Notes on Boundary Conditions for the SVK Model

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1 Introduction

The purpose of these notes is to clarify the traction boundary conditions for the SVK model.

2 Traction BCs

Traction boundary conditions are

$$\mathbf{t}^T = \mathbf{n}^T \sigma, \qquad \mathbf{x} \in \Gamma,$$

where $\mathbf{t}^T = (t_1, t_2)$ is the traction force on the boundary Γ (in physical coordinates), $\mathbf{n}^T = (n_1, n_2)$ is the unit (outward) normal on Γ, and σ is the Cauchy stress. (Matrix notation will be used in these notes.) The following coordinate systems are relevant to the application of the boundary conditions:

$$\begin{aligned} \mathbf{x} &= (x, y) && \text{physical (Eulerian) coordinates,} \\ \mathbf{X} &= (X, Y) && \text{reference (Lagrangian) coordinates,} \\ \mathbf{r} &= (r, s) && \text{computational coordinates.} \end{aligned}$$

Let

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \text{deformation gradient tensor}, \qquad J = \det(F).$$

In terms of the nominal stress P, the traction boundary conditions are

$$\mathbf{t}^T = \mathbf{n}^T \sigma = \frac{1}{J} \mathbf{n}^T F P, \qquad \mathbf{x} \in \Gamma,$$

The problem is to express the traction boundary condition for P in terms of the computational coordinates. Let us assume, for example, that the boundary Γ corresponds to the curve $r = \text{constant}$ and that the outward normal points in the direction of increasing r. Then,

$$\mathbf{n}^T = \alpha \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y} \right), \qquad \alpha = \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right]^{-1/2}.$$

This vector is proportional to the first row vector of the matrix

$$\frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{X}} \right) \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right) = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{X}} \right) F^{-1}$$

Thus,

$$\mathbf{t}^T = \frac{\alpha}{J} \mathbf{k}^T F^{-1} (FP) = \frac{\alpha}{J} \mathbf{k}^T P, \qquad \mathbf{k}^T = \left(\frac{\partial r}{\partial X}, \frac{\partial r}{\partial Y} \right)$$

Here, we observe that $\mathbf{k}^T$ is proportional to the unit (outward) normal vector, $\mathbf{n}_0^T$, to the boundary curve Γ_0 in the reference coordinates. Thus, the form of the traction boundary condition is

$$\mathbf{t}^T = \beta \mathbf{n}_0^T P.$$

where β is a scalar which is related to α, J and the magnitude of the vector $\mathbf{k}$. The final step is to work out a formula for β. Use

$$\frac{\partial \mathbf{x}}{\partial \mathbf{r}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)\left(\frac{\partial \mathbf{X}}{\partial \mathbf{r}}\right) = F\left(\frac{\partial \mathbf{X}}{\partial \mathbf{r}}\right)$$

which implies

$$\det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{r}}\right) = J\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{r}}\right)$$

This formula may be used to eliminate J in the traction boundary condition. We then note that

$$\frac{\alpha}{\det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{r}}\right)} = \left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2\right]^{-1/2}$$

and

$$\det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{r}}\right)\mathbf{k}^T = \left[\left(\frac{\partial X}{\partial s}\right)^2 + \left(\frac{\partial Y}{\partial s}\right)^2\right]^{1/2}\mathbf{n}_0^T$$

so that

$$\beta = \left[\left(\frac{\partial X}{\partial s}\right)^2 + \left(\frac{\partial Y}{\partial s}\right)^2\right]^{1/2}\left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2\right]^{-1/2} = \frac{|d\Gamma_0|}{|d\Gamma|}$$

Here, $|d\Gamma|$ and $|d\Gamma_0|$ are the magnitudes of the line increments of the boundary curves Γ and Γ_0, respectively. This formula agrees with the one derived from the Blue Book. Finally, it is noted that if the traction force is zero, then the boundary condition reduces to

$$0 = \left(\frac{\partial r}{\partial X}\right)P_{11} + \left(\frac{\partial r}{\partial Y}\right)P_{21} \qquad \text{and} \qquad 0 = \left(\frac{\partial r}{\partial X}\right)P_{12} + \left(\frac{\partial r}{\partial Y}\right)P_{22}\,.$$