Notes on Solving Multi-Domain Problems

William D. Henshaw
Centre for Applied Scientific Computing
Lawrence Livermore National Laboratory
Livermore, CA, 94551.
henshaw@llnl.gov
http://www.llnl.gov/casc/Overture

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Abstract:
This document holds notes and results on solving multi-domain problems. We consider iteration strategies for solving the interface equations without forming the full coupled implicit systems.

Contents

1 Solving Multi-Domain Problems by Iteration.

We are interested in solving multi-physics, multi-domain problems. One simple example is the solution to a problem of heat condition between solids with different thermal conductivities. Another example is coupling fluid flow with heat condition in adjacent solids.

1.1 One-dimensional Interface Problem

We begin by considering the solution to the one-dimensional two-domain problem for Poisson's equation, on the domain $\Omega = [-a, b] = \Omega_1 \bigcup \Omega_2$, with $\Omega_1 = [-a, 0]$, $\Omega_2 = [0, b]$, and with interface Ω_I at $x = 0$,

$$\partial_x(\kappa^{(m)}\partial_x u^{(m)}) = f \qquad m = 1, 2, \quad \text{for } x \in \Omega_m, \tag{1}$$

$$\left[u^{(m)}(0)\right] = 0 \qquad \text{for } x \in \Omega_I, \tag{2}$$

$$\left[\kappa^{(m)}\partial_x u^{(m)}(0)\right] = 0 \qquad \text{for } x \in \Omega_I, \tag{3}$$

$$u^{(1)}(-a) = g(a), \quad u^{(2)}(b) = g(b)\ . \tag{4}$$

Note that $\kappa^{(m)} > 0$, $a > 0$ and $b > 0$. We also assume that $\kappa^{(m)}$ is constant.

Suppose that we have a good solution method for each sub-domain and that we do not want to solve the full coupled equations as a single system. We can define an iteration to solve the coupled equations. Let $u^j \approx u^{(1)}$ and $v^j \approx u^{(2)}$. Define the iteration

$$\kappa^{(1)}u^j_{xx} = f \qquad \kappa^{(2)}v^j_{xx} = f \tag{5}$$

$$u^j(-a) = g(a) \qquad v^j(b) = g(b) \tag{6}$$

$$\kappa^{(1)}u^j_x(0) = \kappa^{(2)}v^{j-1}_x(0) \qquad v^j(0) = u^j(0) \tag{7}$$

where we have applied the *Neumann BC* on u^j and the *Dirichlet* condition on v^j.

Note: Another option would be to use a mixed interface condition

$$a_{11}\kappa^{(1)}u^j_x(0) + a_{12}u^j(0) = a_{11}\kappa^{(2)}v^{j-1}_x(0) + a_{12}v^j(0),$$
$$a_{21}\kappa^{(2)}v^j_x(0) + a_{22}v^j(0) = a_{21}\kappa^{(1)}u^{j-1}_x(0) + a_{22}u^j(0),$$

where we require the coefficients a_{ij} to form a non-singular matrix, $det(a_{ij}) \neq 0$. Upon convergence we will have satisfied both jump conditions. **This option still needs to be analysed.**

We can analyze the convergence properties of this iteration. We first construct particular solutions, $u^p(x)$ and $v^p(x)$ that satisfy the equations and boundary conditions, and satisfy some Dirichlet conditions at the interface,

$$\kappa^{(1)}u^p_{xx} = f \qquad \kappa^{(2)}v^p_{xx} = f, \tag{8}$$

$$u^p(-a) = g(-a) \qquad v^p(b) = g(b), \tag{9}$$

$$u^p(0) = 0 \qquad v^p(0) = 0\ . \tag{10}$$

Letting $\hat{u}^j = u^j - u^p$ and $\hat{v}^j = u^j - v^p$ then $(\hat{u}^j, \hat{v}^j)$ satisfy

$$\kappa^{(1)}\hat{u}^j_{xx} = 0 \qquad \kappa^{(2)}\hat{v}^j_{xx} = 0 \tag{11}$$

$$\hat{u}^j(-a) = 0 \qquad \hat{v}^j(b) = 0 \tag{12}$$

$$\kappa^{(1)}\hat{u}^j_x(0) = \kappa^{(2)}\hat{v}^{j-1}_x(0) - \left[\kappa_m u^p_x(0)\right] \qquad \hat{v}^j(0) = \hat{u}^j(0) \tag{13}$$

The solutions to the interior equations and boundary conditions are of the form

$$\hat{u}^j = A_j(x + a)\quad , \qquad \text{and} \quad \hat{v}^j = B_j(x - b),$$

If we choose initial guesses as $u^0 = u^p$ and $v^0 = v^p$ then $A_0 = 0$ and B_0=0. Substitution into the interface conditions gives

$$\kappa^{(1)} A_j = \kappa^{(2)} B_{j-1} - \Big[\kappa_m u^p_x(0)\Big]$$
$$B_j\ (0-b) = A_j\ (0+a)$$

and thus

$$A_j = -\frac{a}{\kappa^{(1)}}\frac{\kappa^{(2)}}{b} A_{j-1} - \frac{1}{\kappa^{(1)}}\Big[\kappa_m u^p_x(0)\Big].$$

This iteration converges provided

$$\left|\frac{a}{\kappa^{(1)}}\frac{\kappa^{(2)}}{b}\right| < 1 \tag{14}$$

In general this is not a good situation since the iteration does not converge for general values of $\kappa^{(1)}/a$ and $\kappa^{(2)}/b$. It will work fine if $\kappa^{(1)}/a \gg \kappa^{(2)}/b$. This result indicates that we should apply the *Neumann* interface BC on $u^{(1)}$ and the *Dirichlet* on $u^{(2)}$ if $\kappa^{(1)}/a > \kappa^{(2)}/b$, but that we should apply the *Neumann* interface BC on $u^{(2)}$ and *Dirichlet* on $u^{(1)}$, if $\kappa^{(1)}/a < \kappa^{(2)}/b$.

We can define an under-relaxed iteration with relaxation parameter ω by

$$A_j = (1-\omega)A_{j-1} + \omega\frac{1}{\kappa^{(1)}}\Big(\kappa^{(2)} B_{j-1} - \Big[\kappa_m u^p_x(0)\Big]\Big)$$
$$B_j = -\frac{a}{b} A_j$$

in which case

$$A_j = \Big[1 - \omega\big(1 + \frac{a}{\kappa^{(1)}}\frac{\kappa^{(2)}}{b}\big)\Big] A_{j-1} - \omega\frac{1}{\kappa^{(1)}}\Big[\kappa_m u^p_x(0)\Big].$$

This iteration will converge provided $0 < \omega < 2\omega_{\rm opt}$ where the optimal value for ω is

$$\omega_{\rm opt} = \frac{1}{1 + \frac{a}{\kappa^{(1)}}\frac{\kappa^{(2)}}{b}}$$

If $\omega = \omega_{\rm opt}$ then the iteration converges in one iteration to the solution

$$A = -\frac{a^{-1}}{\frac{\kappa^{(1)}}{a} + \frac{\kappa^{(2)}}{b}}\Big[\kappa_m u^p_x(0)\Big],$$
$$B = \frac{b^{-1}}{\frac{\kappa^{(1)}}{a} + \frac{\kappa^{(2)}}{b}}\Big[\kappa_m u^p_x(0)\Big].$$

1.2 One-dimensional Interface Problem with Mixed Interface Conditions

Consider now the situation when we use a mixture of the jump conditions on each side of the interface. In this case the interface conditions are

$$\begin{aligned}\alpha\kappa^{(1)}u_x^j(0)+\beta u^j(0)&=\alpha\kappa^{(2)}v_x^{j-1}(0)+\beta v^j(0),\\ \beta\kappa^{(2)}v_x^j(0)-\alpha v^j(0)&=\beta\kappa^{(1)}u_x^{j-1}(0)-\alpha u^j(0),\end{aligned}$$

where $\beta=1-\alpha$. Proceeding as before we get

$$\begin{aligned}\alpha\kappa^{(1)}A_j+\beta aA_j&=\alpha\kappa^{(2)}B_{j-1}-b\beta B_{j-1}-\alpha\Big[\kappa_m u_x^p(0)\Big]\\ \beta\kappa^{(2)}B_j+\alpha bB_j&=\beta\kappa^{(1)}A_j-\alpha aA_j+\beta\Big[\kappa_m u_x^p(0)\Big]\end{aligned}$$

and thus

$$\begin{aligned}B_j&=\left[\frac{\beta\kappa^{(1)}-\alpha a}{\beta\kappa^{(2)}+\alpha b}\right]A_j+\frac{\beta}{\beta\kappa^{(2)}+\alpha b}\Big[\kappa_m u_x^p(0)\Big]\\ A_j&=\left[\frac{\alpha\kappa^{(2)}-\beta b}{\alpha\kappa^{(1)}+\beta a}\right]\left[\frac{\beta\kappa^{(1)}-\alpha a}{\beta\kappa^{(2)}+\alpha b}\right]A_{j-1}+\Big(-\alpha+\beta\frac{\alpha\kappa^{(2)}-\beta b}{\beta\kappa^{(2)}+\alpha b}\Big)\frac{1}{\alpha\kappa^{(1)}+\beta a}\Big[\kappa_m u_x^p(0)\Big]\\ &=\left[\frac{\alpha\kappa^{(2)}/b-\beta}{\beta\kappa^{(2)}/b+\alpha}\right]\left[\frac{\beta\kappa^{(1)}/a-\alpha}{\alpha\kappa^{(1)}/a+\beta}\right]A_{j-1}-\frac{(\alpha^2+\beta^2)b}{[\alpha\kappa^{(1)}+\beta a][\beta\kappa^{(2)}+\alpha b]}\Big[\kappa_m u_x^p(0)\Big]\end{aligned}$$

Define the amplification factor

$$\lambda=\left[\frac{\alpha\kappa^{(2)}/b-\beta}{\beta\kappa^{(2)}/b+\alpha}\right]\left[\frac{\beta\kappa^{(1)}/a-\alpha}{\alpha\kappa^{(1)}/a+\beta}\right]$$

This iteration converges provided

$$|\lambda|<1 \tag{15}$$

Note 1: If $\alpha=\beta=\frac{1}{2}$ then the iteration will converge since

$$|\lambda|=\left|\left[\frac{\kappa^{(2)}/b-1}{\kappa^{(2)}/b+1}\right]\left[\frac{\kappa^{(1)}/a-1}{\kappa^{(1)}/a+1}\right]\right|<1 \tag{16}$$

Note 2: If $\kappa^{(1)}/a\gg\kappa^{(2)}/b$ then we probably want to choose $\alpha=1$, $\beta=0$ since

$$|\lambda|=\left|\frac{\kappa^{(2)}/b}{\kappa^{(1)}/a}\right|\ll 1 \tag{17}$$

Note 3: If we use an under-relaxed iteration with parameter ω then the iteration will converge for $0<\omega<2\omega_{\text{opt}}$ where (*check this*)

$$\omega_{\text{opt}}=\frac{1}{1-\lambda} \tag{18}$$

with amplification factor

$$\lambda(\omega)=1-\omega(1-\lambda) \tag{19}$$

1.3 Two-dimensional Interface Problem

Now consider a two dimension interface problem for two adjacent squares, 2π-periodic in the y-direction, on the region $\Omega = [-a, b] \times [0, 2\pi] = \Omega_1 \bigcup \Omega_2$, $\Omega_1 = [-a, 0] \times [0, 2\pi]$, $\Omega_2 = [0, b] \times [0, 2\pi]$, with interface Ω_I at $x = 0$,

$$
\begin{aligned}
\partial_x(\kappa^{(m)}\partial_x u^{(m)}) + \partial_y(\kappa^{(m)}\partial_y u^{(m)}) &= f \qquad m = 1, 2, \quad \text{for } x \in \Omega_m, && (20)\\
\Big[u^{(m)}(0, y)\Big] &= 0 \qquad \text{for } x \in \Omega_I, && (21)\\
\Big[\kappa^{(m)}\partial_x u^{(m)}(0, y)\Big] &= 0 \qquad \text{for } x \in \Omega_I, u^{(1)}(-a, y) = g(a), \quad u^{(2)}(b, y) = g(b)\ . && (22)
\end{aligned}
$$

If we Fourier transform in y, with dual variable k, and subtract out a particular solution, $(u^p(x, y), v^p(x, y))$, (as in the one-dimensional case) we are led to the iteration

$$
\begin{aligned}
\kappa^{(1)}u^j_{xx} - \kappa^{(1)}k^2 u^j &= 0 & \kappa^{(2)}v^j_{xx} - \kappa^{(2)}k^2 v^j &= 0 && (23)\\
u^j(-a, y) &= 0 & v^j(b, y) &= 0 && (24)\\
\kappa^{(1)}u^j_x(0, y) &= \kappa^{(2)}v^{j-1}_x(0, y) + f_I(y) & v^j(0, y) &= u^j(0, y) && (25)\\
f_I(y) &\equiv \Big[\kappa_m u^p_x(0, y)\Big] & & && (26)
\end{aligned}
$$

The solution to these equations is of the form (for $k \neq 0$)

$$
\begin{aligned}
u^j &= A_j(y)\frac{1}{2}\big(e^{k(x+a)} - e^{-k(x+a)}\big) = A_j(y)\sinh(k(x+a))\\
v^j &= B_j(y)\frac{1}{2}\big(e^{k(x-b)} - e^{-k(x-b)}\big) = B_j(y)\sinh(k(x-b))
\end{aligned}
$$

For $k = 0$ the solution is of the same form as the one-dimensional problem.

Substitution into the interface conditions gives

$$
\begin{aligned}
\kappa^{(1)}A_j k\cosh(ka) &= \kappa^{(2)}B_{j-1}k\cosh(kb) + f_I(y)\\
B_j\ \sinh(-kb) &= A_j\ \sinh(ka)
\end{aligned}
$$

giving

$$B_j = -\frac{\sinh(ka)}{\sinh(kb)}A_j$$

and

$$A_j = -\frac{\tanh(ka)}{\kappa^{(1)}}\frac{\kappa^{(2)}}{\tanh(kb)}A_{j-1} + \frac{1}{k\ \cosh(ka)\kappa^{(1)}}f_I(y)$$

We can define an under-relaxed iteration with relaxation parameter ω by

$$
\begin{aligned}
A_j &= (1-\omega)A_{j-1} + \omega\Big(-\frac{\tanh(ka)}{\kappa^{(1)}}\frac{\kappa^{(2)}}{\tanh(kb)}A_{j-1} + \frac{1}{k\ \cosh(ka)\kappa^{(1)}}f_I(y)\Big)\\
B_j &= -\frac{\sinh(ka)}{\sinh(kb)}A_j
\end{aligned}
$$

in which case

$$A_j = \Big[1 - \omega\big(1 + \frac{\tanh(ka)}{\kappa^{(1)}}\frac{\kappa^{(2)}}{\tanh(kb)}\big)\Big]A_{j-1} - \omega\frac{1}{k\ \cosh(ka)\kappa^{(1)}}f_I(y).$$

This iteration will converge provided $0 < \omega < 2\omega_{\text{opt}}$ where the optimal value for ω is

$$\omega_{\text{opt}} = \frac{1}{1 + \frac{\tanh(ka)}{\kappa^{(1)}} \frac{\kappa^{(2)}}{\tanh(kb)}}.$$

For $ka \gg 1$ and $kb \gg 1$, the optimal value tends to $\omega_{\text{opt}} \to 1/(1 + \frac{a}{b}\frac{\kappa^{(2)}}{\kappa^{(1)}})$. Thus the optimal ω becomes independent of k for large enough k.

Since in general we don't know a or b we could choose

$$\omega_0 = \frac{1}{1 + \frac{\kappa^{(2)}}{\kappa^{(1)}}}.$$

This is a safe value provided $\omega_0 < 2\omega_{\text{opt}}$.

$$\begin{aligned} &\omega_0 < 2\omega_{\text{opt}} \\ \Rightarrow &\frac{1}{1 + \frac{\kappa^{(2)}}{\kappa^{(1)}}} < \frac{2}{1 + \frac{\tanh(ka)}{\kappa^{(1)}} \frac{\kappa^{(2)}}{\tanh(kb)}} \\ \Rightarrow &\frac{\tanh(ka)}{\tanh(kb)} < \frac{\kappa^{(1)}}{\kappa^{(2)}} + 2 \end{aligned}$$

Thus ω_0 should be a reasonable value to choose provided $\kappa^{(1)}/a \geq \kappa^{(2)}/b$ where a and b are estimates of the linear size of the domain.

1.4 Iteration Strategy for Multiple Domains

The previous analyses suggest a iteration strategy for solving a problem with many sub-domains. With multiple sub-domains we need to decide which interface conditions to apply on the two sides of each interface.

Stratgey: For an interface between domains Ω_1 and Ω_2 we apply a Neumann boundary condition (discretizing the jump condition $[\kappa\partial_n u] = 0$) on the side of the interface with larger value for κ (we really want the larger value of "κ/a" but "a" is not known). The other side of the interface should apply the Dirichlet condition (discretizing the jump condition $[u] = 0$). The relaxation parameter can be chosen as

$$\omega_{\rm opt} \approx \Big(1 + \frac{a^{(1)}}{\kappa^{(1)}}\frac{\kappa^{(2)}}{a^{(2)}}\Big)^{-1},$$

where $a^{(1)}$ and $a^{(2)}$ are rough estimates of the linear size of the domains Ω_1 and Ω_2, respectively.

Example: If we have heat conduction between a gas and a solid where $\kappa_{\rm gas} \ll \kappa_{\rm solid}$, then at the interface we will apply the Neumann BC on the solid and the Dirichlet BC on the gas. Thus, during the iteration to solve the iterface equations, the solid is given the heat-flux from the gas, while the gas is given the temperature from the solid.

Remark: The physical interpretation is that the heat flux condition (*Neumann* interface condition) should be applied on the region where *the temperature equilibrates most quickly.* In the previous example with a gas and solid, if the gas region were tiny, $a_{\rm gas} \ll 1$, then it could be that $\kappa_{\rm gas}/a_{\rm gas} \ll \kappa_{\rm solid}/a_{\rm solid}$. In this case, the temperature in the gase would equilibrate most quickly and thus the heat-flux condition should be applied to the gas.

2 stuff

built upon the **Overture** framework [**?**],[**?**],[**?**].

References