

A decorative blue crosshair consisting of a vertical line and a horizontal line intersecting in the upper-left quadrant of the slide.

CS 228 : Logic in Computer Science

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First-Order Logic : Semantics

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 - ▶ For each k -ary relation R^k in the signature τ , a set of k -tuples from A^k is assigned to $R^{\mathcal{A}}$
 - ▶ The structure \mathcal{A} is finite if A (or $u(\mathcal{A})$) is finite

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 - ▶ $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.

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 - ▶ The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$,
 $Q_a^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, $Q_b^{\mathcal{W}} = \{2, 3, 5, 7\}$,
 - ▶ $<^{\mathcal{W}} = \{(0, 1), (0, 2), \dots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \dots, (7, 8)\}$
uniquely defines the word $W = aabbababa$.
 - ▶ For convenience, we can just write the word instead of the structure.

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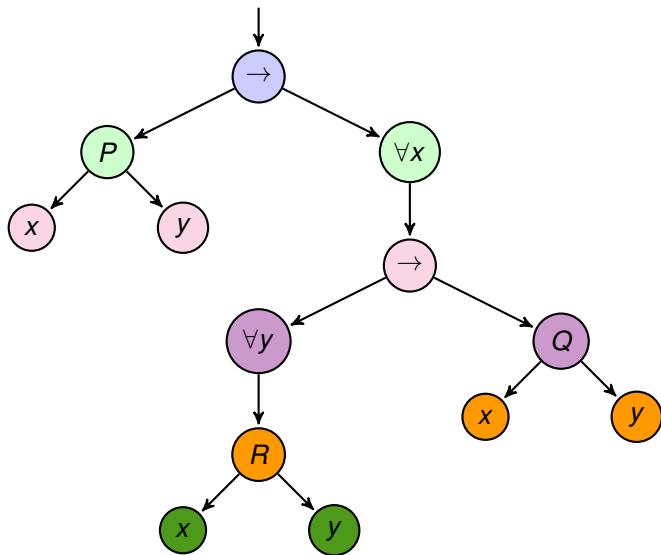
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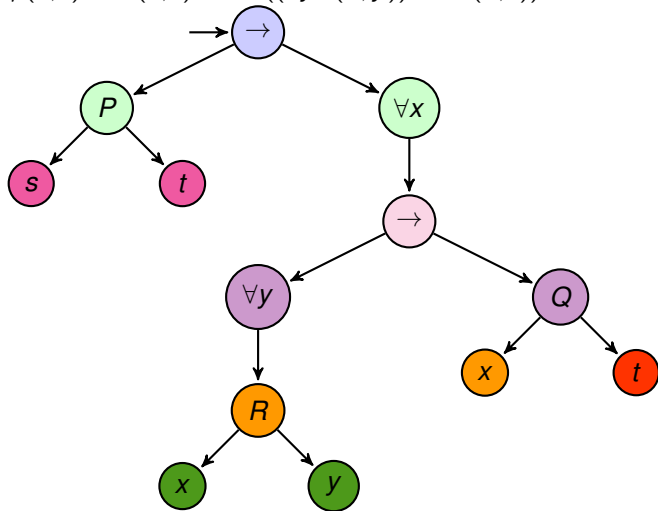
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 - ▶ x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$
- ▶ Given φ , denote by $\varphi(x_1, \dots, x_n)$, that x_1, \dots, x_n are the free variables of φ , also $\text{free}(\varphi)$
- ▶ A **sentence** is a formula φ **none** of whose variables are **free**

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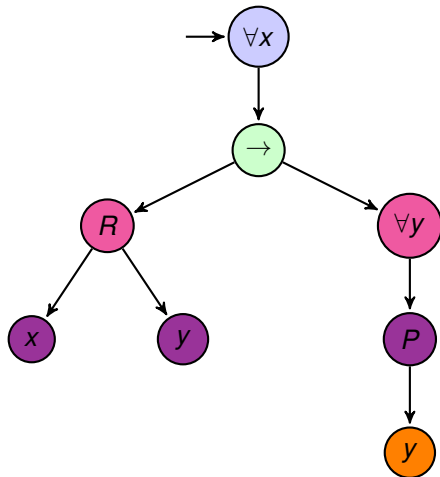


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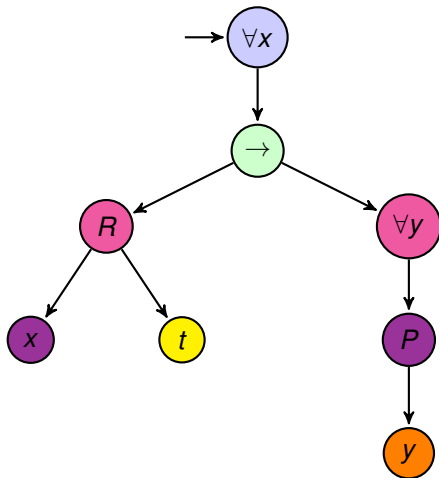
$$\varphi(s, t) = P(s, t) \rightarrow \forall x((\forall y R(x, y)) \rightarrow Q(x, t))$$



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$$\varphi(t) = \forall x(R(x, t) \rightarrow \forall yP(y))$$

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For a τ -structure \mathcal{A} , an assignment over \mathcal{A} is a function $\alpha : \mathcal{V} \rightarrow u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c , then $\alpha(t)$ is $c^{\mathcal{A}}$

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Binding on a Variable

For an assignment α over \mathcal{A} , $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), & y \neq x, \\ a, & y = x \end{cases}$$

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- ▶ $\mathcal{A} \models_{\alpha} (\varphi \rightarrow \psi)$ iff $\mathcal{A} \not\models_{\alpha} \varphi$ or $\mathcal{A} \models_{\alpha} \psi$

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- ▶ $\mathcal{A} \models_{\alpha} (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- ▶ $\mathcal{A} \models_{\alpha} (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x . Thus, assignments matter **only** to free variables.