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MSDM5004
Homework 3 (Part I)

1. Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Show that the numerical scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

is second order both in time and space.

2. Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = u_0(x) & 0 \le x \le 1 \end{cases}$$

where

$$u_0(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} < x \le 1. \end{cases}$$

(1) The analytical solution is given by

$$u(x,t) = \sum_{m=1}^{\infty} a_m e^{-(m\pi)^2 t} \sin m\pi x, \quad \text{where} \quad a_m = 2\int_0^1 u_0(x) \sin m\pi x dx.$$

Compute the coefficient a_m.

(2) Write a code using MATLAB (or some other software) to obtain numerical solution using the explicit scheme. Use $J = 20$, $\Delta x = 0.05$, and (i) $\Delta t = 0.0012$ (ii) $\Delta t = 0.0013$. Plot the numerical solution and the analytical solution at $t = 0, \Delta t, 25\Delta t, 50\Delta t$ (To plot the analytical solution, you can stop the summation at a large number N, when you cannot see difference in the solution curve if N is increased).

3. Write a code to solve the PDE in problem 2 using the Crank-Nicolson method.

Use $J = 20$, $\Delta x = 0.05$, and (i) $\Delta t = 0.0012$ (ii) $\Delta t = 0.0013$, (iii) $\Delta t = 0.012$. Plot the solutions at $t = 0, \Delta t, 25\Delta t, 50\Delta t$.

Ans: 1.

$$T(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_{n-1})}{2\Delta t} - \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{(\Delta x)^2}$$

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \frac{\partial u}{\partial t}(x_j, t_n)\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_n)(\Delta t)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)(\Delta t)^3$$
$$O((\Delta t)^4)$$

$$u(x_j, t_{n-1}) = u(x_j, t_n) - \frac{\partial u}{\partial t}(x_j, t_n)\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t_n)(\Delta t)^2 - \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)(\Delta t)^3$$
$$+ O((\Delta t)^4)$$

$$\Rightarrow \frac{u(x_j, t_{n+1}) - u(x_j, t_{n-1})}{2\Delta t} = \frac{2\frac{\partial u}{\partial t}(x_j, t_n)\Delta t + \frac{1}{3}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)(\Delta t)^3 + O((\Delta t)^4)}{2\Delta t}$$
$$= \frac{\partial u}{\partial t}(x_j, t_n) + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)(\Delta t)^2 + O((\Delta t)^3)$$

$$u(x_{j+1}, t_n) = u(x_j, t_n) + \frac{\partial u}{\partial x}(x_j, t_n)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_n)(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_n)(\Delta x)^3$$
$$+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^4 + O((\Delta x)^5)$$

$$u(x_{j-1}, t_n) = u(x_j, t_n) - \frac{\partial u}{\partial x}(x_j, t_n)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_j, t_n)(\Delta x)^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x_j, t_n)(\Delta x)^3$$
$$+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^4 + O((\Delta x)^5)$$

$$\Rightarrow \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{(\Delta x)^2} = \frac{\frac{\partial^2 u}{\partial x^2}(x_j, t_n)(\Delta x)^2 + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^4 + O((\Delta x)^5)}{(\Delta x)^2}$$

$$= \frac{\partial^2 u}{\partial x^2}(x_j, t_n) + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^2 + O((\Delta x)^3)$$

$$\Rightarrow T(x_j, t_n) = \left(\frac{\partial u}{\partial t}(x_j, t_n) - \frac{\partial^2 u}{\partial x^2}(x_j, t_n)\right)$$

$$+ \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)(\Delta t)^2 + O((\Delta t)^3) - \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^2 + O((\Delta x)^3)$$

$$= \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_j, t_n)\cdot(\Delta t)^2 - \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(x_j, t_n)(\Delta x)^2 + O((\Delta t)^3) + O((\Delta x)^3)$$

$$= O((\Delta t)^2) + O((\Delta x)^2)$$

$\Rightarrow$ second order both in time and space.

2.

(1) $$a_m = 2\int_0^{\frac{1}{2}} 2x \sin m\pi x \, dx + 2\int_{\frac{1}{2}}^{1} (2-2x)\sin m\pi x \, dx$$

$$= 4\left[-\frac{1}{m\pi}x\cos m\pi x\right]_0^{\frac{1}{2}} - 4\int_0^{\frac{1}{2}} -\frac{1}{m\pi}\cos m\pi x \, dx$$

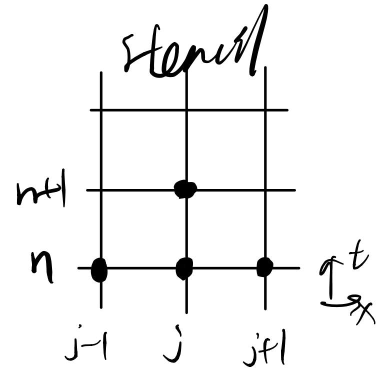
$$+ 4\left[\frac{1}{m\pi}(x-1)\cos m\pi x\right]_{\frac{1}{2}}^{1} - 4\int_{\frac{1}{2}}^{1}\frac{1}{m\pi}\cos m\pi x \, dx$$

$$= -\frac{2}{m\pi}\cos\frac{m\pi}{2} + \frac{4}{(m\pi)^2}\sin\frac{m\pi}{2} + \frac{2}{m\pi}\cos\frac{m\pi}{2} + \frac{4}{(m\pi)^2}\sin\frac{m\pi}{2}$$

$$= \frac{8}{(m\pi)^2}\sin\frac{m\pi}{2}$$

(2) $J=20$

$$\frac{U_j^{n+1}-U_j^{n}}{\Delta t}=\frac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{(\Delta x)^2}$$



$$\Rightarrow U_j^{n+1}=U_j^{n}+\mu\left(U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}\right)$$

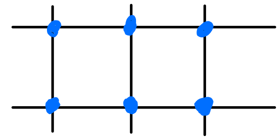
$$\mu:=\frac{\Delta t}{(\Delta x)^2}$$

$$U_j^{0}=u_0(x_j)=\begin{cases}2x_j, & 0\le x_j\le \frac{1}{2}\\ 2-2x_j, & \frac{1}{2}<x_j\le 1\end{cases}$$

$$U_0^{n}=U_J^{n}=0$$

$$\begin{bmatrix}U_1^{n+1}\\ U_2^{n+1}\\ \vdots\\ U_{J-2}^{n+1}\\ U_{J-1}^{n+1}\end{bmatrix}=\begin{bmatrix}1-2\mu & \mu & & & 0\\ \mu & 1-2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \mu & 1-2\mu & \mu\\ & & & \mu & 1-2\mu\end{bmatrix}\begin{bmatrix}U_1^{n}\\ U_2^{n}\\ U_3^{n}\\ \vdots\\ U_{J-2}^{n}\\ U_{J-1}^{n}\end{bmatrix}$$

3. Crank-Nicolson method



$$\frac{U_j^{n+1}-U_j^{n}}{\Delta t}=\frac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{2(\Delta x)^2}+\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{2(\Delta x)^2}$$

$$\Rightarrow -\frac{\mu}{2}U_{j-1}^{n+1}+(1+\mu)U_j^{n+1}-\frac{\mu}{2}U_{j+1}^{n+1}=\frac{\mu}{2}U_{j-1}^{n}+(1-\mu)U_j^{n}+\frac{\mu}{2}U_{j+1}^{n}$$

$$\mu=\frac{\Delta t}{(\Delta x)^2}$$

$$\begin{bmatrix} 1+\mu & -\frac{\mu}{2} & 0 & 0 & & \\ -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} & 0 & & 0 \\ 0 & -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} \\ & & & & -\frac{\mu}{2} & 1+\mu \end{bmatrix}\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{J-2}^{n+1} \\ U_{J-1}^{n+1} \end{bmatrix}=\begin{bmatrix} 1-\mu & \frac{\mu}{2} & \cdots & & \\ \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} & \cdots & 0 \\ 0 & \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} & \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} \\ & & & \frac{\mu}{2} & 1-\mu \end{bmatrix}\begin{bmatrix} U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{J-2}^{n} \\ U_{J-1}^{n} \end{bmatrix}$$

$$1+\mu > \left|-\frac{\mu}{2}-\frac{\mu}{2}\right| = \mu \Rightarrow \text{diagonally dominant}$$

$\Rightarrow$ Thomas algorithm is stable

Thomas algorithm $AU^{n+1}=BU^{n}$

1. $A=LU$
2. solve $Ly=BU^{n}$
3. solve $U\cdot U^{n+1}=y$