Cauchy distribution

The Cauchy distribution $f(x;x_0,\gamma)$ is the distribution of the *x*-intercept of a ray issuing from (x_0,γ) with a uniformly distributed angle. It is also the distribution of the ratio of two independent normally distributed random variables with mean zero.

The Cauchy distribution is often used in statistics as the canonical example of a "pathological" distribution since both its expected value and its variance are *undefined*.

When *U* and *V* are two independent normally distributed random variables with expected value 0 and variance 1, then the ratio V/U has the standard Cauchy distribution.

<u>Proof</u>:

Let *x* denote the ratio V/U, i.e., $x=V/U$. In the two-dimensional space of (U,V), a given x corresponds to a straight line of slope $x=V/U=\tan\theta$. Based on this observation, the CDF of x is given by $F(x)=\frac{1}{2}+\frac{\theta}{\pi}=\frac{1}{2}+\frac{1}{\pi}\arctan x$ with $F(-\infty)=0$ and $F(\infty)=1$. Note that the distribution of (U,V) in the *UV* space is <u>isotropic</u>, and hence the CDF of *x* is $1/2+\theta/\pi$, i.e., the CDF of *x* increases linearly with θ.

From $F(x)=\frac{1}{2}+\frac{1}{\pi}\arctan x$, we have the PDF $f(x)=\frac{1}{\pi}\frac{1}{1+x^2}$, the standard Cauchy distribution.

The Cauchy distribution $f(x;x_0,\gamma)$ is the distribution of the *x*-intercept of a ray issuing from (x_0,γ) with a uniformly distributed angle.

<u>Proof</u>:

A simple geometric observation leads to $F(x;x_0,\gamma)=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x-x_0}{\gamma}$, from which we have

$$f(x;x_0,\gamma)=\frac{1}{\pi\gamma}\frac{1}{1+\left(\frac{x-x_0}{\gamma}\right)^2}=\frac{1}{\pi\gamma}\frac{\gamma^2}{\gamma^2+(x-x_0)^2}.$$

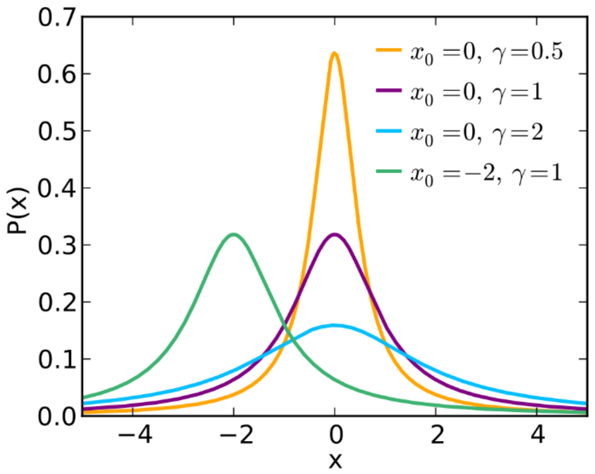


Figure. Probability density function for the Cauchy distribution.

Fourier transform

The Fourier transform of $f(x;0,1)$ is $\int_{-\infty}^{\infty} f(x;0,1)e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+x^2}e^{-ikx}dx = e^{-|k|}$, which can be obtained by using the residue theorem (with different paths chosen for different signs of *k*). More generally, the Fourier transform of $f(x;x_0,\gamma)$ is given by $\int_{-\infty}^{\infty} f(x;x_0,\gamma)e^{-ikx}dx = \int_{-\infty}^{\infty} \frac{1}{\pi\gamma}\frac{\gamma^2}{\gamma^2+(x-x_0)^2}e^{-ik(x-x_0)}e^{-ikx_0}dx = e^{-ikx_0-\gamma|k|}$.

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+x^2}\, e^{-ikx}\, dx$$

to be computed using Residue Theorem.

$$\frac{1}{\pi}\frac{1}{1+x^2} = \frac{1}{2\pi i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

$$= \frac{1}{2\pi i}\frac{2i}{x^2+1} = \frac{1}{\pi}\frac{1}{1+x^2}$$

$k > 0$, using $z = -i$, $e^{-ik(-i)} = e^{-k}$
$= e^{-|k|}$

$$\frac{1}{2\pi i}(-1)(-2\pi i)\, e^{-|k|} = e^{-|k|}$$

$k < 0$, using $z = i$, $e^{-ik(i)} = e^{k} = e^{-|k|}$

$$\frac{1}{2\pi i}(+1)(2\pi i)\, e^{-|k|} = e^{-|k|}$$

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+x^2}\, e^{-ikx}\, dx = e^{-|k|}$$

Inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|} e^{ikx} dk = \frac{1}{\pi}\frac{1}{1+x^2} = f(x;0,1)\,.$$

More generally, $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx_0-\gamma|k|} e^{ikx} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma|k|} e^{ik(x-x_0)} dk = \frac{1}{\pi\gamma}\frac{\gamma^2}{\gamma^2+(x-x_0)^2} = f(x;x_0,\gamma)\,.$

Mean and variance

The Cauchy distribution is an example of a distribution which has *no* mean, variance or higher moments defined. Its mode and median are well defined and are both equal to x_0.

One of the few distributions that is stable

A distribution is said to be stable if a linear combination of two independent random variables with this distribution has the same distribution, up to location and scale parameters. A random variable is said to be stable if its distribution is stable.

It is easy to show that Cauchy distribution is stable by using Fourier transformation.

Outline of the proof:

(1) Consider a sequence of independent and identically distributed random variables $\{X_1, X_2, \cdots, X_n\}$. Let $f(x_i)$ denote the PDF of X_i and Y denote the sample average $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$. Here $f(x_i)$ is understood as the Cauchy distribution $f(x; x_0, \gamma)$.

(2) The PDF of Y, denoted by $g(y)$, is given by $g(y) = \int \prod_{i=1}^{n} dx_i \delta\left(y - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} f(x_i)\right)$ based on the statistical independence.

(3) The Fourier transform of $g(y)$, defined by $g_Y(k) = \int_{-\infty}^{\infty} dy e^{-iky} g(y)$, is given by

$$g_Y(k) = \int_{-\infty}^{\infty} dy e^{-iky} \int \prod_{i=1}^{n} dx_i \delta\left(y - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} f(x_i)\right) = \int \prod_{i=1}^{n} dx_i \exp\left(-i\frac{k}{n}\sum_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} f(x_i)\right),$$

which can be written as

$$g_Y(k) = \left[\int dx_i \exp\left(-i\frac{k}{n} x_i\right) f(x_i)\right]^n = \left[\exp\left(-i\frac{k}{n} x_0 - \gamma\left|\frac{k}{n}\right|\right)\right]^n = \exp\left(-ikx_0 - \gamma|k|\right).$$

(4) From $g_Y(k) = \exp\left(-ikx_0 - \gamma|k|\right)$, we have $g(y) = f(y; x_0, \gamma)$.

Lévy distribution

The Lévy distribution is a continuous probability distribution for a non-negative random variable.

$$f(x;\mu,c) = \sqrt{\frac{c}{2\pi}}\frac{1}{(x-\mu)^{3/2}}\exp\left[-\frac{c}{2(x-\mu)}\right],$$

where μ is the location parameter and c is the scale parameter. Normalization of $f(x;\mu,c)$ can be readily verified.

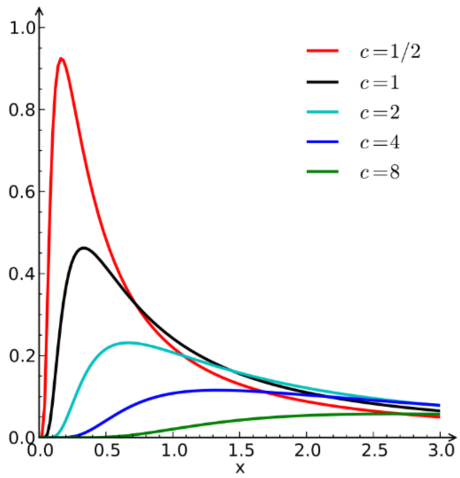


Figure. A few Levy PDFs.

First passage time of a 1D Brownian particle

One of the simplest and omnipresent stochastic systems is that of the Brownian particle in one dimension. *This system describes the motion of a particle which moves stochastically in one dimensional space, with equal probability of moving to the left or to the right*. Given that Brownian motion is used often as a tool to understand more complex phenomena, it is important to understand the probability of the first passage time of the Brownian particle of reaching some position distant from its start location. This is done as follows.

Diffusion

Diffusion equation in 1D

$$\frac{\partial}{\partial t} f(x,t) = D \frac{\partial^2}{\partial x^2} f(x,t)$$

$$f_0(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}$$

This solution is normalized, $\int_{-\infty}^{\infty} f_0(x,t)\,dx = 1$

Initial condition : $f_0(x,0) = \delta(x)$

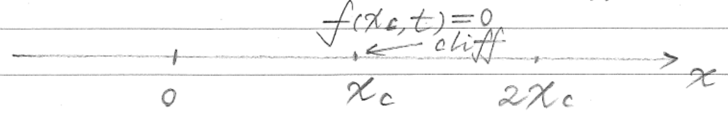
$$\frac{\partial}{\partial t} f_0(x,t) = -\frac{1}{2}\frac{1}{t} f_0 + \frac{x^2}{4Dt^2} f_0$$
$$= \left(-\frac{1}{2}\frac{1}{t} + \frac{x^2}{4Dt^2}\right) f_0$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2Dt} f_0 + \left(-\frac{x}{2Dt}\right)^2 f_0$$

$$D\frac{\partial^2 f}{\partial x^2} = \left(-\frac{1}{2t} + \frac{x^2}{4Dt^2}\right) f_0$$

solution verified.

With absorption (cliff) at $x_c > 0$,



we need a new solution as follows.

$$f(x,t) = f_0(x,t) - f_0(x-2x_c,t)$$
for $x \in (-\infty, x_c]$

$$f(x,0) = f_0(x,0) - f_0(x-2x_c,0)$$
$$= \delta(x) - \delta(x-2x_c)$$
$$= \delta(x) \quad \text{for } x \in (-\infty, x_c]$$

$$f(x_c,t) = f_0(x_c,t) - f_0(-x_c,t) = 0.$$

(2)

$$f(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} - \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-2x_c)^2}{4Dt}}$$

$$= f_0(x,t) - f_0(x-2x_c, t)$$

This solution is for $x \in (-\infty, x_c]$
with $f(x_c, t) = 0$
No probability beyond x_c, the cliff.

$S(t)$: survival probability

$$S(t) = \int_{-\infty}^{x_c} f(x,t)\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x_c} e^{-\frac{x^2}{4Dt}} \frac{1}{\sqrt{4Dt}} dx$$

$$- \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x_c} e^{-\frac{(x-2x_c)^2}{4Dt}} \frac{1}{\sqrt{4Dt}} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x_c}{\sqrt{4Dt}}} e^{-y^2} dy - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x_c}{\sqrt{4Dt}}} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{x_c}{\sqrt{4Dt}}}^{\frac{x_c}{\sqrt{4Dt}}} e^{-y^2} dy = \mathrm{erf}\left(\frac{x_c}{\sqrt{4Dt}}\right)$$

$t \to 0$, $S(0) = \mathrm{erf}(\infty) = 1$
$t \to \infty$, $S(\infty) = \mathrm{erf}(0) = 0$
$S(0) = \mathrm{erf}(\infty) = 1$, must be alive
$S(\infty) = \mathrm{erf}(0) = 0$, must be dead.

$t \to 0$, $S(0) = \mathrm{erf}(\infty) = 1$ Surely alive
$t \to \infty$, $S(\infty) = \mathrm{erf}(0) = 0$. Surely dead.

③

$dt > 0$

$$S(t) - S(t+dt) = p(t)\,dt$$

the probability density of falling

$$p(t)\,dt = S(t) - S(t+dt) > 0$$

$$p(t) = -\frac{d}{dt} S(t)$$

$$= -\frac{d}{dt}\,\mathrm{erf}\left(\frac{x_c}{\sqrt{4Dt}}\right)$$

$$= -\frac{d}{dt}\frac{1}{\sqrt{\pi}}\int_{-\frac{x_c}{\sqrt{4Dt}}}^{\frac{x_c}{\sqrt{4Dt}}} e^{-y^2}\,dy \quad > 0$$

$$p(t) = -\frac{1}{\sqrt{\pi}}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right)\frac{x_c}{\sqrt{4D}}\,(2)\,e^{-\frac{x_c^2}{4Dt}}$$

$$= \frac{x_c}{\sqrt{4\pi D}}\,\frac{1}{t^{3/2}}\,e^{-\frac{x_c^2}{4Dt}}$$

$$p(t) = \sqrt{\frac{x_c^2}{4\pi D}}\,\frac{1}{t^{3/2}}\,e^{-\frac{x_c^2}{4Dt}}$$

A Levy PDF, out of the first passage time distribution.

$$\int_0^\infty p(t)\,dt = \int_0^\infty -S'(t)\,dt = S(0) - S(\infty)$$
$$= 1 - 0 = 1$$

$$\int_0^\infty p(t)\,dt = \int_0^\infty -\frac{d}{dt}S(t)\,dt = S(0) - S(\infty) = 1$$

Normalized.