Notes on Nonlinear Solid Mechanics

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February 23, 2014

Contents

1 Nomenclature

ρ	density	(1)
u_i	displacement vector	(2)
ϵ_{ij}	strain tensor	(3)
σ_{ij}	Cauchy stress tensor	(4)
λ	Lamé constant	(5)
μ	Lamé constant, shear modulus, G	(6)
$\nu = \lambda/(2(\lambda+\nu)$	Poisson's ratio	(7)
$K = \lambda + 2\mu/3$	Bulk modulus	(8)

2 Governing Equations

These notes are based on *Nonlinear Finite Elements for Continua and Structures* by T. Belytschko, W.K. Liu and B. Moran [1].

Deformation and Motion (Section 3.2)

$$\Omega_0 \qquad \text{reference configuration} \tag{9}$$

$$\Omega \qquad \text{current configuration} \tag{10}$$

$$\mathbf{X} = \sum_i X_i \mathbf{e}_i \qquad \text{position of a material pt in } \Omega_0 \tag{11}$$

$$\mathbf{x} = \sum_i x_i \mathbf{e}_i \qquad \text{position of a pt in } \Omega \tag{12}$$

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t) \qquad \text{motion of the body} \tag{13}$$

$$\mathbf{u}(\mathbf{X}, t) = \boldsymbol{\phi}(\mathbf{X}, t) - \mathbf{X} = \mathbf{x} - \mathbf{X} \qquad \text{displacement vector} \tag{14}$$

$$\mathbf{v}(\mathbf{X}, t) = \partial_t \boldsymbol{\phi}(\mathbf{X}, t) = \partial_t \mathbf{u}(\mathbf{X}, t) \qquad \text{velocity of a material pt} \tag{15}$$

$$\mathbf{a}(\mathbf{X}, t) = \partial_t \mathbf{v}(\mathbf{X}, t) \qquad \text{acceleration of a material pt} \tag{16}$$

$$D_t \mathbf{v}(\mathbf{x}, t) = \partial_t \mathbf{v}(\mathbf{x}, t) + \mathbf{v} \cdot \nabla \mathbf{v} \qquad \text{acceleration (Eulerian)} \tag{17}$$

$$\mathbf{F} = \partial \boldsymbol{\phi} / \partial \mathbf{X} = \partial x_i / \partial X_j \qquad \text{deformation gradient} \tag{18}$$

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \tag{19}$$

$$\mathbf{F} = \mathbf{I} + \partial \mathbf{u} / \partial \mathbf{X} \tag{20}$$

$$J = \det(\mathbf{F}), \quad \partial_t J(\mathbf{X}, t) = J \nabla \cdot \mathbf{v} \qquad \text{Jacobian determinant} \tag{21}$$

$$\int_\Omega f(\mathbf{x}, t) d\Omega = \int_{\Omega_0} f(\mathbf{X}, t) J d\Omega_0 \qquad \text{integrals} \tag{22}$$

Rigid Body Motion (3.2.8)

$$\mathbf{x}_t(t) \qquad \text{translation} \tag{23}$$

$$\mathbf{R}(t) \qquad \text{rotation tensor, } \mathbf{R}^T \mathbf{R} = I \tag{24}$$

$$x_{RB}(\mathbf{X}, t) = \mathbf{R}(t)\mathbf{X} + \mathbf{x}_t(t) \qquad \text{rigid body motion} \tag{25}$$

Strain Measures (3.3), use $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$, $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$, $dS^2 = d\mathbf{X} \cdot d\mathbf{X}$,

$$ds^2 - dS^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X} \qquad \text{Green strain, } \mathbf{E} \tag{26}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \qquad \text{Green strain} \tag{27}$$

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \qquad \text{right Cauchy-Green deformation tensor} \tag{28}$$

$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T \qquad \text{left Cauchy-Green deformation tensor} \tag{29}$$

$$\mathbf{E} = \frac{1}{2}((\nabla_0 \mathbf{u})^T + \nabla_0 \mathbf{u} + \nabla_0 \mathbf{u} \cdot (\nabla_0 \mathbf{u})^T) \qquad \text{Green strain} \tag{30}$$

$$\mathbf{E} = \frac{1}{2}(\partial u_i / \partial X_j + \partial u_j / \partial X_i + \partial u_k / \partial X_i \partial u_k / \partial X_j) \qquad \text{Green strain} \tag{31}$$

$$\nabla_0 = \partial / \partial X_i \qquad \text{(left) material gradient} \tag{32}$$

Rate of deformation (3.3.2)

$$\mathbf{L} = \partial \mathbf{v}(\mathbf{x},t)/\partial \mathbf{x} = (\nabla \mathbf{v})^T = \mathbf{D} + \mathbf{W} \qquad \text{velocity gradient, } \mathbf{L} \text{ (Eulerian)} \tag{33}$$

$$d\mathbf{v} = \mathbf{L} \cdot d\mathbf{x} \tag{34}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i) \qquad \text{rate of deformation (velocity strain)} \tag{35}$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \frac{1}{2}(\partial v_i/\partial x_j - \partial v_j/\partial x_i) \qquad \text{spin tensor} \tag{36}$$

$$\partial_t(ds^2) = 2\ d\mathbf{x} \cdot \mathbf{D} \cdot d\mathbf{x} \tag{37}$$

$$\mathbf{L} = \partial \mathbf{v}(\mathbf{x},t)/\partial \mathbf{X}\ \partial \mathbf{X}/\partial \mathbf{x} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{38}$$

$$\dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}, \qquad \textit{pull back operation: } \mathbf{x} \to \mathbf{X} \tag{39}$$

$$\mathbf{D} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1} \qquad \textit{push forward operation: } \mathbf{X} \to \mathbf{x} \tag{40}$$

Stress Measures (3.4)

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F} \cdot \mathbf{P}\ = J^{-1}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T \qquad \text{Stress: Cauchy: } \boldsymbol{\sigma}\text{, PK2: } \mathbf{S}\text{, nominal: } \mathbf{P} \tag{41}$$

$$\mathbf{P} = \mathbf{S}\mathbf{F}^T \qquad \mathbf{S} : \text{ 2nd Piola-Kirchhoff (PK2) stress} \tag{42}$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} d\Gamma = \mathbf{t} d\Gamma = d\mathbf{f} \qquad \text{traction } \mathbf{t} \tag{43}$$

$$\mathbf{n}_0 \cdot \mathbf{P} d\Gamma_0 = \mathbf{t}_0 d\Gamma_0 = d\mathbf{f} \tag{44}$$

$$\mathbf{n}_0 \cdot \mathbf{S} d\Gamma_0 = \mathbf{F}^{-1}\mathbf{t}_0 d\Gamma_0 = \mathbf{F}^{-1} d\mathbf{f} \tag{45}$$

$$\mathbf{n} d\Gamma = J\mathbf{n}_0 \cdot \mathbf{F}^{-1} d\Gamma_0 \qquad \text{Nanson's relation} \tag{46}$$

$$\tag{47}$$

Material time derivatives of integrals and Reynold's transport theorem (3.5.3) (for any material region Ω)

$$D_t \int_\Omega f d\Omega = \int_{\Omega_0} \partial_t (f(\mathbf{X},t) J(\mathbf{X},t) d\Omega_0 \tag{48}$$

$$D_t \int_\Omega f d\Omega = \int_\Omega (f_t + \nabla \cdot (f\mathbf{v})) d\Omega \qquad \text{Reynold's transport theorem} \tag{49}$$

$$\tag{50}$$

Eulerian Conservation Equations (3.5)

$$D_t \rho + \rho \nabla \cdot (\mathbf{v}) = 0 \qquad \text{Mass conservation} \tag{51}$$

$$\rho D_t \mathbf{v} = \nabla \cdot (\boldsymbol{\sigma}) + \rho \mathbf{b} \qquad \text{Linear momentum} \tag{52}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad \text{Angular momentum} \tag{53}$$

$$\rho D_t w^{int} = \mathbf{D} : \boldsymbol{\sigma} - \nabla \cdot q + \rho s \qquad \text{Energy} \tag{54}$$

Lagrangian Conservation Equations (3.6) ($\tilde{q} = J^{-1}\mathbf{F}^T \cdot \mathbf{q}$) **check these**

$$\rho(\mathbf{X},t) J(\mathbf{X},t) = \rho_0(\mathbf{X}) \qquad \text{Mass conservation} \tag{55}$$

$$\rho_0 \partial_t \mathbf{v}(\mathbf{X},t) = \nabla_0 \cdot \mathbf{P} + \rho_0 \mathbf{b} \qquad \text{Linear momentum} \tag{56}$$

$$\mathbf{F}\mathbf{P} = \mathbf{P}^T\mathbf{F}^T \qquad \text{Angular momentum} \tag{57}$$

$$\rho \partial_t w^{int}(\mathbf{X},t) = \dot{\mathbf{F}}^T : \mathbf{P} - \nabla_0 \cdot \tilde{q} + \rho s \qquad \text{Energy} \tag{58}$$

Constitutive Models (5)

$$w = w(E) \qquad w : \text{ elastic strain energy (potential)} \tag{59}$$

$$w(E) = \psi(2\mathbf{E} + I) = \psi(\mathbf{C}) \qquad \psi : \text{ stored energy potential (Hyper-elastic materials), } \mathbf{C} = \mathbf{F}^T\mathbf{F}) \tag{60}$$

$$\tag{61}$$

2.1 Nanson's relation

Here is a derivation of Nanson's relation. We start from the transformation between volume elements,

$$dv = J\,dV,$$

where $dv = dx_1dx_2dx_3$, $dV = dX_1dX_2dX_3$ and $J = \det(F)$. Suppose the volume element dv is formed from the dot product of an oriented area $d\mathbf{a} = da\,\mathbf{n}$ and a line element $d\mathbf{l}$, (and similarly for dV), then

$$dv = d\mathbf{a}^T d\mathbf{l} = da\,\mathbf{n}^T d\mathbf{l}, \qquad \text{(x-volume element in terms of area element and line element)}$$
$$dV = d\mathbf{A}^T d\mathbf{L} = dA\,\mathbf{N}^T d\mathbf{L}, \qquad \text{(X-volume element in terms of area element and line element)}$$

Usng the transformation rule for line elements,

$$d\mathbf{l} = F d\mathbf{L} \qquad \text{(transformation between line elements)}$$

it follows that $dv = JdV$ implies

$$da\,\mathbf{n}^T F\,d\mathbf{L} = J\,dA\,\mathbf{N}^T\,d\mathbf{L}$$

and thus

$$da\,\mathbf{n}^T F = J\,dA\,\mathbf{N}^T,$$

Defining $\beta = dA/da$ gives the relations

$$\begin{aligned}
&F^T\mathbf{n} = \beta J\,\mathbf{N},\\
&\beta = J^{-1}\,\mathbf{N}^T F^T\mathbf{n} = J^{-1}\,\mathbf{n}^T F\mathbf{N}\\
&\mathbf{n} = \beta J\,F^{-T}\mathbf{N},
\end{aligned}$$

2.2 Time derivatives of the Jacobian determinant

Consider an transformation $\mathbf{x} = \mathbf{g}(\mathbf{r}, t)$ from $\mathbb{R}^n \rightarrow^n$m and let $H = [h_{ij}]$, $h_{ij} = \partial g_i/\partial r_j$ be the Jacobian matrix and $J = \det(H)$ be the Jacobian (determinant). We wish to compute $\partial J/\partial t$. The determinant is given by the Leibnitz formula,

$$J = \det(H) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\,h_{1,\sigma(1)}\,h_{2,\sigma(2)}\,\ldots\,h_{n,\sigma(n)}$$

where the sum is over all permutations σ of $\{1, 2, 3, \ldots n\}$. Thus the time derivative is

$$\frac{\partial J}{\partial t} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\,\partial_t h_{1,\sigma(1)}\,h_{2,\sigma(2)}\,\ldots\,h_{n,\sigma(n)} \tag{62}$$
$$+ \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\,h_{1,\sigma(1)}\,\partial_t h_{2,\sigma(2)}\,\ldots\,h_{n,\sigma(n)} \tag{63}$$
$$\ldots \tag{64}$$
$$+ \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\,h_{1,\sigma(1)}\,h_{2,\sigma(2)}\,\ldots\,\partial_t h_{n,\sigma(n)}. \tag{65}$$

Letting $w_i = \partial g_i/\partial t$, then by the chain rule

$$\frac{\partial h_{ij}}{\partial t} = \frac{\partial w_i}{\partial r_j} = \sum_k \frac{\partial w_i}{\partial x_k}\,\frac{\partial x_k}{\partial r_j} = \sum_k \frac{\partial w_i}{\partial x_k}\,h_{kj}$$

Use this last expression in the first term (other terms will be similiar) in the expansion (62)

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \frac{\partial h_{1,\sigma(1)}}{\partial t}\, h_{2,\sigma(2)}\, \ldots\, h_{n,\sigma(n)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \Big(\sum_k \frac{\partial w_1}{\partial x_k}\, h_{k\sigma(1)} \Big)\, h_{2,\sigma(2)}\, \ldots\, h_{n,\sigma(n)}, \tag{66}$$

$$= \sum_k \frac{\partial w_1}{\partial x_k} \Big\{ \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, h_{k\sigma(1)}\, h_{2,\sigma(2)}\, \ldots\, h_{n,\sigma(n)} \Big\}, \tag{67}$$

$$= \frac{\partial w_1}{\partial x_1} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, h_{1\sigma(1)}\, h_{2,\sigma(2)}\, \ldots\, h_{n,\sigma(n)}. \tag{68}$$

$$= \frac{\partial w_1}{\partial x_1} J \tag{69}$$

where we have used the fact that the determinant is zero when two rows are equal and thus only the term $k=1$ remains in (67). Therefore

$$\frac{\partial J}{\partial t} = \Big(\sum_i \frac{\partial w_i}{\partial x_i} \Big) J \;= (\nabla_{\mathbf{x}} \cdot \mathbf{w})\, J \tag{70}$$

2.3 General transformation

Consider the continuity and momentum equations for the solid (we drop the bars here) in the Eulerian frame

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} = 0,$$
$$\rho \Big[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \Big] = \frac{\partial \sigma_{ji}}{\partial x_j}$$

with summation convention. Now we make a general moving coordinate transformation, $\mathbf{x} = \mathbf{g}(\mathbf{r}, t)$. Under this transformation the equations become

$$\frac{\partial \rho}{\partial t} + (v_j - w_j) \frac{\partial r_k}{\partial x_j} \frac{\partial \rho}{\partial r_k} + \frac{\rho}{J} \frac{\partial}{\partial r_j} \Big(J \frac{\partial r_j}{\partial x_k} v_k \Big) = 0,$$
$$\rho \Big[\frac{\partial v_i}{\partial t} + (v_j - w_j) \frac{\partial r_k}{\partial x_j} \frac{\partial v_i}{\partial r_k} \Big] = \frac{1}{J} \frac{\partial}{\partial r_j} \Big(J \frac{\partial r_j}{\partial x_k} \sigma_{ki} \Big)$$

where $\mathbf{w} = \partial \mathbf{g}/\partial t$ is the grid velocity. We could also write this in fully conservative form ...

We also have *CHECK*

$$\frac{\partial J(\mathbf{r}, t)}{\partial t} = J \frac{\partial w_j(\mathbf{r}, t)}{\partial x_j} \;= J \nabla_{\mathbf{x}} \cdot \mathbf{w},$$
$$\frac{\partial J(\mathbf{x}, t)}{\partial t} + w_j \frac{\partial J(\mathbf{x}, t)}{\partial x_j} = J \frac{\partial w_j(\mathbf{x}, t)}{\partial x_j},$$

Let $G = \partial \mathbf{x}/\partial \mathbf{r}$, $G_{ij} = \partial x_i/\partial r_j$, be the Jacobian matrix of the transformation, then

$$G = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial r_1} & \frac{\partial \mathbf{x}}{\partial r_2} & \frac{\partial \mathbf{x}}{\partial r_3} \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial x_1} & \frac{\partial \mathbf{r}}{\partial x_2} & \frac{\partial \mathbf{r}}{\partial x_3} \end{bmatrix} \;= \begin{bmatrix} \nabla_{\mathbf{x}} r_1^T \\ \nabla_{\mathbf{x}} r_2^T \\ \nabla_{\mathbf{x}} r_3^T \end{bmatrix} \;= \begin{bmatrix} \alpha_1 \mathbf{n}_1^T \\ \alpha_2 \mathbf{n}_2^T \\ \alpha_3 \mathbf{n}_3^T \end{bmatrix}, \qquad \alpha_i = 1/|\nabla_{\mathbf{x}} r_i|.$$

The columns of G, $\frac{\partial \mathbf{x}}{\partial r_i}$ are vectors in the directions of the tangents, $\mathbf{t}_i$, to the coordinate directions. The rows of G^{-1} are vectors in the directions of the normals, $\mathbf{n}_i$, to the coordinate planes, i.e. $\nabla_{\mathbf{x}} r_i$ is proportional to the normal to the coordinate plane $r_i = \text{constant}$.

3 Kirchoff material: large rotation, small strain

We consider the case of large rotations and small strains. The most general Kirchoff material (or St. Venant- Kirchoff material), is

$$\mathbf{S} = \mathbf{C} : \mathbf{E}, \quad S_{ij} = C_{ijkl} E_{kl}, \tag{71}$$

where $\mathbf{C}$ is the fourth-order tensor of *elastic modulii* and $\mathbf{S}$ is the PKII stress. The corresponding rate equation is

$$\dot{\mathbf{S}} = \mathbf{C}^{SE} : \dot{\mathbf{E}}, \tag{72}$$

and $\mathbf{C}^{SE} = \mathbf{C}$ is called the *tangent modulus tensor*. The strain energy is

$$w = \frac{1}{2}\mathbf{E} : \mathbf{C} : \mathbf{E} = \frac{1}{2} E_{ij} C_{ijkl} E_{kl}, \tag{73}$$

with

$$S_{ij} = \frac{\partial w}{\partial E_{ij}}, \quad C_{ijkl} = \frac{\partial^2 w}{\partial E_{ij} \partial E_{kl}}. \tag{74}$$

The isotropic Kirchoff material is

$$S_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}, \quad \mathbf{S} = \lambda tr(\mathbf{E}) + 2\mu \mathbf{E}. \tag{75}$$

Note that for a pure rotation and translation that the PKII stress $\mathbf{S}$ is zero ($\mathbf{R}^T\mathbf{R} = \mathbf{I}$):

$$\mathbf{x} = \mathbf{R}(\mathbf{X} - \mathbf{c}(t)) + \mathbf{c}(t), \tag{76}$$

$$\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X} = \mathbf{R}, \tag{77}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - I) = 0, \tag{78}$$

$$\mathbf{S} = 0. \tag{79}$$

The Eulerian equations of motion for a Kirchoff material are

$$\rho D_t \mathbf{v} = \nabla \cdot (\boldsymbol{\sigma}), \tag{80}$$

$$\boldsymbol{\sigma} = J^{-1} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T, \tag{81}$$

$$\mathbf{S} = \lambda tr(\mathbf{E}) I + 2\mu \mathbf{E}. \tag{82}$$

The Lagrangian equations are

$$\rho_0 \partial_t^2 \mathbf{u} = \nabla_{\mathbf{X}} \cdot (\mathbf{P}), \tag{83}$$

$$\mathbf{P} = \mathbf{S}\mathbf{F}^T, \quad \mathbf{S} = \lambda tr(\mathbf{E}) + 2\mu \mathbf{E}. \tag{84}$$

or since $F_{ij} = \delta_{ij} + \partial u_i / \partial X_j$

$$\rho_0 \partial_t^2 u_i = \frac{\partial P_{li}}{\partial X_l} \quad = \frac{\partial P_{li}}{\partial F_{jm}} \frac{\partial F_{jm}}{\partial X_l} \quad = \frac{\partial P_{li}}{\partial F_{jm}} \frac{\partial^2 u_j}{\partial X_l \partial X_m}. \tag{85}$$

In detail:

$$E_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \Big) \tag{86}$$

$$E_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\Big\{ \Big(\frac{\partial u_1}{\partial x_1} \Big)^2 + \Big(\frac{\partial u_2}{\partial x_1} \Big)^2 \Big\} \tag{87}$$

$$E_{12} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_2} \Big) \tag{88}$$

$$E_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}\Big\{ \Big(\frac{\partial u_1}{\partial x_2} \Big)^2 + \Big(\frac{\partial u_2}{\partial x_2} \Big)^2 \Big\} \tag{89}$$

$$S_{11} = \lambda\Big(E_{11} + E_{22} + E_{33}\Big) + 2\mu\Big(E_{11}\Big), \tag{90}$$

$$S_{12} = 2\mu\Big(E_{12}\Big) \tag{91}$$

We can linearize about a state $\mathbf{u}^0$, $\mathbf{F}^0$ and look for solutions of the form $\mathbf{u} = \exp(i(\mathbf{k}\cdot\mathbf{x} - \omega t))\widehat{\mathbf{u}}$,

$$-\rho_0\omega^2\hat{u}_i = \frac{\partial P_{li}}{\partial F_{jm}} k_l k_m \hat{u}_j. \tag{92}$$

We can thus look for eigenvalues $c = \omega/k$ satisfying

$$\det(\mathbf{A} - \rho_0 c^2\mathbf{I}) = 0, \tag{93}$$

$$A_{ij} = \frac{\partial P_{li}}{\partial F_{jm}} \hat{k}_l \hat{k}_m. \tag{94}$$

Question: is $\mathbf{A}$ symmetric ? Apparently yes in 2D (from the maple program eigs.maple). This means the eigenvalues will always be real. But are they positive ?

Now

$$P_{ij} = S_{ik}F^T_{kj} = S_{ik}F_{jk}, \tag{95}$$

$$P_{ji} = S_{jk}F_{ik}, \tag{96}$$

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \tag{97}$$

$$S_{jk} = \lambda E_{nn}\delta_{jk} + 2\mu E_{jk}, \tag{98}$$

and thus *check*

$$\frac{\partial E_{ij}}{\partial F_{lm}} = \frac{1}{2}(\delta_{im}F_{lj} + \delta_{jm}F_{li}), \tag{99}$$

$$\frac{\partial S_{ij}}{\partial E_{lm}} = \lambda\delta_{lm}\delta_{ij} + 2\mu\delta_{il}\delta_{jm}, \tag{100}$$

We work out the eigenvalues with the maple program eigs.maple.

The eigenvalues of $\mathbf{A}$ for general small displacements, or for large rotations with a small perturbation (see more below), are the same as for linear elasticity:

$$\rho_0 c_1^2 = \lambda + 2\mu + O(\mathbf{u}^2_{\mathbf{X}}) \tag{101}$$

$$\rho_0 c_2^2 = \mu + O(\mathbf{u}^2_{\mathbf{X}}) \tag{102}$$

3.1 Perturbation of a rigid body motion

Consider a small perturbation from a rigid body motion (translating-rotating state),

$$\mathbf{x} = \mathbf{R}(t)\mathbf{X} + \mathbf{c}(t) + \mathbf{u}, \quad \mathbf{u} \ll 1, \tag{103}$$

$$\mathbf{F} = \mathbf{R} + \frac{\partial\mathbf{u}}{\partial\mathbf{X}}, \tag{104}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \approx \frac{1}{2}(\mathbf{R}^T\frac{\partial\mathbf{u}}{\partial\mathbf{X}} + \frac{\partial\mathbf{u}}{\partial\mathbf{X}}^T\mathbf{R}), \tag{105}$$

$$\mathbf{S} \approx \lambda tr(\mathbf{E})\mathbf{I} + \mu(\mathbf{R}^T\frac{\partial\mathbf{u}}{\partial\mathbf{X}} + \frac{\partial\mathbf{u}}{\partial\mathbf{X}}^T\mathbf{R}), \tag{106}$$

$$\boldsymbol{\sigma} \approx J^{-1}\mathbf{R}\mathbf{S}\mathbf{R}^T, \tag{107}$$

$$\approx J^{-1}\big(tr(\mathbf{E})\mathbf{I} + \mu(\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\mathbf{R}^T + \mathbf{R}\frac{\partial\mathbf{u}}{\partial\mathbf{X}}^T)\big), \tag{108}$$

$$\mathbf{P} = \mathbf{S}\mathbf{F}^T \approx \mathbf{S}\mathbf{R}^T. \tag{109}$$

Note that from $\boldsymbol{\sigma} \approx J^{-1}\mathbf{R}\mathbf{S}\mathbf{R}^T$ it follows that

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &\approx J^{-1}\dot{\mathbf{R}}\mathbf{S}\mathbf{R}^T + J^{-1}\mathbf{R}\mathbf{S}\dot{\mathbf{R}}^T + J^{-1}\mathbf{R}\dot{\mathbf{S}}\mathbf{R}^T + \dot{J^{-1}}\mathbf{R}\mathbf{S}\mathbf{R}^T && (110)\\
&= \dot{\mathbf{R}}\mathbf{R}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}(\dot{\mathbf{R}}\mathbf{R}^T)^T + J^{-1}\mathbf{R}\dot{\mathbf{S}}\mathbf{R}^T + \dot{J^{-1}}\mathbf{R}\mathbf{S}\mathbf{R}^T && (111)\\
&= \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}^T + J^{-1}\mathbf{R}\dot{\mathbf{S}}\mathbf{R}^T + \dot{J^{-1}}\mathbf{R}\mathbf{S}\mathbf{R}^T && (112)\\
\mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^T \approx \dot{\mathbf{F}}\mathbf{F}^{-1} && (113)
\end{aligned}$$

If we consider a small perturbation from a rigid body motion (translating-rotating state),

$$\mathbf{x} = \mathbf{R}(t)\mathbf{X} + \mathbf{c}(t) + \mathbf{u}, \quad \mathbf{u} \ll 1 \tag{114}$$

Then the eigenvalues of the matrix $\mathbf{A}$ for $(k_1, k_2) = (1, 0)$ are (from cgDoc/sm/eigs.maple) (*check this*)

$$\rho_0 c_1^2 = \lambda + 2\mu + \big(\lambda(su_{1Y} + cu_{2Y}) + 3(\lambda + 2\mu)(cu_{1X} - su_{2X})\big) + O(\mathbf{u}_{\mathbf{X}}^2) \tag{115}$$

$$\rho_0 c_2^2 = \mu + (\lambda + 2\mu)\big[(su_{1Y} + cu_{2Y}) + (cu_{1X} - su_{2X})\big] + O(\mathbf{u}_{\mathbf{X}}^2) \tag{116}$$

where $c = \cos(wt)$ and $s = \sin(wt)$ define the entries in the rotation matrix $\mathbf{R}$.

The eigenvalues can be negative for large strains, for example

$$\rho_0 c_1^2 = -\lambda/2 \quad \text{for} \quad k_1 = 1,\ k_2 = 0,\ c = 0,\ s = 1,\ u_{1X} = 0, u_{2X} = 1, u_{1Y} = 0, u_{2Y} = 0 \tag{117}$$

This means the system is not hyperbolic anymore.

3.2 Invariance of the SVK model under a change of variables

The Eulerian equations of motion for a SVK (Kirchoff) material are

$$\begin{aligned}
\rho D_t \mathbf{v} &= \nabla_{\mathbf{x}} \cdot (\boldsymbol{\sigma}), && (118)\\
\boldsymbol{\sigma} &= J^{-1}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T, && (119)\\
\mathbf{S} &= \lambda tr(\mathbf{E}) + 2\mu\mathbf{E}, && (120)\\
\mathbf{E} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - I) && (121)
\end{aligned}$$

NOTE: in matrix-vector notation, $\nabla_{\mathbf{x}} \cdot (\boldsymbol{\sigma})$, really means

$$\nabla_{\mathbf{x}} \cdot (\boldsymbol{\sigma}) = \begin{bmatrix} \partial_x\sigma_{11} + \partial_y\sigma_{21} + \partial_z\sigma_{31} \\ \partial_x\sigma_{12} + \partial_y\sigma_{22} + \partial_z\sigma_{32} \\ \partial_x\sigma_{13} + \partial_y\sigma_{23} + \partial_z\sigma_{33} \end{bmatrix} = ((\nabla_{\mathbf{x}})^T\boldsymbol{\sigma})^T \tag{122}$$

Consider a change of variables where we rotate the dependent and independent variables by a *constant* rotation matrix $\mathbf{R}$, (with $\mathbf{R}^T\mathbf{R} = I$)

$$\widetilde{\mathbf{x}} = \mathbf{R}\mathbf{x}, \quad \widetilde{\mathbf{X}} = \mathbf{R}\mathbf{X}, \quad \widetilde{\mathbf{u}} = \mathbf{R}\mathbf{u}, \quad \widetilde{\mathbf{v}} = \mathbf{R}\mathbf{v}, \tag{123}$$

$$\tag{124}$$

Claim:

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{R}^T \frac{\partial \widetilde{\mathbf{x}}}{\partial \widetilde{\mathbf{X}}}\mathbf{R} \tag{125}$$

Proof: Since

$$x_i = (R^T)_{ik}\tilde{x}_k, \quad \tilde{X}_l = R_{lp}X_p, \qquad \text{(implied sums)}, \tag{126}$$

then by the chain rule

$$\frac{\partial x_i}{\partial X_j} = (R^T)_{ik}\frac{\partial \tilde{x}_k}{\partial X_j} \quad = (R^T)_{ik}\frac{\partial \tilde{x}_k}{\partial \tilde{X}_l}\frac{\partial \tilde{X}_l}{\partial X_j} \quad = (R^T)_{ik}\frac{\partial \tilde{x}_k}{\partial \tilde{X}_l}R_{lj} \tag{127}$$

and this last expression is the entry ij in the matrix $\mathbf{R}^T \frac{\partial \widetilde{\mathbf{x}}}{\partial \widetilde{\mathbf{X}}}\mathbf{R}$.

Therefore we have (note that $J = \widetilde{J}$ since $\det(\mathbf{F}) = \det(\widetilde{\mathbf{F}})$),

$$\mathbf{F} = \mathbf{R}^T\widetilde{\mathbf{F}}\mathbf{R}, \quad \mathbf{E} = \mathbf{R}^T\widetilde{\mathbf{E}}\mathbf{R}, \quad \mathbf{S} = \mathbf{R}^T\widetilde{\mathbf{S}}\mathbf{R}, \tag{128}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{R}^T\mathbf{F}\widetilde{\mathbf{S}}\widetilde{\mathbf{F}}^T\mathbf{R} = \mathbf{R}^T\widetilde{\boldsymbol{\sigma}}\mathbf{R}, \tag{129}$$

$$\widetilde{\boldsymbol{\sigma}} = \widetilde{J}\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{S}}\cdot\widetilde{\mathbf{F}}^T. \tag{130}$$

Therefore

$$\mathbf{R}^T\rho D_t\widetilde{\mathbf{v}} = \nabla_{\mathbf{x}}\cdot(\mathbf{R}^T\widetilde{\boldsymbol{\sigma}}\mathbf{R}) \quad = \nabla_{\widetilde{\mathbf{x}}}\cdot(|\mathbf{R}|\mathbf{R}\mathbf{R}^T\widetilde{\boldsymbol{\sigma}}\mathbf{R}) \quad = \nabla_{\widetilde{\mathbf{x}}}\cdot(\widetilde{\boldsymbol{\sigma}}\mathbf{R}) \tag{131}$$

Multiplying through by $\mathbf{R}$ and using (122) gives

$$\rho D_t\widetilde{\mathbf{v}} = \nabla_{\widetilde{\mathbf{x}}}\cdot(\widetilde{\boldsymbol{\sigma}}) \tag{132}$$

and thus, since $\widetilde{\boldsymbol{\sigma}} = \widetilde{J}\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{S}}\cdot\widetilde{\mathbf{F}}^T$, the equations are the same in the transformed variables.

4 Constituitive Models

4.1 Elastic strain energy function

The elastic strain energy function (or *elastic strain energy potential*) is denoted by w and is a measure of the potential energy in a strained elastic material. We will consider materials for which

$$w = w(\mathbf{E}).$$

The elastic strain energy function is used to define constituitive models for a class of materials. For Kirchoff materials, w is a positive definite quadratic function of $\mathbf{E}$,

$$w = \frac{1}{2}\mathbf{E}\,\underline{\underline{\mathbf{C}}}\,\mathbf{E} = \frac{1}{2}C_{ijkl}E_{ij}E_{kl}, \quad C_{ijkl} = \frac{\partial^2 w}{\partial E_{ij}\partial E_{kl}} \qquad \text{(Kirchoff materials).}$$
$$\mathbf{S} = \underline{\underline{\mathbf{C}}}\,\mathbf{E}, \quad S_{ij} = C_{ijkl}E_{kl} \qquad \text{(Generalized Hooke's law: Kirchoff materials).}$$

The *minor* symmetries of C_{ijkl} (valid for Kirchoff materials) follow from $\mathbf{S} = \underline{\underline{\mathbf{C}}}\,\mathbf{E}$, and the symmetry of $\mathbf{E}$ and $\mathbf{S}$: $C_{ijkl} = C_{jikl}$, $(i \leftrightarrow j)$, and $C_{ijkl} = C_{ijlk}$ $(k \leftrightarrow l)$. The *major* symmetries (valid for $w = w(\mathbf{E})$) follow from the equality of mixed partials of w, $C_{ijkl} = C_{kjil}$ $(i \leftrightarrow k)$ and $C_{ijkl} = C_{ilkj}$ $(j \leftrightarrow l)$. For Kirchoff and hyperelastic materials (i.e. materials where $w = w(\mathbf{E})$), the PK2 stress, $\mathbf{S}$, is related to w by

$$\mathbf{S} = \frac{\partial w(\mathbf{E})}{\partial \mathbf{E}}.$$

It follows that the nominal stress $\mathbf{P}$ is given by

$$P_{ij} = \frac{\partial w}{\partial F_{ji}}, \quad \mathbf{P} = \frac{\partial w}{\partial \mathbf{F}^T} \tag{133}$$

Equation (133) can be derived using the relations

$$\begin{aligned}
\frac{\partial w}{\partial F_{ij}} &= \frac{\partial w}{\partial E_{kl}}\frac{\partial E_{kl}}{\partial E_{ij}}, \\
E_{kl} &= \frac{1}{2}(F_{\mu k}F_{\mu l} - \delta_{kl}), \qquad \Big(\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - I)\Big), \\
\frac{\partial E_{kl}}{\partial E_{ij}} &= \frac{1}{2}\Big(\delta_{\mu i}\delta_{jk}\,F_{\mu l} + F_{\mu k}\,\delta_{\mu i}\delta_{lj}\Big) \ = \frac{1}{2}\Big(\delta_{jk}F_{il} + \delta_{lj}F_{ik}\Big)
\end{aligned}$$

Whence

$$\begin{aligned}
\frac{\partial w}{\partial F_{ij}} &= \frac{\partial w}{\partial E_{kl}}\,\frac{1}{2}\Big(\delta_{jk}F_{il} + \delta_{lj}F_{ik}\Big) \quad = \frac{1}{2}\Big(\frac{\partial w}{\partial E_{jl}}F_{il} + \frac{\partial w}{\partial E_{kj}}F_{ik}\Big) \\
&= \frac{1}{2}\Big(S_{jl}F_{il} + S_{kj}F_{ik}\Big) = \frac{1}{2}\Big(S_{jl}F_{il} + S_{jk}F_{ik}\Big) \ = \frac{1}{2}(P_{ji} + P_{ji}) \\
&= P_{ji}
\end{aligned}$$

where we have used $\mathbf{P} = \mathbf{S}\mathbf{F}^T$ (i.e. $P_{ij} = S_{ik}F_{jk}$).

Examples:

$$w = \frac{\lambda}{2}\big[\mathrm{tr}(\mathbf{E})\big]^2 + \mu\,\mathrm{tr}(\mathbf{E}^2) \qquad \text{(Saint-Venant Kirchoff),}$$
$$w(\mathbf{E}) = \psi(\mathbf{C}) = \frac{1}{2}\lambda_0(\ln(J))^2 - \mu_0\ln(J) + \frac{1}{2}\mu_0(\mathrm{tr}(\mathbf{C}) - 3), \qquad \text{(Neo-Hookean)}$$

where $J = \det(\mathbf{F})$, $\mathbf{C} = \mathbf{F}^T\mathbf{F} = 2\mathbf{E} + I$ and $w(\mathbf{E}) = \psi(2\mathbf{E} + I)$.

4.2 Hyperelastic (Green elastic) models

Hyperelastic materials are those for which the work is independent of the load path (e.g. rubber like material). They are characterized by by a stored (strain) energy function $\psi(\mathbf{C})$, ($w(\mathbf{E}) = \psi(2\mathbf{E} + I)$, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$),

$$\mathbf{S} = 2\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}} = \frac{\partial w(\mathbf{E})}{\partial\mathbf{E}}$$

4.3 Models for large strain and deformation

In this section we consider how some material models behave for large strains and deformations.

In the article *Invertible finite elements for robust simulation of large deformation* by Irving, Teran and Fediw [2], they note that the SVK model behaves poorly for large strains and they consider some alternatives.

Recall that in material coordinates $\mathbf{X}$ the equations of motion are

$$\begin{aligned}
\rho_0\partial_t^2\mathbf{u} &= \nabla_{\mathbf{X}}\cdot\mathbf{P}, &(134)\\
\rho_0\partial_t^2 u_i &= \frac{\partial}{\partial X_j}P_{ji} = \frac{\partial P_{ji}}{\partial F_{kl}}\frac{\partial F_{kl}}{\partial X_j} &(135)\\
&= K_{jikl}\frac{\partial^2 u_k}{\partial X_l\partial X_j}, \quad \text{(c.f. Don's } K_{ijkl}), &(136)
\end{aligned}$$

where $\mathbf{P} = \mathbf{P}(\mathbf{F})$ ($\mathbf{F} = \mathbf{I} + \partial\mathbf{u}/\partial\mathbf{x}$) is the nominal stress (first Piola-Kirchoff stress). Freezing coefficients and substituting $\mathbf{u} = e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}\hat{\mathbf{u}}$ gives the (matrix) dispersion relation

$$\rho_0\omega^2\hat{u}_i = K_{jikl}k_l k_j\hat{u}_k, \tag{137}$$

whose eigenvalues are the wave speeds.

The SVK model defines $\mathbf{P}$ as a function of $\mathbf{F}$ by

$$\begin{aligned}
\mathbf{P} &= \mathbf{S}\mathbf{F}^T, &(138)\\
\mathbf{S} &= \lambda tr(\mathbf{E})I + 2\mu\mathbf{E}, &(139)\\
\mathbf{E} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - I). &(140)
\end{aligned}$$

The *rotated-linear* (RL) model [2] can be defined in terms of the singular value decomposition (SVD) of the deformation gradient tensor $\mathbf{F}$,

$$\begin{aligned}
\mathbf{F} &= \mathbf{U}\hat{\mathbf{F}}\mathbf{V}^T \quad (SVD), &(141)\\
\hat{\mathbf{P}} &= \lambda tr(\hat{\mathbf{F}} - \mathbf{I})\mathbf{I} + 2\mu(\hat{\mathbf{F}} - \mathbf{I}), &(142)\\
\mathbf{P} &= \mathbf{U}\hat{\mathbf{P}}\mathbf{V}^T &(143)
\end{aligned}$$

The *neo-Hookean* model [1] is defined in terms of the *right Cauchy Green deformation tensor*, $\mathbf{C}$, (not to be confused with the tensor of elastic modulii!)

$$\begin{aligned}
\mathbf{C} &= \mathbf{F}^T\mathbf{F} \quad \text{(right Cauchy Green deformation tensor)}, &(144)\\
\mathbf{S} &= \lambda\ln(J)\mathbf{C}^{-1} + \mu(\mathbf{I} - \mathbf{C}^{-1}), \qquad J = \det(\mathbf{F}), &(145)\\
&= \lambda\ln(J)\mathbf{F}^{-1}\mathbf{F}^{-T} + \mu(\mathbf{I} - \mathbf{F}^{-1}\mathbf{F}^{-T}), &(146)\\
\mathbf{P} &= \mathbf{S}\mathbf{F}^T = \lambda\ln(J)\mathbf{F}^{-1} + \mu(\mathbf{F}^T - \mathbf{F}^{-1}). &(147)
\end{aligned}$$

Note that for small $\partial\mathbf{u}/\partial\mathbf{X}$, $\ln(J) \approx tr(\partial\mathbf{u}/\partial\mathbf{X})$, and $\mathbf{F}^{-1} \approx \mathbf{I} - \partial\mathbf{u}/\partial\mathbf{X}$.

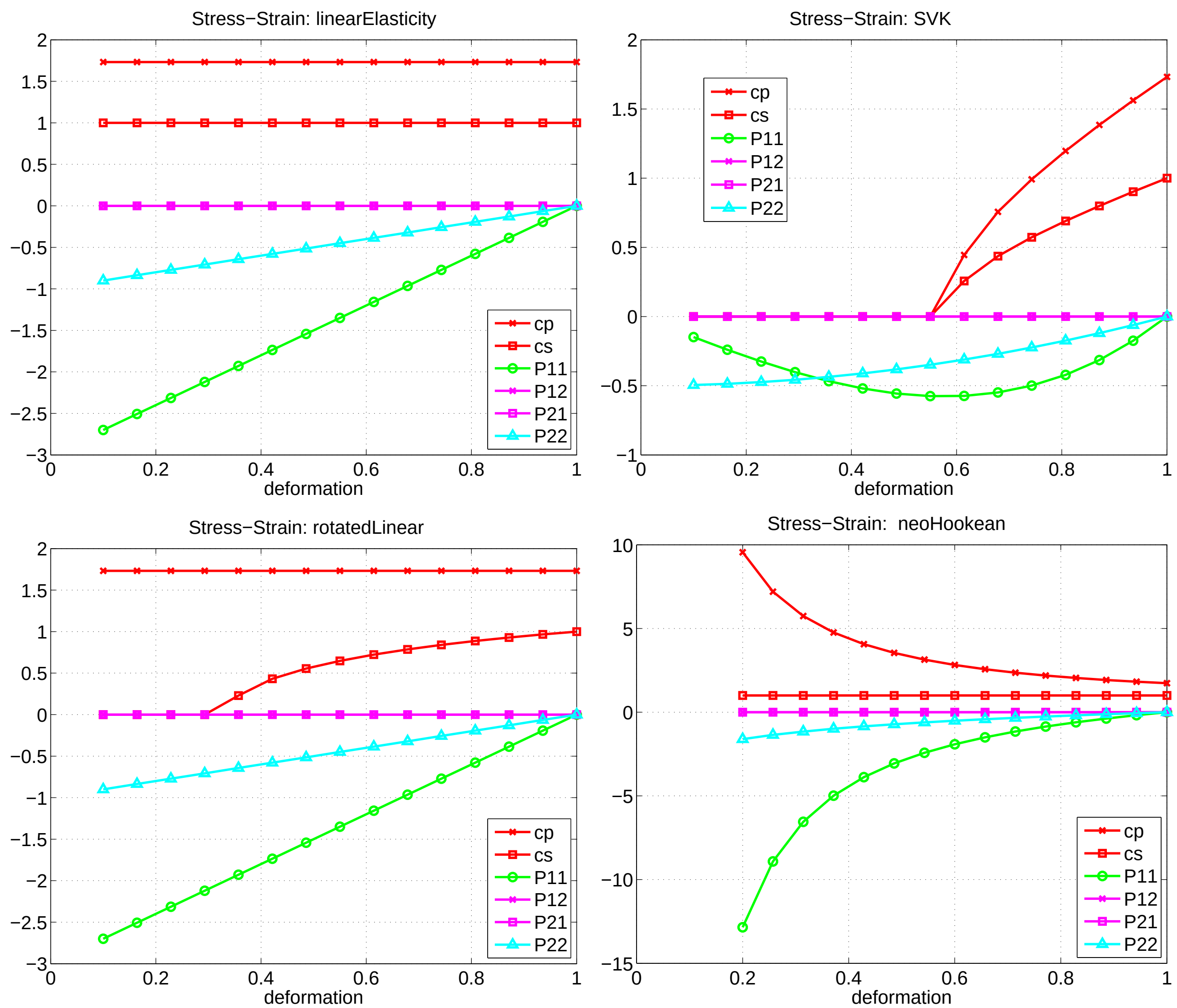

Figure 1: Stress strain relationships.

The different models are compared in Figure 1 for $\rho_0 = \lambda = \mu = 1$. We consider a material under compression with $\partial u_1/\partial x_1 \le 0$ and all other components of $\partial \mathbf{u}/\partial \mathbf{x}$ being zero. We plot the components of $\mathbf{P}$ as a function of the deformation $f = 1 + \partial u_1/\partial x_1$ with $f = 1$ corresponding to no deformation and $f = 0$ corresponding to a material element whose volume has been compressed to zero. We also plot the linearized wave speeds, c_p and c_s which are the eigenvalues of the matrix A (that corresponds to a second-order wave equation in the x-direction),

$$A_{ij} = \frac{\partial P_{1i}}{\partial F_{j1}}, \tag{148}$$

$$\rho_0 \partial_t^2 \mathbf{u} = A \partial_x^2 \mathbf{u}. \tag{149}$$

For linear elasiticity $c_p = \sqrt{(\lambda + 2\mu)/\rho_0} = \sqrt{3} \approx 1.73$ and $c_s = \sqrt{\mu/\rho_0} = 1$.

Note 1: For the SVK model the wave speeds become imaginary for $f < .577$ (see below) ($u_x < -.423$).

Note 2: For the rotated-linear model, c_s goes imaginary for $f < .3?$ ($u_x < -.7?$). Also note that P_{11} and P_{22} are fine but it is $c_s^2 = A_{22} = \partial P_{12}/\partial F_{21}$ that goes bad.

Note 3: For the neo-Hookean model c_p and c_s remain real for $f > 0$ but c_p goes to infinity for $f \to 0$

(meaning a small time step would be needed).

Limited Models. We note that for some problems of interest the models are only intended to be accurate for small strains $\partial\mathbf{u}/\partial\mathbf{x}$ relative to possibly large rotations. However we want models that remain well defined for a wider range of strains so that our codes are robust.

Limited neo-Hookean model: Question: can we remove the stiffness in the neo-Hookean model for $f \to 0$? Consider the SVD decomposition of $\mathbf{F} = \mathbf{U}\hat{\mathbf{F}}\mathbf{V}^T$ and let $\hat{\mathbf{F}} = \text{diag}(\sigma_1, \sigma_2)$. Then for the neo-Hookean model

$$\mathbf{P} = \mathbf{S}\mathbf{F}^T = \lambda \ln(J)\mathbf{F}^{-1} + \mu(\mathbf{F}^T - \mathbf{F}^{-1}), \tag{150}$$

$$= \lambda \ln(J)\mathbf{V}\hat{\mathbf{F}}^{-1}\mathbf{U}^T + \mu(\mathbf{V}\hat{\mathbf{F}}\mathbf{U}^T - \mathbf{V}\hat{\mathbf{F}}^{-1}\mathbf{U}^T), \tag{151}$$

$$= \mathbf{V}\hat{\mathbf{P}}\mathbf{U}^T, \tag{152}$$

$$\hat{\mathbf{P}} = \lambda \ln(\sigma_1\sigma_2)\text{diag}(\sigma_1^{-1}, \sigma_2^{-1}) + \mu(\text{diag}(\sigma_1, \sigma_2) - \text{diag}(\sigma_1^{-1}, \sigma_2^{-1})) \tag{153}$$

$$= \begin{bmatrix} \lambda \ln(\sigma_1\sigma_2)\sigma_1^{-1} + \mu(\sigma_1 - \sigma_1^{-1}) & 0 \\ 0 & \lambda \ln(\sigma_1\sigma_2)\sigma_2^{-1} + \mu(\sigma_2 - \sigma_2^{-1}) \end{bmatrix} \tag{154}$$

(note that $J = \det(\mathbf{F}) = \det(\hat{\mathbf{F}}) = \sigma_1\sigma_2$). We could limit the size of $\hat{\mathbf{P}}$ as $\sigma_i \to 0$ but we also need to make sure that $c_p^2 = \partial P_{11}/\partial F_{11}$ and $c_s^2 = \partial P_{12}/\partial F_{21}$ behave in a reasonable way.

Figure 2 shows a first attempt at limiting the neo-Hookean model by changing $\mathbf{F}$ by altering the singular values σ_i. Here we assume that $\sigma_1 \geq \sigma_2$.

$$\xi = \sigma_c - \sigma_2, \tag{155}$$

$$\sigma_1 = \sigma_1 + 1.5\xi^2, \qquad \text{if } \xi > 0, \tag{156}$$

$$\sigma_2 = \sigma_c - .4\xi^{1/2}, \qquad \text{if } \xi > 0, \tag{157}$$

In the above formula we increase both σ_1 and σ_2 for $\sigma_2 < \sigma_c$ ($\sigma_c = .5$ in the figure). The result is that c_p and c_s remain real and of reasonable size. There is still a bit of a discontinuity in the slopes at $\sigma_2 = \sigma_c$. ****FIX ME: Find a logical way to limit.**

Limited RL: Question: can we limit the RL model so the wave speeds remain real? Figure 2 also shows a limited RL model that used the limiter ($\sigma_c = .75$)

$$\xi = \sigma_c - \sigma_2, \tag{158}$$

$$\sigma_1 = \sigma_1 + .35\xi, \qquad \text{if } \xi > 0, \tag{159}$$

$$\sigma_2 = \sigma_c - .15\xi, \qquad \text{if } \xi > 0, \tag{160}$$

Some analysis. For the case of a one-dimensional compression the wave speeds c_p^2 and c_s^2 are eigenvalues of the matrix A: (where in the 1D case the off-diagonal terms will be zero)

$$A_{ij} = K_{1ij1} = \frac{\partial P_{1i}}{\partial F_{j1}} \tag{161}$$

For the SVK model we have (*check me*)

$$c_p^2 = A_{11} = \frac{\partial P_{11}}{\partial F_{11}} = (\lambda + 2\mu)F_{11}^2 + \mu F_{12}^2 + S_{11}, \tag{162}$$

$$c_s^2 = A_{22} = \frac{\partial P_{12}}{\partial F_{21}} = (\lambda + 2\mu)F_{21}^2 + \mu F_{22}^2 + S_{11}, \tag{163}$$

$$S_{11} = \frac{1}{2}\lambda(F_{11}^2 + F_{12}^2 + F_{22}^2 + F_{21}^2 - 2) + \mu(F_{11}^2 + F_{21}^2 - 1), \tag{164}$$

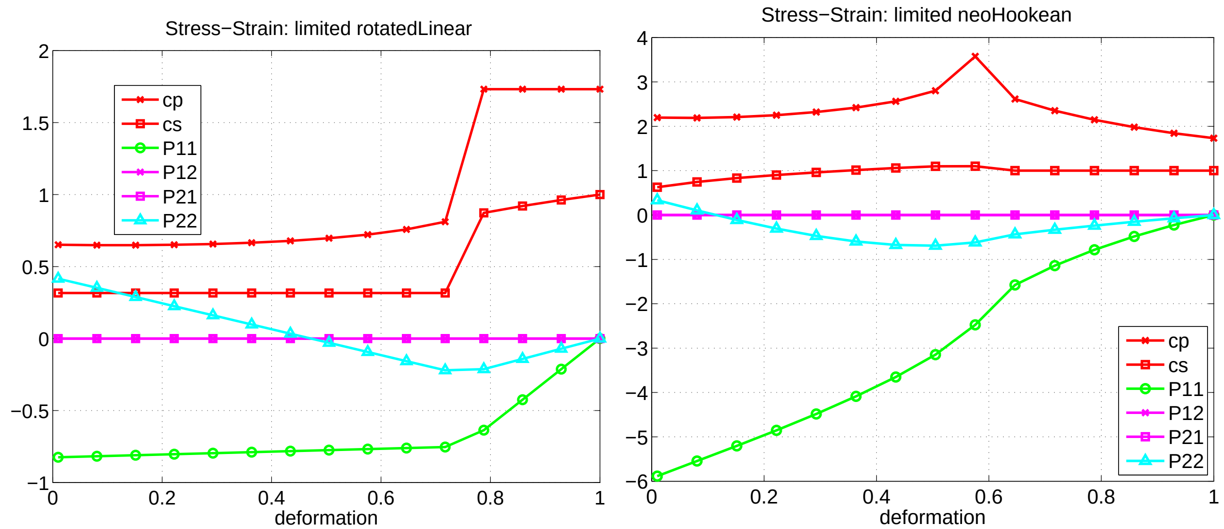

Figure 2: Limited stress-strain models.

Note that for $F_{11} \to 0$, $F_{22} = 1$, $F_{12} = F_{21} = 0$ (i.e. large one-dimensional compression), A_{11} and A_{22} become negative since $S_{11} \sim -\lambda - \mu$ is negative. In particular $A_{11} = 0$ at $F_{11} = 1/\sqrt{3} \approx .577$ and $A_{22} = 0$ at $F_{11} = \sqrt{\lambda/(\lambda + 2\mu)}$. The scheme is no-longer hyperbolic when this occurs.

For the neo-Hooklean model we have (check me)

$$P_{11} = \lambda \ln(J) \frac{F_{22}}{J} + \mu(F_{11} - \frac{F_{22}}{J}, \tag{165}$$

$$P_{12} = \lambda \ln(J) \frac{(-F_{21})}{J} + \mu(F_{21} + \frac{F_{12}}{J}), \tag{166}$$

$$J = F_{11}F_{22} - F_{12}F_{21}. \tag{167}$$

whence, (*check me*)

$$c_p^2 = \frac{\partial P_{11}}{\partial F_{11}} = \lambda\Big((1 - \ln(J))\frac{F_{22}^2}{J^2}\Big) + \mu(1 + \frac{F_{22}^2}{J^2}), \tag{168}$$

$$c_s^2 = \frac{\partial P_{12}}{\partial F_{21}} = \lambda\Big((1 - \ln(J))\frac{F_{12}F_{21}}{J^2} - \ln(J)\frac{1}{J}\Big) + \mu(1 + \frac{F_{12}^2}{J}) \tag{169}$$

We see that for a material under compression, $0 < J < 1$, $\ln(J) < 0$ and both c_p^2 and c_s^2 remain positive.

For the **rotated-linear** model, ****Finish me****

$$\mathbf{F} = \mathbf{U}\hat{\mathbf{F}}\mathbf{V}^T \quad (SVD), \tag{170}$$

$$\hat{\mathbf{P}} = \lambda tr(\hat{\mathbf{F}} - \mathbf{I})\mathbf{I} + 2\mu(\hat{\mathbf{F}} - \mathbf{I}), \tag{171}$$

$$= \operatorname{diag}(\hat{P}_1, \hat{P}_2), \tag{172}$$

$$\mathbf{P} = \mathbf{U}\hat{\mathbf{P}}\mathbf{V}^T \tag{173}$$

$$P_{ij} = U_{im}\hat{P}_m V_{jm} \tag{174}$$

where $\hat{\mathbf{F}} = \operatorname{diag}(\sigma_1, \sigma_2)$ with

$$\sigma_1^2 = \sigma_1^2(\mathbf{F}) \tag{175}$$

and

$$\frac{\partial P_{ij}}{\partial F_{kl}} = \tag{176}$$

5 Equations solved by the HEMP code

These notes are based on the discussion in *Computer Simulation of Dynamic Phenomena* by Mark Wilkins [3].

HEMP: Hydrodynamic, Elastic, Magneto and Plastic

Chapt. 3. Hooke's law (stress-strain relationship)

$$
\begin{aligned}
&\sigma_i \quad \text{(principal components of the stress tensor)}\\
\dot\sigma_{ii} &= \lambda\frac{\dot V}{V} + 2\mu\dot\epsilon_{ii}\\
&= (\lambda + 2\mu/3)\frac{\dot V}{V} + 2\mu\Big(\dot\epsilon_{ij} - \frac{1}{3}\frac{\dot V}{V}\delta_{ij}\Big)
\end{aligned}
$$

We are using natural strain (referring to the current configuration rather than the original).

$$
\begin{aligned}
\dot\sigma_{ij} &= -\dot P\delta_{ij} + \dot s_{ij}\\
-\dot P &= K\frac{\dot V}{V}\\
K &= \lambda + 2\mu/3 \quad \text{(bulk modulus)}\\
\dot s_{ij} &= 2\mu\Big(\dot\epsilon_{ij} - \frac{1}{3}\frac{\dot V}{V}\delta_{ij}\Big) \quad \text{(stress deviators)}\\
P &= -\frac{1}{3}\sum_i \sigma_{ii}\\
\frac{\dot V}{V} &= \sum_i \dot\epsilon_{ii}, \qquad \text{(continuity: } \frac{\dot V}{V} = \nabla\cdot\mathbf{U}\text{)}\\
\sum_i s_{ii} &= 0
\end{aligned}
$$

The strains should be corrected for the rigid body motion (which should not contribute to the strain):

$$
\dot s_{ij} = 2\mu\Big(\dot\epsilon_{ij} - \frac{1}{3}\frac{\dot V}{V}\delta_{ij}\Big) + \dot\delta_{ij}
$$

Section 3.1.2 Rigid body equations

$$
\begin{aligned}
\dot\delta_{xx} &= -2\dot\omega_z s_{xy} + 2\dot\omega_y s_{zx}\\
\dot\omega_x &= \frac{1}{2}\big[\partial\dot z/\partial y - \partial\dot y/\partial z\big]\\
&\text{etc.}
\end{aligned}
$$

Section 3.2 Plastic Flow Region: For a material undergoing a perfect plastic flow, the principal component of the stress deviator will satisfy

$$
\begin{aligned}
f(s_1, s_2, s_3) &= 0, \qquad \text{principal stress deviators lie on this surface} && (177)\\
\dot\epsilon_i^p &= \dot\lambda s_i, \qquad \text{the plastic strain proportional to } s_i && (178)\\
\sum_i \epsilon_i^p &= 0, \qquad \text{plastic incompressibility} && (179)\\
\epsilon_i &= \epsilon_i^e + \epsilon_i^p \qquad \text{total strain a sum of elastic and plastic} && (180)
\end{aligned}
$$

Von Mises generalized condition for plastic flow

$$\dot{\epsilon}_i^p = \dot{\lambda}\partial f/\partial\sigma_i, \qquad \text{when } f = \dot{f} = 0$$

Section 3.2.2 Von Mises Yield Condition:

$$\begin{aligned}
\sigma_{eq} &= Y^0, \qquad \text{yield surface} \\
\sigma_{eq} &= \sqrt{\frac{3}{2}}\sqrt{2J_2} = \sqrt{\frac{3}{2}}\sqrt{s_1^2 + s_2^2 + s_3^2} \qquad \text{equivalent stress} \\
2J_2 &= \sum_{ij} s_{ij}^2 = \sum_i s_i^2 \qquad J_2 \text{ is the second invariant of } s_{ij}
\end{aligned}$$

Implementing the plastic yield condition: If the updated equivalent stress exceeds the yield stress, then we scale the stress deviators so that the resulting equivalent stress lies on the yield surface:

$$\begin{aligned}
\sigma_{eq}^* &= \sqrt{3/2}\sqrt{(s_1^*)^2 + (s_2^*)^2 + (s_3^*)^2} > Y^0 \qquad \text{(updated equivalent stress)} \\
s_i^{n+1} &= m s_i^*, \qquad \text{(scale the stress deviators)} \\
m &= Y^0/\sigma_{eq}^* \\
\Rightarrow \sigma_{eq}^{n+1} &= \sqrt{3/2}\sqrt{(s_1^{n+1})^2 + (s_2^{n+1})^2 + (s_3^{n+1})^2} = m\ \sigma_{eq}^* = Y^0
\end{aligned}$$

The plastic strain increment is then

$$\Delta\epsilon_i^p = \frac{1}{2\mu}(s_i^* - s_i^{n+1}) = \frac{1}{2\mu}(\frac{1}{m} - 1)s_i^{n+1} \tag{181}$$

$$\sum_i \Delta\epsilon_i^p = 0, \qquad \text{plastic incompressibility} \tag{182}$$

Note that (181) follows the rule given by equation (178).

Section 3.3.1 Strain Hardening: The Yield stress Y^0 is more generally a function of the plastic strain, Temperature etc. Here is an example of a stress dependent yield,

$$Y = Y^0(1 + \beta\epsilon^p)^n$$

Section 3.4.1 Maxwell Solid model describes a visco-elastic-plastic material

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^V = \frac{\dot{s}_{ij}}{2\mu} + \frac{s_{ij}}{2\eta}, \qquad \eta = \text{coefficient of viscosity}$$

5.1 Hemp Equations

Here are the continuous form of the equations used by Hemp.

$$\rho = \text{actual density}, \quad \rho_0 = \text{reference density of the EOS} \tag{183}$$

$$V = \rho_0/\rho, \qquad \text{(relative volume, non-dimensional, see pressure EOS)} \tag{184}$$

$$M = \frac{\rho_0}{V^0} V(0), \qquad \text{(mass, } V^0\text{= initial relative volume)} \tag{185}$$

$$\frac{d}{dt} M = 0, \qquad \text{(conservation of mass)} \tag{186}$$

$$\rho \frac{d}{dt} \dot{x}_\alpha = \partial_\beta \sigma_{\alpha\beta}, \quad \partial_\beta \equiv \partial/\partial x_\beta, \qquad \text{(conservation of momentum)} \tag{187}$$

$$\frac{d}{dt} E = -(P+q)\dot{V} + V\big[s_{\alpha\beta}\dot{\epsilon}_{\alpha\beta}\big], \qquad \text{(conservation of energy)} \tag{188}$$

$$\sigma_{\alpha\beta} = -(P+q)\delta_{\alpha\beta} + s_{\alpha\beta} \tag{189}$$

$$\dot{\epsilon}_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \dot{x_\beta} + \partial_\beta \dot{x_\alpha}) \tag{190}$$

$$\dot{s}_{\alpha\beta} = 2\mu\Big(\dot{\epsilon}_{\alpha\beta} - \frac{1}{3}\frac{\dot{V}}{V}\delta_{\alpha\beta}\Big) \tag{191}$$

$$P = a(\eta-1) + b(\eta-1)^2 + c(\eta-1)^3 + d\eta E, \qquad \text{(pressure EOS)} \tag{192}$$

$$\eta = 1/V = \rho/\rho_0 \tag{193}$$

$$\sqrt{2J} - \sqrt{2/3}\; Y \leq 0, \qquad \text{(Von Mises Yield Condition)} \tag{194}$$

$$q = C_0^2 \rho L^2 \dot{s}^2 + C_L \rho L a |\dot{s}|, \qquad \text{(artificial viscosity)} \tag{195}$$

Note: Wilkins defines $\dot{\epsilon}_{\alpha\beta}$ without the $\frac{1}{2}$ for $i \neq j$.
Note: E is the internal energy per **original** volume, $E = \rho_0 e$. The energy equation can be also written as

$$\rho \frac{d}{dt}(E/\rho_0) = -(P+q)\dot{V}/V + \big[s_{\alpha\beta}\dot{\epsilon}_{\alpha\beta}\big] \tag{196}$$

$$= -(P+q)\nabla\cdot\mathbf{U} + \big[s_{\alpha\beta}\dot{\epsilon}_{\alpha\beta}\big] \tag{197}$$

Compare this to the usual Eulerian equation for the internal energy,

$$\rho\frac{De}{Dt} = -P\nabla\cdot\mathbf{U} + \boldsymbol{\tau} : \nabla\mathbf{U} - \nabla\cdot\mathbf{q} \tag{198}$$

or the conservation equation for the total energy

$$\frac{\partial E^T}{\partial t} = -\nabla\cdot((E^T+P)\mathbf{U}) + \nabla\cdot(\mathbf{U}\cdot\boldsymbol{\tau}) - \nabla\cdot(\mathbf{q}) \tag{199}$$

$$E^T = \rho e + \frac{1}{2}\rho\mathbf{U}\cdot\mathbf{U} \tag{200}$$

5.2 Hemp Discretization

Here is the Hemp approximation in semi-discrete form

$$U_\alpha^{n+\frac{1}{2}} = U_\alpha^{n-\frac{1}{2}} + \frac{\Delta t}{\rho^n}\partial_\beta(\sigma_{\alpha\beta}^n) \tag{201}$$

$$x_\alpha^{n+1} = x_\alpha^n + \Delta t^{n+\frac{1}{2}} U_\alpha^{n+\frac{1}{2}} \tag{202}$$

$$v^{n+1} = \text{Volume element from } x_\alpha^{n+1} \tag{203}$$

$$V^{n+1} = (\rho_0/M)v^{n+1}, \quad \rho^{n+1} = \rho_0/V^{n+1} \tag{204}$$

$$\dot{\epsilon}_{\alpha\beta}^{n+\frac{1}{2}} = \frac{1}{2}(\partial_\beta U_\alpha^{n+\frac{1}{2}} + \partial_\alpha U_\beta^{n+\frac{1}{2}}) \tag{205}$$

$$s_{\alpha\beta}^{n+1} = s_{\alpha\beta}^n + \Delta t\ 2\mu\Big(\dot{\epsilon}_{\alpha\beta}^{n+\frac{1}{2}} - \frac{1}{3}\frac{\dot{V}^{n+\frac{1}{2}}}{V^{n+\frac{1}{2}}}\delta_{\alpha\beta}\Big) \tag{206}$$

$$(E^{n+1} - E^n)/\Delta t = -\Big(\frac{1}{2}(P^{n+1} + P^n) + \bar{q}\Big)(V^{n+1} - V^n)/\Delta t + V^{n+\frac{1}{2}}\big[s_{\alpha\beta}\dot{\epsilon}_{\alpha\beta}\big]^{n+\frac{1}{2}} \tag{207}$$

$$P^{n+1} = A(\eta^{n+1}) + B(\eta^{n+1})E^{n+1}, \quad \eta^{n+1} = 1/V^{n+1} \quad \text{(coupled with } E^{n+1}) \tag{208}$$

$$\sigma_{\alpha\beta}^{n+1} = -(P^{n+1} + q^{n+\frac{1}{2}})\delta_{\alpha\beta} + s_{\alpha\beta}^{n+1} \tag{209}$$

When the plastic yield condition is taken into account, equation (206) is replaced by

$$s_{\alpha\beta}^* = s_{\alpha\beta}^n + \Delta t\ 2\mu\Big(\dot{\epsilon}_{\alpha\beta}^{n+\frac{1}{2}} - \frac{1}{3}\frac{\dot{V}^{n+\frac{1}{2}}}{V^{n+\frac{1}{2}}}\delta_{\alpha\beta}\Big) \tag{210}$$

$$2J_2^* = \sum_{\alpha\beta}(s_{\alpha\beta}^*)^2 \tag{211}$$

$$m^* = \sqrt{2/3}Y^0/(2J_2^*) \qquad (m^* > 1 : \text{elastic}, \ m^* < 1 : \text{plastic}) \tag{212}$$

$$s_{\alpha\beta}^{n+1} = \min(1, m^*)\ s_{\alpha\beta}^* \tag{213}$$

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