Nonlinear Elasticity Notes

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1 Governing equations

The purpose of these notes is to describe the necessary equations for a Saint Venant-Kirchhoff material (or Kirchhoff material for short). The equations and some of the notation follows the "Blue" book.

1.1 Stress and Strain

Let $\mathbf{x}'$ measure position in the lab (Eulerian) frame and let $\mathbf{x}$ measure position in the material reference (Lagrangian) frame. Thus,

$$\mathbf{x}'(\mathbf{x},t) = \mathbf{x} + \mathbf{u}(\mathbf{x},t)$$

where $\mathbf{u}$ is displacement. The deformation tensor is

$$F(\mathbf{x},t) = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}}$$

and the Green strain tensor is

$$E(\mathbf{x},t) = \frac{1}{2}\left(F^T F - I\right)$$

For a Kirchhoff material, the PK2 stress S_{ij} is given by

$$S_{ij} = \lambda(\operatorname{tr} E)\delta_{ij} + 2\mu E_{ij}$$

where the Lamé constants λ and μ are related to Young's modulus E and Poisson's ratio ν by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

(see Blue book, p.228). The PK2 stress, in turn, is related to the nominal stress P by

$$P = SF^T$$

(see Blue book, p.103). These two stress measures can also be related to the usual Cauchy stress, see Box 3.2 on p.103.

1.2 Equations of Motion

In the Lagrangian frame, the momentum equation is

$$\rho_0 \frac{\partial}{\partial t} v_i = \frac{\partial}{\partial x_j} P_{ji} \tag{1}$$

where ρ_0 is the density of the reference material and $\mathbf{v}$ is velocity. The latter is related to displacement by

$$\frac{\partial}{\partial t} u_i = v_i \tag{2}$$

From the equations above, we have

$$P = SF^T, \qquad S = (\lambda(\operatorname{tr} E)\, I + 2\mu E)\,, \qquad E = \frac{1}{2}\left(F^T F - I\right)$$

so that P is a function of F which, in turn, is a function of the gradient of the displacement $\mathbf{u}$. Thus, differentiating P with respect to time t gives an equation of the form

$$\frac{\partial}{\partial t} P_{ij} = K_{ijk\ell}(F) \frac{\partial v_k}{\partial x_\ell} \tag{3}$$

The next task is to work out the tensor $K_{ijk\ell}(F)$, the derivative of P_{ij} with respect to $F_{k\ell}$ (which depends on F). Use

$$K_{ijk\ell} = \frac{\partial P_{ij}}{\partial F_{k\ell}} = \frac{\partial}{\partial F_{k\ell}} (S_{im} F_{jm}) = \left(\frac{\partial S_{im}}{\partial E_{\alpha\beta}} \frac{\partial E_{\alpha\beta}}{\partial F_{k\ell}} \right) F_{jm} + S_{im} \left(\frac{\partial F_{jm}}{\partial F_{k\ell}} \right)$$

and the formulas

$$\frac{\partial S_{im}}{\partial E_{\alpha\beta}} = \lambda \delta_{\alpha\beta} \delta_{im} + 2\mu \delta_{i\alpha} \delta_{m\beta}, \qquad \frac{\partial E_{\alpha\beta}}{\partial F_{k\ell}} = \frac{1}{2} (\delta_{\alpha\ell} F_{k\beta} + \delta_{\beta\ell} F_{k\alpha})$$

(thanks Bill) to get

$$K_{ijk\ell} = \lambda F_{ji} F_{k\ell} + \mu F_{j\ell} F_{ki} + \mu \left(F_{jm} F_{km}\right) \delta_{i\ell} + S_{i\ell} \delta_{jk} \tag{4}$$

The equations of motion consist of (1), (2) and (3) with (4).

1.3 Quasi-linear form and eigen-structure

Let us consider the first-order equations in (1) and (3) for velocity and stress. These equations have the quasi-linear form

$$\frac{\partial}{\partial t} \mathbf{w} + A_1 \frac{\partial}{\partial x_1} \mathbf{w} + A_2 \frac{\partial}{\partial x_2} \mathbf{w} = 0$$

where

$$\mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ P_{11} \\ P_{12} \\ P_{21} \\ P_{22} \end{bmatrix}, \qquad A_\ell = \begin{bmatrix} 0 & 0 & -\delta_{1\ell}/\rho_0 & 0 & -\delta_{2\ell}/\rho_0 & 0 \\ 0 & 0 & 0 & -\delta_{1\ell}/\rho_0 & 0 & -\delta_{2\ell}/\rho_0 \\ -K_{111\ell} & -K_{112\ell} & 0 & 0 & 0 & 0 \\ -K_{121\ell} & -K_{122\ell} & 0 & 0 & 0 & 0 \\ -K_{211\ell} & -K_{212\ell} & 0 & 0 & 0 & 0 \\ -K_{221\ell} & -K_{222\ell} & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \ell = 1, 2$$

We are interested in the eigenvalues and eigenvectors of the matrix

$$C = \alpha_1 A_1 + \alpha_2 A_2 = \begin{bmatrix} 0 & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 & 0 \\ 0 & 0 & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 \\ -c_{3,1} & -c_{3,2} & 0 & 0 & 0 & 0 \\ -c_{4,1} & -c_{4,2} & 0 & 0 & 0 & 0 \\ -c_{5,1} & -c_{5,2} & 0 & 0 & 0 & 0 \\ -c_{6,1} & -c_{6,2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

where (α_1, α_2) are metrics given by a further mapping from the Lagrangian reference domain to a unit computational domain, and

$$\left.\begin{array}{ll} c_{3,k} = \alpha_1 K_{11k1} + \alpha_2 K_{11k2} & c_{4,k} = \alpha_1 K_{12k1} + \alpha_2 K_{12k2} \\ c_{5,k} = \alpha_1 K_{21k1} + \alpha_2 K_{21k2} & c_{6,k} = \alpha_1 K_{22k1} + \alpha_2 K_{22k2} \end{array}\right\} \qquad k = 1 \text{ or } 2$$

The eigenvalue problem for C has the form

$$(C - \lambda I)\xi = \begin{bmatrix} -\lambda & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 & 0 \\ 0 & -\lambda & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 \\ -c_{3,1} & -c_{3,2} & -\lambda & 0 & 0 & 0 \\ -c_{4,1} & -c_{4,2} & 0 & -\lambda & 0 & 0 \\ -c_{5,1} & -c_{5,2} & 0 & 0 & -\lambda & 0 \\ -c_{6,1} & -c_{6,2} & 0 & 0 & 0 & -\lambda \end{bmatrix} \xi = 0$$

The bottom four equations from this system give

$$-c_{i,1}\xi_1 - c_{i,2}\xi_2 - \lambda\xi_i = 0, \qquad i = 3,4,5,6$$

If $\lambda \neq 0$, then

$$\xi_i = -\frac{1}{\lambda}\left(c_{i,1}\xi_1 + c_{i,2}\xi_2\right), \qquad i = 3,4,5,6 \tag{5}$$

Now, the top two equations for the system give

$$-\lambda\xi_1 - \alpha_1\xi_3/\rho_0 - \alpha_2\xi_5/\rho_0 = 0$$
$$-\lambda\xi_2 - \alpha_1\xi_4/\rho_0 - \alpha_2\xi_6/\rho_0 = 0$$

Eliminating ξ_i , $i = 3,4,5,6$, from these two equations gives

$$\begin{aligned}(\alpha_1 c_{3,1} + \alpha_2 c_{5,1} - \rho_0\lambda^2)\xi_1 + (\alpha_1 c_{3,2} + \alpha_2 c_{5,2})\xi_2 = 0\\ (\alpha_1 c_{4,1} + \alpha_2 c_{6,1})\xi_1 + (\alpha_1 c_{4,2} + \alpha_2 c_{6,2} - \rho_0\lambda^2)\xi_2 = 0\end{aligned} \tag{6}$$

Nontrivial solutions for ξ exist if

$$(\beta_1 - \rho_0\lambda^2)(\beta_4 - \rho_0\lambda^2) - \beta_2\beta_3 = 0$$

where

$$\beta_1 = \alpha_1 c_{3,1} + \alpha_2 c_{5,1}, \qquad \beta_2 = \alpha_1 c_{3,2} + \alpha_2 c_{5,2}, \qquad \beta_3 = \alpha_1 c_{4,1} + \alpha_2 c_{6,1}, \qquad \beta_4 = \alpha_1 c_{4,2} + \alpha_2 c_{6,2}$$

Solving for $\rho_0\lambda^2$ gives

$$\rho_0\lambda^2 = \frac{1}{2}(\beta_1 + \beta_4) \pm \frac{1}{2}\sqrt{(\beta_1 - \beta_4)^2 + 4\beta_2\beta_3} \tag{7}$$

which gives two pairs of nonzero eigenvalues. There are two conditions which ensure that these four eigenvalues are real. They are

$$(\beta_1 - \beta_4)^2 + 4\beta_2\beta_3 \geq 0 \qquad \beta_1 + \beta_4 \geq \sqrt{(\beta_1 - \beta_4)^2 + 4\beta_2\beta_3}$$

The remaining two eigenvalues are zero.

(Check this. I am not sure that it is correct. DWS) Note that for the case $\alpha_1 = 1$ and $\alpha_2 = 0$, we have

$$\beta_1 = K_{1111} = (\lambda + 2\mu)F_{11}^2 + \mu F_{12}^2, \qquad \beta_4 = K_{1221} = (\lambda + 2\mu)F_{11}^2 + \mu F_{12}^2,$$

and

$$\beta_2 = K_{1121} = \beta_3 = K_{1211} = (\lambda + 2\mu)F_{11}F_{21} + \mu F_{12}F_{22}$$

so that

$$(\beta_1 - \beta_4)^2 + 4\beta_2\beta_3 \geq 0$$

and thus $\rho_0\lambda^2$ is real.

The components of the (right) eigenvectors belonging to the four nonzero eigenvalues may be obtained as follows. Given λ from (7) it is straightforward to find nontrivial values for ξ_1 and ξ_2 satisfying (6). These two components may then be used in (5) to obtain ξ_i , $i = 3,4,5,6$. These four eigenvectors pair up and have the form

$$\xi = [r_1, r_2, -r_3, -r_4, -r_5, -r_6]^T, \qquad \xi = [r_1, r_2, r_3, r_4, r_5, r_6]^T$$

for the two eigenvalues given by

$$\rho_0\lambda^2 = \frac{1}{2}(\beta_1 + \beta_4) + \frac{1}{2}\sqrt{(\beta_1 - \beta_4)^2 + 4\beta_2\beta_3}$$

say, and

$$\xi = [s_1, s_2, -s_3, -s_4, -s_5, -s_6]^T, \qquad \xi = [s_1, s_2, s_3, s_4, s_5, s_6]^T$$

for the two eigenvalues given by

$$\rho_0\lambda^2 = \frac{1}{2}(\beta_1+\beta_4) - \frac{1}{2}\sqrt{(\beta_1-\beta_4)^2+4\beta_2\beta_3}$$

The eigenvectors corresponding to the two zero eigenvalues are

$$\xi = [0,0,\alpha_2,0,-\alpha_1,0]^T, \qquad \xi = [0,0,0,\alpha_2,0,-\alpha_1]^T$$

by inspection. Thus, the matrix of right eigenvectors has the form

$$R = \begin{bmatrix} r_1 & s_1 & 0 & 0 & s_1 & r_1 \\ r_2 & s_2 & 0 & 0 & s_2 & r_2 \\ -r_3 & -s_3 & \alpha_2 & 0 & s_3 & r_3 \\ -r_4 & -s_4 & 0 & \alpha_2 & s_4 & r_4 \\ -r_5 & -s_5 & -\alpha_1 & 0 & s_5 & r_5 \\ -r_6 & -s_6 & 0 & -\alpha_1 & s_6 & r_6 \end{bmatrix}$$

We also need the row vectors η_i^T , $i = 1,2,\ldots,6$ of R^{-1} . They are

$$\eta_1^T = \left[\frac{s_2}{2D_1}, -\frac{s_1}{2D_1}, -\frac{\alpha_1(\alpha_1 s_4+\alpha_2 s_6)}{2D_2}, \frac{\alpha_1(\alpha_1 s_3+\alpha_2 s_5)}{2D_2}, -\frac{\alpha_2(\alpha_1 s_4+\alpha_2 s_6)}{2D_2}, \frac{\alpha_2(\alpha_1 s_3+\alpha_2 s_5)}{2D_2}\right]$$

$$\eta_2^T = \left[-\frac{r_2}{2D_1}, \frac{r_1}{2D_1}, \frac{\alpha_1(\alpha_1 r_4+\alpha_2 r_6)}{2D_2}, -\frac{\alpha_1(\alpha_1 r_3+\alpha_2 r_5)}{2D_2}, \frac{\alpha_2(\alpha_1 r_4+\alpha_2 r_6)}{2D_2}, -\frac{\alpha_2(\alpha_1 r_3+\alpha_2 r_5)}{2D_2}\right]$$

$$\begin{aligned}\eta_3^T = \Bigg[0,0,&\frac{\alpha_1(s_4 r_5 - s_5 r_4)+\alpha_2(r_5 s_6 - r_6 s_5)}{D_2}, \frac{\alpha_1(r_3 s_5 - r_5 s_3)}{D_2}, \\ &\frac{\alpha_1(s_3 r_4 - s_4 r_3)+\alpha_2(s_3 r_6 - s_6 r_3)}{D_2}, \frac{\alpha_2(r_3 s_5 - r_5 s_3)}{D_2}\Bigg]\end{aligned}$$

$$\begin{aligned}\eta_4^T = \Bigg[0,0,&\frac{\alpha_1(s_4 r_6 - s_6 r_4)}{D_2}, \frac{\alpha_1(r_3 s_6 - r_6 s_3)+\alpha_2(r_5 s_6 - r_6 s_5)}{D_2}, \\ &\frac{\alpha_2(s_4 r_6 - s_6 r_4)}{D_2}, \frac{\alpha_1(s_3 r_4 - s_4 r_3)+\alpha_2(r_4 s_5 - r_5 s_4)}{D_2}\Bigg]\end{aligned}$$

$$\eta_5^T = \left[-\frac{r_2}{2D_1}, \frac{r_1}{2D_1}, -\frac{\alpha_1(\alpha_1 r_4+\alpha_2 r_6)}{2D_2}, \frac{\alpha_1(\alpha_1 r_3+\alpha_2 r_5)}{2D_2}, -\frac{\alpha_2(\alpha_1 r_4+\alpha_2 r_6)}{2D_2}, \frac{\alpha_2(\alpha_1 r_3+\alpha_2 r_5)}{2D_2}\right]$$

$$\eta_6^T = \left[\frac{s_2}{2D_1}, -\frac{s_1}{2D_1}, \frac{\alpha_1(\alpha_1 s_4+\alpha_2 s_6)}{2D_2}, -\frac{\alpha_1(\alpha_1 s_3+\alpha_2 s_5)}{2D_2}, \frac{\alpha_2(\alpha_1 s_4+\alpha_2 s_6)}{2D_2}, -\frac{\alpha_2(\alpha_1 s_3+\alpha_2 s_5)}{2D_2}\right]$$

where

$$D_1 = r_1 s_2 - r_2 s_1$$

and

$$D_2 = \alpha_1^2(r_3 s_4 - r_4 s_3) + \alpha_1\alpha_2(r_3 s_6 - r_6 s_3) + \alpha_1\alpha_2(s_4 r_5 - s_5 r_4) + \alpha_2^2(r_5 s_6 - r_6 s_5)$$

Note: The formulas above for the left eigenvectors $(\eta_1^T, \eta_2^T, \eta_5^T, \eta_6^T)$ corresponding to non-zero eigenvalues can be simplified somewhat by noting that the system

$$\eta^T(C-\lambda I) = \eta^T \begin{bmatrix} -\lambda & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 & 0 \\ 0 & -\lambda & 0 & -\alpha_1/\rho_0 & 0 & -\alpha_2/\rho_0 \\ -c_{3,1} & -c_{3,2} & -\lambda & 0 & 0 & 0 \\ -c_{4,1} & -c_{4,2} & 0 & -\lambda & 0 & 0 \\ -c_{5,1} & -c_{5,2} & 0 & 0 & -\lambda & 0 \\ -c_{6,1} & -c_{6,2} & 0 & 0 & 0 & -\lambda \end{bmatrix} = 0 \qquad (8)$$

implies that the third, fourth, fifth and sixth components of η^T are given in terms of the first and second components by

$$\eta_3 = -\frac{\alpha_1}{\rho_0\lambda}\eta_1, \qquad \eta_4 = -\frac{\alpha_1}{\rho_0\lambda}\eta_2, \qquad \eta_5 = -\frac{\alpha_2}{\rho_0\lambda}\eta_1, \qquad \eta_6 = -\frac{\alpha_2}{\rho_0\lambda}\eta_2, \tag{9}$$

So, once the first and second components are found using the formulas for $(\eta_1^T, \eta_2^T, \eta_5^T, \eta_6^T)$ above, the four remaining components may be determined using the more compact formulas in (9). It can be noted that the formulas in (9) come from the last four components of the homogeneous system in (8). One may also consider the first two components in (8), but this just leads to the equations found previously in (6) which determine the four non-zero eigenvalues.

1.4 Godunov method

The governing equations consist of conservation equations for momentum, and evolution equations for displacement and stress, given in (1), (2) and (3), respectively. In terms of the mapped coordinates (r, s), the conservation equations for momentum may be written in the conservation form

$$\frac{\partial}{\partial t}\mathbf{v} + \frac{1}{J}\frac{\partial}{\partial r}\mathbf{f} + \frac{1}{J}\frac{\partial}{\partial s}\mathbf{g} = 0, \tag{10}$$

where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad \mathbf{f} = -\frac{J}{\rho_0}\begin{bmatrix} r_x P_{11} + r_y P_{21} \\ r_x P_{12} + r_y P_{22} \end{bmatrix}, \qquad \mathbf{g} = -\frac{J}{\rho_0}\begin{bmatrix} s_x P_{11} + s_y P_{21} \\ s_x P_{12} + s_y P_{22} \end{bmatrix},$$

and $J = r_x s_y - r_y s_x$ is the jacobian of the mapping. The mapped equations for stress have the quasi-linear form

$$\frac{\partial}{\partial t}\mathbf{P} + B_1\frac{\partial}{\partial r}\mathbf{v} + B_2\frac{\partial}{\partial s}\mathbf{v} = 0, \tag{11}$$

where

$$\mathbf{P} = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{21} \\ P_{22} \end{bmatrix}, \qquad B_1 = -r_x\begin{bmatrix} K_{1111} & K_{1121} \\ K_{1211} & K_{1221} \\ K_{2111} & K_{2121} \\ K_{2211} & K_{2221} \end{bmatrix} - r_y\begin{bmatrix} K_{1112} & K_{1122} \\ K_{1212} & K_{1222} \\ K_{2112} & K_{2122} \\ K_{2212} & K_{2222} \end{bmatrix},$$

$$B_2 = -s_x\begin{bmatrix} K_{1111} & K_{1121} \\ K_{1211} & K_{1221} \\ K_{2111} & K_{2121} \\ K_{2211} & K_{2221} \end{bmatrix} - s_y\begin{bmatrix} K_{1112} & K_{1122} \\ K_{1212} & K_{1222} \\ K_{2112} & K_{2122} \\ K_{2212} & K_{2222} \end{bmatrix},$$

First-order discretizations of (10) and (11) are

$$\frac{\mathbf{v}_{i,j}^{n+1} - \mathbf{v}_{i,j}^n}{\Delta t} + \frac{\mathbf{f}_{i+1/2,j}^n - \mathbf{f}_{i-1/2,j}^n}{J_{i,j}\Delta r} + \frac{\mathbf{g}_{i,j+1/2}^n - \mathbf{g}_{i,j-1/2}^n}{J_{i,j}\Delta s} = 0, \tag{12}$$

$$\frac{\mathbf{P}_{i,j}^{n+1} - \mathbf{P}_{i,j}^n}{\Delta t} + B_{1,i,j}^n\left(\frac{\mathbf{v}_{i+1/2,j}^n - \mathbf{v}_{i-1/2,j}^n}{\Delta r}\right) + B_{2,i,j}^n\left(\frac{\mathbf{v}_{i,j+1/2}^n - \mathbf{v}_{i,j-1/2}^n}{\Delta s}\right) = 0, \tag{13}$$

respectively. The coefficient matrices in (13) are $B_{1,i,j}^n = B_1(F_{i,j}^n)$ and $B_{2,i,j}^n = B_2(F_{i,j}^n)$, and these are given in terms of $F_{i,j}^n = I + (\mathbf{u_x})_{i,j}^n$. The gradient of $\mathbf{u}$ is computed in mapped coordinates using standard centered differences. The displacement is advanced on the grid using

$$\frac{\mathbf{u}_{i,j}^{n+1} - \mathbf{u}_{i,j}^n}{\Delta t} = \mathbf{v}_{i,j}^n. \tag{14}$$

The fluxes $\mathbf{f}^n_{i\pm1/2,j}$ and $\mathbf{g}^n_{i,j\pm1/2}$ in (12), and the velocities $\mathbf{v}^n_{i\pm1/2,j}$ and $\mathbf{v}^n_{i,j\pm1/2}$ in (13) are found by solving Riemann problems in each (mapped) coordinate direction. For example, let us consider the following Riemann problem in the r direction, namely

$$\frac{\partial}{\partial t}\mathbf{v} + \frac{1}{J}\frac{\partial}{\partial r}\mathbf{f} = 0, \qquad \frac{\partial}{\partial t}\mathbf{P} + B_1\frac{\partial}{\partial r}\mathbf{v} = 0,$$

with

$$\mathbf{v}(r,0) = \left\{ \begin{array}{ll} \mathbf{v}_L & \text{if } r<0 \\ \mathbf{v}_R & \text{if } r>0 \end{array} \right., \qquad \mathbf{P}(r,0) = \left\{ \begin{array}{ll} \mathbf{P}_L & \text{if } r<0 \\ \mathbf{P}_R & \text{if } r>0 \end{array} \right.$$

This problem is solved approximately by considering the linearized equation

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{v} \\ \mathbf{P} \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \mathbf{v} \\ \mathbf{P} \end{bmatrix} = 0, \tag{15}$$

where $C_{11}=C_{22}=0$, and

$$C_{12} = -\frac{J}{\rho_0}\begin{bmatrix} r_x & 0 & r_y & 0 \\ 0 & r_x & 0 & r_y \end{bmatrix}, \qquad C_{21} = \frac{1}{2}\left(B_1(F_L)+B_1(F_R)\right).$$

For the calculation of $\mathbf{f}^n_{i-1/2,j}$ and $\mathbf{v}^n_{i-1/2,j}$, say, we use $(\mathbf{v}_L,\mathbf{P}_L,F_L) = (\mathbf{v}_{i-1,j},\mathbf{P}_{i-1,j},F_{i-1,j})$ on the left and $(\mathbf{v}_R,\mathbf{P}_R,F_R) = (\mathbf{v}_{i,j},\mathbf{P}_{i,j},F_{i,j})$ on the right. Let λ_k be the eigenvalues of the coefficient matrix in (15), and let $\mathbf{r}_k = [\mathbf{r}_{1,k},\mathbf{r}_{2,k}]^T$ be the corresponding eigenvectors. Further, let

$$\alpha = R^{-1}\begin{bmatrix} \mathbf{v}_R - \mathbf{v}_L \\ \mathbf{P}_R - \mathbf{P}_L \end{bmatrix}$$

be the vector of wave strengths. Using these quantities, we take

$$\mathbf{f}^n_{i-1/2,j} = \mathbf{f}_L + \sum_{\lambda_k<0}\lambda_k\alpha_k\mathbf{r}_{1,k}, \qquad \mathbf{v}^n_{i-1/2,j} = \mathbf{v}_L + \sum_{\lambda_k<0}\alpha_k\mathbf{r}_{1,k}$$

Similar steps are used for the remaining fluxes and velocities.

The second-order extension of the numerical approach follows the usual steps. First-order Taylor expansions are obtained, and the quasi-linear form of the equations is used to replace time derivatives with spatial derivatives. The spatial derivatives are evaluated using slope-limited differences of characteristic variables. Further details are suppressed here.

1.5 Boundary conditions

(The comments on the traction boundary conditions should be checked and updated.) The governing equations are defined on $\mathbf{x}=(x,y)\in\Omega$ in a reference (Lagrangian) frame. The equations are solved for displacement $\mathbf{u}(\mathbf{x}$,t), and the formula

$$\mathbf{x}' = \mathbf{x} + \mathbf{u}(\mathbf{x},t)$$

determines the corresponding domain $\mathbf{x}'=(x',y')\in\Omega'$ in a lab (Eulerian) frame. Let Γ be a portion of the boundary of Ω and let Γ' be the corresponding boundary of Ω'. Displacement boundary conditions are

$$\mathbf{u}(\mathbf{x},t) = \mathbf{d}, \qquad \text{a given displacement for } \mathbf{x}\in\Gamma$$

and traction boundary conditions are

$$\mathbf{n}'\cdot\sigma(\mathbf{x},t) = \tau, \qquad \text{a given stress for } \mathbf{x}\in\Gamma$$

where $\mathbf{n}'$ is the outward normal to Γ' and σ is the Cauchy stress. Using

$$F = \frac{\partial\mathbf{x}'}{\partial\mathbf{x}}, \qquad \sigma = \frac{1}{J}F\cdot P, \qquad J=\det(F)$$

where F is the gradient tensor and P is the nominal stress tensor, we may write the traction boundary condition as

$$\mathbf{z} \cdot P(\mathbf{x}, t) = \tau, \qquad \mathbf{z} = \frac{1}{J}\mathbf{n}' \cdot F, \qquad \text{for } \mathbf{x} \in \Gamma$$

Within the overlapping grid framework, the domain Ω is described by a set of component grids, each defined by a mapping $\mathbf{x} = \mathbf{x}(\mathbf{r})$ from a computational space $\mathbf{r} = (r, s) \in [0,1]^2$ to $\mathbf{x}$. Let us assume that the boundary Γ is described by $\mathbf{x} = \mathbf{x}(r, s)$ for a particular component grid with $r \in [0,1]$ and $s = \text{fixed} = s_0$, where $s_0 = 0$ or 1. For this case, the corresponding boundary Γ' in the lab frame is described by

$$\mathbf{x}' = \mathbf{x}(r, s_0) + \mathbf{u}(\mathbf{x}(r, s_0), t), \qquad r \in [0,1]$$

A vector tangent to Γ' is given by

$$\mathbf{t}' = \frac{d\mathbf{x}'}{dr} = F \cdot \frac{d\mathbf{x}}{dr}$$

which has components

$$t_1' = F_{11}x_r + F_{12}y_r, \qquad t_2' = F_{21}x_r + F_{22}y_r$$

Thus, we can write $\mathbf{n}' = (n_1', n_2')$ as

$$n_1' = \frac{t_2'}{[(t_1')^2 + (t_2')^2]^{1/2}}, \qquad n_2' = -\frac{t_1'}{[(t_1')^2 + (t_2')^2]^{1/2}},$$