Contents

ECE 8 Robotics Notes 11-8-22

- Review last weeks Equilibrium notes
- Stability or Equilibrium point
- Recall the cruise control model
  - the fixed IC and find v_eq
  - is an attractive point, such that the model trajectories converge to this point
  - $v_eq = v_d$

Lecture 15:

Example 9: Ball moving on a Terrain

- Consider a ball that is initialized with zero velocity at a positive height and moves along the following terrain
- See Figure 1
- Assume friction does exist
  - if we have an initial energy, then that energy is consumed by the friction
  - aka the energy decreases due to friction Cases for the ball rolling: Depends on where the ball is initially released from
  - BUT friction can cause the ball from not reaching certain points along the terrain
- Equilibrium point of a ball in a "well" is at the bottom of the well
  - equilibrium point is a value of our key variable (in this case velocity or position or energy) and it remains there after a time, forever
  - if we push the ball in this case slightly, it still remains near the equilibrium point

Now consider a ball on a Hill

- See Figure 2
- there is an equilibrium point at the top of the hill but it is not very stable
  - any push will cause it to fall to either side of the hill
  - if the ball has no velocity at the top of the hill, then it will stay at the top for all time. This is why we call it an equilibrium point

Conclusions

- Ball in Well behavior is a stable equilibrium
  - trajectories stay near the eq. point

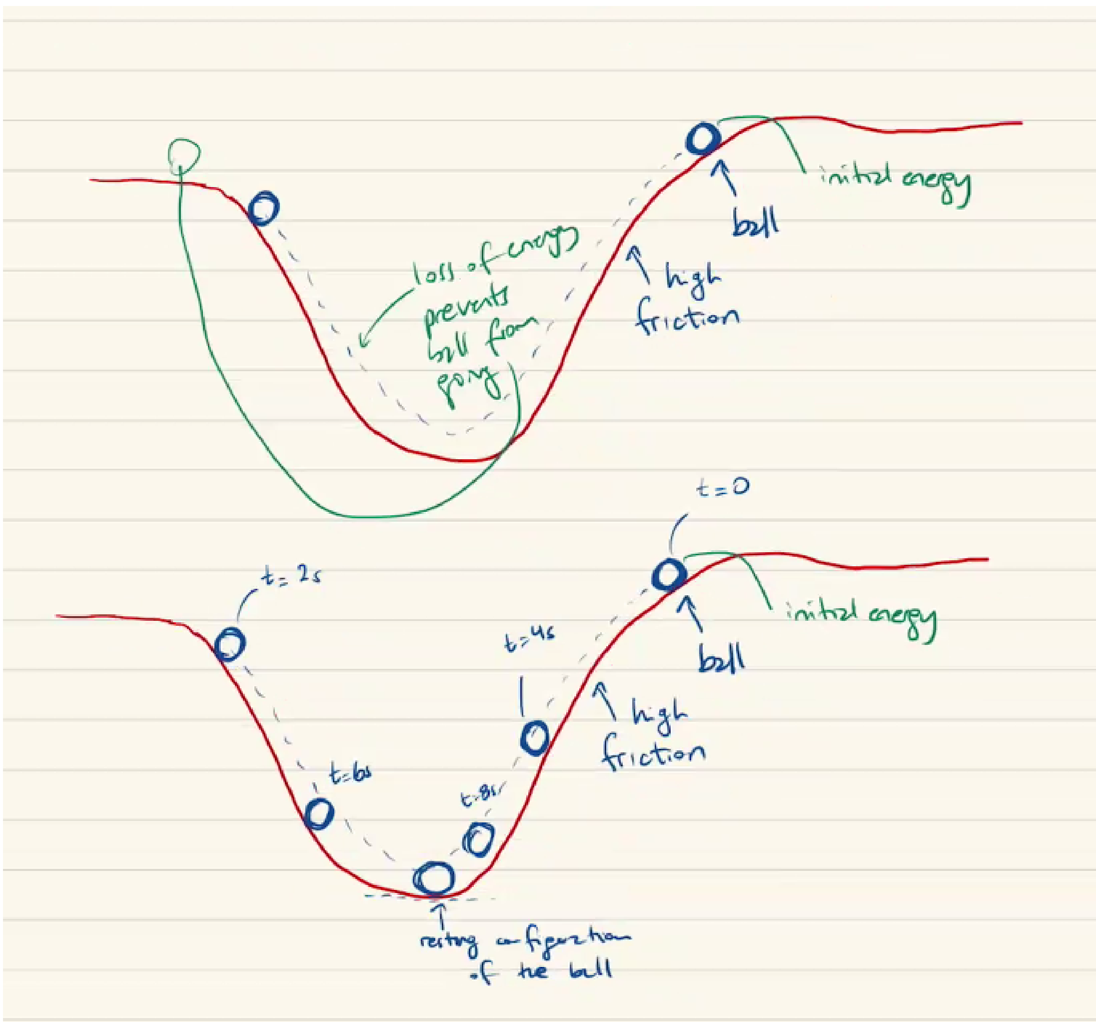

Figure 1: Ball rolling down a well has an equilibrium point at the bottom of the well. Any push about the equilibrium point will still keep the ball at the base of the well.

- Ball on Hill behavior is unstable equilibrium
  - trajectories do not remain near the eq. point

Definition of Stability of an Equilibrium point

Given a general discrete closed-loop system

$$x_{k+1} = f(x_k)$$

$f(x_k)$ models the dynamics of the full system.

A point x_{eq} is a stable equilibrium point if for all initial conditions (IC) nearby x_{eq} the resulting trajectories remain nearby x_{eq}.

Revisit Example 7 : Cruise Control

- From Lect. 14 We know that,
$$v_{eq} = v_d$$
- the equilibrium point is equal to our desired point, but is it stable or unstable?
- From our simulations during lecture 14 we had the following plots (see figure 3):

In Class Activities

`IMPORTANT: THIS IS GOING TO BE ON THE QUIZ`

1. Consider the discrete time model
$$x_{k+1} = \frac{1}{2}x_k$$

- a. Calculate the equilibrium point for the system
  - from lecture: 14 (See figure 4)
- Solution:
  - We have $x_{k+1} = f(x_k)$ but $f(x_k) = \frac{1}{2}x_k$ so we get $x_{eq} = x_k$ because (see Figure) $x_{eq} = f(x_{eq})$
  - To solve for the exact value we do the following:
$$x_{eq} = f(x_{eq}) = \frac{1}{2}x_{eq}x_{eq} = \frac{1}{2}x_{eq}$$
  Combine like terms
$$0 = \frac{1}{2}x_{eq} - x_{eq}0 = -\frac{1}{2}x_{eq}$$
  but this is only true (LHS = RHS) if $x_{eq} = 0$

`-b. Numerically: Is the equilibrium point attractive? (stable)`

- from lecture:14 (see figure 5)
- We need to show that every trajectory to $x_{k+1} = (1/2) \cdot x_k$ such that the limit
$$\lim_{k \to \infty} x_k = x_{eq}$$
  - Consider $x_0 = 1$ out IC, then we get
$$x_1 = 1/2x_2 = 1/4x_3 = 1/8 \dots x_k = 1/2^k$$
- test this:

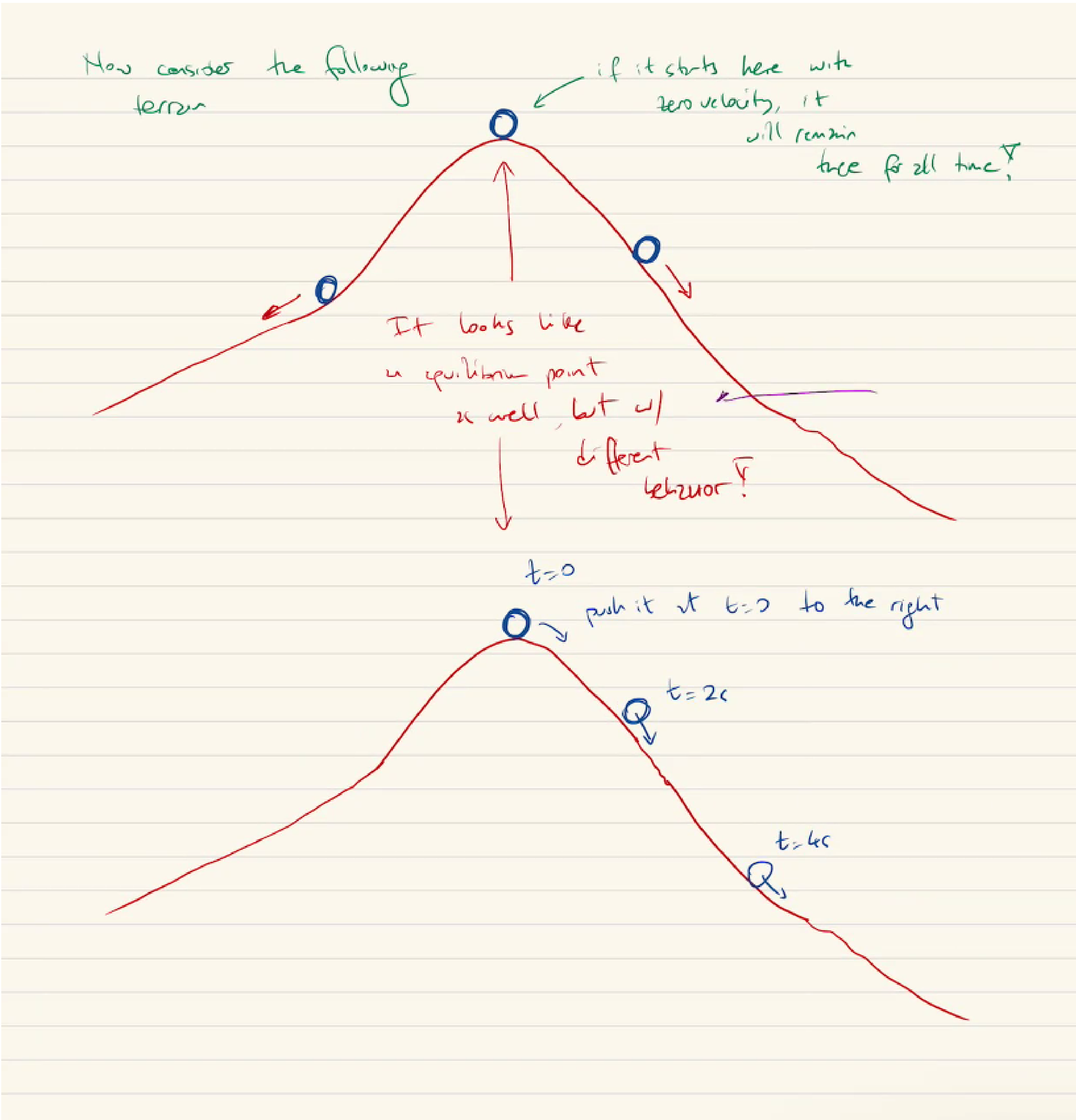

Figure 2: Ball on a hill. Equilibrium point is at the top of the hill but it is unstable. The ball falls to either side depending on a small push.

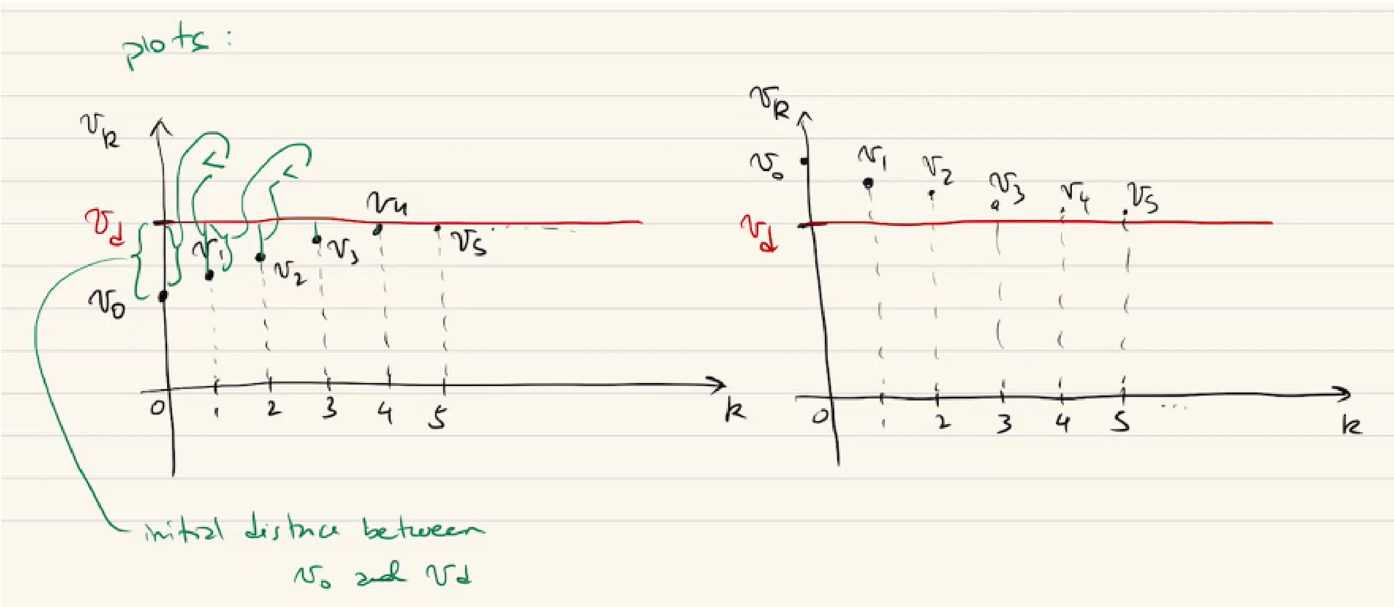

Figure 3: Plots of equilibrium velocity v_d and trajectories $v_0, v_1, \ldots$. Notice that the distance between the red line for v_d and th points in black are shrinking. This means the trajectories are converging to v_d which means this equilibrium point is stable.

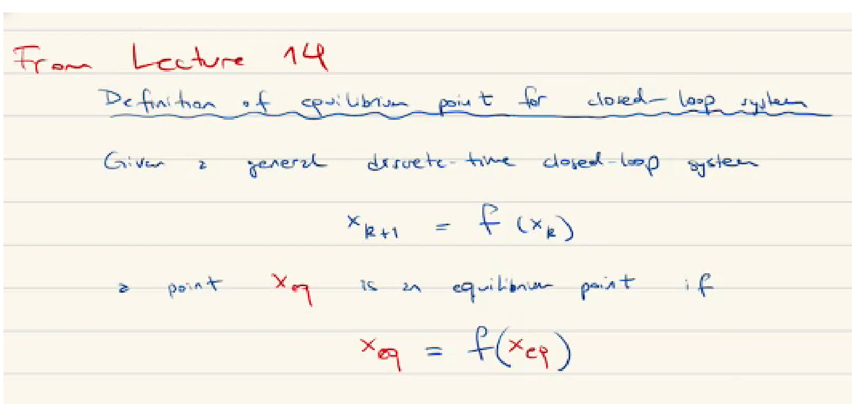

Figure 4: Notes from lecture 14, Used to solve this problem 1a

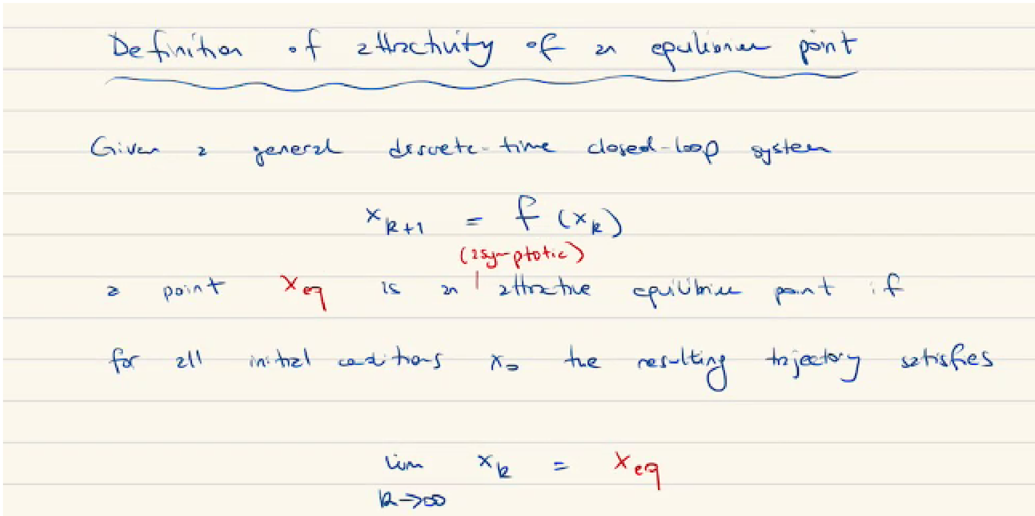

Figure 5: Notes from lecture 14, Used to solve this problem 1b

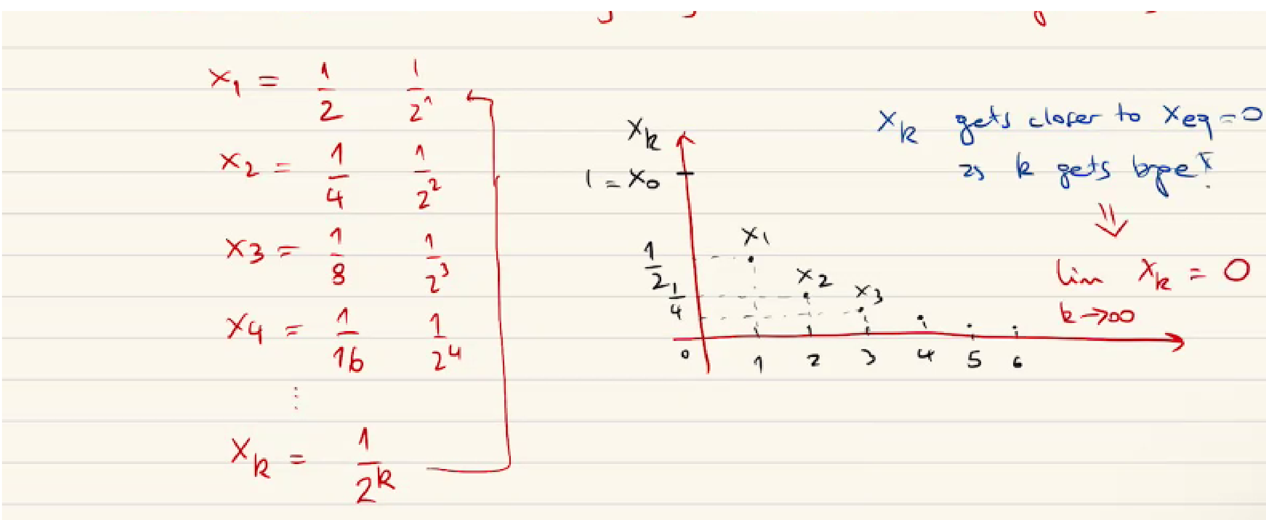

Figure 6: Diagram of Plotted points from Problem 1b. The limit as k goes to infinity.

- $k=0$ is 1/2, $k=1$ then 1/4, we can plot these values
- See Figure 6

- Want To Show: this equation is true regardless of what our IC is (x_0) (regardless from where we start)
- This is true, (proof not given, proof is only shown with the diagram),
- Due to how the points behave on our graph we see they "converge" to zero so then $x_{eq}=0$ is attractive

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- `c. Is the equilibrium point stable? (this follows from 1b)`
  - from lecture: 15
  - Solution:
    - The point $x_{eq}=0$ is stable because from the answer to 1b. we see that the initial distance of x_0 to x_{eq} is the largest distance along the trajectory.
    - if the trajectory starts close to x_{eq} then it remains close by x_{eq}

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2. Consider the cruise control model

$$v_{k+1} = v_k + \Delta \frac{-bv_k}{m}$$

- a. `Find` v_{eq}

`IMPORTANT: KNOW THIS FOR QUIZ`

- found in previous lecture
- value: (let $v_{eq}=v_k$)

$$v_{eq} = f(v_{eq})v_{eq} = v_k + \Delta \frac{-bv_k}{m} \dots v_{eq} = v_k = 0$$

- b. `is it attractive?`
  - Prove that for each IC v_0, we have

$$\lim_{k\to\infty} v_k = v_{eq} = 0$$

  - for the resulting trajectories of the given model, also given by v_k
  - To do this we use simulations
  - Q: what values of b, m, Δ should be used?
  - unless b, m, Δ are fixed we cannot answer this, because there are too many different systems to simulate

- c. `is it stable?`

`Same issue:`

- Q: what values of b, m, Δ should be used?
- unless b, m, Δ are fixed we cannot answer this, because there are too many different systems to simulate

Stability and attractively of an equilibrium point (special case)

If the discrete time model is of the form

$$x_{k+1} = a \cdot x_k$$

where $a \in \mathbf{R}$ and x_k is a trajectory with x_k a real number (not a vector of dimension 2 or larger) then the equilibrium point for the system

$$x_{k+1} = a \cdot x_k$$

is both stable and attractive (asymptotically stable) if

$$a \neq 0 |a| < 1 \text{absolute value of a is smaller than 1}$$

hold.

For the cruise control

$$v_{k+1} = v_k + \Delta\frac{-bv_k}{m}$$

$$v_{k+1} = v_k - (\Delta\frac{b}{m})v_k$$

Let $a = -(\Delta\frac{b}{m})$ and apply the conditions a mentioned above, and this will allow us to test our model's equilibrium points.

Closing Remarks

- Quiz on Thursday
- Think about/consider:
  - $x_{k+1} = 2x_k$
  - Determine eq. points
  - stability
  - attractively
  - notice this value is larger than 1, and may break the conditions we did previously
- No quiz until after Thanks giving
- A catch up quiz will come up and can be taken at any point before finals
- If we are not in person on Thursday we will have the quiz on canvas
- Lab is due Monday after Thanksgiving ! (More than 12 days to solve)
- 3 quizzes (2 required, 1 Extra credit)
- 2 more labs
- experiments on the last weeks of class

Lab 4 due: Nov. 28th

- Working with CoppeliaSim !
  - make sure it works before the holiday
- simulate the movement of quadcopter
- record data
- store data
- Use MATLAB to plot the trajectories
- this lab covers your ability to code, not the physics