Limits Continued

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Introduction

In the previous section, I introduced the idea of a limit as a general concept. In this section, I'm going to show some more advanced examples. More specifically, I'll introduce the idea of a limit that does not exist, as well as limits when infinity is involved.

Review of One Sided Limits

There are two types of one sided limits, left sided limits and right sided limits. Let's say we have some function $f(x)$ and x is approaching some value a.

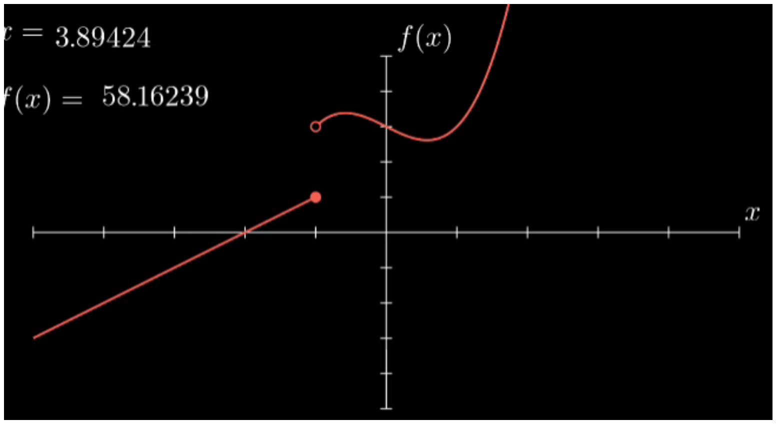
A left sided limit, or "the limit as x approaches a from the left", is the value the output of the function gets closer and closer to as x gets closer and closer to a from the left. In other words, if you plug in numbers less than a and make them bigger and bigger without reaching a, what value does the output of the function approach?

A right sided limit, or "the limit as x approaches a from the right", is the value the output of the function gets closer and closer to as x gets closer and closer to a from the right. In other words, if you plug in numbers bigger than a and make them smaller and smaller without reaching a, what value does the output of the function approach?

Example 1

Consider the following function $f(x)$, along with its graph below.

$$f(x) = \begin{cases} x+2 & x \leq -1 \\ x^3 - x + 3 & x > -1 \end{cases}$$



Given the above graph, what is the limit as x approaches -1 from the left and what is the limit as x approaches -1 from the right?

Now remember that a limit is not asking you what the value of the function is at -1, it's asking you what the value of the function approaches as x gets closer and closer to -1.

In this case it's first asking for the limit as x approaches -1 from the left, which is another way of saying "If you plug in x values less than -1 and make them bigger and bigger without reaching -1, what value does the output of $f(x)$ approach?" We can figure this out by watching the following animation:

{animation}

Thus, $\lim\limits_{x \to -1^-} f(x) = 1$

Next it's asking for the limit as x approaches -1 from the right, which is another way of saying, "If you plug in x values greater than -1 and make them smaller and smaller without reaching -1, what value does the output of $f(x)$ approach?" We can figure this out by watching the following animation:

{animation}
Thus, $\lim\limits_{x \to -1^+} f(x) = 3$

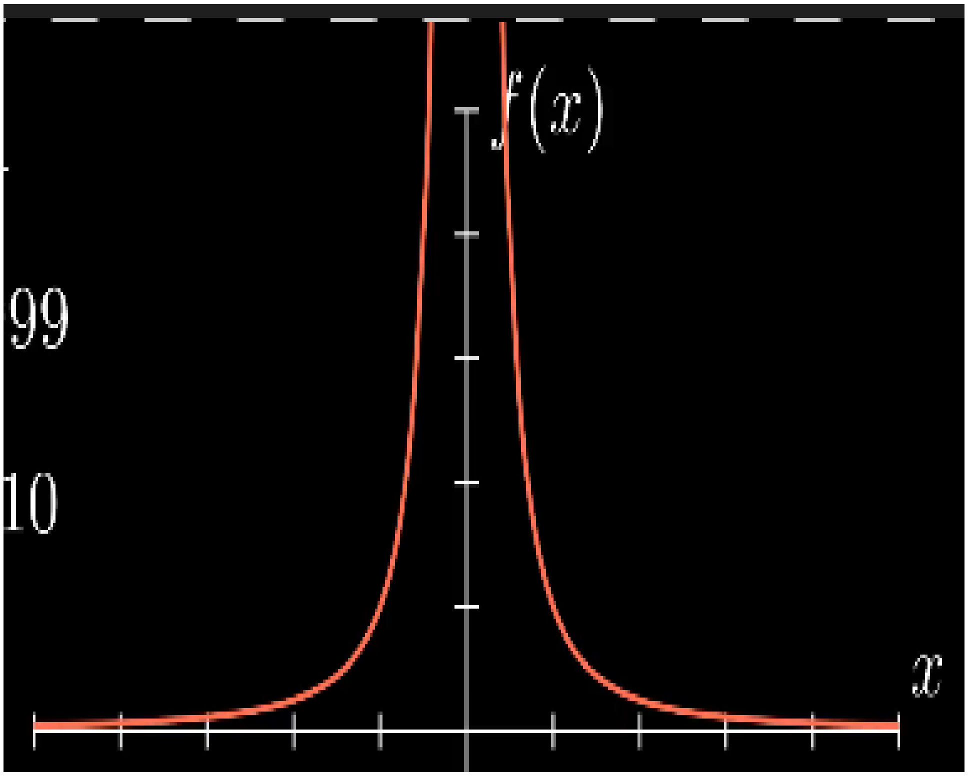
What does $\lim\limits_{x \to -1^+} f(x)$ equal? Well since $\lim\limits_{x \to -1^+} f(x) \neq \lim\limits_{x \to -1^+} f(x)$, we say that the limit "does not exist" or "DNE" for short.

Limits at infinity

Next, I'm going to be introducing the idea of a limits that involve infinity. More specifically, I'm going to be talking about what it means when a limit equals positive or negative infinity, as well as what it means when x approaches positive or negative infinity. Additionally, I'm going to be introducing a concept known as an asymptote, which is important when dealing with functions that go to infinity.

Limits that equal Positive/Negative Infinity

Consider the graph of the function $g(x) = \frac{1}{x^2}$:



What is $\lim\limits_{x \to 0} g(x)$?

Remember that to find $\lim\limits_{x \to a} g(x)$ you first must find $\lim\limits_{x \to a^+} g(x)$ and $\lim\limits_{x \to a^-} g(x)$, and if they equal each other, then $\lim\limits_{x \to a} g(x)$ equals that value, otherwise the limit does not exist.

Let's start by finding $\lim\limits_{x \to 0^-} g(x)$.

{animation}

Thus $\lim\limits_{x \to 0^-} g(x) = \infty$.

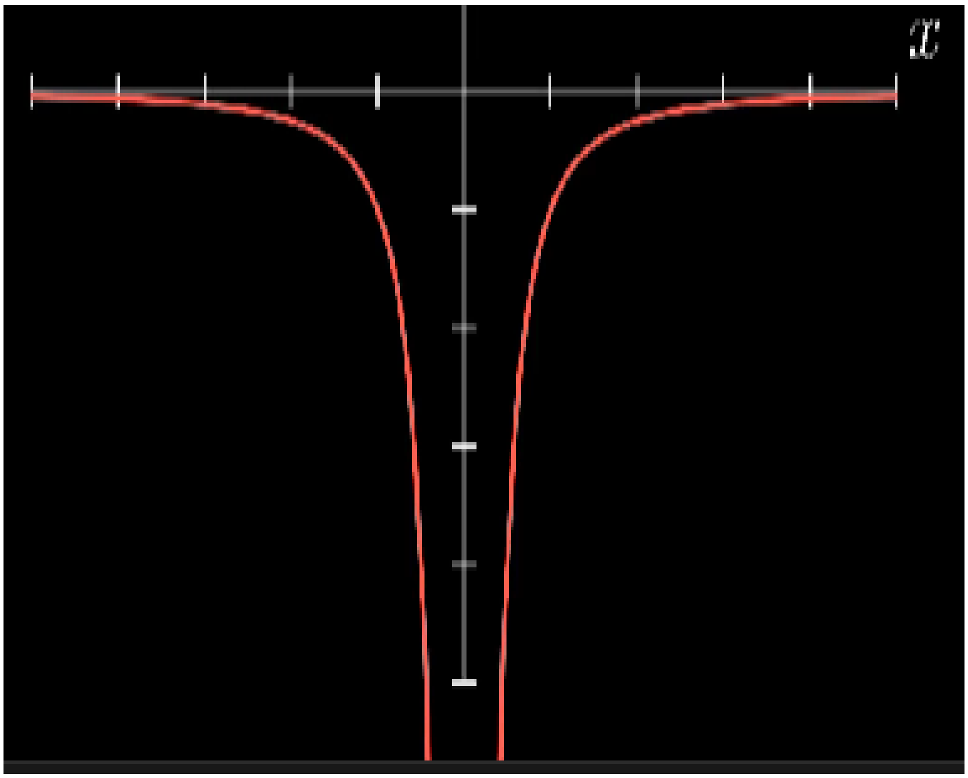
Next let's find $\lim\limits_{x \to 0^+} g(x)$

{animation}

Thus $\lim_{x \to 0^+} g(x) = \infty$

Since $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} g(x) = \infty$, $\lim_{x \to 0} g(x) = \infty$.

Next consider the graph of the function $h(x) = -\frac{1}{x^2}$



What is $\lim_{x \to 0} h(x)$?

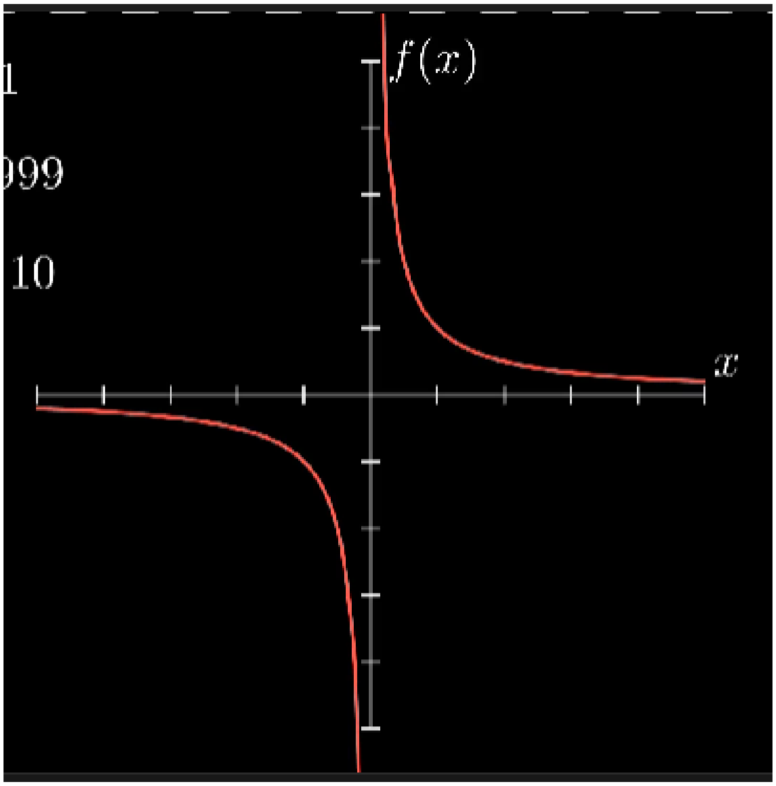
We can once again figure this out by finding $\lim_{x \to 0-} h(x)$ and $\lim_{x \to 0+} h(x)$

{animation}
Thus, $\lim_{x \to 0-} h(x) = -\infty$

{animation}
$\lim_{x \to 0+} h(x) = -\infty$

Since $\lim_{x \to 0-} h(x) = \lim_{x \to 0+} h(x) = -\infty$, $\lim_{x \to 0} h(x) = -\infty$

What if we had the function $j(x) = \frac{1}{x}$?



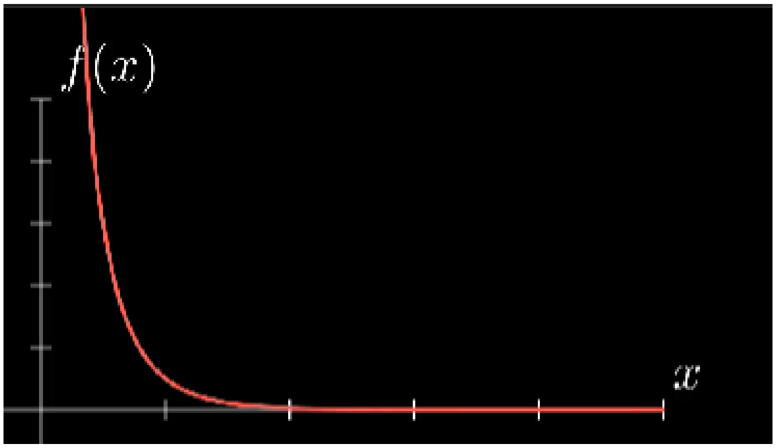
What is $\lim_{x \to 0} j(x)$?

We can figure this out by one again calculating the left and right sided limits:

{animation}
So $\lim_{x \to 0^-} j(x) = -\infty$ and $\lim_{x \to 0^+} j(x) = \infty$. Since the left and right sided limits do not equal each other, $\lim_{x \to 0} j(x)$ DNE.

Let's move onto another category of limits involving infinity. Instead of looking at limits that equal positive or negative infinity we're going to look at limits where the value x approaches is positive or negative infinity.

We'll once again use the function the function $h(x) = \frac{1}{x^2}$:



What is $\lim_{x\to\infty} h(x)$? Let's find out:

{animation}

Thus, $\lim_{x\to\infty} h(x) = 0$. This is because as x grows without bound, $h(x)$ gets extremely small. While the animation doesn't show it, you can think of it like this. For all numbers greater than zero, there exists a point on the function who's height is less than that number. Even if you pick an extremely small positive number (e.g. 0.000000001) there will eventually be a point where the height of $h(x)$ dips below that value, meaning the limit cannot equal any positive number. Thus it makes sense to say that $\lim_{x\to\infty} h(x) = 0$.

What about $\lim_{x\to-\infty} h(x)$?

{animation}
$\lim_{x\to\infty} h(x)$ also equals 0. This is the case for the exact same reasons as $\lim_{x\to\infty} h(x)$ but mirrored over the x-axis.