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(a) Solution:

$$f(x|\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\quad f_Y(y)=\frac{1}{\sqrt{2\pi\cdot 4}}e^{-\frac{(x-1)^2}{8}}$$

$$F_Z(z)=P(Z\le z)=P(X+Y\le z)$$

$$=P(Y=y,\ X\le z-y)$$

$$=\int_{-\infty}^{+\infty}P(X\le z-y)f_Y(y)\,dy$$

$$\Rightarrow f_Z(z)=\int_{-\infty}^{+\infty}f_X(z-y)f_Y(y)\,dy$$

$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{+\infty}e^{-\frac{(y-z)^2}{2}-\frac{(y-1)^2}{4}}dy$$

$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{+\infty}e^{-\frac{z^2-2yz+y^2}{2}-\frac{y^2-2y+1}{4}}dy$$

$$=\frac{1}{\sqrt{8\pi}}\int_{-\infty}^{+\infty}e^{-\frac{(3y-2z-1)^2+2(z-1)^2}{12}}dy$$

$$= \frac{1}{\sqrt{8\pi}} e^{\frac{(z-1)^2}{6}} \int_{-\infty}^{+\infty} e^{-\frac{\left(y-\frac{2z+1}{3}\right)^2}{4/3}} dy$$

$$= \frac{1}{\sqrt{6\pi}} e^{-\frac{(z-1)^2}{6}}$$

$$\Rightarrow \mu = 1,\ \sigma^2 = 3$$

$$\Rightarrow z \sim N(1, 3)$$

(b)