Descriptive Statistics

Fall 2024

Instructor:

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Thanks to Prof Ajit Rajawade for the initial version of the slides

Terminology

- **Population**: The collection of all elements which we wish to study, example: data about occurrence of tuberculosis all over the world
- In this case, "population" refers to the set of people in the entire world.
- The population is often too large to examine/study.
- So we study a subset of the population – called as a **sample**.
- In an experiment, we basically collect **values** for one or more **attributes or variables** of each member of the sample.

Examples of samples

variable / attribute

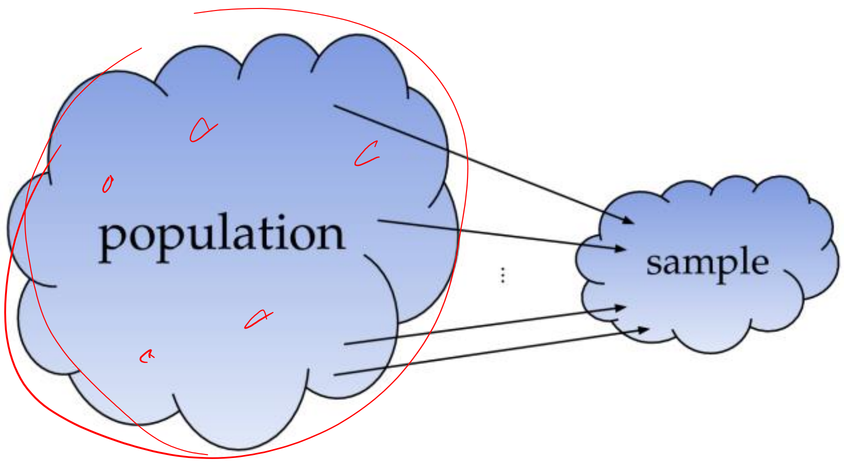
index	username	country	age	ezlvl	time	points	finished
0	mary	us	38	0	124.94	418	0
1	jane	ca	21	0	331.64	1149	1
2	emil	fr	52	1	324.61	1321	1
3	ivan	ca	50	1	39.51	226	0
4	hasan	tr	26	1	253.19	815	0
5	jordan	us	45	0	28.49	206	0
6	sanjay	ca	27	1	585.88	2344	1
7	lena	uk	23	0	408.76	1745	1
8	shuo	cn	24	1	194.77	1043	0
9	r0byn	us	59	0	255.55	1102	0
10	anna	pl	18	0	303.66	1209	1
11	joro	bg	22	1	381.97	1491	1

observation

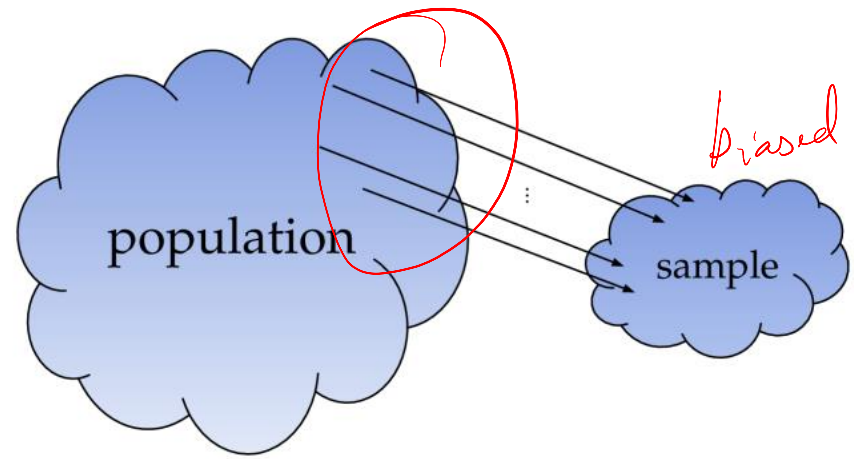
Sample

Table 1.1: A data table that contains observations of seven variables for 12 players of a computer game. Each row in this table corresponds to one player. Each column corresponds to one characteristic that was measured for all the players.

Population and Samples



(a) Representative sample selection



(b) Biased sample selection

Data Representation and Visualization

Need for data visualization

- The raw dataset or tables may be too large. Cannot make sense of the data just by inspecting raw table of numbers.

- Even if data is not too large, patterns emerge sometimes only under right type of visualization.

Outline

- Visualizing values of each variable separately
- Visualizing pairs of variables.
- Multi-dimensional data

Terminology

- **Discrete data:** Data whose values are restricted to a finite or countably infinite set. Eg: letter grades at IITB, genders, marital status (single, married, divorced), income brackets in India for tax purposes

- **Continuous data:** Data whose values belong to an uncountably infinite set (Eg: a person's height, temperature of a place, speed of a car at a time instant).

Raw data

- Example: Country of winners of any competition
- Example: Grades of students in CS 215

21 AA

25 AB

01 AP

09 BB

11 BC

02 DX

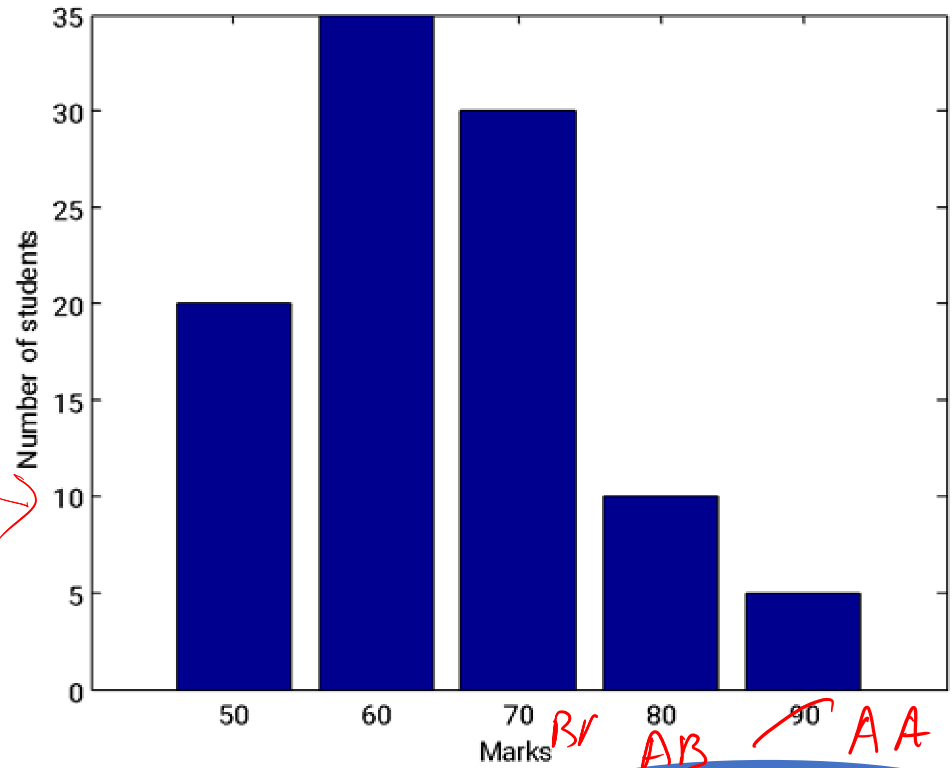
03 CC

24 AA

Frequency Tables

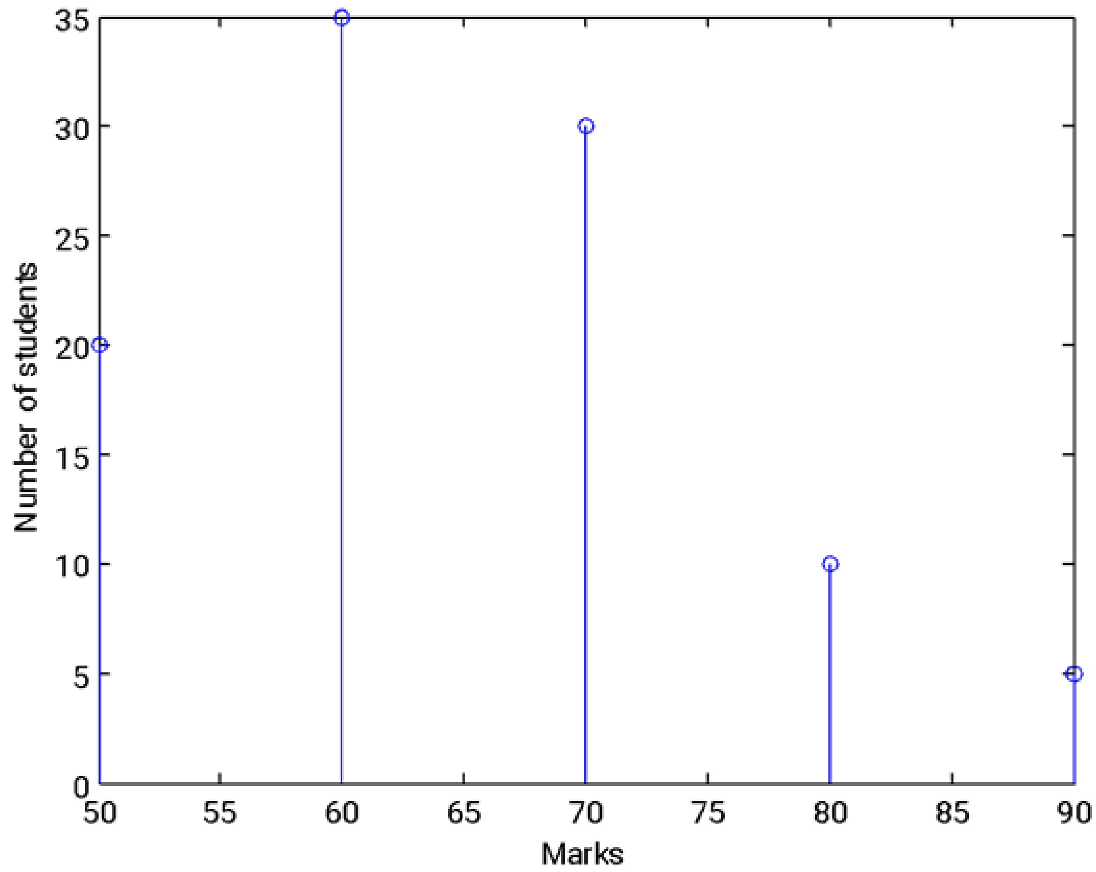
- The frequency table can be visualized using a **line graph** or a **bar graph** or a **frequency polygon**.

Grade	Number of students
AA	5
AB	10
BB	30
BC	35
CC	20



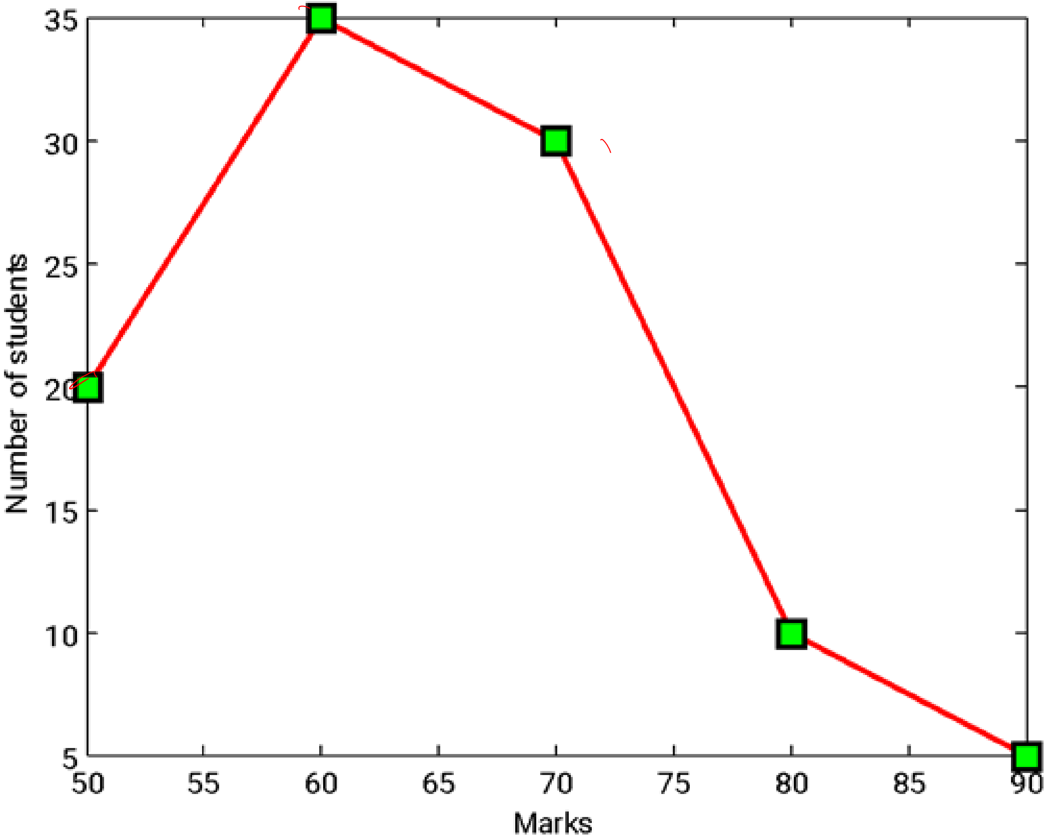
A **bar graph** plots the distinct data values on the X axis and their frequency on the Y axis by means of the height of a thick vertical bar!

Grade	Number of students
AA	5
AB	10
BB	30
BC	35
CC	20



A **line diagram** plots the distinct data values on the X axis and their frequency on the Y axis by means of the height of a vertical line!

Grade	Number of students
AA	5
AB	10
BB	30
BC	35
CC	20



A **frequency polygon** plots the frequency of each data value on the Y axis, and connects consecutive plotted points by means of a line.

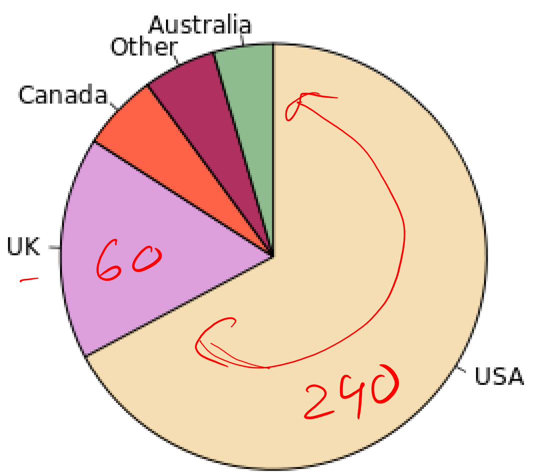
Relative frequency tables

- Sometimes the actual frequencies are not important.
- We may be interested only in the *percentage* or *fraction* of those frequencies for each data value – i.e. *relative frequencies*.

Grade	Fraction of number of students
AA	0.05
AB	0.10
BB	0.30
BC	0.35
CC	0.20

Pie charts

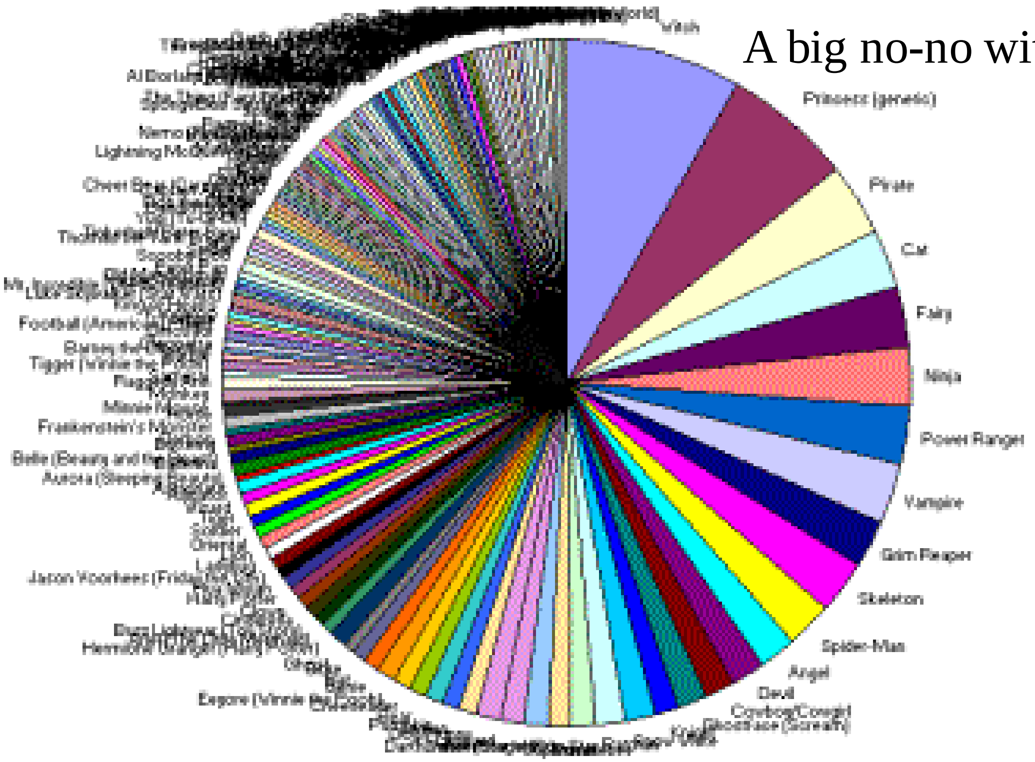
- For a small number of distinct data values which are non-numerical, one can use a **pie-chart** (it can also be used for numerical values).
- It consists of a circle divided into sectors corresponding to each data value.
- The area of each sector = relative frequency for that data value.



Population of native English speakers:
https://en.wikipedia.org/wiki/Pie_chart

$$\frac{240}{360} = \frac{2}{3}$$

Pie charts can be confusing



A big no-no with too many categories.

http://stephenturbek.com/articles/2009/06/better-charts-from-simple-questions.html

Dealing with continuous data

	Marks
23	85
01	20
03	65
10	75

- Example: temperature of a place at a time instant, speed of a car at a given time instant, weight or height of an animal, etc.

continuous

- The raw data: marks in final exams.

Visualizing numerical data

- Reduce to a known problem
  - Group into bins/intervals
  - Draw histograms of each bin.

Dealing with continuous data

- Let the sample points be $\{x_i\}$, $1 <= i <= N$.
- Let there be some K ($K << N$) bins, where the j^{th} bin has interval $[a_j, b_j)$.
- Thus frequency f_j for the j^{th} bin is defined as follows:

$$f_j = |\{x_i : a_j \leq x_i < b_j, 1 \leq i \leq N\}|$$

- Such frequency tables are also called **histograms** and they can also be used to store relative frequency instead of frequency.

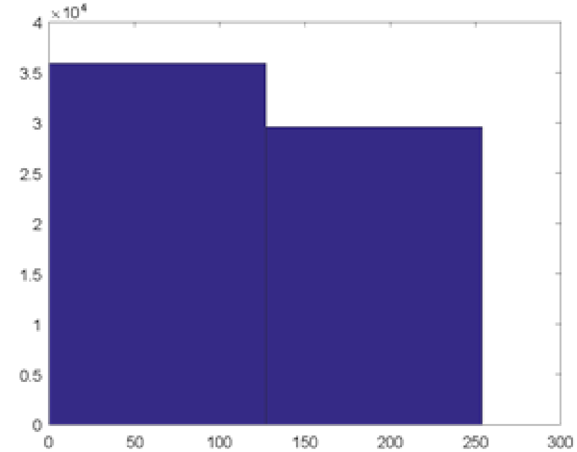
The histogram binning problem

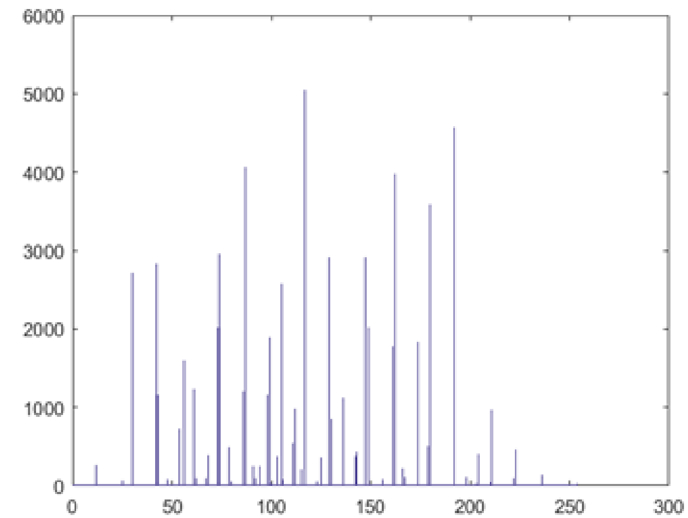
- If you have too few bins (each bin is very wide), there is very little idea you get about the data distribution from the histogram.

- Extreme: only one bin to represent all intensities in an image.

- If you have many bins (all will be narrow), then there are very points falling into each bin. Again there is very little idea you get about the data distribution from the histogram.

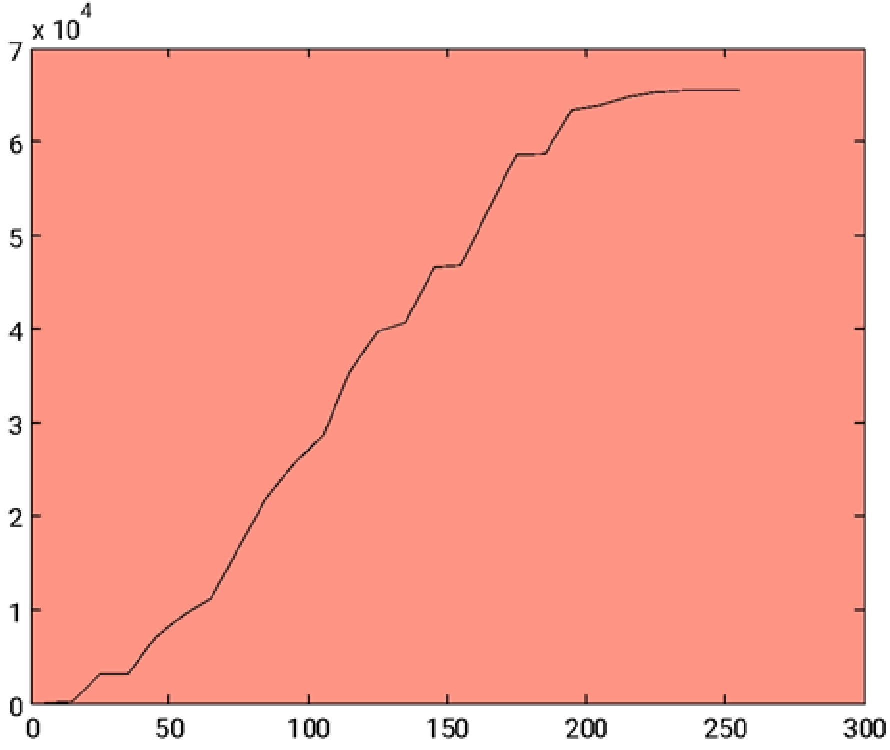
- Extreme: For intensities from a 512 x 512 image, if you had 512^2 histogram bins.





Cumulative frequency plot

- The **cumulative** (relative) **frequency plot** tells you the (proportion) number of sample points whose value is *less than or equal to* a given data value.



The cumulative frequency plot for the frequency plot from two slides back!

Summarizing Data

08	02	22	97	38	15	00	40	00	75	04	05	07	78	52	12	50	77	91	08
49	49	99	40	17	81	18	57	60	87	17	40	98	43	69	48	04	56	62	00
81	49	31	73	55	79	14	29	93	71	40	67	53	88	30	03	49	13	36	65
52	70	95	23	04	60	11	42	69	24	68	56	01	32	56	71	37	02	36	91
22	31	16	71	51	67	63	89	41	92	36	54	22	40	40	28	66	33	13	80
24	47	32	60	99	03	45	02	44	75	33	53	78	36	84	20	35	17	12	50
32	98	81	28	64	23	67	10	26	38	40	67	59	54	70	66	18	38	64	70
67	26	20	68	02	62	12	20	95	63	94	39	63	08	40	91	66	49	94	21
24	55	58	05	66	73	99	26	97	17	78	78	96	83	14	88	34	89	63	72
21	36	23	09	75	00	76	44	20	45	35	14	00	61	33	97	34	31	33	95
78	17	53	28	22	75	31	67	15	94	03	80	04	62	16	14	09	53	56	92
16	39	05	42	96	35	31	47	55	58	88	24	00	17	54	24	36	29	85	57
86	56	00	48	35	71	89	07	05	44	44	37	44	60	21	58	51	54	17	58
19	80	81	68	05	94	47	69	28	73	92	13	86	52	17	77	04	89	55	40
04	52	08	83	97	35	99	16	07	97	57	32	16	26	26	79	33	27	98	66
88	36	68	87	57	62	20	72	03	46	33	67	46	55	12	32	63	93	53	69
04	42	16	73	38	25	39	11	24	94	72	18	08	46	29	32	40	62	76	36
20	69	36	41	72	30	23	88	34	62	99	69	82	67	59	85	74	04	36	16
20	73	35	29	78	31	90	01	74	31	49	71	48	86	81	16	23	57	05	54
01	70	54	71	83	51	54	69	16	92	33	48	61	43	52	01	89	19	67	48

Summarizing a sample-set

- There are some values that can be considered "representative" of the entire sample-set. Such values are called as a "statistic".
- The most common statistic is the sample (arithmetic) **mean**:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- It is basically what is commonly regarded as "average value".

Summarizing a sample-set

- Another common statistic is the sample **median**, which is the “middle value”.

- We sort the data array **A** from smallest to largest. If N is odd, then the median is the value at the $(N+1)/2$ position in the sorted array.

- If N is even, the median can take any value in the interval $(A[N/2], A[N/2+1])$ – why?

Properties of the mean and median

- Consider each sample point x_i were replaced by $ax_i + b$ for some constants a and b.

- What happens to the mean? What happens to the median?

- Consider each sample point x_i were replaced by its square.

- What happens to the mean? What happens to the median?

Properties of the mean and median

- ***Question:*** Consider a set of sample points x_1, x_2, ..., x_N. For what value *y*, is the sum total of the **squared** difference with every sample point, the least? That is, what is:

$$\arg\min_y \sum_{i=1}^{N} (y - x_i)^2 \, ?$$

Total squared deviation (or total squared loss)

Answer: mean

- ***Question:*** For what value *y*, is the sum total of the **absolute** difference with every sample point, the least? That is, what is:

$$\arg\min_y \sum_{i=1}^{N} | y - x_i | \, ?$$

Total absolute deviation (or total absolute loss)

Answer: median

Properties of the mean and median

- The mean need not be a member of the original sample-set.

- The median is always a member of the original sample-set if N is odd.

- The median is not unique and will not be a member of the set if N is even.

Properties of the mean and median

- Consider a set of sample points $x_1, x_2, \ldots, x_N$. Let us say that some of these values get grossly corrupted.
- What happens to the mean?
- What happens to the median?

Example

- Let A ={1,2,3,4,6}
- Mean (A) = 3.2, median (A) = 3
- Now consider A = {1,2,3,4,20}
- Mean (A) = 6, median(A) = 3.

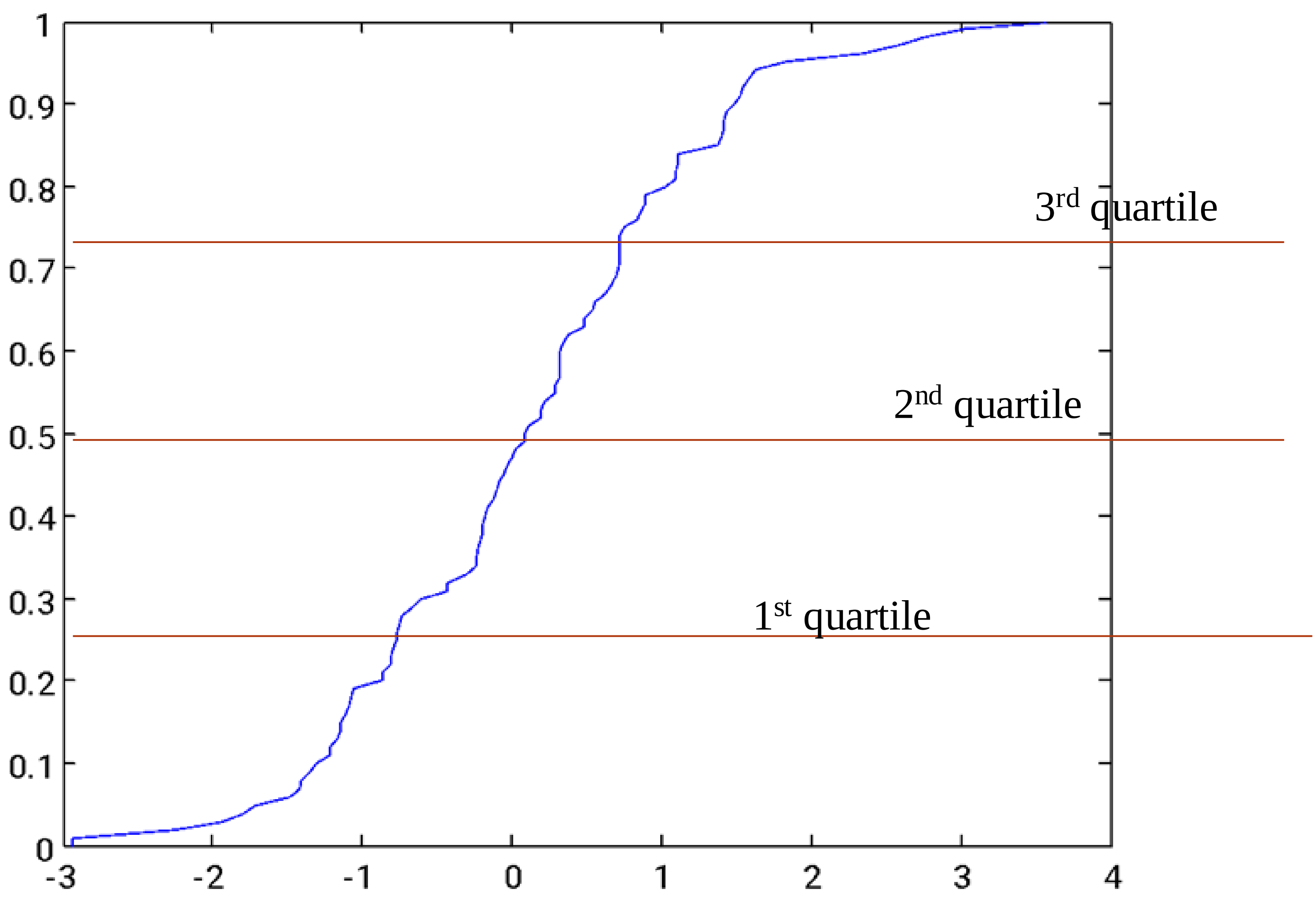
Concept of percentiles

- The sample 100p percentile ($0 \le p \le 1$) is defined as the data value y such that 100p% of the data have a value less than or equal to y, and 100(1-p)% of the data have a larger value.

- For a data set with n sample points, the sample 100p percentile is that value such that at least np of the values are less than or equal to it. And at least n(1-p) of the values are greater than it.

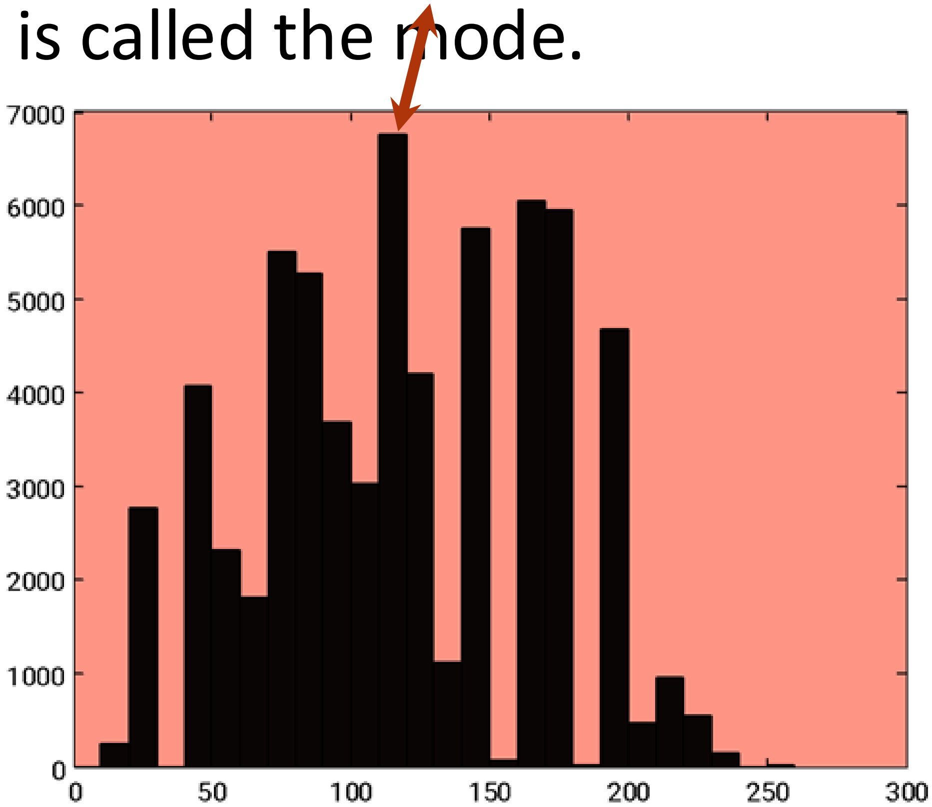
Concept of quantiles

- The sample 25 percentile = first quartile.
- The sample 50 percentile = second quartile.
- The sample 75 percentile = third quartile.
- Quantiles can be inferred from the cumulative relative frequency plot (how?).
- Or by sorting the data values (how?).


3rd quartile
2nd quartile
1st quartile
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-3
-2
-1
0
1
2
3
4

Concept of mode

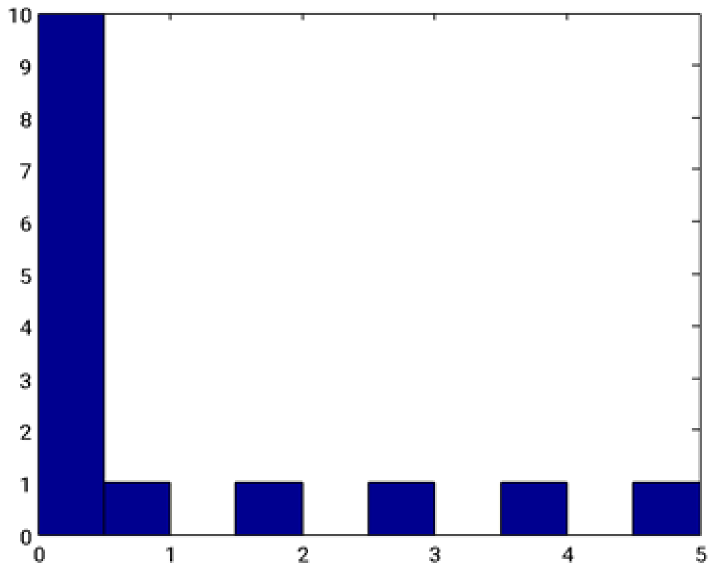
- The value that occurs with the highest frequency is called the mode.

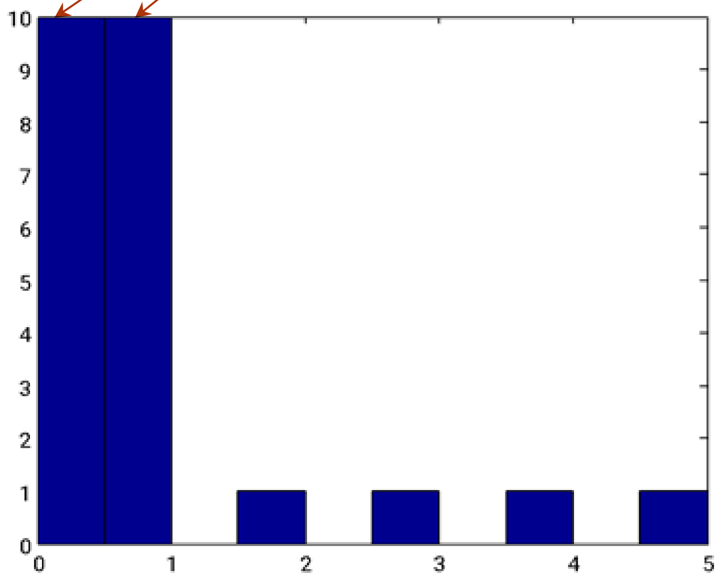


Concept of mode

- The mode may not be unique, in which case all the highest frequency values are called **modal values**.

Mode at 0





Histogram for finding mean

- Given the histogram, the mean of a sample can be approximated as follows:

$$\bar{x} \approx \frac{\sum_{j=1}^{K} f_j (a_j + b_j)/2}{N}$$

- Here f_j is the frequency of the *j*th bin.

Histogram for finding median

- Given the histogram, the median of a sample is the value at which you can split the histogram into two regions of equal areas.

- Keep adding areas from the leftmost bins till you reach more than $N/2$ – now you know the bin in which the median will lie – the median is the midpoint of the bin.

- More useful for histograms whose “bins” contain single values.

Variance and Standard deviation

- The **variance** is (approximately) the average value of the squared distance between the sample points and the sample mean. The formula is:

$$\text{variance} = s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(\bar{x} - x_i)^2$$

The division by N-1 instead of N is for a very technical reason which we will understand after many lectures. As such, the variance is computed usually when N is large so the numerical difference is not much.

- The variance measures the "spread of the data around the sample mean".
- Its positive square-root is called as the **standard deviation**.

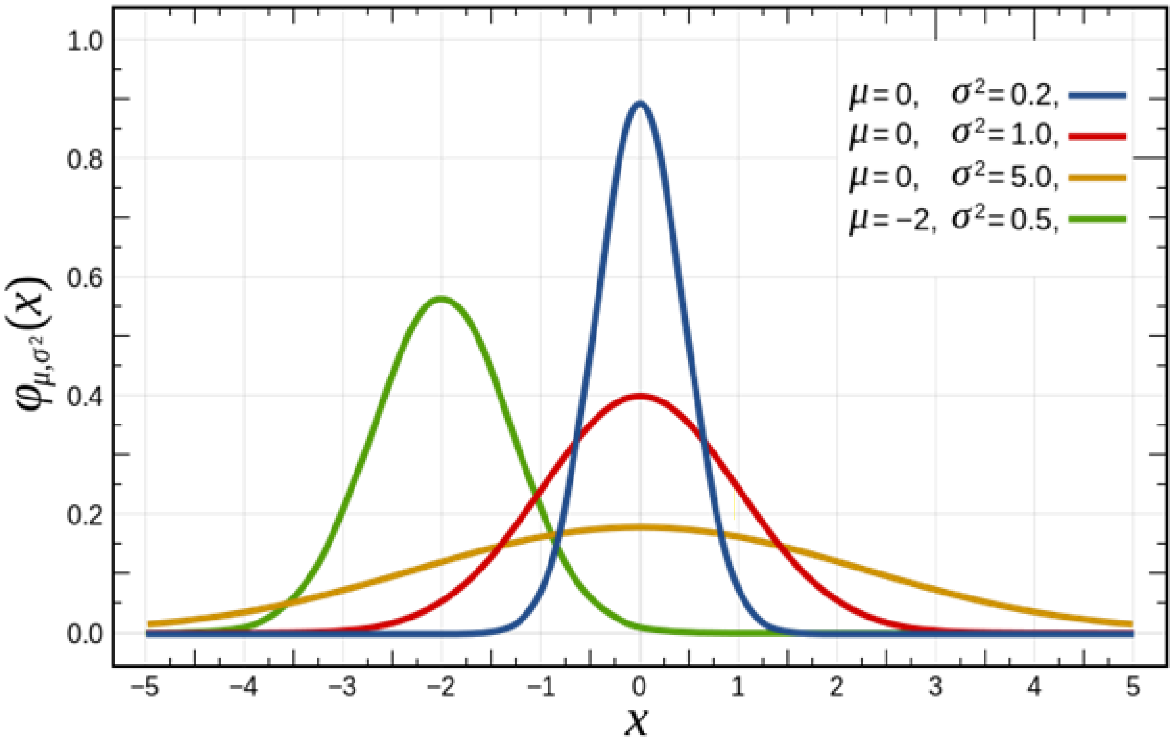


Image source

Variance and Standard deviation: Properties

- Consider each sample point x_i were replaced by $ax_i + b$ for some constants a and b. What happens to the standard deviation?

Standard deviation: practical application 1

- Let us say a factory manufactures a product which is required to have a certain weight *w*.
- In practice, the weight of each instance of the product will deviate from *w*.
- In such a case, we need to see whether the average weight is close to (or equal to *w*).
- But we also need to see that the standard deviation is small.
- In fact, the standard deviation can be used to predict how likely it is that the product weight will deviate significantly from the mean.

Standard deviation: practical application 2

- In the definition of diseases such as osteoporosis (low bone density)
- A person whose bone density is less than 2.5σ below the average bone density for that age-group, gender and geographical region, is said to be suffering from osteoporosis. Here σ is the standard de[illegible] bone density of that particular po[illegible]

Image source

Chebyshev's inequality

- Suppose I told you that the average marks for this course was 75 (out of 100). And that the variance of the marks was 25.

- Can you say something about how many students secured marks from 65 to 85?

- You obviously cannot predict the exact number – but you can say **something** about this number.

- That something is given by Chebyshev's inequality.

Chebyshev's inequality: and Chebyshev

https://en.wikipedia.org/wiki/Pafnuty_Chebyshev

Russian mathematician:
Stellar contributions in probability and statistics, geometry, mechanics

Two-sided Chebyshev's inequality:

The proportion of sample points k or more than k $(k>0)$ standard deviations away from the sample mean is less than $1/k^2$.

Chebyshev's inequality: and Chebyshev

Two-sided Chebyshev's inequality:

The proportion of sample points *k* or more than *k (k>0)* standard deviations away from the sample mean is less than or equal to $1/k^2$.

$$S_k = \{x_i : |x_i - \overline{x}| \geq k\sigma\}$$

$$\frac{|S_k|}{N} < \frac{1}{k^2}$$

Proof: on the board!
And in the book.

Chebyshev's inequality

- Applying this inequality to the previous problem, we see that the fraction of students who got less than 65 or more than 85 marks is as follows:

$$S_k = \{x_i : |x_i - \bar{x}| \geq k\sigma\}$$

$$\frac{|S_k|}{N} \leq \frac{1}{k^2}$$

$$\bar{x} = 75$$

$$\sigma = 5$$

$$k = 2$$

$$\frac{|S_k|}{N} \leq \frac{1}{4}$$

- So the fraction of students who got from 65 to 85 is more than 1-0.25 = 0.75.

Chebyshev's inequality

1	Kerala	93.91
2	Lakshadweep	92.28
3	Mizoram	91.58
4	Tripura	87.75
5	Goa	87.40
6	Daman & Diu	87.07
7	Puducherry	86.55
8	Chandigarh	86.43
9	Delhi	86.34
10	Andaman & Nicobar Islands	86.27
11	Himachal Pradesh	83.78
12	Maharashtra	82.91

Mean = 87.69
Std. dev. = 3.306

Fraction of states with literacy rate in the range $(\mu-1.5\sigma, \mu+1.5\sigma)$ is $11/12 \approx 91\%$

As predicted by Chebyshev's inequality, it is **at least** $1-1/(1.5*1.5) \approx 0.55$

The bounds predicted by this inequality are loose – but they are correct!

https://en.wikipedia.org/wiki/Indian_states_ranking_by_literacy_rate

One-sided Chebyshev's inequality

- Also called the Chebyshev-Cantelli inequality.

The proportion of sample points k or more than k *(k>0)* standard deviations away from the sample mean **and greater than the sample mean** is less than or equal to $1/(1+k^2)$.

$$S_k = \{x_i : x_i - \bar{x} \geq k\sigma\}$$

$$\frac{|S_k|}{N} \leq \frac{1}{1+k^2}$$

Notice: no absolute value!

Proof: on the board!
And in the book.

One-sided Chebyshev's inequality (Another form)

- Also called the Chebyshev-Cantelli inequality.

The proportion of sample points k or more than k *(k>0)* standard deviations away from the sample mean **and less than the sample mean** is less than or equal to $1/(1+k^2)$.

$$S_k = \{x_i : x_i - \bar{x} \le -k\sigma\}$$

$$\frac{|S_k|}{N} \le \frac{1}{1+k^2}$$

Notice: no absolute value!

Proof: on the board!
And in the book.

Correlation between different data values

- Sometimes each sample-point can have a pair of attributes.

- And it may so happen that large values of the first attribute are accompanied with large (or small) values of the second attribute for a large number of sample-points.

Correlation between different data values

- Example 1: Populations with higher levels of fat intake show higher incidence of heart disease.
- Example 2: People with higher levels of education often have higher incomes.
- Example 3: Literacy Rate in India as a function of time?

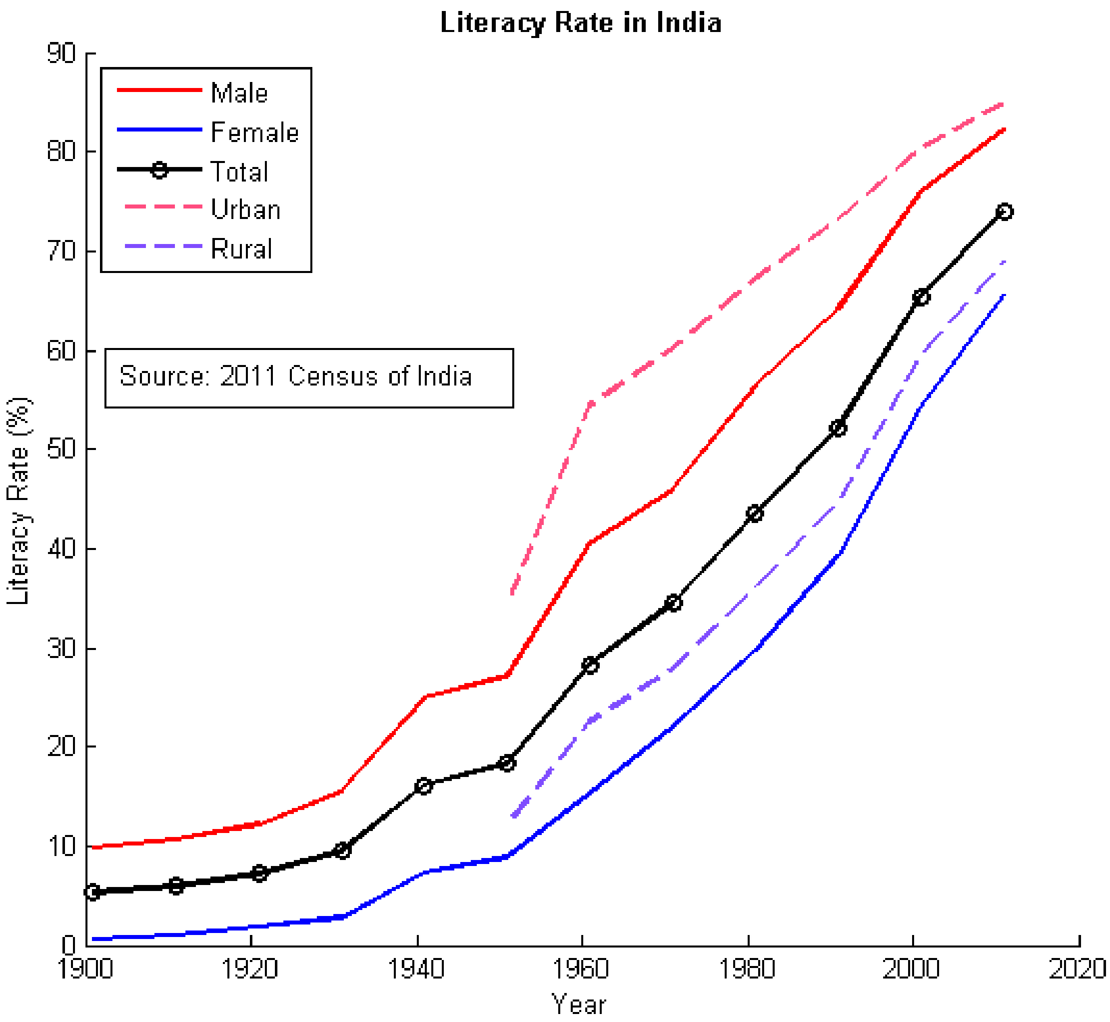


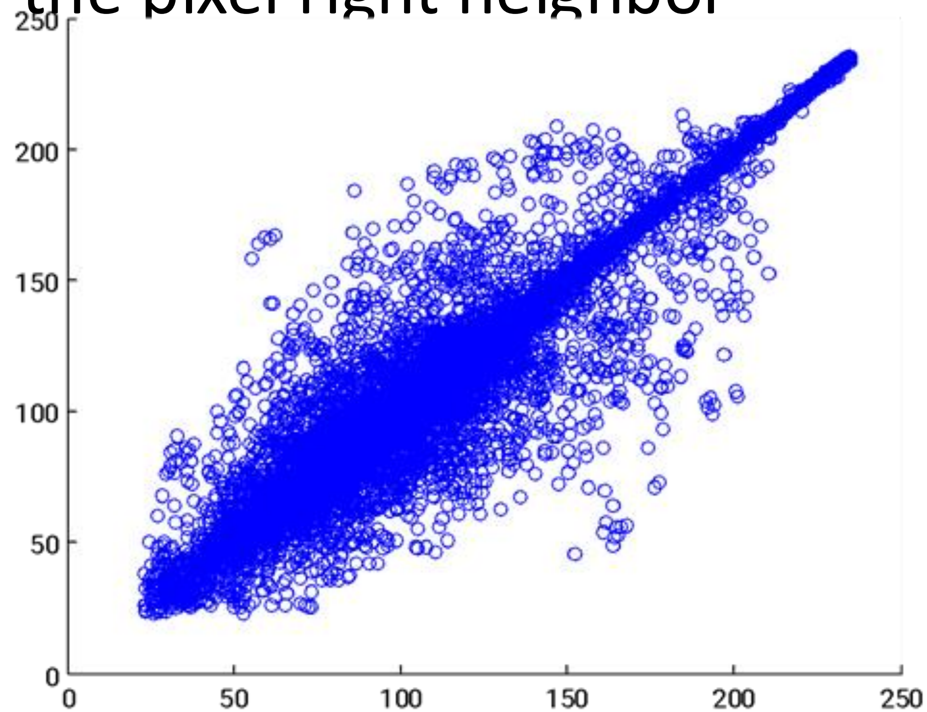
Image source

Visualizing such relationships?

- Can be done by means of a scatter plot
- X axis: values of attribute 1, Y axis: values of attribute 2
- Plot a marker at each such data point. The marker may be a small circle, a +, a *, and so on.

Visualizing such relationships?

- Image processing example: pixel intensity value f the pixel right neighbor



Correlation coefficient

- Let the sample-points be given as (x_i, y_i), $1 <= i <= N$.
- Let the sample standard deviations be σ_x and σ_y, and the sample means be μ_x and μ_y.
- The **correlation-coefficient** is given as:

$$r(x,y) = \frac{\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)}{\sqrt{\sum_{i=1}^{N}(x_i - \mu_x)^2 \sum_{i=1}^{N}(y_i - \mu_y)^2}} = \frac{\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)}{(N-1)\sigma_x \sigma_y}$$

Correlation coefficient

- The correlation-coefficient is given as:

$$r(x,y)=\frac{\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)}{\sqrt{\sum_{i=1}^{N}(x_i-\mu_x)^2\sum_{i=1}^{N}(y_i-\mu_y)^2}}=\frac{\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)}{(N-1)\sigma_x\sigma_y}$$

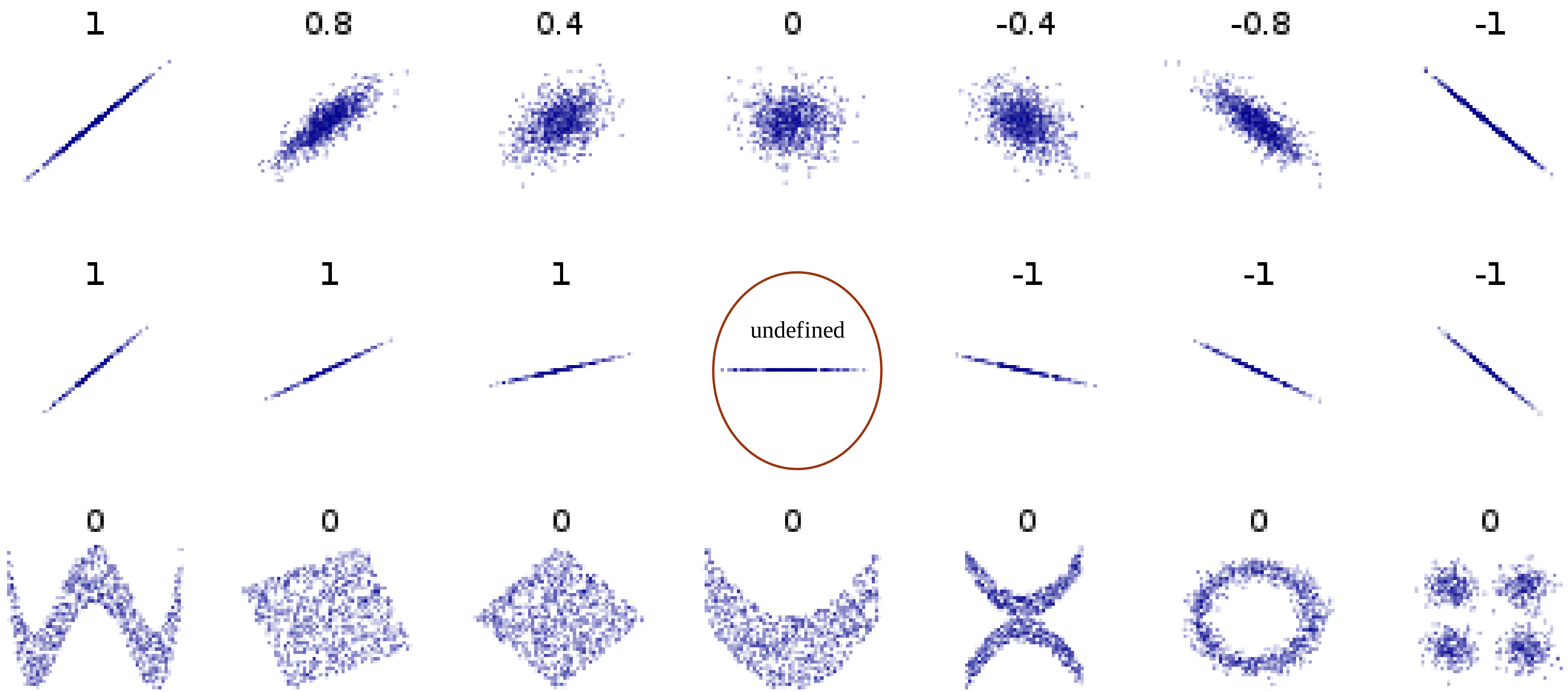
- *r* > 0 means the data are **positively correlated** (one attribute being higher implies the other is higher)
- *r* < 0 means the data are **negatively correlated** (one attribute being higher implies the other is lower)
- *r* = 0 means the data are **uncorrelated** (there is no such relationship!)
- *r* is **undefined** if the standard deviation of either ***x*** or ***y*** is 0.

Correlation coefficient: Properties

- The correlation-coefficient is given as:

$$r(x,y) = \frac{\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)}{\sqrt{\sum_{i=1}^{N}(x_i - \mu_x)^2 \sum_{i=1}^{N}(y_i - \mu_y)^2}} = \frac{\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)}{(N-1)\sigma_x \sigma_y}$$

- -1 <= *r* <= 1 always!



Correlation coefficient values for various toy datasets in 2D:
for each dataset, a scatter plot is provided

https://en.wikipedia.org/wiki/Correlation_and_dependence

Correlation coefficient: geometric interpretation

- Consider the N values $x_1, x_2, \ldots, x_N$. We will assemble them into a vector $\boldsymbol{x}$ (1D array) of N elements.

- We will also create vector $\boldsymbol{y}$ from $y_1, y_2, \ldots, y_N$.

- Now create vectors $\boldsymbol{x}$-μ_x and $\boldsymbol{y}$-μ_y – by deducting μ_x from each element of $\boldsymbol{x}$, and μ_y from each element of $\boldsymbol{y}$.

- Note that you may be used to vectors in 2D or 3D, but in statistics or machine learning, we frequently use vectors in N-D!

Correlation coefficient: geometric interpretation

- Then $r(\boldsymbol{x}, \boldsymbol{y})$ is basically the cosine of the angle between $\boldsymbol{x}$-μ_x and $\boldsymbol{y}$-μ_y!

$$r(\mathbf{x}-\mu_{\mathbf{x}}, \mathbf{y}-\mu_{\mathbf{y}}) = \cos\theta = \frac{(\mathbf{x}-\mu_{\mathbf{x}})\bullet(\mathbf{y}-\mu_{\mathbf{y}})}{\|\mathbf{x}-\mu_{\mathbf{x}}\|_2 \|\mathbf{y}-\mu_{\mathbf{y}}\|_2}$$

$$(\mathbf{x}-\mu_{\mathbf{x}})\bullet(\mathbf{y}-\mu_{\mathbf{y}}) = \sum_{i=1}^{N}(x_i-\mu_{\mathbf{x}})(y_i-\mu_{\mathbf{y}})$$

$$\|\mathbf{x}-\mu_{\mathbf{x}}\|_2 = \sqrt{\sum_{i=1}^{N}(x_i-\mu_{\mathbf{x}})^2}, \|\mathbf{y}-\mu_{\mathbf{y}}\|_2 = \sqrt{\sum_{i=1}^{N}(y_i-\mu_{\mathbf{y}})^2}$$

Vector magnitude - also called the L2-norm of the vector.

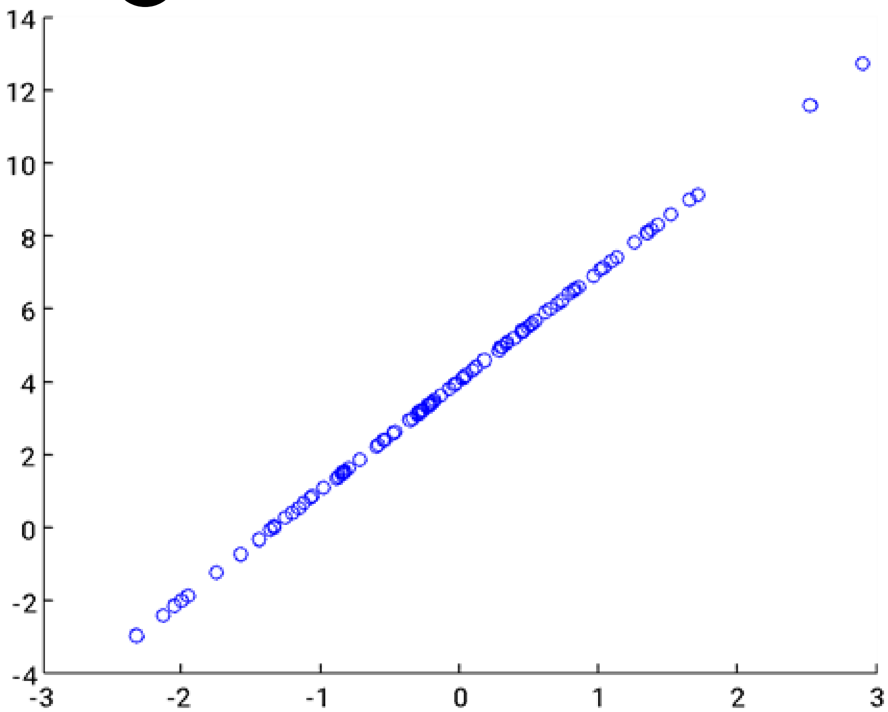
- Note that the cosine of an angle has a value from -1 to +1.

Correlation coefficient: Properties

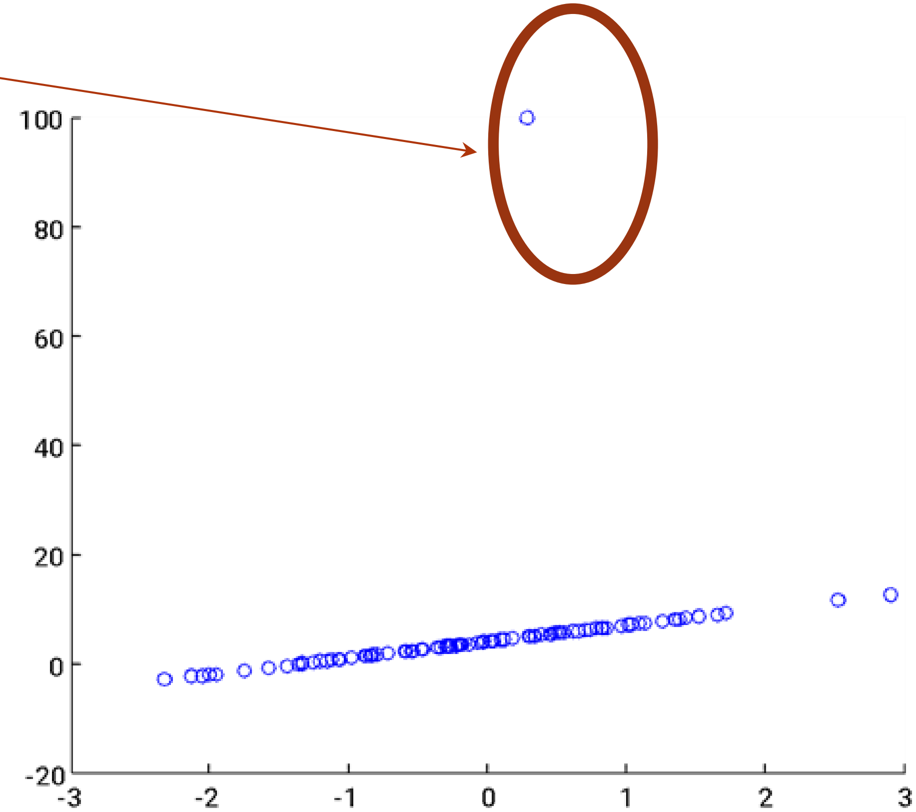
- In the following, we have a,b,c,d constant.
- If $y_i = a+bx_i$ where $b > 0$, then $r(x,y) = 1$.
- If $y_i = a+bx_i$ where $b < 0$, then $r(x,y) = -1$.
- If r is the correlation coefficient of data pairs as (x_i,y_i), $1 <= i <= N$, then it is also the correlation coefficient of data pairs $(b+ax_i,d+cy_i)$ when a and c have the same sign.

Correlation coefficient: a word of caution

- Sensitive to outliers!



r = 1



r = 0.33

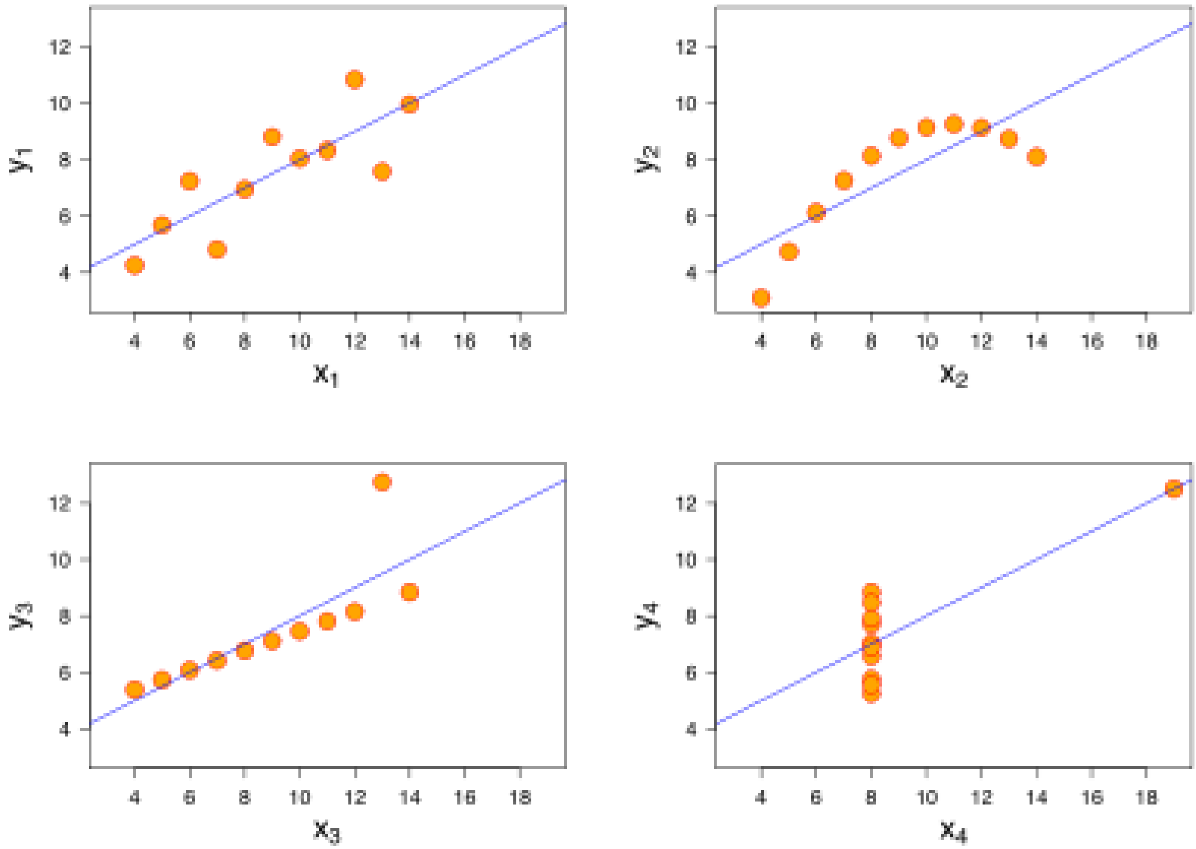
Caution with correlation: Anscombe's quartet

- The correlation coefficient can be a misleading value, and graphical examination of the data is important.

- This was illustrated beautifully by a British statistician named Frank Anscombe – by showing four examples that graphically appear very different – even though they produce identical correlation coefficients.

- These examples are famously called Anscombe's quartet.

Caution with correlation: Anscombe's quartet



In each of these examples, the following quantities were the same:

- Mean and variance of *x*
- Mean and variance of *y*
- Correlation coefficient $r(x,y)$

But the data are graphically very different!

Image source

Reflective (or Uncentered) correlation coefficient

- A version of the correlation coefficient in which you do not deduct the mean values from the vectors!

$$r(\boldsymbol{x},\boldsymbol{y}) = \frac{\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)}{\sqrt{\sum_{i=1}^{N}(x_i-\mu_x)^2\sum_{i=1}^{N}(y_i-\mu_y)^2}} \neq r_{uncentered}(\boldsymbol{x},\boldsymbol{y}) = \frac{\sum_{i=1}^{N}x_i y_i}{\sqrt{\sum_{i=1}^{N}x_i^{\,2}\sum_{i=1}^{N}y_i^{\,2}}}$$

- Uncentered c.c. is not "translation invariant":

$$r_{uncentered}(\boldsymbol{x},\boldsymbol{y}) \neq r_{uncentered}(\boldsymbol{x}+a,\boldsymbol{y}+b)$$

Correlation does not **necessarily** imply causation

- A high correlation between two attributes does not mean that one causes the other.

- Example 1: Fast rotating windmills are observed when the wind speed is high. Hence can one say that the windmill rotation produces speedy wind? (a wind**mill** in the literal sense ☺)

Correlation does not **necessarily** imply causation

- In example 1, the cause and effect were swapped. High wind speed leads to fast rotation and not vice-versa.

- Example 2: High sale of ice-cream is correlated with larger occurrence of drowning. Hence can one say that ice-cream causes drowning?

- In this case, there is a third factor that is highly correlated with both – ice-cream sales, as well as drowning. Ice-cream sales and swimming activities are on the rise in the summer!

Correlation does not **necessarily** imply causation

- The above statement does not mean that correlation is *never* associated with causation (example: increase in age does cause increase in height in children or adolescents) – just that it is not *sufficient* to establish causation.

- Consider the argument: “High correlation between tobacco usage and lung cancer occurrence does **not** imply that smoking causes lung cancer.”

Correlation does not **necessarily** imply causation – but it may!

- However multiple observational studies that eliminate other possible causes do lead to the conclusion that smoking causes cancer!

- ❑ higher tobacco dosage associated with higher occurrence of cancer
- ❑ stopping smoking associated with lower occurrence of cancer
- ❑ higher duration of smoking associated with higher occurrence of cancer
- ❑ unfiltered (as opposed to filtered) cigarettes associated with higher occurrence of cancer

- See https://www.sciencebasedmedicine.org/evidence-in-medicine-correlation-and-causation/ and http://www.americanscientist.org/issues/pub/what-everyone-should-know-about-statistical-correlation for more details.