Logic in CS Autumn 2024

Problem Sheet 5

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1. Consider the formula $\varphi = \forall x \exists y R(x, y) \wedge \exists y \forall x \neg R(x, y)$. Show that φ is satisfiable over a structure whose universe is infinite and countable.

2. Let τ be a signature consisting of a binary relation P and a unary relation F. Let $\mathcal{F}$ be a structure consisting of a universe of people, $P(x, y)$ is interpreted on $\mathcal{F}$ as "x is a parent of y" and $F(x)$ is interpreted as "x is female". Given the τ-structure $\mathcal{F}$,
   (a) Define a formula $\varphi_B(x, y)$ which says x is a brother of y
   (b) Define a formula $\varphi_A(x, y)$ which says x is an aunt of y
   (c) Define a formula $\varphi_C(x, y)$ which says x and y are cousins
   (d) Define a formula $\varphi_O(x)$ which says x is an only child
   (e) Give an example of a family relationship that cannot be defined by a formula

3. Consider the signature τ that has the binary functions $+, \times$. Let $\mathcal{N}$ be the structure over τ having as universe the set $\mathbb{N}$ of natural numbers and which interprets $+, \times$ in the usual way. Construct FO formulae $\mathsf{Zero}(x), \mathsf{One}(x), \mathsf{Even}(x), \mathsf{Odd}(x)$ and $\mathsf{Prime}(x)$ using τ such that
   - For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Zero}(a)$ iff a is zero.
   - For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{One}(a)$ iff a is one.
   - For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Even}(a)$ iff a is even.
   - For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Odd}(a)$ iff a is odd.
   - For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Prime}(a)$ iff a is prime.

   Goldbach's conjecture says that every even integer greater than 2 is the sum of two primes. Whether or not this is true is an open question in number theory. State Goldbach's conjecture as a FO-sentence over τ.

4. A group is a structure $(G, +, 0)$ where G is a set, $0 \in G$ is a special element called the identity and $+ : G \times G \to G$ is a binary operation such that
   (a) The operation $+$ is associative
   (b) The constant 0 is a right-identity for the operation $+$
   (c) Every element in G has a right inverse: for each $x \in G$, we can find $y \in G$ such that $x + y = 0$
   (d) For any three elements $x, y, z \in G$, if $x + z = y + z$, then $x = y$

Using a signature $\tau = (c, \mathsf{op})$ where c is a constant and op is a binary function symbol write (a)-(d) in FO.

5. Let τ be a signature consisting of the binary function symbol $+$ and a constant 0. We denote by $x + y$ the function $+(x, y)$. Consider the following sentences:

$$\varphi_1 := \forall x \forall y \forall z\ [(x + (y + z)) = ((x + y) + z)]$$

$$\varphi_2 := \forall x\ [(x + 0) = x \wedge (0 + x) = x]$$

$$\varphi_3 := \forall x\ [\exists y\ (x + y = 0) \wedge \exists z (z + x) = 0]$$

Let ψ be the conjunction of the three sentences.

(a) Show that ψ is satisfiable by exhibiting a τ-structure.

(b) Show that ψ is not valid.

(c) Let α be the sentence $\forall x \forall y\ ((x + y) = (y + x))$. Does α follow as a consequence of ψ? That is, is it the case that $\psi \rightarrow \alpha$?

(d) Show that ψ is not equivalent to any of $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_3$ and $\varphi_1 \wedge \varphi_3$.

6. Explain the difference between the first order prefixes $\exists x \forall y \exists z$ and $\forall x \exists y \forall z$.

7. Show that the sentences $\forall x \exists y \forall z\ (E(x, y) \wedge E(x, z) \wedge E(y, z))$ and $\exists x \forall y \exists z\ (E(x, y) \wedge E(x, z) \wedge E(y, z))$ are not equivalent by exhibiting a graph which satisfies one but not both of the sentences.

8. For each $n \in \mathbb{N}$, $\exists^{\geq n}$ denotes a counting quantifier. Intuitively, $\exists^{\geq n}$ means that "there exist atleast n such that". FO with counting quantifiers is the logic obtained by adding these quantifiers (for each $n \in \mathbb{N}$) to the fixed symbols of FO. The syntax and semantics are as follows:

Syntax : For any formula φ of FO with counting quantifiers, $\exists^{\geq n} x\ \varphi$ is also a formula.

Semantics : $\mathcal{A} \models \exists^{\geq n} x\ \varphi$ iff $\mathcal{A} \models \varphi(a_i)$ for each of n distinct elements $a_1, a_2, \ldots, a_n$ from the universe $u(\mathcal{A})$.

(a) Using counting quantifiers, define a sentence φ_{45} such that $\mathcal{A} \models \varphi_{45}$ iff $|u(\mathcal{A})| = 45$.

(b) Define a FO sentence φ (not using counting quantifiers) that is equivalent to the sentence $\exists^{\geq n} x\ (x = x)$.

9. Write an FO formula that will evaluate to true only over a structure that has atleast n elements and atmost m elements.