Some SBP identities. Let the inner product (u,v) be defined as

$$(u,v) = h\sum_{j=1}^{\infty} \omega_j u_j v_j.$$

The index 1 refers to the first physical gridpoint at the boundary. Introduce difference operators

$$hD_+u_i = u_{i+1} - u_i, \tag{1}$$
$$hD_-u_i = u_i - u_{i-1}, \tag{2}$$
$$2hD_0u_i = u_{i+1} - u_{i-1}, \tag{3}$$

$$\widetilde{D_0}u_i = \begin{cases} D_+u_i, & i = 1, \\ D_0u_i, & i \geq 2. \end{cases}$$

We have

$$(D_+u,v) + (u,D_-v) = \\ -\omega_1 v_0 u_1 + (\omega_1 - \omega_2)v_1u_2 + \ldots + (\omega_i - \omega_{i+1})v_iu_{i+1}. \tag{4}$$

$$(D_0u,v) + (u,D_0v) = \\ -\frac{\omega_1}{2}(v_0u_1 + v_1u_0) + \frac{\omega_1 - \omega_2}{2}(v_1u_2 + v_2u_1) + \ldots + \frac{\omega_i - \omega_{i+1}}{2}(v_iu_{i+1} + v_{i+1}u_i). \tag{5}$$

$$(\widetilde{D_0}u,v) + (u,\widetilde{D_0}v) = \\ -2\omega_1v_1u_1 + (\omega_1 - \frac{\omega_2}{2})(v_1u_2 + v_2u_1) + \frac{\omega_2 - \omega_3}{2}(v_2u_3 + v_3u_2) + \ldots + \frac{\omega_i - \omega_{i+1}}{2}(v_iu_{i+1} + v_{i+1}u_i). \tag{6}$$

$$(D_+D_-u,v) - (u,D_+D_-v) = \\ -\frac{\omega_1}{h}(v_0u_1 - v_1u_0) + \frac{\omega_1 - \omega_2}{h}(v_1u_2 - v_2u_1) + \ldots + \frac{\omega_i - \omega_{i+1}}{h}(v_iu_{i+1} - v_{i+1}u_i). \tag{7}$$

$$(D_0D_+D_-u,v) - (u,D_0D_+D_-v) = \\ -\frac{\omega_1}{2h^2}((v_{-1} - 2v_0)u_1 + v_1(u_{-1} - 2u_0)) - \frac{\omega_2}{2h^2}(v_0u_2 + v_2u_0) + \ldots \\ -\frac{\omega_i - \omega_{i+1}}{2h^2}(2v_iu_{i+1} + 2v_{i+1}u_i) + \frac{\omega_i - \omega_{i+2}}{2h^2}(v_iu_{i+2} + v_{i+2}u_i). \tag{8}$$

$$
\begin{aligned}
(\widetilde{D_0}D_+D_-u, v) - (u, \widetilde{D_0}D_+D_-v) = \\
&- \frac{\omega_1}{2h^2}((2v_0 - 6v_1)u_1 + v_1(2u_0 - 6u_1)) - \frac{\omega_2}{2h^2}(v_0u_2 + v_2u_0) \\
&- \frac{6\omega_1 - 2\omega_2}{2h^2}(v_1u_2 + v_2u_1) + \frac{2\omega_1 - \omega_3}{2h^2}(v_1u_3 + v_3u_1) + \dots \\
&- \frac{\omega_i - \omega_{i+1}}{2h^2}(2v_iu_{i+1} + 2v_{i+1}u_i) + \frac{\omega_i - \omega_{i+2}}{2h^2}(v_iu_{i+2} + v_{i+2}u_i). \qquad (9)
\end{aligned}
$$

1 Some Discretizations

We study the problem $u_{xx} + a(u_{xy} + u_{yx})$ with boundary conditions $u_x + au_y$. We start with the simplest second order discretization using centered finite differences. With $\omega_1 = 1/2$ we would like to show that

$$
(u, D_+D_-v + a(D_0v_y + \partial_y D_0v)) = (v, D_+D_-u + a(D_0u_y + \partial_y D_0v)), \qquad (10)
$$

subjected to some discrete version of the boundary conditions. Starting with the left term we find

$$
\begin{aligned}
(u, D_+D_-v + D_0v_y) = -(D_-v, D_-u) - \frac{1}{2}\left(u_0D_-v_1 + u_1D_-v_2\right) \\
- a(D_0u, v_y) - \frac{a}{4}\left(u_0v_{y1} + u_1v_{y0} + u_1v_{y2} + u_2v_{y1}\right) \\
- a(u_y, D_0v).
\end{aligned}
$$

The following identities are useful

$$
\begin{aligned}
u_0 &= u_1 - hD_-u_1, \\
u_2 &= u_1 + hD_+u_1.
\end{aligned}
$$

Now we can rewrite

$$
\begin{gathered}
-\frac{1}{2}\left((u_1 - hD_-u_1)D_-v_1 + u_1D_-v_2\right) \\
-\frac{a}{4}\left((u_1 - hD_-u_1)v_{y1} + u_1v_{y0} + u_1v_{y2} + (u_1 + hD_+u_1)v_{y1}\right) = \\
-u_1 \underbrace{\left(\frac{1}{2}\left(D_-v_1 + D_-v_2\right) + \frac{a}{4}\left(v_{y0} + 2v_{y1} + v_{y2}\right)\right)}_{\gamma_1} \\
+\frac{h}{2}D_-u_1D_-v_1 + \frac{hav_{y1}}{4}\left(D_-u_1 - D_+u_1\right).
\end{gathered}
$$

We now continue with the second term

$$
\begin{aligned}
(v, D_+D_-u + a(D_0u_y + \partial_y D_0u)) = -(D_-v, D_-u) - a(D_0v, u_y) - a(v_y, D_0u) \\
- \frac{1}{2}\left(v_0D_-u_1 + v_1D_-u_2\right) - \frac{a}{4}\left(v_0u_{y1} + v_1u_{y0} + v_1u_{y2} + v_2u_{y1}\right).
\end{aligned}
$$

The boundary contributions become

$$
\begin{gathered}
-\frac{1}{2}\left((v_1 - hD_-v_1)D_-u_1 + v_1 D_-u_2\right) \\
-\frac{a}{4}\left((v_1 - hD_-v_1)u_{y1} + v_1 u_{y0} + v_1 u_{y2} + (v_1 + hD_+v_1)u_{y1}\right) = \\
-v_1 \underbrace{\left(\frac{1}{2}\left(D_-u_1 + D_-u_2\right) + \frac{a}{4}\left(u_{y0} + 2u_{y1} + u_{y2}\right)\right)}_{\gamma_2} = \\
+\frac{1}{2}hD_-v_1 D_-u_1 + \frac{au_{y1}}{4}\left(hD_-v_1 - hD_+v_1\right).
\end{gathered}
$$

Zeroing out γ_1 and γ_2 as the boundary conditions we find the following expression for the equality (10)

$$
\begin{aligned}
&-(D_-u, D_-v) - a(D_0u, v_y) - a(u_y, D_0v) \\
&+\frac{h}{2}D_-u_1 D_-v_1 - \frac{h^2 a v_{y1}}{4}D_+D_-u_1 = -(D_-v, D_-u) - a(D_0v, u_y) - a(v_y, D_0u) \\
&\qquad\qquad + \frac{1}{2}hD_-v_1 D_-u_1 - \frac{h^2 a u_{y1}}{4}D_+D_-v_1.
\end{aligned}
$$

Only if $a = 0$ or $u_1 = 0$ do we get a self-adjoint discretization....

Discretization 2

We try with $w_1 = 1$. The left term

$$
\begin{aligned}
(u, D_+D_-v + D_0v_y) = &-(D_-v, D_-u) - u_0 D_-v_1 \\
&- a(u_y, D_0v) - a(D_0u, v_y) - \frac{a}{2}\left(u_0 v_{y1} + (u_0 + hD_+u_0)v_{y0}\right)
\end{aligned}
$$

The right term

$$
\begin{aligned}
(D_+D_-u + D_0u_y, v) = &-(D_-u, D_-v) - v_0 D_-u_1 \\
&- a(v_y, D_0u) - a(D_0v, u_y) - \frac{a}{2}\left(v_0 u_{y1} + (v_0 + hD_+v_0)u_{y0}\right).
\end{aligned}
$$

This won't work either....