
sq256	8	1.1×10^{-5}	1.3×10^{-5}
sq512	16	4.3×10^{-6}	5.2×10^{-6}
rate		1.57	1.59

Table 4: SM, traveling.nc.l1, $t = 0.2$, pv=nc diss=10, norm=l1, traveling-wave, Sun Apr 26 10:32:44 2009

grid	N	u_1	u_2
sq32	1	2.6×10^{-4}	2.9×10^{-4}
sq64	2	9.8×10^{-5}	1.1×10^{-4}
sq128	4	3.5×10^{-5}	4.1×10^{-5}
sq256	8	1.3×10^{-5}	1.6×10^{-5}
sq512	16	5.2×10^{-6}	6.8×10^{-6}
rate		1.43	1.36

Table 5: SM, traveling.c.l1, $t = 0.2$, pv=c diss=10, norm=l1, traveling-wave, Sun Apr 26 10:44:43 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
sq32	1	1.1×10^{-2}	1.2×10^{-2}	1.4×10^{-2}	1.1×10^{-2}	1.1×10^{-2}	2.2×10^{-2}	2.1×10^{-4}	2.8×10^{-4}
sq64	2	7.5×10^{-3}	7.8×10^{-3}	9.6×10^{-3}	7.2×10^{-3}	7.2×10^{-3}	1.4×10^{-2}	7.9×10^{-5}	1.1×10^{-4}
sq128	4	4.8×10^{-3}	4.9×10^{-3}	6.0×10^{-3}	4.6×10^{-3}	4.6×10^{-3}	9.0×10^{-3}	3.1×10^{-5}	4.1×10^{-5}
sq256	8	3.0×10^{-3}	3.0×10^{-3}	3.7×10^{-3}	2.9×10^{-3}	2.9×10^{-3}	5.5×10^{-3}	1.2×10^{-5}	1.6×10^{-5}
rate		0.63	0.66	0.66	0.64	0.64	0.68	1.37	1.39

Table 6: SM, traveling.g.l1, $t = 0.2$, pv=g diss=10, norm=l1, traveling-wave, Sun Apr 26 10:37:56 2009

5 Rayleigh Surface Wave

The Rayleigh surface wave is a traveling surface wave that decays exponentially fast into the bulk of the solid. Consider the elastic half-space $y > 0$. We look for traveling wave solutions of the form

$$u_1 = Ae^{-by}e^{ik(x-ct)}, \tag{99}$$

$$u_2 = Be^{-by}e^{ik(x-ct)}, \tag{100}$$

$$u_3 = 0. \tag{101}$$

where $b > 0$ and c is the speed of the wave. Substituting these forms into the elasticity equations gives

$$\big(c_s^2 b^2 + (c^2 - c_p^2)k^2\big)A - i(c_p^2 - c_s^2)bkB = 0 \tag{102}$$

$$-i\big(c_p^2 - c_s^2\big)bkA + \big(c_p^2 b^2 + (c^2 - c_s^2)k^2\big)B = 0. \tag{103}$$

The condition for non-trivial solutions implies

$$\left(c_p^2 b^2 - (c_p^2 - c^2)k^2\right)\left(c_s^2 b^2 - (c_s^2 - c^2)k^2\right) = 0, \tag{104}$$

with solutions

$$\begin{aligned} b_1 &= k\big(1 - \frac{c^2}{c_p^2}\big)^{1/2}, \quad \frac{B_1}{A_1} = -\frac{b_1}{ik}, \\ b_2 &= k\big(1 - \frac{c^2}{c_s^2}\big)^{1/2}, \quad \frac{B_2}{A_2} = \frac{ik}{b_2}. \end{aligned} \tag{105}$$

For b to be real we require $c < c_s < c_p$. Thus the Rayeligh wave speed is less than the shear wave speed. The general solution is thus of the form

$$u_1 = \Big(A_1 e^{-b_1 y} + A_2 e^{-b_2 y}\Big)e^{ik(x-ct)} \tag{106}$$

$$u_2 = \Big(-\frac{b_1}{ik}A_1 e^{-b_1 y} + \frac{ik}{b_2}A_2 e^{-b_2 y}\Big)e^{ik(x-ct)}. \tag{107}$$

The traction boundary conditions at $y = 0$ are

$$\partial_y u_1 + \partial_x u_2 = 0, \qquad \lambda(\partial_x u_1 + \partial_y u_2) + 2\mu\partial_y u_2 = 0. \tag{108}$$

Substituting the equations for u_1 and u_2 into the boundary conditions gives (using the expresssions (105) for b_1 and b_2)

$$2b_1 A_1 + (2 - \chi)k^2\frac{A_2}{b_2} = 0, \tag{109}$$

$$(2 - \chi)A_1 + 2A_2 = 0, \tag{110}$$

where

$$\chi \equiv \frac{c^2}{c_s^2}. \tag{111}$$

The condition for non-trivial solutions results in the following equation for χ,

$$(2 - \chi)^2 = 4\sqrt{1 - \chi}\sqrt{1 - \gamma\chi}, \tag{112}$$

$$\gamma = \frac{\mu}{\lambda + 2\mu} = \frac{c_s^2}{c_p^2}. \tag{113}$$

The solution to this formula (giving the Rayleigh wave speed) can be written as (see “On formulas for the Rayleigh wave speed”, by Pham Chi Vinh and R.W. Ogden, Wave Motion, **39** (2004), pp. 191-197.)

$$\frac{\rho c^2}{\mu} = 4(1-\gamma)\big(2 - \frac{4}{3}\gamma + \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}\big)^{-1}, \tag{114}$$

$$R = \frac{2}{27}(27 - 90\gamma + 99\gamma^2 - 32\gamma^3), \tag{115}$$

$$D = \frac{4}{27}(1-\gamma)^2(11 - 62\gamma + 107\gamma^2 - 64\gamma^3), \tag{116}$$

where the principal roots must be taken. Note that the formula (114) requires complex arithmetic. In addition we have the relation

$$A_2 = (\frac{\chi}{2} - 1)A_1. \tag{117}$$

In summary the Rayleigh wave is of the form

$$u_1 = A\Big(e^{-b_1 y} + (\frac{\chi}{2} - 1)e^{-b_2 y}\Big)e^{ik(x-ct)}, \tag{118}$$

$$u_2 = A\Big(\frac{ib_1}{k}e^{-b_1 y} + \frac{ik}{b_2}(\frac{\chi}{2} - 1)e^{-b_2 y}\Big)e^{ik(x-ct)}. \tag{119}$$

or in the real form **check me**

$$u_1 = \Big(e^{-b_1 y} + (\frac{\chi}{2} - 1)e^{-b_2 y}\Big)(A\cos(k(x-ct)) + B\sin(k(x-ct))) \tag{120}$$

$$u_2 = \Big(\frac{b_1}{k}e^{-b_1 y} + \frac{k}{b_2}(\frac{\chi}{2} - 1)e^{-b_2 y}\Big)(-A\sin(k(x-ct)) + B\cos(k(x-ct))). \tag{121}$$

with b_1 and b_2 given by (105). Also note that for a solid in the lower half space $y < 0$, one makes the transformation $y \to -y$ and $u_2 \to -u_2$ as the equations and boundary conditions are invariant under this transformation.

Note: Since the wave speed c is independent of k (c only depends on $\gamma = \mu/(\lambda + 2\mu)$), we can form a superposition of different wave numbers to find a Rayleigh wave for any surface shape. In particular, for a given surface shape $u_2(x, y = 0, t = 0) = f(x)$ with a discrete Fourier series

$$f(x) = \sum_k a_k e^{ikx} \tag{122}$$

we choose

$$A_k = a_k\Big(\frac{b_1}{k} - \frac{k}{b_2}(\frac{\chi}{2} - 1)\Big)^{-1}, \tag{123}$$

so that $u_2(x, y = 0, t = 0) = f(x)$, and the solution will be

$$u_1 = \sum_k \Big\{A_k\Big(e^{-b_1 y} + (\frac{\chi}{2} - 1)e^{-b_2 y}\Big)e^{ik(x-ct)}\Big\}, \tag{124}$$

$$u_2 = \sum_k \Big\{A_k\Big(\frac{ib_1}{k}e^{-b_1 y} + \frac{ik}{b_2}(\frac{\chi}{2} - 1)e^{-b_2 y}\Big)e^{ik(x-ct)}\Big\}. \tag{125}$$

with the height of the surface satisfying $u_2(x, y = 0, t) = f(x - ct)$

6 Orthogonal Coordinates

Given an orthogonal coordinate system (r_1, r_2, r_3) with scale factors h_i, and unit coordinate vectors $\hat{\mathbf{r}}_i = \hat{\mathbf{r}}_i(r_1, r_2, r_3)$,

$$\delta \mathbf{x} = \sum_i h_i \ \delta r_i \ \hat{\mathbf{r}}_i \ , \quad \frac{\partial \mathbf{x}}{\partial r_i} = h_i \hat{\mathbf{r}}_i \ , \quad \hat{\mathbf{r}}_i = \frac{\partial \mathbf{x}}{\partial r_i} \Big/ \left| \frac{\partial \mathbf{x}}{\partial r_i} \right|$$

then the gradient of a scalar function F is given by (see for example Batchelor[1])

$$\nabla F = \sum_i \frac{\hat{\mathbf{r}}_i}{h_i} \frac{\partial}{\partial r_i}.$$

Given a vector function

$$\mathbf{F} = \sum_i F_i \hat{\mathbf{r}}_i,$$

the divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \sum_i \frac{\hat{\mathbf{r}}_i}{h_i} \cdot \frac{\partial}{\partial r_i} \mathbf{F} \quad = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial r_i} \left(h_{i+1} h_{i+2} F_i \right)$$

where the subscript of h_{i+1} is wrapped modulo 3. The curl is

$$\nabla \times \mathbf{F} = \sum_i \ \frac{\hat{\mathbf{r}}_i}{h_{i+1} h_{i+2}} \left\{ \frac{\partial (h_{i+2} F_{i+2})}{\partial r_{i+1}} - \frac{\partial (h_{i+1} F_{i+1})}{\partial r_{i+2}} \right\}$$

The Laplacian is

$$\Delta F = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial r_i} \left(\frac{h_{i+1} h_{i+2}}{h_i} \frac{\partial}{\partial r_i} F_i \right)$$

The infinitesimal strain tensor is

$$\mathbf{e} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

with curvilinear components

$$e_{ij} = \hat{\mathbf{r}}_i \cdot \mathbf{e} \cdot \hat{\mathbf{r}}_j,$$

This gives

$$\begin{aligned} e_{ii} &= \frac{1}{h_i} \frac{\partial u_i}{\partial r_i} + \sum_{j \neq i} \frac{u_j}{h_i h_j} \frac{\partial h_i}{\partial r_j}, \\ e_{ij} &= \frac{h_j}{2h_i} \frac{\partial}{\partial r_j} \left(\frac{u_j}{h_j} \right) + \frac{h_i}{2h_j} \frac{\partial}{\partial r_i} \left(\frac{u_i}{h_i} \right), \qquad \text{for } i \neq j. \end{aligned}$$

The stress tensor for linear elasticity is

$$\boldsymbol{\sigma} = \lambda (\nabla \cdot \mathbf{u}) \mathbf{I} + 2\mu \mathbf{e},$$

with components

$$\sigma_{ij} = \lambda \nabla \cdot \mathbf{u} \delta_{ij} + 2\mu e_{ij}.$$

7 Eigenfunctions of an Annulus

7.1 Exact Solutions in Polar Coordinates

For polar coordinates in two dimensions we have $r_1 = r$, $r_2 = \theta$, $h_1 = 1$, and $h_2 = r$. Then

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta(u_\theta),$$
$$\nabla \times \mathbf{u} = \{(1/r)\partial_r(ru_\theta) - (1/r)\partial_\theta(u_r)\}\, \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}.$$

The components of the strain tensor are

$$e_{rr} = \partial_r u_r, \quad e_{\theta\theta} = \frac{1}{r}\partial_\theta u_\theta + u_r/r, \quad e_{r\theta} = \frac{r}{2}\partial_r(u_\theta/r) + (1/2r)\partial_\theta u_r.$$

The components of the stress tensor are

$$\sigma_{rr} = \lambda\left(\frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta(u_\theta)\right) + 2\mu\partial_r u_r, \tag{126}$$
$$= (\lambda + 2\mu)\partial_r u_r + \lambda u_r/r + \lambda\left(\frac{1}{r}\partial_\theta(u_\theta)\right) \tag{127}$$
$$\sigma_{\theta\theta} = \lambda\left(\frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta(u_\theta)\right) + 2\mu\left(\frac{1}{r}\partial_\theta u_\theta + u_r/r\right), \tag{128}$$
$$\sigma_{r\theta} = 2\mu\left(\frac{r}{2}\partial_r(u_\theta/r) + (1/2r)\partial_\theta u_r\right) \tag{129}$$

Let us now look for solutions in an annular region with inner radius r_0 and outer radius r_1 (tube) with no angular variations ($u_\theta = 0$, $\partial_\theta f = 0$). The equations of motion are

$$\rho\partial_t^2 u_r = (\lambda + 2\mu)\partial_r(\nabla \cdot \mathbf{u}), \tag{130}$$
$$= (\lambda + 2\mu)\partial_r\left(\frac{1}{r}\partial_r(ru_r)\right). \tag{131}$$

This form can be seem from using $\Delta\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$ in

$$\rho\partial_t^2 \mathbf{u} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u},$$
$$= (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}.$$

7.2 Steady Solutions on the Annulus

The time independent solution to (131) is of the form

$$u_r = Ar + \frac{B}{r}.$$

If we have a constant pressure p_0 on the inner surface at $r = r_0$ and a constant pressure p_1 on the outer surface at $r = r_1$ then the boundary conditions are

$$\sigma_{rr} = (\lambda + 2\mu)\partial_r u_r + \lambda u_r/r = -p_m, \qquad \text{for } r = r_m,\ m = 0, 1.$$

The solution is (see Love [4], page 144),

$$A = \frac{p_1 r_1^2 - p_0 r_0^2}{2(\lambda + \mu)(r_0^2 - r_1^2)}, \qquad B = \frac{(p_1 - p_0) r_0^2 r_1^2}{2\mu(r_0^2 - r_1^2)}.$$

We could also look for solutions with a specified displacement at each boundary,

$$u_r(r_a) = u_a, \quad u_r(r_b) = u_b,$$

with solution

$$A = ??, \quad B = ??$$

7.3 Time-Harmonic Solution on the Annulus

We now look for time-harmonic eigenfunctions of the form $u_r = Ue^{-i\omega t}$. These will satisfy

$$-\omega^2 \rho U = (\lambda + 2\mu)\partial_r(\nabla \cdot \mathbf{u}), \tag{132}$$

$$= (\lambda + 2\mu)\partial_r\left(\frac{1}{r}\partial_r(rU)\right). \tag{133}$$

or

$$r^2U'' + rU' + (\alpha^2 r^2 - 1)U = 0,$$

where $\alpha^2 = \omega^2\rho/(\lambda + 2\mu)$. The solution to this equation is

$$U = AJ_1(\alpha r) + BY_1(\alpha r),$$

where J_1 and Y_1 are the Bessel functions of the first and second kind.

Let us first look for some solutions with $U(r_a) = U(r_b) = 0$. Then

$$AJ_1(\alpha r_a) + BY_1(\alpha r_a) = 0,$$
$$AJ_1(\alpha r_b) + BY_1(\alpha r_b) = 0.$$

For non-trivial solutions we need α to satisfy

$$W_d(\alpha) = J_1(\alpha r_a)Y_1(\alpha r_b) - J_1(\alpha r_b)Y_1(\alpha r_a) = 0.$$

This equation has an infinite sequence of solutions $\{\alpha_n\}_{n=1}^{\infty}$. Given the value of r_b/r_a the values for α_n can be computed. Table 7 gives a few values for α, A and B computed with the maple program annulusEigs.maple.

n	α_n	A_n	B_n
1	6.39315676162127e+00	3.70054796926959e-01	-2.62717605998727e-01
2	1.26246990207465e+01	-2.44533726961558e-01	2.04902796341409e-01

Table 7: Eigenvalues for the annulus with displacement boundary conditions, $r_a = \frac{1}{2}$, $r_b = 1$

Now consider using traction boundary conditions, $\sigma_{rr}(r) = 0$ at $r = r_a$ and $r = r_b$ where (using (127))

$$r\ \sigma_{rr} = A\left[(\lambda + 2\mu)\alpha r J_1'(\alpha r) + \lambda J_1(\alpha r)\right] + B\left[(\lambda + 2\mu)\alpha r Y_1'(\alpha r) + \lambda J_1(\alpha r)\right]$$
$$\equiv AG_J(\alpha r) + BG_Y(\alpha r)$$

The condition for a non-trivial solution is

$$W_n(\alpha) = G_J(\alpha r_a)G_Y(\alpha r_b) - G_J(\alpha r_b)G_Y(\alpha r_a) = 0.$$

Given the ratios r_b/r_a and λ/μ we can compute the roots α_n to this equation, see table 8.

n	α_n	A_n	B_n
1	1.31135301901399e+00	1.86192468503469e+00	-1.14016375430058e+00
2	6.46336546990778e+00	2.10897392919530e+00	3.67967002089678e+00

Table 8: Eigenvalues for the annulus with traction boundary conditions, $r_a = \frac{1}{2}$, $r_b = 1$, $\lambda/\mu = 1$

7.4 Annulus Eigenfunction Convergence Results

Here are some numerical results for when solving for the eigenfunctions of an annulus. These tables computed using `conv.p annulusEigen.conv [args]` in the `cg/sm/conv` dir. Results are shown for mode 1 ($n = 1$ in tables 7 and 8.)

Results for displacement boundary conditions.

1. Table 9 : SOS, non-conservative, dirichlet BC : rate($\mathbf{u}$) = 1.79.
2. Table 10 : SOS, conservative, dirichlet BC : rate($\mathbf{u}$) = 1.80.
3. Table 11 : FOS, Godunov, dirichlet BC : rate($\mathbf{u}$) = 2.79, rate($\mathbf{v}$) = 2.52, rate($\boldsymbol{\sigma}$) = 2.59.
4. Table 12 : FOS, Hemp, dirichlet BC : rate($\mathbf{u}$) = 1.88, rate($\mathbf{v}$) = 2.52, rate($\boldsymbol{\sigma}$) = 1.88.

grid	N	u_1	u_2
an1	1	3.3×10^{-2}	3.2×10^{-2}
an2	2	1.0×10^{-2}	1.0×10^{-2}
an4	4	2.9×10^{-3}	2.9×10^{-3}
an8	8	7.8×10^{-4}	7.7×10^{-4}
rate		1.80	1.79

Table 9: SM, annulusEigen.nc.d.mode1, bcn=d, mode=1, pv=nc, diss=1, $t = 0.5$, TZ, Sun Apr 26 13:32:36 2009

grid	N	u_1	u_2
an1	1	2.7×10^{-2}	2.0×10^{-2}
an2	2	8.3×10^{-3}	5.8×10^{-3}
an4	4	2.4×10^{-3}	1.6×10^{-3}
an8	8	6.4×10^{-4}	4.2×10^{-4}
rate		1.80	1.87

Table 10: SM, annulusEigen.c.d.mode1, bcn=d, mode=1, pv=c, diss=1, $t = 0.5$, TZ, Sun Apr 26 13:32:50 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
an1	1	2.3×10^{-1}	2.3×10^{-1}	1.7×10^{-1}	5.9×10^{-2}	5.9×10^{-2}	1.7×10^{-1}	4.8×10^{-3}	4.7×10^{-3}
an2	2	4.2×10^{-2}	4.2×10^{-2}	1.6×10^{-2}	6.2×10^{-3}	6.2×10^{-3}	1.6×10^{-2}	3.0×10^{-4}	2.9×10^{-4}
an4	4	7.6×10^{-3}	7.5×10^{-3}	3.0×10^{-3}	1.1×10^{-3}	1.1×10^{-3}	3.0×10^{-3}	5.8×10^{-5}	5.7×10^{-5}
an8	8	1.2×10^{-3}	1.2×10^{-3}	7.1×10^{-4}	2.6×10^{-4}	2.6×10^{-4}	7.1×10^{-4}	1.3×10^{-5}	1.3×10^{-5}
rate		2.53	2.52	2.62	2.59	2.59	2.61	2.79	2.79

Table 11: SM, annulusEigen.g.d.mode1, bcn=d, mode=1, pv=g, diss=1, $t = 0.5$, TZ, Sun Apr 26 13:33:15 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{22}	u_1	u_2
an1	1	1.6×10^{-1}	1.5×10^{-1}	1.3×10^{-1}	1.2×10^{-1}	1.3×10^{-1}	2.0×10^{-2}	2.0×10^{-2}
an2	2	4.8×10^{-2}	4.8×10^{-2}	3.7×10^{-2}	3.3×10^{-2}	3.7×10^{-2}	5.8×10^{-3}	5.7×10^{-3}
an4	4	1.4×10^{-2}	1.4×10^{-2}	1.0×10^{-2}	9.0×10^{-3}	1.0×10^{-2}	1.5×10^{-3}	1.5×10^{-3}
an8	8	4.0×10^{-3}	3.9×10^{-3}	2.6×10^{-3}	2.3×10^{-3}	2.6×10^{-3}	4.0×10^{-4}	4.0×10^{-4}
rate		1.77	1.76	1.88	1.88	1.88	1.89	1.88

Table 12: SM, annulusEigen.h.d.mode1, pv=h, bcn=d, mode=1, diss=1, $t = 0.5$, TZ, Tue May 19 7:02:10 2009

Results for traction boundary conditions.

1. Table 13 : SOS, non-conservative, traction BC : rate($\mathbf{u}$) = 1.85.
2. Table 14 : SOS, conservative, traction BC : rate($\mathbf{u}$) = 1.95.
3. Table 15 : FOS, Godunov, traction BC : rate($\mathbf{u}$) = 1.86, rate($\mathbf{v}$) = 1.86, rate($\boldsymbol{\sigma}$) = 1.90.
4. Table 16 : FOS, Hemp, traction BC : rate($\mathbf{u}$) = 2.21, rate($\mathbf{v}$) = 1.76, rate($\boldsymbol{\sigma}$) = 2.05.

grid	N	u_1	u_2
an1	1	2.5×10^{-3}	5.4×10^{-3}
an2	2	7.4×10^{-4}	1.6×10^{-3}
an4	4	2.0×10^{-4}	4.3×10^{-4}
an8	8	5.3×10^{-5}	1.1×10^{-4}
rate		1.85	1.87

Table 13: SM, annulusEigen.nc.sf.mode1, pv=nc, bcn=sf, mode=1, diss=1, $t = 0.5$, TZ, Sun Apr 26 13:35:13 2009

grid	N	u_1	u_2
an1	1	2.5×10^{-4}	3.7×10^{-4}
an2	2	5.6×10^{-5}	7.1×10^{-5}
an4	4	1.5×10^{-5}	1.7×10^{-5}
an8	8	4.2×10^{-6}	4.2×10^{-6}
rate		1.95	2.15

Table 14: SM, annulusEigen.c.sf.mode1, pv=c, bcn=sf, mode=1, diss=1, $t = 0.5$, TZ, Sun Apr 26 13:35:27 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
an1	1	6.9×10^{-3}	6.7×10^{-3}	10.0×10^{-3}	4.6×10^{-3}	4.6×10^{-3}	1.0×10^{-2}	1.9×10^{-3}	1.8×10^{-3}
an2	2	2.0×10^{-3}	2.0×10^{-3}	2.4×10^{-3}	1.2×10^{-3}	1.2×10^{-3}	2.4×10^{-3}	5.3×10^{-4}	5.2×10^{-4}
an4	4	5.4×10^{-4}	5.4×10^{-4}	6.0×10^{-4}	3.4×10^{-4}	3.4×10^{-4}	6.0×10^{-4}	1.4×10^{-4}	1.4×10^{-4}
an8	8	1.4×10^{-4}	1.4×10^{-4}	1.5×10^{-4}	8.8×10^{-5}	8.8×10^{-5}	1.5×10^{-4}	3.8×10^{-5}	3.8×10^{-5}
rate		1.87	1.86	2.02	1.90	1.90	2.03	1.87	1.86

Table 15: SM, annulusEigen.g.sf.mode1, bcn=sf, mode=1, diss=1 $t = 0.5$, TZ, Sun Apr 26 13:21:39 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{22}	u_1	u_2
an1	1	1.2×10^{-3}	1.2×10^{-3}	1.6×10^{-3}	1.5×10^{-3}	1.6×10^{-3}	2.0×10^{-4}	1.9×10^{-4}
an2	2	3.1×10^{-4}	3.1×10^{-4}	4.1×10^{-4}	3.8×10^{-4}	4.1×10^{-4}	4.5×10^{-5}	4.5×10^{-5}
an4	4	7.4×10^{-5}	7.4×10^{-5}	8.0×10^{-5}	7.2×10^{-5}	8.0×10^{-5}	9.3×10^{-6}	9.2×10^{-6}
an8	8	1.8×10^{-5}	1.8×10^{-5}	1.4×10^{-5}	1.2×10^{-5}	1.4×10^{-5}	2.0×10^{-6}	2.0×10^{-6}
rate		2.03	2.02	2.28	2.33	2.28	2.21	2.21

Table 16: SM, annulusEigen.h.sf.mode1, pv=h, bcn=sf, mode=1, diss=1, $t = 0.5$, TZ, Tue May 19 7:03:36 2009

8 Vibrational modes of a sphere

Chapter XII of Love [4] discusses vibrational mode of a sphere governed by the equations of linear elasticity.

There are solutions to the elastic wave equation of the form $u_j = A\cos(\omega t)\hat{u}_j$, $j = 1, 2, 3$, where

$$\hat{u}_j = -\frac{1}{h^2}\frac{\partial}{\partial x_j}\delta + \sum_n \Big\{ \psi_n(\kappa r)\left(x_{j+1}\frac{\partial \chi_n}{\partial x_{j+2}} - x_{j+2}\frac{\partial \chi_n}{\partial x_{j+1}} + \frac{\partial \phi_{n+1}}{\partial x_j}\right) \tag{134}$$

$$- \frac{n+1}{n+2}\psi_{n+2}(\kappa r) k^2 r^{2n+5}\frac{\partial}{\partial x_j}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\Big\}, \tag{135}$$

$$\delta = \sum_n \omega_n \psi_n(hr), \tag{136}$$

where

$$\begin{aligned}
\kappa^2 &= \omega^2\rho/\mu, \quad h^2 = \omega^2\rho/(\lambda+2\mu), \\
\psi_n(x) &= \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin(x)}{x}\right) \\
&= (-1)^n \frac{1}{2}\sqrt{2\pi}x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(x) \\
\omega_n, \chi_n, \phi_n &= \text{solid spherical harmonic functions of order } n. \\
&= r^n \ e^{il\phi} P_n^l(\cos\theta)
\end{aligned}$$

and δ is a solution to the equation $(\nabla^2 + h^2)\delta = 0$ (i.e. compression wave). The relationships between ω_n, χ_n and ϕ_n will be determined by the boundary conditions. Also note the relationships

$$x\frac{d\psi_{n-1}(x)}{dx} = x^2\psi_n(x) = -\psi_{n-2}(x) - (2n-1)\psi_{n-1}(x).$$

and

$$\psi_1(x) = \frac{x\cos(x) - \sin(x)}{x^3}, \tag{137}$$

$$\psi_2(x) = -\frac{x^2\sin(x) + 3x\cos(x) - 3\sin(x)}{x^5}, \tag{138}$$

$$\psi_3(x) = -\frac{x^3\cos(x) - 6x^2\sin(x) - 15x\cos(x) + 15\sin(x)}{x^7}, \tag{139}$$

$$P_0^0(x) = 1, \tag{140}$$

$$P_1^0(x) = x, \quad P_1^1(x) = -(1-x^2)^{1/2}, \tag{141}$$

$$P_2^0(x) = \frac{1}{2}(3x^2-1), \quad P_2^1(x) = -3x(1-x^2)^{1/2}, \quad P_2^2(x) = 3(1-x^2), \tag{142}$$

and

$$\psi_n(x) = \frac{(-1)^n}{1.3.5\ldots(2n+1)}\left\{1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2.4.(2n+3)(2n+5)} - \ldots\right\}.$$

Application of the traction free boundary condition at $r = a$ leads to the conditions

$$a_n\omega_n + c_n\phi_n = 0, \tag{143}$$

$$b_n\omega_n + d_n\phi_n = 0, \tag{144}$$

$$p_n\left(x_{j+1}\frac{\partial\chi_n}{\partial x_{j+2}} - x_{j+2}\frac{\partial\chi_n}{\partial x_{j+1}}\right) = 0, \qquad j = 1, 2, 3 \tag{145}$$

where

$$p_n = (n-1)\psi_n(\kappa a) + \kappa a \psi_n'(\kappa a), \tag{146}$$

$$a_n = \frac{1}{(2n+1)h^2}\left\{\kappa^2 a^2 \psi_n(ha) + 2(n-1)\psi_{n-1}(ha)\right\}, \tag{147}$$

$$b_n = -\frac{1}{2n+1}\left\{\frac{\kappa^2}{h^2}\psi_n(ha) + \frac{2(n+2)}{ha}\psi_n'(ha)\right\}, \tag{148}$$

$$c_n = \kappa^2 a^2 \psi_n(\kappa a) + 2(n-1)\psi_{n-1}(\kappa a), \tag{149}$$

$$d_n = \kappa^2 \frac{n}{n+1}\left\{\psi_n(\kappa a) + \frac{2(n+2)}{\kappa a}\psi_n'(\kappa a)\right\}. \tag{150}$$

This leads to two classes of solutions

Class I : $p_n = 0$ and $\omega_n = \phi_n \equiv 0$. The zeros of the frequency equation $p_n = 0$ determine values of κ (or ω).

Class II : $\chi_n \equiv 0$, $\phi_n = (-a_n/c_n)\omega_n = (-b_n/d_n)\omega_n$ with frequency equation $a_n d_n - b_n c_n = 0$.

8.1 Vibrations of the first class

One class of solutions, *vibrations of the first class* is given by

$$\mathbf{u} = A\cos(\omega t)\psi_n(\kappa r)\begin{bmatrix} y\partial_z\chi_n - z\partial_y\chi_n \\ z\partial_x\chi_n - x\partial_z\chi_n \\ x\partial_y\chi_n - y\partial_x\chi_n \end{bmatrix} \tag{151}$$

where possible values for κ (and thus ω) are determined as roots of

$$p_n = (n-1)\psi_n(\kappa a) + \kappa a \psi_n'(\kappa a) = 0.$$

The above solution satisfies stress-free boundary conditions at $r = a$ where a is the radius of the sphere. This solution also satisfies $\nabla \cdot \mathbf{u} = 0$.

Case $n = 0$ leads to $\mathbf{u} = 0$.

Case I: $n = 1$, $m = 0$:

$$\chi_1 = rP_1^0(\cos\theta) = r\cos\theta = z$$

$$\mathbf{u} = A\cos(\omega t)\psi_1(\kappa r)\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$$

$$\psi_1'(\kappa a) = 0$$

Roots $\kappa a/\pi = 1.83456604098843,\ 2.89503202144422, \ldots$.

Case I(b): $n = 1$, $m = 1$: (Just a rotated version of Case I)

$$\chi_1 = (a\cos\phi + b\sin\phi)\ rP_1^1(\cos\theta) = -(a\cos\phi + b\sin\phi)\ r\sin\theta = -(ax + by)$$

$$\mathbf{u} = A\cos(\omega t)\psi_1(\kappa r)\begin{bmatrix} bz \\ -az \\ -bx + ay \end{bmatrix}$$

$$\psi_1'(\kappa a) = 0$$

Case I(c): $n = 1$, general case for some constants a, b, c:

$$\mathbf{u} = A\cos(\omega t)\psi_1(\kappa r)\begin{bmatrix} ay - cz \\ -ax + bz \\ -by + cx \end{bmatrix}$$

Case II: $n = 2$ $m = 0$:

$$\begin{aligned}
\psi_2(x) &= -\frac{3x\cos(x) + (x^2 - 3)\sin(x)}{x^5} \\
\chi_2 &= r^2 P_2^0(\cos\theta) = r^2(3\cos^2\theta - 1) \\
&= (3z^2 - (x^2 + y^2 + z^2)) = 2z^2 - x^2 - y^2 \\
\mathbf{u} &= A\cos(\omega t)\psi_2(\kappa r)\begin{bmatrix} 6yz \\ -6xz \\ 0 \end{bmatrix}
\end{aligned}$$

$$\psi_2(\kappa a) + \kappa a \psi_2'(\kappa a) = 0$$

Roots $\kappa a/\pi = .796135239733,\ 2.2714621464, \ldots.$

8.2 Vibrations of the second class

A second class of solutions, *vibrations of the second class*, is of the form (134) with $\chi_n \equiv 0$, (and keeping only one term in the sums),

$$\hat{u}_j = -\frac{1}{h^2}\frac{\partial}{\partial x_j}(\omega_n \psi_n(hr)) + \psi_{n-1}(\kappa r)\frac{\partial \phi_n}{\partial x_j} - \frac{n}{n+1}\kappa^2\psi_{n+1}(\kappa r) r^{2n+3}\frac{\partial}{\partial x_j}\left(\frac{\phi_n}{r^{2n+1}}\right). \tag{152}$$

Using the identities

$$x_j\chi_n = \frac{r^2}{2n+1}\left\{\frac{\partial \chi_n}{\partial x_j} - r^{2n+1},\frac{\partial}{\partial x_j}\left(\frac{\chi_n}{r^{2n+1}}\right)\right\}, \tag{153}$$

$$\frac{d\psi_{n-1}(x)}{dx} = x\psi_n(x) \tag{154}$$

the solution can also be written as

$$\hat{u}_j^{(n)} = -\frac{1}{h^2}\left\{\psi_n(hr)\frac{\partial \omega_n}{\partial x_j} + h^2 x_j \psi_{n+1}(hr)\omega_n\right\} \tag{155}$$

$$+ \psi_{n-1}(\kappa r)\frac{\partial \phi_n}{\partial x_j} - \frac{n}{n+1}\kappa^2\psi_{n+1}(\kappa r)\left\{r^2\frac{\partial \phi_n}{\partial x_j} - (2n+1)x_j\phi_n\right\} \tag{156}$$

This latter form is convenient for numerical evaluation. The spatial derivatives are defined by (** check me **)

$$\frac{\partial}{\partial x_\alpha}\hat{u}_j^{(n)} = -\frac{1}{h^2}\Big\{h^2 x_\alpha \psi_{n+1}(hr)\frac{\partial \omega_n}{\partial x_j} + \psi_n(hr)\frac{\partial^2 \omega_n}{\partial x_j \partial x_\alpha} \tag{157}$$

$$+ h^2\delta_{j\alpha}\psi_{n+1}(hr)\omega_n + h^4 x_j x_\alpha \psi_{n+2}(hr)\omega_n + h^2 x_j \psi_{n+1}(hr)\frac{\partial \omega_n}{\partial x_\alpha}\Big\} \tag{158}$$

$$+ \kappa^2 x_\alpha \psi_n(\kappa r)\frac{\partial \phi_n}{\partial x_j} + \psi_{n-1}(\kappa r)\frac{\partial^2 \phi_n}{\partial x_j \partial x_\alpha} - \frac{n}{n+1}\kappa^4 x_\alpha \psi_{n+2}(\kappa r)\left\{r^2\frac{\partial \phi_n}{\partial x_j} - (2n+1)x_j\phi_n\right\} \tag{159}$$

$$- \frac{n}{n+1}\kappa^2\psi_{n+1}(\kappa r)\left\{2x_\alpha\frac{\partial \phi_n}{\partial x_j} - (2n+1)\delta_{j\alpha}\phi_n + r^2\frac{\partial^2 \phi_n}{\partial x_j \partial x_\alpha} - (2n+1)x_j\frac{\partial \phi_n}{\partial x_\alpha}\right\} \tag{160}$$

8.2.1 Radial vibrations, $n = 0$

When $n = 0$ the solutions are radial vibrations,

$$\hat{u}_j^{(n)} = \frac{x_j}{r}\psi_0'(hr) \tag{161}$$

$$\psi_0(ha) + \frac{4}{\kappa^2 a^2} ha\psi_0'(ha) = 0 \tag{162}$$

with roots in table 17 (from cgDoc/sm/sphere1.maple). These agree with Love, p 285.

m	ha/π
1	8.15966436697752e-01
2	1.92853458475813e+00
3	2.95387153514092e+00
4	3.96577216329668e+00

Table 17: First few roots of the frequency equation for spheroidal vibrations, $n = 0$ with $\lambda = \mu$.

8.2.2 Spheroidal vibrations, $n = 2$

When $n = 2$ the solutions are *spheroidal vibrations.*

In the case $\lambda = \mu$ (i.e. the material satisfies Poisson's condition), the roots are given in table 18 (from cgDoc/sm/sphere2.maple) Note that for each m, ϕ_n and ω_n are related by

$$\phi_n = \mathcal{C}_n^m \omega_n, \tag{163}$$

where the constants $\mathcal{C}_n^m$ are given in the table.

m	$\kappa a/\pi$	$\mathcal{C}_n^m$
1	8.40296489389027e-01	-3.75375159272393e-01
2	1.54866444711667e+00	-7.98905931500425e-02
3	2.65126525857435e+00	4.64019111586348e-02
4	3.11312271792062e+00	-1.61929174684121e-02

Table 18: First few roots of the frequency equation for spheroidal vibrations, $n = 2$ with $\lambda = \mu$. The spherical harmonics are related by $\phi_n = \mathcal{C}_n^m \omega_n$.

Note that (see cgDoc/sm/sphericalHarmonics.maple) the solid spherical harmonics of order 2 are

$$r^2 Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(2z^2 - x^2 - y^2\right) \tag{164}$$

$$r^2 Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\left(z(x + iy)\right) \tag{165}$$

$$r^2 Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left(x^2 - y^2 + 2ixy\right) \tag{166}$$

I think that the solution using $\omega_n = Y_2^1$ or $\omega_n = r^2 Y_2^2$ will just be a rotated version of the solution for $\omega_n = Y_2^0$.

8.3 Sphere Eigenfunction Convergence Results

Here are some numerical results for when solving for the eigenfunctions of a sphere. These tables computed using `conv.p sphereEigen.conv [args]` in the `cg/sm/conv` dir. Results are shown for mode 1 ($n = 1$ in tables 19 and **??**.)

Results for vibrations of the first class.

grid	N	u_1	u_2	u_3
sphere1	1	1.3×10^{-2}	1.4×10^{-2}	1.0×10^{-2}
sphere2	2	2.7×10^{-3}	2.9×10^{-3}	2.2×10^{-3}
sphere4	4	7.0×10^{-4}	7.6×10^{-4}	6.0×10^{-4}
rate		2.11	2.12	2.05

Table 19: SM, sphereEigen.nc.sf.class1.n1.m0, pv=nc, bcn=sf, class=1, n=1, m=0, diss=0.5, $t = 0.5$, TZ, Tue Aug 18 14:41:45 2009

grid	N	u_1	u_2	u_3
sphere1	1	1.4×10^{-2}	1.6×10^{-2}	1.2×10^{-2}
sphere2	2	3.7×10^{-3}	4.2×10^{-3}	3.0×10^{-3}
sphere4	4	9.4×10^{-4}	1.0×10^{-3}	7.8×10^{-4}
rate		1.97	1.99	1.95

Table 20: SM, sphereEigen.c.sf.class1.n1.m0, pv=c, bcn=sf, class=1, n=1, m=0, diss=0.5, $t = 0.5$, TZ, Tue Aug 18 14:43:25 2009

grid	N	u_1	u_2	u_3
sphere1	1	6.2×10^{-3}	2.6×10^{-3}	1.3×10^{-3}
sphere2	2	1.3×10^{-3}	7.1×10^{-4}	2.6×10^{-4}
sphere4	4	2.6×10^{-4}	1.6×10^{-4}	7.7×10^{-5}
rate		2.28	1.99	2.03

Table 21: SM, sphereEigen.nc.sf.class1.n2.m0, pv=nc, bcn=sf, class=1, n=2, m=0, diss=0.5, $t = 0.5$, TZ, Tue Aug 18 14:41:59 2009

grid	N	u_1	u_2	u_3
sphere1	1	2.1×10^{-3}	2.3×10^{-3}	8.5×10^{-4}
sphere2	2	7.3×10^{-4}	7.0×10^{-4}	1.7×10^{-4}
sphere4	4	2.0×10^{-4}	1.7×10^{-4}	3.4×10^{-5}
rate		1.72	1.86	2.32

Table 22: SM, sphereEigen.c.sf.class1.n2.m0, pv=c, bcn=sf, class=1, n=2, m=0, diss=0.5, $t = 0.5$, TZ, Tue Aug 18 14:44:13 2009

Results for vibrations of the second class.

grid	N	u_1	u_2	u_3
sphere1	1	2.3×10^{-3}	5.0×10^{-3}	4.2×10^{-3}
sphere2	2	7.5×10^{-4}	8.9×10^{-4}	8.2×10^{-4}
sphere4	4	1.8×10^{-4}	1.6×10^{-4}	2.9×10^{-4}
rate		1.85	2.49	1.91

Table 23: SM, sphereEigen.nc.sf.class2.n2.m1, pv=nc, bcn=sf, class=2, n=2, m=1, diss=0.5, $t = 0.5$, TZ, Thu Sep 3 7:02:48 2009

grid	N	u_1	u_2	u_3
sphere1	1	1.8×10^{-2}	2.1×10^{-2}	2.2×10^{-2}
sphere2	2	1.9×10^{-3}	2.1×10^{-3}	2.9×10^{-3}
sphere4	4	2.2×10^{-4}	2.1×10^{-4}	3.1×10^{-4}
rate		3.16	3.32	3.08

Table 24: SM, sphereEigen.nc.sf.class2.n2.m1, pv=nc, bcn=sf, class=2, n=2, m=1, diss=0., filter=1 $t = 0.5$, TZ, Thu Sep 3 7:15:38 2009

grid	N	u_1	u_2	u_3
sphere1	1	6.8×10^{-3}	6.7×10^{-3}	5.7×10^{-3}
sphere2	2	1.8×10^{-3}	1.8×10^{-3}	1.6×10^{-3}
sphere4	4	5.0×10^{-4}	5.0×10^{-4}	4.5×10^{-4}
rate		1.88	1.87	1.83

Table 25: SM, sphereEigen.c.sf.class2.n2.m1, pv=c, bcn=sf, class=2, n=2, m=1, diss=0.5, $t = 0.5$, TZ, Thu Sep 3 7:05:56 2009

9 The method of analytic solutions

The *method of analytic solutions* is a very useful technique for constructing exact solutions to check the accuracy of a numerical implementation. This method, also sometimes known as the *method of manufactured solutions* [5], or *twilight-zone forcing* [2] adds forcing functions to the governing equations and boundary conditions. These forcing functions are determined so that some given functions, $\bar{u}(\mathbf{x}, t)$, will be the exact solution to the forced equations. With this approach, the error in the discrete solution can be easily determined. To illustate the technique, consider solving the IBVP for the equations of linear elasticity in second-order form,

$$
\begin{aligned}
&\rho\mathbf{u}_{tt} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\Delta\mathbf{u} + \mathbf{f}, && \text{for } \mathbf{x}\in\Omega,\\
&\mathbf{u}(x,0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1(\mathbf{x}), && \text{for } \mathbf{x}\in\Omega, \text{ at } t=0,\\
&\mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t), && \text{for } \mathbf{x}\in\partial\Omega.
\end{aligned}
$$

Any given smooth function, $\bar{\mathbf{u}}(\mathbf{x}, t)$, will be an exact solution of the IBVP if we set the forcing function, initial conditions and boundary conditions as

$$
\begin{aligned}
&\mathbf{f}(\mathbf{x},t) = \rho\bar{\mathbf{u}}_{tt} - (\lambda+\mu)\nabla(\nabla\cdot\bar{\mathbf{u}}) - \mu\Delta\bar{\mathbf{u}},\\
&\mathbf{u}_0(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x},0), \quad \mathbf{u}_1(\mathbf{x}) = \bar{\mathbf{u}}_t(\mathbf{x},0), \quad \text{and} \quad \mathbf{g}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t).
\end{aligned}
$$

In our numerical implementation, we have a number of choices available for $\bar{\mathbf{u}}$, including polynomials, trigonometric functions, and exponential functions, among others.

For the two-dimensional results given in the tables we use a trigonometric exact solution given by

$$
\begin{aligned}
\bar{u}_1 &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (167)\\
\bar{u}_2 &= \frac{1}{2}\sin(f_x\pi x)\sin(f_y\pi y)\cos(f_t\pi t). && (168)\\
& && (169)
\end{aligned}
$$

For the first-order system we also define exact solutions for the velocities and stresses (**fix me : these are all currently the same function ***)

$$
\begin{aligned}
\bar{v}_1 &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (170)\\
\bar{v}_2 &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (171)\\
\bar{\sigma}_{11} &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (172)\\
\bar{\sigma}_{12} &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (173)\\
\bar{\sigma}_{21} &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (174)\\
\bar{\sigma}_{22} &= \frac{1}{2}\cos(f_x\pi x)\cos(f_y\pi y)\cos(f_t\pi t), && (175)
\end{aligned}
$$

9.1 Results for the cic grid, trigonometric exact solution and displacement boundary conditions

These tables show results for SOS-NC, SOS-C and FOS-G.

grid	N	u_1	u_2
cic1	1	1.0×10^{-2}	8.1×10^{-3}
cic2	2	2.8×10^{-3}	2.1×10^{-3}
cic4	4	6.6×10^{-4}	5.2×10^{-4}
cic8	8	1.7×10^{-4}	1.3×10^{-4}
rate		1.98	1.99

Table 26: SM, tz.cic.nc.d.trig, bcn=d, degreeX=2, degreeT=2, $t = .5$, trig TZ, Tue May 19 11:37:48 2009

grid	N	u_1	r	u_2	r
cic1	1	1.2×10^{-2}		1.2×10^{-2}	
cic2	2	3.1×10^{-3}	3.8	3.1×10^{-3}	4.0
cic4	4	7.7×10^{-4}	4.0	7.6×10^{-4}	4.0
cic8	8	1.9×10^{-4}	4.0	1.9×10^{-4}	4.0
rate		1.97		2.00	

Table 27: SM, tz.cic.nc.d.trig, bcn=d, $t = .5$, trig TZ, fx=1, diss=.1, filter=0 $\lambda = 1$, Thu Sep 3 15:10:47 2009

grid	N	u_1	u_2
cic1	1	9.9×10^{-3}	9.2×10^{-3}
cic2	2	2.5×10^{-3}	2.4×10^{-3}
cic4	4	5.9×10^{-4}	5.9×10^{-4}
cic8	8	1.5×10^{-4}	1.5×10^{-4}
rate		2.02	1.99

Table 28: SM, tz.cic.c.d.trig, bcn=d, degreeX=2, degreeT=2, $t = .5$, trig TZ, Tue May 19 11:42:31 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
cic1	1	1.0×10^{-2}	1.0×10^{-2}	3.3×10^{-2}	2.5×10^{-2}	2.5×10^{-2}	3.4×10^{-2}	4.0×10^{-3}	4.0×10^{-3}
cic2	2	2.6×10^{-3}	2.6×10^{-3}	9.4×10^{-3}	5.5×10^{-3}	5.5×10^{-3}	9.5×10^{-3}	9.6×10^{-4}	1.0×10^{-3}
cic4	4	6.5×10^{-4}	6.5×10^{-4}	2.7×10^{-3}	1.4×10^{-3}	1.4×10^{-3}	2.8×10^{-3}	2.2×10^{-4}	2.2×10^{-4}
cic8	8	1.6×10^{-4}	1.6×10^{-4}	7.5×10^{-4}	3.6×10^{-4}	3.6×10^{-4}	7.5×10^{-4}	5.0×10^{-5}	5.0×10^{-5}
rate		1.98	1.99	1.82	2.03	2.03	1.83	2.11	2.11

Table 29: SM, tz.cic.g.d.trig, bcn=d, degreeX=2, degreeT=2, $t = .5$, trig TZ, Tue May 19 11:46:39 2009

9.2 Results for the cic grid, trigonometric exact solution and traction boundary conditions

These tables show results for SOS-NC, SOS-C and FOS-G.

grid	N	u_1	u_2
cic1	1	1.8×10^{-2}	1.8×10^{-2}
cic2	2	5.3×10^{-3}	4.6×10^{-3}
cic4	4	1.3×10^{-3}	1.2×10^{-3}
cic8	8	3.4×10^{-4}	3.0×10^{-4}
rate		1.92	1.97

Table 30: SM, tz.cic.nc.sf.trig, bcn=sf, $t = .5$, trig TZ, fx=1, Tue May 19 13:07:48 2009

grid	N	u_1	u_2
cic1	1	1.3×10^{-2}	1.7×10^{-2}
cic2	2	3.4×10^{-3}	4.5×10^{-3}
cic4	4	8.6×10^{-4}	1.1×10^{-3}
cic8	8	2.2×10^{-4}	2.9×10^{-4}
rate		1.98	1.95

Table 31: SM, tz.cic.c.sf.trig, bcn=sf, $t = .5$, , Tue May 19 12:44:59 2009

grid	N	v_1	v_2	s_{11}	s_{12}	s_{21}	s_{22}	u_1	u_2
cic1	1	1.2×10^{-2}	1.8×10^{-2}	3.2×10^{-2}	2.0×10^{-2}	2.0×10^{-2}	3.8×10^{-2}	4.3×10^{-3}	5.9×10^{-3}
cic2	2	2.4×10^{-3}	3.2×10^{-3}	9.7×10^{-3}	4.5×10^{-3}	4.5×10^{-3}	1.0×10^{-2}	9.7×10^{-4}	1.2×10^{-3}
cic4	4	5.5×10^{-4}	6.5×10^{-4}	3.0×10^{-3}	1.5×10^{-3}	1.5×10^{-3}	2.9×10^{-3}	2.5×10^{-4}	2.7×10^{-4}
cic8	8	1.4×10^{-4}	1.5×10^{-4}	8.2×10^{-4}	4.3×10^{-4}	4.3×10^{-4}	7.8×10^{-4}	7.7×10^{-5}	7.8×10^{-5}
rate		2.13	2.30	1.75	1.82	1.82	1.86	1.93	2.10

Table 32: SM, tz.cic.g.sf.trig, bcn=sf, $t = .5$, , Tue May 19 12:47:56 2009

9.3 Results for the sphere grid, trigonometric exact solution and displacement boundary conditions

These tables show results for SOS-NC, SOS-C and FOS-G.

grid	N	u_1	u_2	u_3
sphere1	1	4.0×10^{-2}	1.9×10^{-2}	1.8×10^{-2}
sphere2	2	7.8×10^{-3}	5.4×10^{-3}	5.6×10^{-3}
sphere4	4	1.3×10^{-3}	1.2×10^{-3}	1.3×10^{-3}
rate		2.47	1.97	1.87

Table 33: SM, tz.sphere.nc.d.trig, bcn=d, $t = .5$, trig TZ, fx=1, Sat Aug 22 8:02:57 2009

grid	N	$\|u\|$	r
sphere1	1	1.1×10^{-2}	
sphere2	2	2.8×10^{-3}	4.0
sphere4	4	6.9×10^{-4}	4.0
rate		1.99	

Table 34: SM, tz.sphere.c.d.trig, bcn=d, $t = .1$, trig TZ, fx=1, fx=1, diss=.0, filter=1, filterOrder=6, $\lambda = 1$, Mon Sep 28 15:18:56 2009

9.4 Results for the sphere in a box grid, trigonometric exact solution and displacement boundary conditions

These tables show results for SOS-NC, SOS-C and FOS-G.

grid	N	$\|u\|$	r
sib1	1	8.2×10^{-1}	
sib2	2	1.2×10^{-1}	7.0
sib4	4	6.3×10^{-3}	18.7
sib8	8	1.1×10^{-3}	5.7
rate		3.28	

Table 35: SM, tz.sib.nc.d.trig, bcn=d, $t = .25$, trig TZ, fx=1, fx=1, diss=.0, filter=1, filterOrder=6, $\lambda = 1$, Sat Sep 12 7:04:48 2009

grid	N	$\|u\|$	r
sib1	1	8.2×10^{-1}	
sib2	2	1.2×10^{-1}	6.6
sib4	4	6.9×10^{-3}	17.8
sib8	8	1.1×10^{-3}	6.3
rate		3.27	

Table 36: SM, tz.sib.c.d.trig, bcn=d, $t = .25$, trig TZ, fx=1, fx=1, diss=.0, filter=1, filterOrder=6, $\lambda = 1$, Sat Sep 12 13:59:53 2009

grid	N	u	r	v	r	σ	r
sib1	1	9.6×10^{-2}		1.4×10^{-1}		8.3×10^{-1}	
sib2	2	1.2×10^{-2}	8.0	2.7×10^{-2}	5.1	2.1×10^{-1}	3.9
sib4	4	1.5×10^{-3}	8.1	4.7×10^{-3}	5.8	5.2×10^{-2}	4.1
rate		3.01		2.44		1.99	

Table 37: SM, tz.sib.g.d.trig, bcn=d, $t = .25$, trig TZ, fx=1, fx=1, diss=.0, filter=0, filterOrder=6, $\lambda = 1$, Sat Sep 12 18:59:25 2009

9.5 Results for the sphere in a box grid, trigonometric exact solution and traction boundary conditions

These tables show results for SOS-NC, SOS-C and FOS-G.

grid	N	$\|u\|$	r
sib1	1	8.2×10^{-1}	
sib2	2	1.2×10^{-1}	7.0
sib4	4	9.6×10^{-3}	12.2
sib8	8	2.2×10^{-3}	4.4
rate		2.92	

Table 38: SM, tz.sib.nc.sf.trig, bcn=sf, $t = .25$, trig TZ, fx=1, fx=1, diss=.0, filter=1, filterOrder=6, $\lambda = 1$, Sat Sep 12 10:01:48 2009

grid	N	$\|u\|$	r
sib1	1	8.1×10^{-1}	
sib2	2	1.2×10^{-1}	6.6
sib4	4	2.2×10^{-2}	5.7
sib8	8	6.9×10^{-3}	3.1
rate		2.31	

Table 39: SM, tz.sib.c.sf.trig, bcn=sf, $t = .25$, trig TZ, fx=1, fx=1, diss=.0, filter=1, filterOrder=6, $\lambda = 1$, Fri Sep 11 20:20:45 2009

grid	N	u	r	v	r	σ	r
sib1	1	1.2×10^{-1}		5.0×10^{-1}		5.7×10^{-1}	
sib2	2	1.2×10^{-2}	9.9	1.7×10^{-1}	2.8	1.7×10^{-1}	3.3
sib4	4	2.9×10^{-3}	4.1	3.3×10^{-2}	5.3	5.0×10^{-2}	3.5
rate		2.67		1.95		1.75	

Table 40: SM, tz.sib.g.sf.trig, bcn=sf, $t = .1$, trig TZ, fx=1, fx=1, diss=.0, filter=0, filterOrder=6, $\lambda = 1$, Mon Sep 28 18:43:49 2009

10 Energy Estimates

10.1 Summation by Parts Formulae

Define the discrete inner product and norm

$$(u,v)_{r,s} = \sum_{j=r}^{s} \overline{u}_j v_j h, \tag{176}$$

$$\|u\|_{r,s} = (u,u)_{r,s}. \tag{177}$$

Here are some useful summation by parts formula: (Some of these can be found in reference [3]),

$$\begin{aligned}
(u, D_+ v)_{r,s} &= -(D_- u, v)_{r+1,s+1} + \overline{u}_j v_j|_r^{s+1} && (178)\\
&= -(D_- u, v)_{r,s+1} + \overline{u}_{s+1} v_{s+1} - \overline{u}_{r-1} v_r && (179)\\
&= -(D_+ u, v)_{r,s} - h(D_+ u, D_+ v)_{r,s} + \overline{u}_j v_j|_r^{s+1} && (180)
\end{aligned}$$

$$\begin{aligned}
(u, D_- v)_{r,s} &= -(D_+ u, v)_{r-1,s-1} + \overline{u}_j v_j|_{r-1}^{s} && (181)\\
&= -(D_+ u, v)_{r-1,s} + \overline{u}_{s+1} v_s - \overline{u}_{r-1} v_{r-1} && (182)\\
&= -(D_- u, v)_{r,s} + h(D_- u, D_- v)_{r,s} + \overline{u}_j v_j|_{r-1}^{s} && (183)
\end{aligned}$$

$$(u, D_0 v)_{r,s} = -(D_0 u, v)_{r,s} + \frac{1}{2}(\overline{u}_j v_{j+1} + \overline{u}_{j+1} v_j)|_{r-1}^{s} \tag{184}$$

10.2 Discrete Inner-Products

For now on we consider the semi-infinite interval $[0,\infty)$ so that we need only be concerned with the boundary at $x=0$.

Define the continuous inner product on

$$(u,v) = \int_0^\infty \overline{u}(x) v(x) dx. \tag{185}$$

Define the 2nd-order accurate discrete inner product

$$(u,v)_h = \frac{1}{2}\overline{u}_0 v_0 h + (u,v)_{1,\infty} \tag{186}$$

To define high-order accurate discrete inner-products whose weights are 1 except near the boundaries we can use the Euler-Maclaurin summation formula, described in section 10.2.1.

Here, for example, is a fourth-order accurate discrete inner product,

$$(u,v)_{4h} = \frac{3}{8}\overline{u}_0 v_0 h + \frac{7}{6}\overline{u}_1 v_1 h + \frac{23}{24}\overline{u}_2 v_2 h + (u,v)_{3,\infty} \tag{187}$$

10.2.1 Euler-Maclaurin Summation Formula

The Euler-Maclaurin summation formula relates the integral of a function to a trapezoial sum of values and the derivatives at the end-points of the interval,

$$\int_0^n f(x)dx = \Big[\frac{1}{2}f_0 + \sum_{i=1}^{N-1} f_i + \frac{1}{2}f_N\Big] - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\big[f^{(2k-1)}(x)|_0^N\big] \quad + R_p \tag{188}$$

$$|R_p| \le \frac{2}{(2\pi)^{2p}} \int |f^{(2p)(x)}| dx \tag{189}$$

Here B_m are the Bernouilli numbers with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$. Note that the formula is exact for polynomials of degree $2p-1$ or less. The second Euler-Maclaurin summation formula (based on cell-centers) is

$$\int_0^n f(x)dx = \sum_{i=0}^{N-1} f_{i+\frac{1}{2}} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\big(1-2^{-2k+1}\big)\left[f^{(2k-1)}(x)|_0^N\right] \quad + S_p \tag{190}$$

We can use the Euler-Maclaurin formula to determine higher-order approximations to the integral of the form

$$\int_0^n f(x)dx = \sum_{i=0}^{m-1} w_i(f_i + f_{N-i}) + \sum_{i=m}^{N-m} f_i \tag{191}$$

where the quadrature weights are 1 except near the boundary.

For example, taking $p=1$ we obtain the *fourth-order accurate* approximation

$$\int_0^n f(x)dx \approx \Big[\frac{1}{2}f_0 + \sum_{i=1}^{N-1} f_i + \frac{1}{2}f_N\Big] + \frac{1}{12}f'(0) - \frac{1}{12}f'(N). \tag{192}$$

Making the second-order accurate approximation to $f'(0)$ of

$$f'(0) = -\frac{3}{2}f_0 + 2f_1 - \frac{1}{2}f_2 + O(h^2) \tag{193}$$

gives the fourth-order accurate approximation to the integral (**check this**)

$$\int_0^n f(x)dx \approx \frac{3}{8}f_0 + \frac{7}{6}f_1 + \frac{23}{24}f_2 + \sum_{i=3}^{N-3} f_i + ... \tag{194}$$

10.3 Energy estimates for discrete operators

For now on we consider the semi-infinite interval $[0,\infty)$ so that we need only be concerned with the boundary at $x=0$.

Define the continuous inner product

$$(u,v) = \int_0^\infty \overline{u}(x)v(x)dx. \tag{195}$$

Recall the continuous integration by parts formulae,

$$(u,v_x) = -(u_x,v) - \overline{u}(0)\ v(0) \tag{196}$$

$$(u,v_{xx}) = -(u_x,v_x) - \overline{u}(0)\ v_x(0) \tag{197}$$

From (197) we see that the natural boundary conditions for u_{xx} are $u=0$ or $u_x=0$.

Define the 2nd-order accurate discrete inner product

$$(u,v)_h = \frac{1}{2}\overline{u}_0 v_0 h + (u,v)_{1,\infty} \tag{198}$$

Consider

$$(u,D_+D_-v)_h = \frac{1}{2}\overline{u}_0 D_+D_-v_0 h + (u,D_+D_-v)_{1,\infty} \tag{199}$$

$$= \frac{1}{2}\overline{u}_0(D_-v_1 - D_-v_0) - (D_+u,D_+v)_{0,\infty} - u_0 D_-v_1 \tag{200}$$

$$= -\frac{1}{2}\overline{u}_0(D_-v_1 + D_-v_0) - (D_+u,D_+v)_{0,\infty} \tag{201}$$

$$= -\overline{u}_0 D_0 v_0 - (D_+u,D_+v)_{0,\infty} \tag{202}$$

Thus we see that the *natural boundary conditions* for the D_+D_- operator are $u_0 = 0$ or $D_0 u_0 = 0$. Note that $(D_+u, D_+v)_{0,\infty}$ is a second-order accurate approximation to $\int_0^\infty u_x v_x dx$ since D_+u and $D+v$ are cell centered quantities.

Now consider

$$(u, D_0 v)_h = \frac{1}{2}\overline{u}_0 D_0 v_0 h + (u, D_0 v)_{1,\infty} \tag{203}$$

$$= \frac{1}{2}\overline{u}_0 D_0 v_0 h - (D_0 u, v)_{1,\infty} - \frac{1}{2}(\overline{u}_0 v_1 + \overline{u}_1 v_0) \tag{204}$$

$$= \frac{1}{2}\overline{u}_0 D_0 v_0 h + \frac{1}{2} D_0 \overline{u}_0 v_0 h - (D_0 u, v)_h - \frac{1}{2}(\overline{u}_0 v_1 + \overline{u}_1 v_0) \tag{205}$$

$$= -\frac{1}{2}\Big(\overline{u}_0 \frac{1}{2}(v_1 + v_{-1}) + \frac{1}{2}(\overline{u}_1 + \overline{u}_{-1}) v_0\Big) - (D_0 u, v)_h \tag{206}$$

which implies

$$(u, D_0 v)_h + (D_0 u, v)_h = -\frac{1}{2}\Big(\overline{u}_0 \frac{1}{2}(v_1 + v_{-1}) + \frac{1}{2}(\overline{u}_1 + \overline{u}_{-1}) v_0\Big) \tag{207}$$

$$(u, D_0 u)_h = -\frac{1}{4}\Big(\overline{u}_0 \frac{1}{2}(u_1 + u_{-1}) + \frac{1}{2}(\overline{u}_1 + \overline{u}_{-1}) u_0\Big) \tag{208}$$

A fourth-order approximation to u_{xx} is $D_+D_-(1 - \frac{h^2}{12} D_+D_-)u$. If we use the 2nd-order accurate inner-product with this operator we get

$$(u, D_+D_-(1 - \frac{h^2}{12} D_+D_-)v)_h = (u, D_+D_- v)_h - (h^2/12)(u, (D_+D_-)^2 v)_h \tag{209}$$

Now consider the term $(u, (D_+D_-)^2 v)_h$. Using summation by parts, ((first use (202) with $v \rightarrow D_+D_- v$)

$$(u, (D_+D_-)^2 v)_h = \frac{1}{2}\overline{u}_0 (D_+D_-)^2 v_0 h + (u, (D_+D_)^2 v)_{1,\infty} \tag{210}$$

$$= -\overline{u}_0 D_0 D_+ D_- v_0 - (D_+ u, D_+(D_+D_-)v)_{0,\infty} \tag{211}$$

$$= -\overline{u}_0 D_0 D_+ D_- v_0 + D_+ \overline{u}_0 D_+ D_- v_0 + (D_+D_- u, D_+D_- v)_{0,\infty} \tag{212}$$

$$= -\overline{u}_0 D_0 D_+ D_- v_0 + D_+ \overline{u}_0 D_+ D_- v_0 - \frac{1}{2} D_+D_- \overline{u}_0 D_+D_- v_0 + (D_+D_- u, D_+D_- v)_h \tag{213}$$

$$= -\overline{u}_0 D_0 D_+ D_- v_0 + D_0 \overline{u}_0 D_+ D_- v_0 + (D_+D_- u, D_+D_- v)_h \tag{214}$$

Therefore

$$(u, D_+D_-(1 - \frac{h^2}{12} D_+D_-)v)_h = -\Big[(D_+u, D_+v)_{0,\infty} + (h^2/12)(D_+D_- u, D_+D_- v)_h\Big] \tag{215}$$

$$- \overline{u}_0 D_0\big(v_0 - (h^2/12) D_+D_- v_0\big) + (h^2/12) D_0 \overline{u}_0 D_+D_- v_0 \tag{216}$$

Question: How accurate is $(D_+u, D_+v)_{0,\infty} + (h^2/12)(D_+D_- u, D_+D_- v)_h$??

From this last expression we see the *Dirichlet* natural-boundary-conditions are

$$u_0 = 0 \quad \text{and} \quad D_+D_- u_0 = 0 \tag{217}$$

In the case when we also have $D_+D_-(1 - \frac{h^2}{12} D_+D_-)u_0 = 0$ it would then follow that $(D_+D_-)^2 u_0 = 0$, and this would be a fourth-order accurate aprpoxiamtion.

The *Neumann* natural-boundary-conditions from (216) are

$$D_0(1 - (h^2/12) D_+D_-)u_0 = 0 \quad \text{and} \quad D_0 u_0 = 0 \tag{218}$$

Note, however, that a fourth-order approximation to $u_x \approx D_0(1-(h^2/6)D_+D_-)u_0$, which differs from(218). However, the conditions (218) imply $D_0 D_+ D_- u_0 = 0$, which in turn implies that $D_0(1-(h^2/6)D_+D_-)u_0 = 0$. (is this correct?)

We now turn to consideration (u, v_x) using a fourth-order approximation to $v_x \approx D_0(1 - (h^2/6)D_+D_-)u_0$,

$$(u, D_0(1 - (h^2/6)D_+D_-)v)_h = (u, D_0 v)_h - (h^2/6)(u, D_0 D_+ D_-)v)_h \tag{219}$$

Now (using (206)),

$$\begin{aligned}
(u, D_0 D_+ D_-)v)_h &= -\overline{u}_0 \frac{1}{4}((D_+D_-)v_1 + (D_+D_-)v_{-1}) - \frac{1}{4}(\overline{u}_1 + \overline{u}_{-1})(D_+D_-)v_0 && (220)\\
&- (D_0 u, D_+ D_- v)_h && (221)\\
& && (222)
\end{aligned}$$

and

$$\begin{aligned}
(D_0 u, D_+ D_- v)_h &= -D_0 \overline{u}_0 D_0 v_0 - (D_0 D_+ u, D_+ v)_{0,\infty} && (223)\\
&= -D_0 \overline{u}_0 D_0 v_0 + (D_0 D_+ D_- u, v)_{1,\infty} + D_0 D_+ \overline{u}_0 v_0 && (224)\\
&= -D_0 \overline{u}_0 D_0 v_0 + D_0 D_+ \overline{u}_0 v_0 - \frac{1}{2} D_0 D_+ D_- \overline{u}_0 v_0 + (D_0 D_+ D_- u, v)_h && (225)\\
&= -D_0 \overline{u}_0 D_0 v_0 + D_0^2 \overline{u}_0 v_0 + (D_0 D_+ D_- u, v)_h && (226)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(u, D_0 D_+ D_-)v)_h + (D_0 D_+ D_- u, v)_h = -\overline{u}_0 \frac{1}{4}((D_+D_-)v_1 + (D_+D_-)v_{-1}) - \frac{1}{4}(\overline{u}_1 + \overline{u}_{-1})(D_+D_-)v_0 \\
+ D_0 \overline{u}_0 D_0 v_0 - D_0^2 \overline{u}_0 v_0
\end{aligned}$$

10.4 Integration by parts formulae for the second-order inner product

Here are some integration by parts formulae for the second-order inner product:

$$(u,v)_h \equiv \frac{1}{2}\overline{u}_0 v_0 h + (u,v)_{1,\infty} \tag{227}$$

$$\begin{align}
(u, D_- v)_h &= -\frac{1}{2}\Big(\overline{u}_0 v_{-1} + \overline{u}_1 v_0\Big) - (D_+ u, v)_h \tag{228}\\
&= -\overline{u}_0 \frac{1}{2}\Big(v_{-1} + v_0\Big) - (D_+ u, v)_{0,\infty} \tag{229}
\end{align}$$

$$\begin{align}
(u, D_+ D_- v)_h &= -\overline{u}_0 D_0 v_0 - (D_+ u, D_+ v)_{0,\infty} \tag{230}\\
&= -\frac{1}{2}\Big(\overline{u}_0 D_+ v_{-1} + \overline{u}_1 D_+ v_0\Big) - (D_+ u, D_+ v)_h \tag{231}
\end{align}$$

check these

$$\begin{align}
(u, D_0 v)_h &= -\frac{1}{4}\Big(\overline{u}_0(v_1 + v_{-1}) + (\overline{u}_1 + \overline{u}_{-1})v_0\Big) - (D_0 u, v)_h \tag{232}\\
& \tag{233}
\end{align}$$

Using $\overline{u}_1 = \overline{u}_0 + hD_+\overline{u}_0$ and $\overline{u}_{-1} = \overline{u}_0 - hD_-\overline{u}_0$ implies

$$\begin{align*}
\overline{u}_{-1} + \overline{u}_1 &= 2\overline{u}_0 + hD_+\overline{u}_0 - hD_-\overline{u}_0\\
&= 2\overline{u}_0 + 2hD_+\overline{u}_0 - 2hD_0\overline{u}_0
\end{align*}$$

whence

$$(u, D_0 v)_h = -\frac{1}{4}\Big(\overline{u}_0(v_1 + v_{-1})\Big) - \frac{1}{2}\overline{u}_0 v_0 - \frac{1}{2}hD_+\overline{u}_0 v_0 - (D_0 u, v)_{1,\infty} \tag{234}$$

Define the operator $\widetilde{D}_0$ by

$$\begin{align}
\widetilde{D}_0 u_i &= D_0 u_i \qquad \text{for } i > 0 \tag{235}\\
&= D_+ u_i \qquad \text{for } i = 0 \tag{236}
\end{align}$$

and then

$$(u, \widetilde{D}_0 v)_h = -\overline{u}_0 v_0 - (\widetilde{D}_0 u, v)_h \tag{237}$$

10.5 Stress free boundary conditions

The stress free boundary conditions on a boundary $x = 0$ are (with $c_p = \lambda + 2\mu$)

$$c_p u_x + \lambda v_y = 0$$
$$\mu v_x + \mu u_y = 0$$

The energy estimate will involve (with $\dot{u} = u_t$),

$$\mathcal{E} = (\dot{u}, L^u) + (\dot{v}, L^v)$$
$$L^u = \partial_x(c_p \partial_x u) + \partial_x(\lambda \partial_y v)$$
$$L^v = \partial_x(\mu \partial_x v) + \partial_x(\mu \partial_y u))$$

Use the second-order accurate approximations

$$L_h^u = D_+(c_p D_- u) + D_0(\lambda \partial_y v),$$
$$L_h^v = D_+(\mu D_- v) + D_0(\mu \partial_y u)).$$

Using the summation by parts forumale (230) and (**??**)

$$\begin{aligned}
(\dot{u}, L_h^u)_h &= c_p(\dot{u}, D_+D_-u) + \lambda(\dot{u}, D_0 \partial_y v) \\
&= -\dot{u}_0(c_p D_0 u_0) - \dot{u}_0 \frac{1}{2}\lambda(\partial_y v_{-1} + \partial_y v_0) - c_p(D_+\dot{u}, D_+u)_{1,\infty}^2 - \lambda(\widetilde{D}_0 \dot{u}, \partial_y v)_h \\
(\dot{v}, L_h^v)_h &= \mu(\dot{v}, D_+D_-v) + \mu(\dot{v}, D_0 \partial_y u) \\
&= -\dot{v}_0(\mu D_0 v_0) - \dot{v}_0 \frac{1}{2}\mu(\partial_y u_{-1} + \partial_y u_0)) - \mu(D_+\dot{v}, D_+v)_{1,\infty}^2 - \mu(\widetilde{D}_0 \dot{v}, \partial_y u)_h
\end{aligned}$$

We would like to have the boundary terms vanish,

$$\dot{u}_0\Big(c_p D_0 u_0 + \frac{1}{2}\lambda(\partial_y v_{-1} + \partial_y v_0)\Big) = 0$$
$$\dot{v}_0\Big(\mu D_0 v_0 + \frac{1}{2}\mu(\partial_y u_{-1} + \partial_y u_0))\Big) = 0$$

If instead we the operator $\widetilde{D}_0$ in the discretization

$$L_h^u = D_+(c_p D_- u) + \widetilde{D}_0(\lambda \partial_y v),$$
$$L_h^v = D_+(\mu D_- v) + \widetilde{D}_0(\mu \partial_y u)).$$

then after summation by parts we get

$$\begin{aligned}
(\dot{u}, L_h^u)_h &= c_p(\dot{u}, D_+D_-u) + \lambda(\dot{u}, \widetilde{D}_0 \partial_y v) \\
&= -\dot{u}_0(c_p D_0 u_0) - \lambda(\dot{u}_0 \partial_y v_0) - c_p(D_+\dot{u}, D_+u)_{1,\infty}^2 - \lambda(\widetilde{D}_0 \dot{u}, \partial_y v)_h \\
(\dot{v}, L_h^v)_h &= \mu(\dot{v}, D_+D_-v) + \mu(\dot{v}, \widetilde{D}_0 \partial_y u) \\
&= -\dot{v}_0(\mu D_0 v_0) - \mu(\dot{v}_0 \partial_y u_0)) - \mu(D_+\dot{v}, D_+v)_{1,\infty}^2 - \mu(\widetilde{D}_0 \dot{v}, \partial_y u)_h
\end{aligned}$$

The boundary terms vanish if

$$\dot{u}_0\Big(c_p D_0 u_0 + \lambda \partial_y v_0\Big) = 0$$
$$\dot{v}_0\Big(D_0 v_0 + \partial_y u_0\Big) = 0$$

which implies that

$$\begin{aligned}
\dot{u}_0 = 0 &\quad \text{or} \quad c_p D_0 u_0 + \lambda \partial_y v_0 = 0 \\
\text{and} \quad \dot{v}_0 = 0 &\quad \text{or} \quad D_0 v_0 + \partial_y u_0 = 0
\end{aligned}$$

Fourth-order accuracy: (assume constant λ and μ)

$$L_h^u = (\lambda + 2\mu)D_+D_-(1 - h^2/12(D_+D_-)u + \lambda D_0(1 - h^2/6(D_+D_-))\partial_y v$$
$$L_h^v = \mu D_+D_-(1 - h^2/12(D_+D_-)v + \mu D_0(1 - h^2/6(D_+D_-))\partial_y u$$

Summation by parts...

$$\begin{aligned}
E_1 = (u, D_+D_-(1 - h^2/12D_+D_-)u) &= -u_0(b_{11}) + D_+D_-u_0(b_{12}) + \\
&- \Big[(D_+u, D_+v)_{0,\infty} + (h^2/12)(D_+D_-u, D_+D_-v)_h\Big] \\
E_2 = (u, D_0(1 - h^2/6D_+D_-)\partial_y v) &= -u_0(b_{21}) + D_+D_-u_0(b_{22}) + cD_0u_0\ D_0\partial_y v_0 - cD_0^2 u_0 \partial_y v_0 \\
b_{11} &= D_0(1 - (h^2/12)D_+D_-)u_0 \\
b_{12} &= D_0 u_0 \\
b_{21} &= \frac{1}{2}(\partial_y v_1 + \partial_y v_{-1}) - ch^2\frac{1}{2}((D_+D_-)\partial_y v_1 + (D_+D_-)\partial_y v_{-1}) \\
b_{22} &=
\end{aligned}$$

If we only had the first two terms on the right-hand-side of E_2, we could eliminate the boundary terms in $(\lambda + 2\mu)E_1 + \lambda E_2$ by setting the coefficients of u_0 and $D_+D_-u_0$ to zero:

$$(\lambda + 2\mu)b_{11} + \lambda b_{21} = 0$$
$$(\lambda + 2\mu)b_{12} + \lambda b_{22} = 0$$

One possible way to remove the extra terms in E_2 would be to use one-sided difference approximations for $(\partial_x(\lambda\partial_y v)$ near the boundary. On the bounadry itself we could maybe use the fact that $v_{xy} = -u_{yy}$.

10.6 Using higher-order discrete inner-products

We consider the use of a more accurate discrete inner-product given by

$$(u, v)_{4h} = (\frac{1}{2} + a)\overline{u}_0 v_0 h + (1 + b)\overline{u}_1 v_1 h + (1 + c)\overline{u}_2 v_2 h + (u, v)_{3,\infty} \tag{238}$$

With $a = b = c = 0$ this reduces to the second-order accurate formula.

We also define a fourth-order accurate inner-product for cell-centred values (imagine that $u_j \approx U(x_j + h/2)$)

$$(u, v)_{4c} = (1 + \tilde{a})\overline{u}_0 v_0 h + (1 + \tilde{b})\overline{u}_1 v_1 h + (1 + \tilde{c})\overline{u}_2 v_2 h + \sum_{j=3}^{\infty} \overline{u}_j v_j h \tag{239}$$

Now using (202)

$$
\begin{aligned}
(u, D_+D_-v)_{4h} &= (\frac{1}{2}+a)\overline{u}_0 D_+D_-v_0 h + (1+b)\overline{u}_1 D_+D_-v_1 h + (1+c)\overline{u}_2 D_+D_-v_2 h \\
&\quad + (u, D_+D_-v)_{3,\infty} \\
&= -\overline{u}_0 D_0 v_0 - (D_+u, D_+v)_{0,\infty} \\
&\quad + a\overline{u}_0 D_+D_-v_0 h + b\overline{u}_1 D_+D_-v_1 h + c\overline{u}_2 D_+D_-v_2 h \\
&= -\overline{u}_0 D_0 v_0 - (D_+u, D_+v)_{4c} \\
&\quad + \tilde{a} D_+u_0 D_+v_0 + \tilde{b} D_+u_1 D_+v_1 + \tilde{c} D_+u_2 D_+v_2 \\
&\quad + a\overline{u}_0 D_+D_-v_0 h + b\overline{u}_1 D_+D_-v_1 h + c\overline{u}_2 D_+D_-v_2 h
\end{aligned}
$$

Consider now (use the results from $(u,(D_+D_-)^2v)_h$) (Note target: $(u, v_{xxxx}) = -uv_{xxx} + u_x v_{xx} + (u_{xx}, v_{xx})$.

$$
\begin{aligned}
(u, (D_+D_-)^2v)_{4h} &= (\frac{1}{2}+a)\overline{u}_0 (D_+D_-)^2 v_0 h + (1+b)\overline{u}_1 (D_+D_-)^2 v_1 h + (1+c)\overline{u}_2 (D_+D_-)^2 v_2 h \\
&\quad + (u, (D_+D_-)^2 v)_{3,\infty} \\
&= -(u_0 - u_{-1}) D_+D_-v_0 - u_0 D_0 D_+D_-v_0 + (D_+D_-u, D_+D_-v)_{0,\infty} \\
&\quad + a\overline{u}_0 (D_+D_-)^2 v_0 h + b\overline{u}_1 (D_+D_-)^2 v_1 h + c\overline{u}_2 (D_+D_-)^2 v_2 h \\
&= -(u_0 - u_{-1}) D_+D_-v_0 - u_0 D_0 D_+D_-v_0 + (D_+D_-u, D_+D_-v)_{4h??} \\
&\quad + a\overline{u}_0 (D_+D_-)^2 v_0 h + b\overline{u}_1 (D_+D_-)^2 v_1 h + c\overline{u}_2 (D_+D_-)^2 v_2 h \\
&\quad - \tilde{a} D_+D_-\overline{u}_0 D_+D_-v_0 h - \tilde{b} D_+D_-\overline{u}_1 D_+D_-v_1 h - \tilde{c} D_+D_-\overline{u}_2 D_+D_-v_2 h
\end{aligned}
$$

10.7 A model problem

Following Nilsson et.al. we consider the model problem

$$u_{tt} = \nabla\cdot\mathbf{f} \qquad \text{for } x>0, \tag{240}$$

$$u_x + au_y = 0 \qquad \text{at } x=0, \tag{241}$$

$$\mathbf{f} = (u_x + au_y, u_y + au_x). \tag{242}$$

After Fourier transforming in y we get (still using u for $\hat{u}$),

$$u_{tt} = Lu \equiv \partial_x(u_x + i\omega a u) - \omega^2 u + i\omega a u_x$$

which we discretize as

$$L_h u \equiv D_+D_-u + 2i\omega a D_0 u - \omega^2 u$$

We look for discrete boundary conditions that will define a self-adjoint operator and thus consider the quantity

$$B = (v, L_h u)_h - (L_h v, u)_h.$$

where $(u,v)_h$ is the second-order discrete inner product. Noting that

$$\begin{aligned}(v, D_+D_-u)_h - (D_+D_-v, u)_h &= -\frac{1}{2h}(\bar{v}_{-1}u_0 - \bar{v}_0 u_{-1}) - \frac{1}{2h}(\bar{v}_0 u_1 - \bar{v}_1 u_0)\\ &= u_0 D_0 \bar{v}_0 - \bar{v}_0 D_0 u_0\end{aligned}$$

and (defining the averaging operator $Eu_i = \frac{1}{2}(u_{i+1} + u_{i-1})$),

$$\begin{aligned}(v, i\omega D_0 u)_h - (i\omega D_0 v, u)_h &= i\omega\Big((v, D_0 u) + (D_0 v, u)\Big)\\ &= -i\omega\frac{1}{2}\Big(\bar{v}_0 E u_0 + (E\bar{v}_0)u_0\Big)\\ &= \frac{1}{2}\Big(-\bar{v}_0 E(i\omega u_0) + E(-i\omega\bar{v}_0)u_0\Big)\end{aligned}$$

And thus

$$B = u_0\Big(D_0\bar{v}_0 + aE(-i\omega\bar{v}_0)\Big) - \bar{v}_0\Big(D_0 u_0 + aE(i\omega u_0)\Big)$$

This implies the operator is self-adjoint with the discrete boundary condition

$$D_0 u_0 + aE(i\omega u_0) = 0$$

Note that the BC for v is the same, $D_0 v_0 + aE(i\omega v_0) = 0$, as required for self-adjointness. In physical space this boundary condition is

$$D_0 u_0 + aE(\partial_y u_0) = 0$$

11 Beam Equations

Ref. Graff.

Euler-Bernoulli beam (small deflections, low frequency respsonse)

$$\rho A \partial_t^2 y + \partial_x^2 \big(EI \partial_x^2 y \big) = q(x, t)$$

Large rotation Euler-Bernoulli beam (using the von Karman strains, ref. wikipedia)

$$\rho A \partial_t^2 y + \partial_x^2 \big(EI \partial_x^2 y \big) - \frac{3}{2} EA (\partial_x y)^2 \partial_x^2 y = q(x, y)$$

Timoshenko beam theory (accounts for shearing effects, high frequency respsonse)

$$\rho A \partial_t^2 y + GA\kappa \Big(\partial_x \psi - \partial_x^2 y \Big) = q(x, y)$$
$$\rho I \partial_t^2 \psi + GA\kappa \Big(\psi - \partial_x y \Big) - EI \partial_x^2 \psi = q(x, y)$$

or upon eliminating ψ

$$\frac{EI}{\rho A} \partial_x^4 y - \frac{I}{A} \Big(1 + \frac{E}{G\kappa} \Big) \partial_x^2 \partial_t^2 y + \partial_t^2 y + \frac{\rho I}{GA\kappa} \partial_t^4 y = \ldots$$

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