Intro to Limits as a Concept

Written by David Green

Introduction

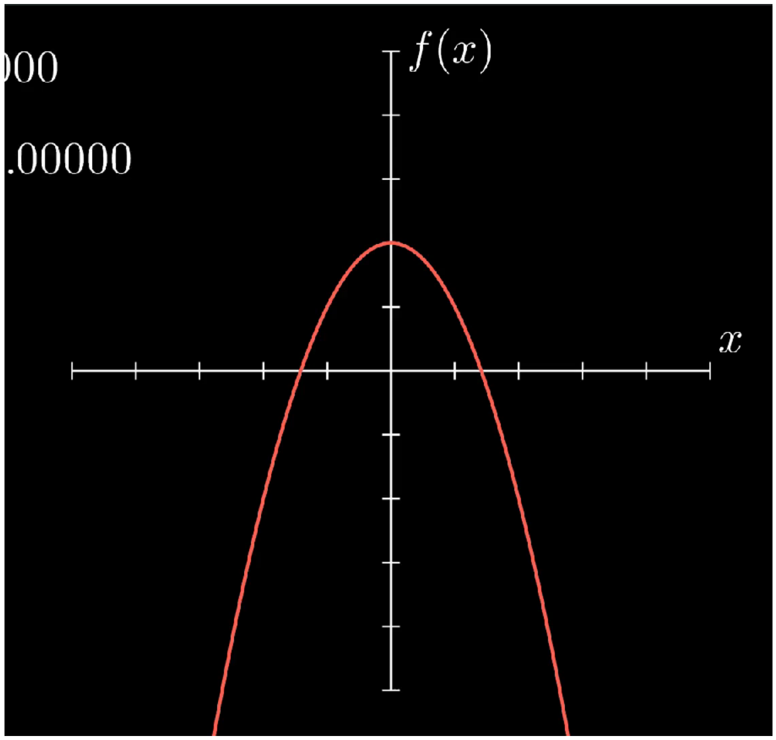
Limits are one of the most fundamental ideas used in calculus. In this section, we're going to be introducing the idea of a limit as a concept. More specifically, I'm going to be explaining what it means in math when we say the word "limit", along with some examples on how we can apply limits to describe the behavior of functions.

What is a Limit?

In previous classes like algebra you've dealt with things called functions, equations that have an input variable and an output variable, where each input value has one output value. An example would be $f(x) = 7x^2 + 3$. Your input value is x and your output value is $f(x)$. In algebra you were tasked with finding the output value of the function given some input x. So for example if you wanted to find $f(2)$ you would say $f(2) = 7(2)^2 + 3 = 31$.

Limits are a bit different though. A limit is not the output value of a function at a specific point but rather the value that the output of a function appears to get closer and closer to as your x value gets closer and closer to that point.

Consider the following graph of the function $f(x) = -x^2 + 2$:



The limit as x approaches 0 from the left of $f(x)$ is equal to 2. In other words, if we start to the left of 0 and bring x closer and closer to 0 without actually reaching 0, then the output of the function (or the height of the function) gets closer and closer to 2.

We can further demonstrate this using the following animation:

As we plug in x values less than 0 that get closer and closer to 0 without x actually reaching 0, then the output gets closer and closer to 2.

Therefore, we can say the limit as x approaches 0 of $f(x)$ from the left equals 2.

The limit as x approaches 0 from the right of $f(x)$ is also equal to 2. In other words, if we start to the right of 0 and bring x closer and closer to 0 without actually reaching 0, then the output of the function (or the height of the function) gets closer and closer to 2.

We can once again further demonstrate this using the following animation:

As we plug in x values greater than 0 that get closer and closer to 0 without x actually reaching 0, then the output gets closer and closer to 2.

Therefore, we can say the limit as x approaches 0 of $f(x)$ from the right equals 2.

Since the limit as x approaches 0 of $f(x)$ from the left $= 2$ and the limit as x approaches 0 of $f(x)$ from the right $= 2$, we can say the limit as x approaches 0 of $f(x) = 2$

We'll summarize these three ideas with the following animation:

You may be confused as to why for both the left and right sided limits the point is stopping just before $x = 0$ at not AT $x = 0$. This is because a limit is not the value the output of the function is equal to at $x = 0$, It's is the value the $f(x)$ value gets closer and closer to as x approaches 0. This may already seem like a bit of a redundant observation since the value of $f(x)$ already is 5 when $x = 2$. So it would make sense that if x gets closer and closer to 2 then the output will get closer and closer to 5. However in the next examples we'll show why this concept is important.

Notation

I've currently expressed the idea of a limit in plain English but from this point on we're going to be using the mathematical notation to represent a limit, which goes as follows:

$$\lim_{x \to a} f(x) = C$$

The number next to the arrow (which we're calling a) represents the value x is approaching, and next to "lim" is the function you are finding the limit of. And finally whatever value the limit is equal to goes on the other side of the equals sign. When you see this notation you would verbally say it as "The limit as x approaches a of $f(x)$ equals C"

If we want to refer to just a left sided limit or right sided limit specifically, then we can use the following notation:

$$\text{Left sided limit: } \lim_{x \to a^-} f(x) = C$$

$$\text{Right sided limit: } \lim_{x \to a^+} f(x) = C$$

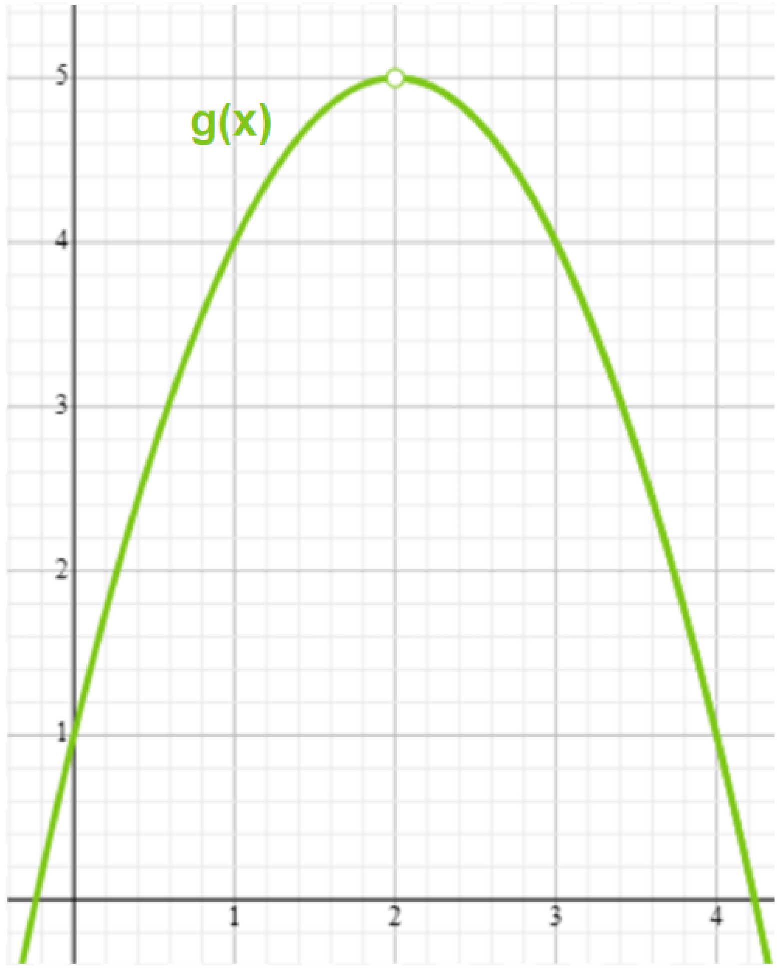
In our previous example we could say " $\lim_{x \to 0^-} f(x) = 2$ and $\lim_{x \to 0^+} f(x) = 2$. Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 2$, $\lim_{x \to 0} f(x) = 2$."

Why do we care about Limits?

In the previous example the idea of getting closer and closer to 2 may seem kinda redundant, don't we already know what the value of the function is at 2? Why do we need to get closer and closer to 2 if we can already just plug in 2? You're right, in this case it may seem unnecessary to engage in this kind of process, but the reason I'm showing this example is to introduce the idea of a limit within a simpler environment. The reason why limits are useful is because not all values that x approaches can be found by just plugging in the value. This is due to discontinuities, which we'll show off in the next examples.

Finding limits on discontinuous functions

Let's take the graph from our previous example and modify it slightly:



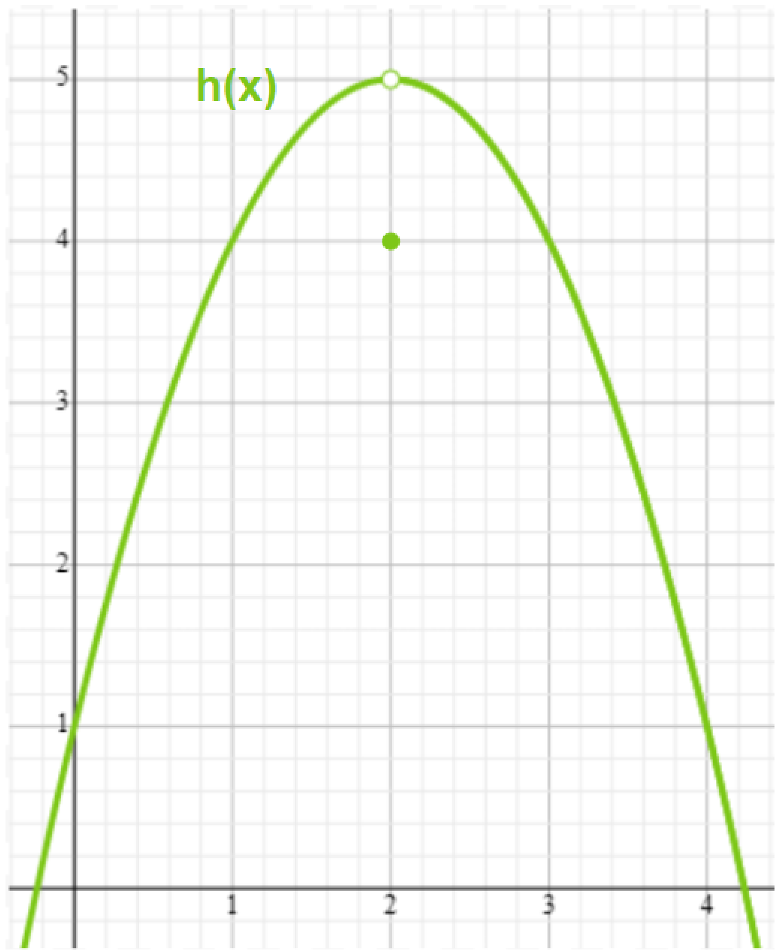
Let's call this new function $g(x)$. Now, what is $\lim_{x \to 2} g(x)$?

With this function we can't find the limit by just plugging in 2, since there's a hole in the graph when $x = 2$, meaning it's undefined at 2. But you can still find the limit. This is because a limit is not asking you what the value of the function is at 2, it's asking what value the output of the function approaches as the x value gets closer and closer to 2.

If you eyeball it, it seems that the output of the function is getting closer and closer to 5 as x gets closer and closer to 2.

Let's use the following animation to find the limit:
Since $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = 2$, $\lim_{x \to 0} g(x) = 2$.

Let's modify the graph again:



Let's call this function $h(x)$. Now once again, what is $\lim_{x\to 2} h(x)$?

This graph is almost the same as before, except now the function is defined at 0. In this case, $f(2) = 1$. What does the limit equal now?

{insert animation}

The answer is the same as before, $\lim_{x\to 0} h(x) = 2$. This is because although $f(0) = 1$, the output of the function does not approach 1 as x gets closer and closer to 0. It instead seems to get closer and closer to 2.

The point that I'm trying to hammer down with these examples is that a limit is not asking you what the value of a function is at a point, it's asking you what the value of the function approaches as you get closer and closer to that point. It doesn't matter whether or not the output at a certain x value is defined or undefined or even connected to the rest of the graph, a limit is only asking you how the function behaves when you get closer and closer to that x value without actually reaching it.

In the three examples I showed, the x value at 0 were all different but the limit as x approaches 0 for all three functions were exactly the same. This is because the value that the graphs approached as x got closer and closer to 0 were the same.

Key Takeaways

1. A limit is the output a function appears to get closer and closer to as x gets closer and closer to some value without reaching it.

2. The notation for a limit is $\lim\limits_{x \to a} f(x) = C$, where a is the value x is getting closer and closer to, $f(x)$ is your function, and C is the value the limit is equal to.

3. You can find a limit on a graph by eyeballing it and seeing what value the height of function seems to get closer and closer to as x gets closer and closer to a.

4. You can also find a limit by making a table and plugging in values slightly less than a and values slightly greater than a to see what values their outputs seem to be getting closer and closer to.