Quiz 1 on September 21, 2021 You may write on both sides of the paper.

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1. Consider a one-dimensional random walk starting from the origin $x=0$. The step size is δ, i.e. $x(t+1)-x(t)=\pm\delta$ where $t=0,1,2,3,\cdots$ measures the number of steps completed. Find the probability density function (PDF) of x at $t=N$ when N steps have been completed. Here we assume that the step size δ is very small and the number of steps N is very large, hence the distribution of x can be treated as a continuous one.

Solution: $y = x/\delta$

$\Rightarrow y(0)=0, \quad y(t+1)-y(t)=\pm 1$

N steps $\Rightarrow -N, -N+2, \cdots, N-2, N$ position.

$\frac{N-(-N)}{2}+1 = N+1$ possibilities.

Assuming: n forward, $N-n$ backward. $(n=1,2,\cdots,N)$.

$$p(y=-N+2n, N) = p(x=(2n-N)\delta, N)$$

$$= \frac{N!}{n!(N-n)!}\cdot\left(\frac{1}{2}\right)^N.$$

$(\ln N! = N\ln N - N)$

$$n=\frac{N+m}{2} \quad \approx e^{N\ln N-\frac{N+m}{2}\ln\frac{N+m}{2}-\frac{N-m}{2}\ln\frac{N-m}{2}}\cdot\left(\frac{1}{2}\right)^N$$

$$= e^{N\ln 2-\frac{m^2}{2N}}\cdot\left(\frac{1}{2}\right)^N$$

$$x=(2n-N)\delta = m\delta \Rightarrow m = x/\delta$$

$$\Rightarrow f(x, t=N) \approx e^{N\ln 2-\frac{x^2}{2N\delta^2}}\cdot\frac{1}{2^N}$$

$$= e^{-\frac{x^2}{2N\delta^2}}$$

because higher-order terms are neglected,

$$\int e^{-\frac{x^2}{2N\delta^2}}dx \neq 1 \Rightarrow f(x, t=N) = \frac{1}{\sqrt{2\pi N\delta^2}}e^{-\frac{x^2}{2N\delta^2}}$$

2. Consider the stochastic equation of motion $\frac{d}{dt}x(t)=\xi(t)$, in which ξ is a white noise satisfying the autocorrelation $\langle\xi(t_2)\xi(t_1)\rangle = 2D\delta(t_2-t_1)$. (a) Find the mean square displacement $\langle[x(t)]^2\rangle$ as a function of time t with the initial condition $x(0)=0$. (b) Find the probability density function of x at time t based on $\langle[x(t)]^2\rangle$ found in (a).

Solution:

(a) $\frac{d}{dt}x(t)=\xi(t) \Rightarrow x(t)=\int\xi(t)dt$

$$\Rightarrow [x(t)]^2=\left(\int\xi(t)dt\right)^2$$

$$=\int\int\xi(t_1)dt_1\,\xi(t_2)dt_2$$

$$\Rightarrow \langle[x(t)]^2\rangle=\left\langle\int\int\xi(t_1)\xi(t_2)dt_1dt_2\right\rangle$$

$$=\int\int\langle\xi(t_1)\xi(t_2)\rangle dt_1dt_2$$

$$=2D\int\int\delta(t_1-t_2)dt_1dt_2$$

$$=2D\int 1dt.$$

$$=2Dt.$$

(b) ∵ $\langle x \rangle = \langle \int \xi(t)\,dt \rangle = \int \langle \xi(t) \rangle dt$
$= 0$

$\langle x^2 \rangle = 2Dt$

$A(x) = \dfrac{\langle x \rangle - x(0)}{t} = 0$

$B(x) = \dfrac{\langle x^2 \rangle - x^2(0)}{t} = 2D$.

Fokker-Planck equation:

$$\frac{\partial P_1(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x)\cdot P_1(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x)\cdot P_1(x,t)]$$

$$\overset{A(x)=0,\,B(x)=2D}{=\!=} D\frac{\partial^2 P_1(x,t)}{\partial x^2}$$

$$\Rightarrow P_1 \propto e^{-\frac{x^2}{4Dt}}.$$

$$K\int P_1(x,t)\,dx = 1 \Rightarrow K\int e^{-\frac{x^2}{4Dt}}dx = 1$$

$$\Rightarrow K = \frac{1}{\sqrt{4\pi Dt}} \Rightarrow P(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$