Notes on Solving the Equations of Solid Mechanics as a First Order System

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1 The velocity-stress formation (first order system)

The equations of elasticity can be written as a first order system for the velocity and stress

$$\rho_0 \partial_t v_i = \partial_{x_j} \sigma_{ij} + \rho_0 f_i \tag{1}$$
$$\partial_t \sigma_{ij} = \lambda \partial_{x_k} v_k \delta_{ij} + \mu(\partial_{x_j} v_i + \partial_{x_i} v_j) \tag{2}$$
$$\partial_t u_i = v_i \tag{3}$$

The displacement can be integrated from the velocity.

1.1 Boundary conditions for the velocity stress formulation

At a displacement boundary we impose specified values for the displacement which implies

$$u_i(\mathbf{x},t) = g_i(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega_d \tag{4}$$
$$v_i(\mathbf{x},t) = \dot{g}_i(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega_d \tag{5}$$
$$\partial_{x_j} \sigma_{ij} = \rho_0 \ddot{g}_i(\mathbf{x},t) - \rho_0 f_i, \qquad \mathbf{x} \in \partial\Omega_d \tag{6}$$

Here the last equation imposes Neumann like conditions on the normal components of the stress.

At a traction boundary we impose specified values for the traction $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$,

$$n_j \sigma_{ij} = g_i(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega_t \tag{7}$$
$$n_j \Big[\lambda \partial_{x_k} u_k \delta_{ij} + \mu(\partial_{x_j} u_i + \partial_{x_i} u_j) \Big] = g_i(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega_d \tag{8}$$
$$n_j \Big[\lambda \partial_{x_k} v_k \delta_{ij} + \mu(\partial_{x_j} v_i + \partial_{x_i} v_j) \Big] = \dot{g}_i(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega_d \tag{9}$$

Here the last two equations impose Neumann like conditions on the displacement and velocities.

1.2 Discrete Displacement Boundary conditions for the velocity stress formulation

At a displacement boundary we have a condition on the stresses of the form

$$\partial_{x_j}\sigma_{ij} = G_i(\mathbf{x},t) \tag{10}$$

In two-dimensions this takes the form

$$\partial_x\sigma_{11} + \partial_y\sigma_{21} = G_1(\mathbf{x},t), \tag{11}$$
$$\partial_x\sigma_{12} + \partial_y\sigma_{22} = G_2(\mathbf{x},t). \tag{12}$$

On a boundary $x = 0$ of a Cartesian grid this gives Neumann boundary conditions for σ_{11} and σ_{12},

$$\partial_x\sigma_{11} = G_1(0,y,t) - \partial_y\sigma_{21}, \tag{13}$$
$$\partial_x\sigma_{12} = G_2(0,y,t) - \partial_y\sigma_{22}. \tag{14}$$

On a boundary $y = 0$ of a Cartesian grid this gives Neumann boundary conditions for σ_{21} and σ_{22},

$$\partial_y\sigma_{21} = G_1(x,0,t) - \partial_x\sigma_{11}, \tag{15}$$
$$\partial_y\sigma_{22} = G_2(x,0,t) - \partial_x\sigma_{12}. \tag{16}$$

At a corner $\mathbf{x} = (0,0)$ we get only two conditions for σ_{11} and σ_{22} (say)

$$\partial_x\sigma_{11} = G_1(0,y,t) - \partial_y\sigma_{21}, \tag{17}$$
$$\partial_y\sigma_{22} = G_2(x,0,t) - \partial_x\sigma_{12}. \tag{18}$$

However, at a corner we can use the definition of the stresses in terms of the displacement since on the corner we know the first spatial derivatives of displacement in terms of the tangential derivatives of the boundary data $g_i(\mathbf{x},t)$,

$$\sigma_{ij} = \lambda\partial_{x_k}u_k\delta_{ij} + \mu(\partial_{x_j}u_i + \partial_{x_i}u_j), \tag{19}$$
$$\partial_{x_j}u_i(0,0,t) = \partial_{x_j}g_i(0,0,t). \tag{20}$$

That is we use $x-$derivatives of the boundary data from the boundary $y = 0$ to get u_{1x} and u_{2x}, and $y-$derivatives of the boundary data from the boundary $x = 0$ to get u_{1y} and u_{2y}. Equations (19) and (20) can be used as Dirichlet conditions for all stress components at the corner point $\mathbf{i} = (0,0)$. We could also use the additional relationships obtained by differentiating (11),(12) in the tangential direction to the boundary,

$$\partial_x^2\sigma_{12} = \partial_x G_2(x,0,t) - \partial_{xy}\sigma_{22}, \tag{21}$$
$$\partial_y^2\sigma_{21} = \partial_y G_1(0,y,t) - \partial_{xy}\sigma_{11}, \tag{22}$$
$$\partial_x^2\sigma_{11} = \partial_x G_1(x,0,t) - \partial_{xy}\sigma_{21}, \tag{23}$$
$$\partial_y^2\sigma_{22} = \partial_y G_2(0,y,t) - \partial_{xy}\sigma_{12}. \tag{24}$$

The first equation above could be used to get σ_{12} at the ghost point $\mathbf{i} = (-1,0)$ while the second equation could give σ_{21} at ghost point $\mathbf{i} = (0,-1)$.

1.3 Boundary conditions at a corner between a displacement and traction boundary

Consider a Cartesian grid with a displacement boundary at $y = 0$ and a traction boundary on $x = 0$.

$$u_i(x,0,t) = g_i(x,0,t) \tag{25}$$
$$\sigma_{11}(0,y,t) = \lambda(\partial_x u_1 + \partial_y u_2) + 2\mu\partial_x u_1 = G_1(0,y,t), \tag{26}$$
$$\sigma_{12}(0,y,t) = \mu(\partial_y u_1 + \partial_x u_2) = G_2(0,y,t). \tag{27}$$

We can differentiate the displacement boundary condition in x to get $\partial_x u_i$ and use the stress conditions to compute $\partial_y u_i$

$$\partial_x u_i(x,0,t) = \partial_x g_i(x,0,t), \tag{28}$$
$$\lambda \partial_y u_2(0,y,t) = G_1(0,y,t) - (\lambda + 2\mu)\partial_x u_1, \tag{29}$$
$$\mu \partial_y u_2(0,y,t) = G_2(0,y,t) - \mu \partial_x u_2, \tag{30}$$

and thus we know enough to compute the other two components of the stress tensor at the corner

$$\sigma_{22}(0,0,t) = \lambda(\partial_x u_1 + \partial_y u_2) + 2\mu \partial_y u_2 \tag{31}$$
$$\sigma_{21}(0,0,t) = \mu(\partial_y u_1 + \partial_x u_2) \tag{32}$$

Therefore at the corner point we can compute all components of the displacement and stress in terms of the given boundary data functions.

1.4 Boundary conditions at a corner between two traction boundaries

Consider a Cartesian grid with a traction boundary at $y = 0$ and a traction boundary on $x = 0$.

$$\sigma_{11}(0,y,t) = \lambda(\partial_x u_1 + \partial_y u_2) + 2\mu \partial_x u_1 = G_1(0,y,t), \tag{33}$$
$$\sigma_{12}(0,y,t) = \mu(\partial_y u_1 + \partial_x u_2) = G_2(0,y,t), \tag{34}$$
$$\sigma_{21}(x,0,t) = \mu(\partial_y u_1 + \partial_x u_2) = G_3(x,0,t), \tag{35}$$
$$\sigma_{22}(x,0,t) = \lambda(\partial_x u_1 + \partial_y u_2) + 2\mu \partial_y u_2 = G_4(x,0,t)\ . \tag{36}$$

We can combine (33) and (36) to give equations for $\partial_x u_1$ and $\partial_y u_2$ at the corner:

$$\partial_x u_1(0,0,t) = \frac{\lambda + 2\mu}{4(\lambda + \mu)\mu}\Big(G_1 - \lambda G_4\Big), \tag{37}$$
$$\partial_y u_2(0,0,t) = \frac{\lambda + 2\mu}{4(\lambda + \mu)\mu}\Big(G_4 - \lambda G_1\Big)\ . \tag{38}$$

The above two conditions can be used to get the two ghost values $U_1(-1,0)$ and $U_2(0,1)$ for the discrete approximations to u_1 and u_2. Unfortunately we only have one equation for $\partial_y u_1$ and $\partial_x u_2$ at the corner so we need an addition condition. If we differentiate (34) with respect to y and (36) w.r.t x we get

$$\partial_y^2 u_1 = -\partial_y \partial_x u_2 + \frac{1}{\mu}\partial_y G_2(0,y,t), \tag{39}$$
$$= \frac{\lambda}{\lambda + 2\mu}\partial_x^2 u_1 - \frac{1}{\lambda + 2\mu}\partial_x G_4 + \frac{1}{\mu}\partial_y G_2(0,y,t). \tag{40}$$

The above condition can be used to compute the ghost value $U_1(0,-1)$ (using the already computed value of $U_1(-1,0)$ to compute an approximation ot $\partial_x^2 u_1$. Similarly if we differentiate (35) with respect to x and (33) w.r.t y we get the condition

$$\partial_x^2 u_2 = -\partial_x \partial_y u_1 + \frac{1}{\mu}\partial_x G_3(0,y,t), \tag{41}$$
$$= \frac{\lambda}{\lambda + 2\mu}\partial_y^2 u_2 - \frac{1}{\lambda + 2\mu}\partial_y G_1 + \frac{1}{\mu}\partial_x G_3(0,y,t). \tag{42}$$

and this can be used to get $U_2(-1,0)$.