Symmetric Boundary Conditions for Linear Elasticity

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1 Introduction

The purpose of these notes is to describe the application of boundary conditions for the first-order formulation of the equations of linear elasticity (FOS). The FOS form advances the components of displacement, velocity and stress. The new boundary conditions ensure that the stress tensor remains symmetric.

The governing equations are given in Section 2 which is followed by a discussion of the boundary conditions in Section 3.

2 Governing equations

Consider an elastic solid that occupies the space $\mathbf{x} \in \Omega$ at time $t = 0$. It is assumed that the solid is a homogeneous isotropic material, and that its displacement $\mathbf{u}(\mathbf{x}, t)$ is governed by

$$\rho_0 \frac{\partial^2 u_j}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 f_j, \qquad \mathbf{x} \in \Omega, \quad t > 0, \tag{1}$$

where ρ_0 is the density of the material, $\mathbf{f}$ is an acceleration due to an applied body force, and the components of the Cauchy stress tensor σ_{ij} are given by

$$\sigma_{ij} = \lambda \left(\epsilon_{kk}\right) \delta_{ij} + 2\mu\epsilon_{ij}, \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2}$$

Here, ϵ_{ij} is the (linear) strain tensor, δ_{ij} is the identity tensor, and λ and μ are Lamé constants, related to Young's modulus E and Poisson's ratio ν by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{3}$$

The initial conditions for the second-order equations in (1) are

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{4}$$

where $\mathbf{u}_0(\mathbf{x})$ and $\mathbf{v}_0(\mathbf{x})$ are the initial displacement and velocity of the solid, respectively. The boundary conditions for (1) are applied for $\mathbf{x} \in \partial\Omega$ and may take various forms. For these notes, we restrict the discussion to two dimensions, and consider the boundary conditions

$$\begin{array}{ll}
\mathbf{u} = \mathbf{g}(\mathbf{x}, t) & \text{displacement boundary conditions,} \\[4pt]
\sigma \mathbf{n} = \mathbf{h}(\mathbf{x}, t) & \text{traction boundary conditions,} \\[4pt]
\left.\begin{array}{l} \mathbf{n}^T \sigma \mathbf{p} = k(\mathbf{x}, t) \\ \mathbf{n}^T \mathbf{u} = q(\mathbf{x}, t) \end{array}\right\} & \text{slip-wall boundary conditions.}
\end{array} \tag{5}$$

Here, $\mathbf{n}$ is the unit outward normal on the boundary, $\mathbf{p}$ is the unit tangent vector on the boundary, and $\mathbf{g}$, $\mathbf{h}$, k and q are the given displacement, traction stress, tangential stress and normal displacement at the boundary, respectively.

The first-order form of the governing equations are

$$\frac{\partial u_j}{\partial t} = v_j, \qquad \frac{\partial v_j}{\partial t} = \frac{1}{\rho_0}\frac{\partial \sigma_{ij}}{\partial x_i} + f_j, \qquad \frac{\partial \sigma_{ij}}{\partial t} = \lambda\left(\dot{\epsilon}_{kk}\right)\delta_{ij} + 2\mu\dot{\epsilon}_{ij}, \tag{6}$$

where $\mathbf{v}(\mathbf{x},t)$ is velocity and the components of the rate of strain tensor $\dot{\epsilon}_{ij}$ are given by

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

In two-dimensional space $\mathbf{x} = (x,y)$ with the body force taken to be zero, the equations in (6) for velocity and stress have the form

$$\frac{\partial}{\partial t}\mathbf{w} + A\frac{\partial}{\partial x}\mathbf{w} + B\frac{\partial}{\partial y}\mathbf{w} = 0, \tag{7}$$

where

$$\mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & -1/\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho_0 & 0 & 0 \\ -\lambda-2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -1/\rho_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho_0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda-2\mu & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 Boundary conditions

Let us consider two-dimensional space for simplicity. In this case, the system of first-order equations in (6) involves 8 dependent variables, i.e. 2 components of displacement, 2 components of velocity, and 4 components of stress. For linear elasticity the stress tensor is symmetric, i.e. $\sigma_{12} = \sigma_{21}$, and a constraint on the application of the boundary conditions is that the stress tensor remains symmetric.

The discussion is organized as follows. We first describe the application of the primary Dirichlet conditions for displacement, traction and slip-wall boundary conditions (the given physical boundary conditions), and then describe the application of secondary (compatibility) conditions for the three cases.

3.1 Primary Dirichlet conditions

Let us first consider the displacement boundary conditions given in (5). For the boundary $r(x,y) = 0$, say, the primary Dirichlet conditions for displacement imply

$$\mathbf{u} = \mathbf{g}, \qquad \mathbf{v} = \frac{\partial}{\partial t}\mathbf{g}, \qquad \text{on } r = 0\,.$$

There is no issue of breaking the symmetry of the stress tensor here. There is an issue, however, for the traction and slip-wall cases which are discussed next.

For the case of a traction boundary, the primary Dirichlet conditions involve the components of stress. Again, consider the boundary $r(x,y) = 0$ and assume that $\mathbf{n}$ is the outward normal to the boundary. If $\tilde{\sigma}$ is the current value for stress, then set

$$\sigma = \tilde{\sigma} + \mathbf{a}\mathbf{n}^T + \mathbf{n}\mathbf{a}^T, \qquad \text{on } r = 0\,, \tag{8}$$

where $\mathbf{a}$ is a vector to be determined such that $\sigma\mathbf{n} = \mathbf{h}$. Note that the update of $\tilde{\sigma}$ is symmetric independent of the choice for $\mathbf{a}$. Right multiplication with $\mathbf{n}$ gives

$$\sigma\mathbf{n} = \tilde{\sigma}\mathbf{n} + \mathbf{a} + \mathbf{n}(\mathbf{a}^T\mathbf{n}) = \mathbf{h}. \tag{9}$$

It is noted that left multiplication with $\mathbf{n}^T$ gives

$$\mathbf{n}^T\sigma = \mathbf{n}^T\tilde{\sigma} + (\mathbf{n}^T\mathbf{a})\mathbf{n}^T + \mathbf{a}^T.$$

Taking the transpose of this equation and noting that σ and $\tilde{\sigma}$ are symmetric gives (9) back again thus maintaining symmetry. Solving (9) for the components of $\mathbf{a}$ gives

$$a_1 = \frac{1}{2}\left(\left(1+n_2^2\right)\tilde{h}_1 - n_1 n_2 \tilde{h}_2\right), \qquad a_2 = \frac{1}{2}\left(\left(1+n_1^2\right)\tilde{h}_2 - n_1 n_2 \tilde{h}_1\right), \tag{10}$$

where $(\tilde{h}_1, \tilde{h}_2)$ are the components of $\mathbf{h} - \tilde{\sigma}\mathbf{n}$. Thus, the primary condition for a traction boundary is (8) with $\mathbf{a}$ given by (10).

For the case of a slip wall boundary on $r(x,y) = 0$, say, the primary Dirichlet conditions are on certain components of stress and displacement/velocity. For stress, set

$$\sigma = \tilde{\sigma} + \alpha\left(\mathbf{n}\mathbf{p}^T + \mathbf{p}\mathbf{n}^T\right), \qquad \text{on } r = 0, \tag{11}$$

where $\mathbf{n}$ is the normal vector, $\mathbf{p}$ is the tangent vector, and α is a scalar to be determined so that $\mathbf{n}^T\sigma\mathbf{p} = k$. Again, the update of $\tilde{\sigma}$ is symmetric. Left multiplication with $\mathbf{n}^T$ and right multiplication with $\mathbf{p}$ gives

$$\mathbf{n}^T\sigma\mathbf{p} = \mathbf{n}^T\tilde{\sigma}\mathbf{p} + \alpha = k$$

so that

$$\alpha = k - \mathbf{n}^T\tilde{\sigma}\mathbf{p} \tag{12}$$

It is noted that left multiplication with $\mathbf{p}^T$ and right multiplication with $\mathbf{n}$ gives the same result since σ and $\tilde{\sigma}$ are symmetric. For displacement/velocity set

$$\mathbf{u} = \tilde{\mathbf{u}} + \beta\mathbf{n}, \qquad \mathbf{v} = \tilde{\mathbf{v}} + \gamma\mathbf{n}, \qquad \text{on } r = 0, \tag{13}$$

where

$$\beta = q - \mathbf{n}^T\tilde{\mathbf{u}}, \qquad \gamma = \frac{\partial q}{\partial t} - \mathbf{n}^T\tilde{\mathbf{v}}. \tag{14}$$

Thus, the primary conditions for a slip wall boundary are (11) with α given by (12) and (13) with β and γ given by (14).

3.2 Secondary Neumann conditions

For the case of a displacement boundary, secondary Neumann conditions can be derived for the components of stress. We begin with the momentum equations. These are

$$\rho_0\frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + f_1, \qquad \rho_0\frac{\partial v_2}{\partial t} = \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + f_2$$

For the case of the boundary $r(x,y) = 0$ where $\mathbf{u} = \mathbf{g}(\mathbf{x},t)$, we consider the mapped equations

$$\begin{aligned} r_x\frac{\partial \sigma_{11}}{\partial r} + r_y\frac{\partial \sigma_{12}}{\partial r} &= \rho_0\frac{\partial^2 g_1}{\partial t^2} - s_x\frac{\partial \sigma_{11}}{\partial s} + s_y\frac{\partial \sigma_{12}}{\partial s} - f_1 \\ r_x\frac{\partial \sigma_{12}}{\partial r} + r_y\frac{\partial \sigma_{22}}{\partial r} &= \rho_0\frac{\partial^2 g_2}{\partial t^2} - s_x\frac{\partial \sigma_{12}}{\partial s} + s_y\frac{\partial \sigma_{22}}{\partial s} - f_2 \end{aligned} \tag{15}$$

Using a standard discretization centered on the boundary, (15) give two equations for the three components of stress in the first line of ghost cells. A third equation is given by extrapolation of the scalar quantity $\mathbf{p}^T\sigma\mathbf{p}$, where $\mathbf{p} \propto (-r_y, r_x)$ is the tangent vector, i.e.

$$r_y^2\sigma_{11} - 2r_y r_x\sigma_{12} + r_x^2\sigma_{22} = \text{extrapolated} \tag{16}$$

The equations in (15) and (16), then, lead to three linear equations for the three components of stress in the ghost cells of the form

$$\begin{bmatrix} r_x & r_y & 0 \\ 0 & r_x & r_y \\ r_y^2 & -2r_y r_x & r_x^2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where (r_x, r_y) is evaluated on the boundary, $(\sigma_{11}, \sigma_{12}, \sigma_{22})$ are ghost-cell values, and (d_1, d_2, d_3) are known values from (15) and (16). The solution of the linear system is

$$\begin{aligned} \sigma_{11} &= \frac{1}{D}\left[r_x\left(r_x^2 + r_y^2\right)d_1 + r_x r_y\left(r_y d_1 - r_x d_2\right) + r_y^3 d_3\right] \\ \sigma_{12} &= \frac{1}{D}\left[r_y^3 d_1 + r_x^3 d_2 - r_x r_y d_3\right] \\ \sigma_{22} &= \frac{1}{D}\left[r_y\left(r_x^2 + r_y^2\right)d_2 + r_x r_y\left(r_x d_2 - r_y d_1\right) + r_x^3 d_3\right] \end{aligned}$$

where

$$D = \left(r_x^2 + r_y^2\right)^2$$

There is no change to the application of secondary Neumann boundary conditions for the case of traction and slip-wall boundary conditions. These set ghost values for displacement and velocity, and there is no issue of breaking the symmetry of the stress tensor.

3.3 Secondary Dirichlet conditions

For the traction and slip-wall cases, secondary Dirichlet conditions are applied to update components of stress that were not set for the primary Dirichlet conditions. For both cases, approximate values for the components of stress are computed using the defining equations for stress in (2) and finite difference approximations for the derivatives of displacement on the boundary. Let $\tilde{\sigma}$ be the current value of the stress tensor on the boundary, and let $\hat{\sigma}$ be the stress tensor with components determined by finite difference approximations. For the traction case, set

$$\sigma = \tilde{\sigma} + \theta \mathbf{p}\mathbf{p}^T$$

where $\mathbf{p}$ is the unit tangent vector on the boundary and θ is given by

$$\theta = \mathbf{p}^T\left(\hat{\sigma} - \tilde{\sigma}\right)\mathbf{p}$$

so that the tangent-tangent component of stress is specified. (The normal-normal and normal-tangent components were set in the primary Dirichlet conditions.) For the slip-wall case, set

$$\sigma = \tilde{\sigma} + \theta \mathbf{p}\mathbf{p}^T + \phi \mathbf{n}\mathbf{n}^T$$

where $\mathbf{n}$ is the unit normal vector on the boundary, θ is given as before, and ϕ is given by

$$\phi = \mathbf{n}^T\left(\hat{\sigma} - \tilde{\sigma}\right)\mathbf{n}$$

so that the tangent-tangent and normal-normal components of stress are specified.