Fluid-Solid Interface Conditions

Solid equations:

$$\rho_0 \frac{\partial}{\partial t} v_i = \frac{\partial}{\partial x_j} P_{ji}$$

$$\frac{\partial}{\partial t} P_{ij} = K_{ijkl} \frac{\partial}{\partial x_l} v_k .$$

Mapped eqns.

$$\rho_0 v_{1t} = P_{11r} r_x + P_{11s} s_x + P_{21r} r_y + P_{21s} s_y$$

$$\rho_0 v_{2t} = P_{12r} r_x + P_{12s} s_x + P_{22r} r_y + P_{22s} s_y$$

and

$$P_{ijt} = K_{ij11} (v_{1r} r_x + v_{1s} s_x)$$
$$+ K_{ij12} (v_{1r} r_y + v_{1s} s_y)$$
$$+ K_{ij21} (v_{2r} r_x + v_{2s} s_x)$$
$$+ K_{ij22} (v_{2r} r_y + v_{2s} s_y)$$

Assume $\underline{r}$ = const $\Rightarrow$ interface

$$\rho_0 v_{1t} = r_x P_{11r} + r_y P_{21r}$$

$$\rho_0 v_{2t} = r_x P_{12r} + r_y P_{22r}$$

and

$$P_{ijt} = r_x K_{ij11} v_{1r} + r_y K_{ij12} v_{1r}$$
$$+ r_x K_{ij21} v_{2r} + r_y K_{ij22} v_{2r}.$$

Define

$$\tilde{P}_{11} = r_x P_{11} + r_y P_{21}$$

$$\tilde{P}_{12} = r_x P_{12} + r_y P_{22}$$

These are components of ~~the~~ a vector proportional to the traction $\underline{t}$.

So,

$$\rho_0 v_{1t} = \tilde{P}_{11r}$$ ignoring lower-order terms.

$$\rho_0 v_{2t} = \tilde{P}_{12r}$$

and

$$\tilde{P}_{11t} = T_{11} v_{1r} + T_{12} v_{2r}$$

$$\tilde{P}_{12t} = T_{21} v_{1r} + T_{22} v_{2r}$$

where

$$T_{11} = r_x \left(K_{1111} r_x + K_{1112} r_y \right) + r_y \left(K_{2111} r_x + K_{2112} r_y \right)$$

$$T_{12} = r_x \left(K_{1121} r_x + K_{1122} r_y \right) + r_y \left(K_{2121} r_x + K_{2122} r_y \right)$$

$$T_{21} = r_x \left(K_{1211} r_x + K_{1212} r_y \right) + r_y \left(K_{2211} r_x + K_{2212} r_y \right)$$

$$T_{22} = r_x \left(K_{1221} r_x + K_{1222} r_y \right) + r_y \left(K_{2221} r_x + K_{2222} r_y \right)$$

Matrix form,

$$\underline{q}_t + \underline{\underline{A}}\,\underline{q}_r = 0.$$

where

$$\underline{q} = \begin{bmatrix} v_1 \\ v_2 \\ \tilde{P}_{11} \\ \tilde{P}_{12} \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -\frac{1}{\rho_0} & 0 \\ 0 & 0 & 0 & -\frac{1}{\rho_0} \\ -T_{11} & -T_{12} & 0 & 0 \\ -T_{21} & -T_{22} & 0 & 0 \end{bmatrix}.$$

Eigenvalue problem

$$(\underline{\underline{A}} - \kappa \underline{\underline{I}})\underline{z} = \begin{bmatrix} -\kappa & 0 & -\frac{1}{\rho_0} & 0 \\ 0 & -\kappa & 0 & -\frac{1}{\rho_0} \\ -T_{11} & -T_{12} & -\kappa & 0 \\ -T_{21} & -T_{22} & 0 & -\kappa \end{bmatrix} \underline{z} = 0.$$

Have

$$-\kappa z_1 - \frac{1}{\rho_0} z_3 = 0 \quad \Rightarrow \quad z_3 = -\rho_0 \kappa z_1$$

$$-\kappa z_2 - \frac{1}{\rho_0} z_4 = 0 \quad \Rightarrow \quad z_4 = -\rho_0 \kappa z_2$$

Then

$$-T_{11} z_1 - T_{12} z_2 - \kappa z_3 = 0$$

$$-T_{21} z_1 - T_{22} z_2 - \kappa z_4 = 0.$$

Eliminate z_3, z_4 to give

$$(\rho_0 \kappa^2 - T_{11}) z_1 - T_{12} z_2 = 0$$

$$-T_{21} z_1 + (\rho_0 \kappa^2 - T_{22}) z_2 = 0.$$

Solving gives

$$\kappa^2 = c_s^2 \text{ or } c_p^2.$$

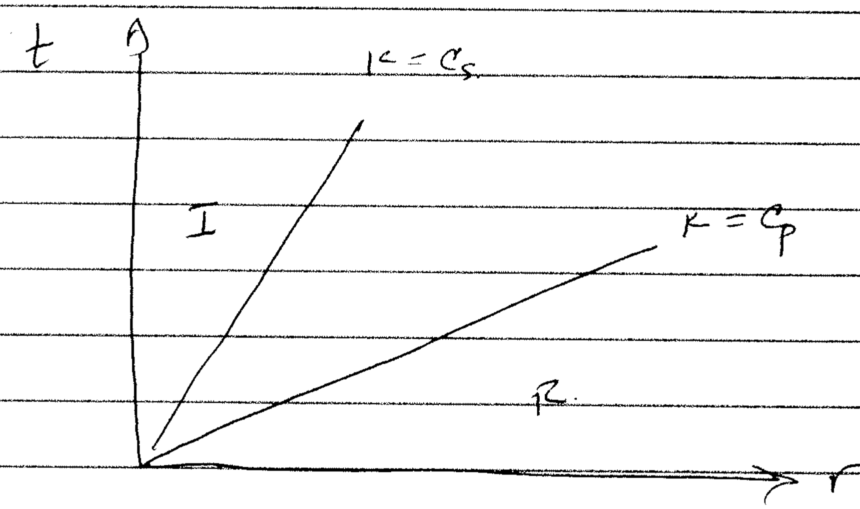
and

$$(z_1, z_2) = (z_1, z_2)_s \text{ or } (z_1, z_2)_p$$

$$\text{with } z_1^2 + z_2^2 = 1$$

Cases.

1)



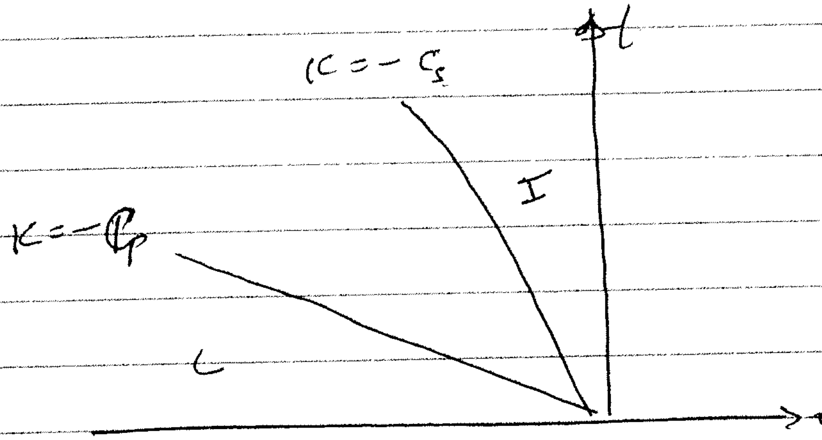
$$V_{1I} = V_{1R} - \xi_p z_{1p} - \xi_s z_{1s}$$

$$V_{2I} = V_{2R} - \xi_p z_{2p} - \xi_s z_{2s}$$

$$\tilde{P}_{11I} = \tilde{P}_{11R} + \xi_p \rho_0 c_p z_{1p} + \xi_s \rho_0 c_s z_{1s}$$

$$\tilde{P}_{12I} = \tilde{P}_{12R} + \xi_p \rho_0 c_p z_{2p} + \xi_s \rho_0 c_s z_{2s}$$

2)



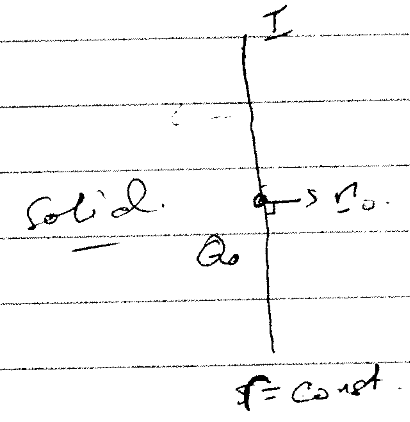
$$v_{1I} = v_{1L} + \xi_p z_{1p} + \xi_s z_{1s}$$

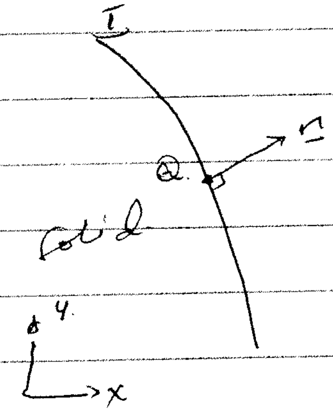
$$v_{2I} = v_{2L} + \xi_p z_{2p} + \xi_s z_{2s}$$

$$\tilde{P}_{11_I} = \tilde{P}_{11_L} + \xi_p \rho_0 c_p z_{1p} + \xi_s \rho_0 c_s z_{1s}$$

$$\tilde{P}_{12_I} = \tilde{P}_{12_L} + \xi_p \rho_0 c_p z_{2p} + \xi_s \rho_0 c_s z_{2s}.$$

Picture

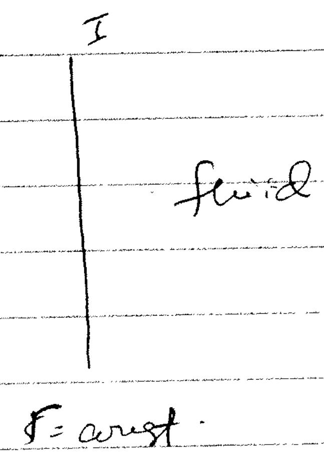


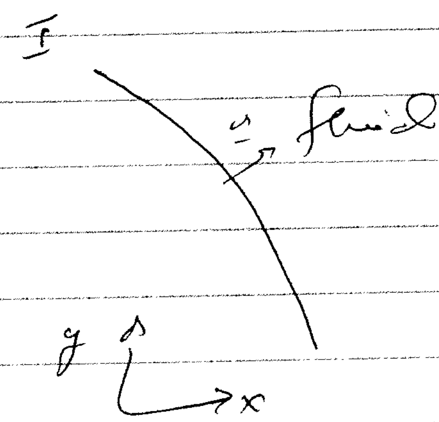


ref.   phys.

(mapped to a unit computational domain)

fluid





mapped unit ξ domain   phys.

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\frac{Dp}{Dt} - \gamma\frac{p}{\rho^{\gamma+1}}\frac{D\rho}{Dt}$$

Fluid eqns.

$$\frac{D\rho}{Dt} + \rho(u_x + v_y) = 0.$$

$$\frac{D}{Dt}\underline{v} + \frac{1}{\rho}\nabla p = 0.$$

$$\frac{D}{Dt}p - a^2\frac{D}{Dt}\rho = 0. \Rightarrow \frac{D}{Dt}p + \rho a^2\nabla\cdot\underline{v} = 0$$

mapping : $(x,y,t) \to (r,s,\tau)$

$$\rho_t = \rho_r r_t + \rho_s s_t + \rho_\tau \tau_t$$

$$x = x(r,s,\tau) = x(r(x,y,t), s(\quad), \tau(t))$$

$$y = y(r,s,\tau)$$

$$t = \tau$$

$$x_t = x_r r_t + x_s s_t + x_\tau = 0$$

$$\cancel{y_t = y_r r_t + y_s s_t + y_\tau}$$

$$1 = x_r r_x + x_s s_x + \cancel{x_\tau \tau_x}$$

$$0 = x_r r_y + x_s s_y$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_x & s_x & 0 \\ r_y & s_y & 0 \\ r_t & s_t & 1 \end{pmatrix} \begin{pmatrix} x_r \\ x_s \\ x_\tau \end{pmatrix}$$

$$r = r\left(x(r,s,\tau),\ y(r,s,\tau),\ t(\tau)\right)$$

$$1 = r_x x_r + r_y y_r$$

$$0 = r_x x_s + r_y y_s$$

$$0 = r_x x_\tau + r_y y_\tau + r_t$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_r & y_r & 0 \\ x_s & y_s & 0 \\ x_\tau & y_\tau & 1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_t \end{pmatrix}$$

$$r_t = \frac{\begin{vmatrix} x_r & y_r & 1 \\ x_s & y_s & 0 \\ x_\tau & y_\tau & 0 \end{vmatrix}}{x_r y_s - x_s y_r} = \frac{x_s y_\tau - y_s x_\tau}{x_r y_s - x_s y_r}$$

$$= -r_y y_\tau - r_x x_\tau$$

$$\rho_t = \rho_\tau - (r_x x_\tau + r_y y_\tau)\rho_r - (s_x x_\tau + s_y y_\tau)\rho_s.$$

$$\rho_x = \rho_r r_x + \rho_s s_x + \rho_\tau \cancel{\tau_x}^{\,0.}$$

$$\rho_y = \rho_r r_y + \rho_s s_y + \rho_\tau \cancel{\tau_y}^{\,0.}$$

So,

$$\frac{D}{Dt}\rho = \rho_\tau + \left[r_x u + r_y v - r_x x_\tau - r_y y_\tau \right]\rho_r + \left[s_x u + s_y v - s_x x_\tau - s_y y_\tau \right]\rho_s$$

$$\nabla \cdot \underline{v} = u_x + v_y = u_r r_x + u_s s_x + v_r r_y + v_s s_y$$

r - direction only. (use $\tau \to t$)

$$\rho_t + \left[r_x (u - \dot{x}) + r_y (v - \dot{y}) \right] \rho_r$$

$$+ r_x u_r + r_y v_r = 0.$$

$$u_t + [\quad] u_r + \frac{1}{\rho} r_x p_r = 0$$

$$v_t + [\quad] v_r + \frac{1}{\rho} r_y p_r = 0.$$

$$p_t + [\quad] p_r + \rho a^2 (r_x u_r + r_y v_r) = 0.$$

Matrix form.

$$\underline{q}_t + \underline{\underline{A}}\, \underline{q}_r = 0.$$

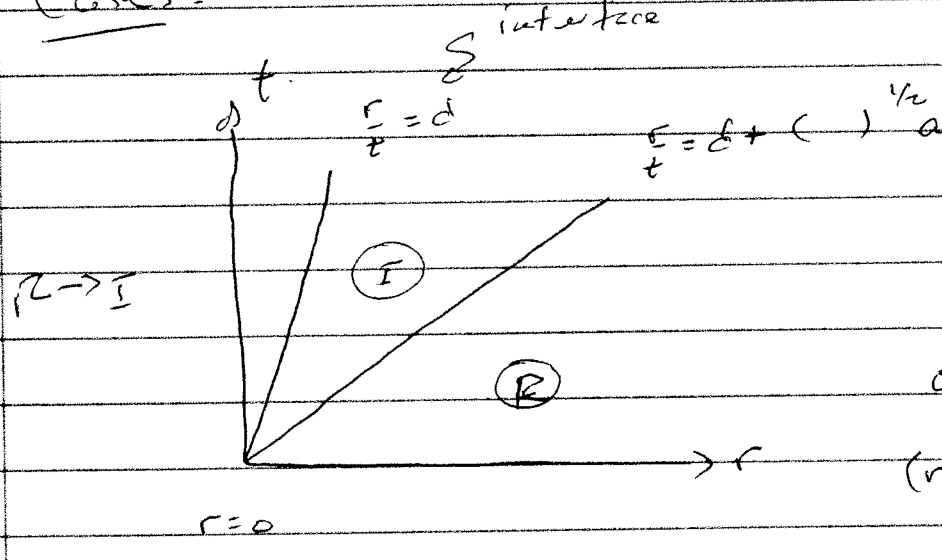
$$q = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad A = \begin{bmatrix} d & r_x & r_y & 0 \\ 0 & d & 0 & r_x/\rho \\ 0 & 0 & d & r_y/\rho \\ 0 & r_x \rho a^2 & r_y \rho a^2 & d \end{bmatrix}$$

Matrix of right eigenvectors.

$$\underset{=}{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -\alpha_1 a & 0 & \alpha_2 & \alpha_1 a \\ -\alpha_2 a & 0 & -\alpha_1 & \alpha_2 a \\ \rho a^2 & 0 & 0 & \rho a^2 \end{bmatrix}.$$

$$\alpha_1 = \frac{r_x}{(r_x^2 + r_y^2)^{1/2}}, \quad \alpha_2 = \frac{r_y}{(r_x^2 + r_y^2)^{1/2}}.$$

Cases.



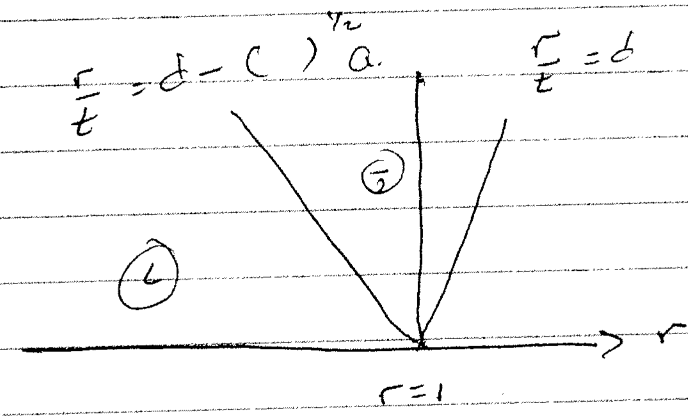
outward normal
$(n_1, n_2) = -(\alpha_1, \alpha_2)$

$$u_I = u_R \pm \alpha_1 a \xi.$$

$$v_I = v_R \pm \alpha_2 a \xi. \qquad \xi = \text{wave strength}$$

$$p_I = p_R \pm \rho a^2 \xi.$$

$L \to I$



outward normal

$(n_1, n_2) = (\alpha_1, \alpha_2)$

$$u_I = u_L - \alpha_1 a \xi.$$

$$v_I = v_L - \alpha_2 a \xi.$$

$$p_I = p_L + \rho a^2 \xi$$

Both cases, can take

$$\left.\begin{aligned} u_I &= u_{L,R} + n_1 \xi. \\ v_I &= v_{L,R} + n_2 \xi. \\ p_I &= p_{L,R} - \rho a \xi. \end{aligned}\right\}$$

And

$$\rho_I = \rho_{L,R} - \xi / a.$$