Chapter 14 Zero-Knowledge Technique

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2. Soundness: if the statement is false, no cheating prover can convince the honest verifier that it is true, except with some small probability
3. Zero-Knowledgeness: if the statement is true, no cheating verifier learns anything other than the statement is true

More Basics

The first two are properties of more general interactive proof systems. The most common use of a ZKP is in authentication systems where one party wants to be able to prove its identity to a second party via some secret information (such as a password) but doesn't want the second party to learn anything about the secret.

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Commonly, we refer to the prover as Peggy and the verifier as Victor (this is not just in our textbook). Also, sometimes, we refer to the parties as Alice and Bob.

A Basic Example Protocol

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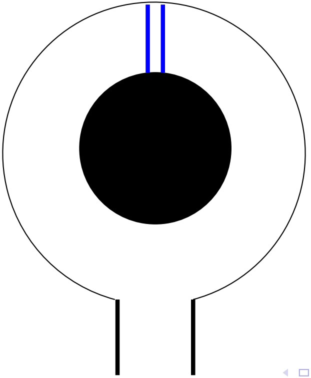
In the story, Peggy has uncovered the secret word that opens a magic door in the back of a cave. The cave is shaped like a circle with the entrance on one side and the magic door blocking the opposite side.



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Victor says he will pay her for the secret word but not until he is sure she really knows it. Peggy says she will tell him the secret word but not until she gets the money. They devise a scheme by which Peggy can prove she knows the secret word without telling it to Victor.

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First, Victor waits outside the cave as Peggy goes in. She randomly enters either the path on the left or the one on the right. Then Victor enters the cave and shouts the name of the path he wants her to use to return, either the left or the right at random. Providing she really does know the secret word, this is easy; she opens the door, if necessary, as returns along the desired path. Note that Victor does now know which path Peggy has gone down.

A Basic Example Protocol

However, suppose she did not know the secret word. Then, she would only be able to return by the named path if Victor were to name the path that she had just entered. Since Victor would want to choose the path at random, she would have a 50% chance of guessing correctly. If they were to repeat this trick many times, say 20 times in a row, her chance of anticipating all of Victor's requests becomes very small (2^{-20}). So, if Peggy reliably appears at the exit Victor names, he can conclude she very likely has the secret word.

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If there was another observer, there is no way to convince them that Peggy knows the secret word since she and Victor could have planned the sequence.

Color Example

The classic example is the following: Imagine your friend is color-blind. You have two billiard balls; one is red, one is green, but they are otherwise identical. To your friend they seem completely identical, and he is skeptical that they are actually distinguishable. You want to prove to him that they are in fact differently-colored. On the other hand, you do not want him to learn which is red and which is green.

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Here is the proof system. You give the two balls to your friend so that he is holding one in each hand. You can see the balls at this point, but you don't tell him which is which. Your friend then puts both hands behind his back. Next, he either switches the balls between his hands, or leaves them be, with probability $\frac{1}{2}$ each. Finally, he brings them out from behind his back. You now have to 'guess' whether or not he switched the balls.

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By looking at their colors, you can of course say with certainty whether or not he switched them. On the other hand, if they were the same color and hence indistinguishable, there is no way you could guess correctly with probability higher than $\frac{1}{2}$.

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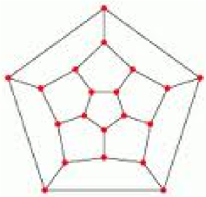
If you and your friend repeat this 'proof' t times (for large t), your friend should become convinced that the balls are indeed differently colored; otherwise, the probability that you would have succeeded at identifying all the switches and non-switches is at most $\left(\frac{1}{2}\right)^t$. Furthermore, the proof is 'zero-knowledge' because your friend never learns which ball is green and which is red; indeed, he gains no knowledge about how to distinguish the balls.

A More Practical Example

Peggy knows a Hamiltonian cycle for a large graph G, which is Peggy's public key. Victor knows G, since it is public, but not the cycle. Peggy will prove she knows the cycle without revealing it. However, Peggy does not want to simply reveal the Hamiltonian cycle or any other information to Victor. Maybe she is the only one who knows the information? Maybe he wants to buy the information but wants verification first? Here is why the Hamiltonian cycle creates a problem:

Sir William Rowan Hamilton (1805-1865) was an Irish physicist, astronomer and mathematician. His major work was in reformulating Newtonian mechanics, which was renamed Hamiltonian mechanics. As a result of his work was his invention of a puzzle known as the Icosian Game in 1857. One question the game posed is whether it was possible to start at a vertex of the above graph and return to that vertex by visiting every other vertex once each. This idea became known as a Hamiltonian circuit.



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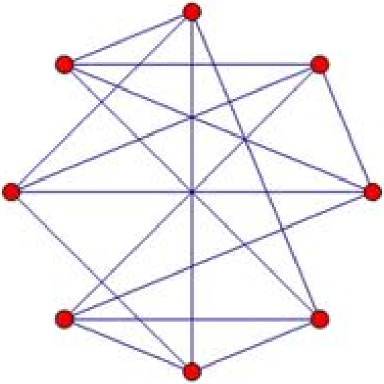
The question is, when does a graph have a Hamiltonian cycle? a Hamiltonian path?

Is This Graph Hamiltonian?



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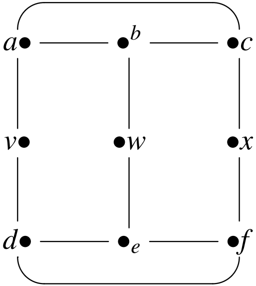

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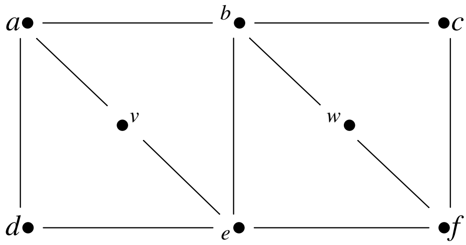

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If we add edges to G, we eventually obtain a complete graph, which has a Hamiltonian circuit.

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But $deg_G(a) \leq deg_H(a)$ for all a, so $(**)$ implies $(*)$.

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$$\bullet_y \;—\; \bullet_{a_1} \;—\; \bullet_{a_2} \;—\; \ldots \;—\; \bullet_{a_{i-1}} \;—\; \bullet_{a_i} \;—\; \bullet_{a_{i+1}} \;—\; \ldots \;—\; \bullet_{a_{n-2}} \;—\; \bullet_x$$

Ore's Theorem

Now, $V(H) = \{x, y, a_1, a_2, \ldots, a_{n-2}\}$. Moreover, we note that for $i > 1$,

$$\{y, a_i\} \in E(H) \Rightarrow \{x, a_{i-1}\} \notin E(H) \qquad (***)$$

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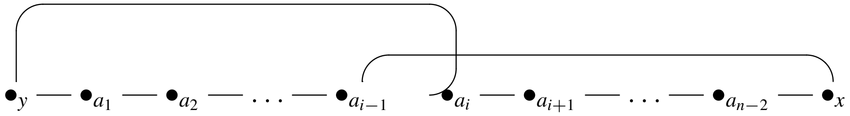

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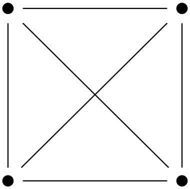

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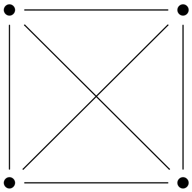

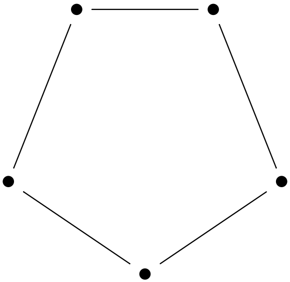

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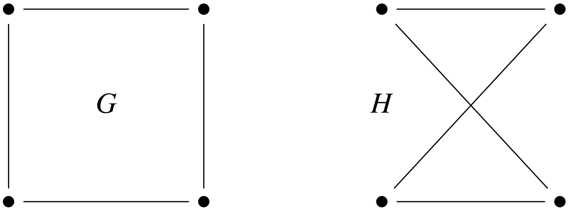

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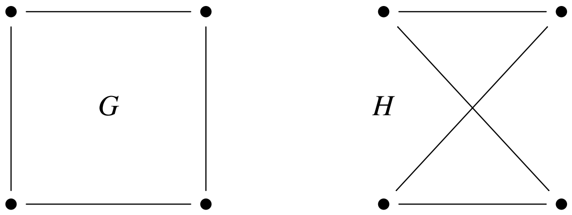

Or, H differs from G in that the vertices have different names.
Since it is trivial to translate a Hamiltonian cycle between isomorphic graphs with known isomorphism, if Peggy knows a Hamiltonian cycle in G, she must also know one for H.

Peggy labels the vertices of H with random numbers and then for each edge of H she writes on a small piece of paper the two vertices incident to the edge and then puts these pieces of paper upside down on a table. The purpose of this commitment is that Peggy is not able to change H while at the same time Victor has no information about H.

Hamiltonian Protocol

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Victor then chooses one of two questions to ask Peggy. He can either ask her to show the isomorphism between H and G, or he can ask her to show a Hamiltonian cycle in H.

If Peggy is asked to show that the two graphs are isomorphic, she first uncovers all of H (by turning over all of the pieces of paper she put on the table) and then provides the vertex translations that map H to G. Victor can verify that they are isomorphic.

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If Peggy is asked to prove she knows a Hamiltonian cycle in H, she translates her Hamiltonian cycle in G onto H and only uncovers the edges on the Hamiltonian cycle. This is enough for Victor to verify that H is Hamiltonian.

Why This Is ZKP: Completeness

During each round, Peggy does not know which question will be asked until after giving Victor H. Therefore, in order to be able to answer both, H must be isomorphic to G and she must have a Hamiltonian cycle in H. Because only someone who knows a Hamiltonian cycle n G would have been able to answer both questions, Victor (after a sufficient number of rounds) becomes convinced that Peggy does know this information.

Why This Is ZKP: Zero-Knowledgeness

Peggy's answers do not reveal the original Hamiltonian cycle in G. Each round, Victor will only learn H isomorphic to G or a Hamiltonian cycle in H. He would need both answers for a single H to discover the cycle in G, so the information remains unknown as long as Peggy can generate a unique H every round.

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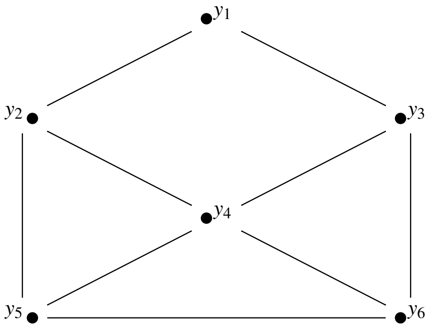
For example, if Peggy knew ahead of time that Victor would ask to see a Hamiltonian cycle in H, she could generate a Hamiltonian cycle in an unrelated graph. Similarly, if Peggy knew in advance that Victor would ask to see an isomorphism, then she would simply generate an isomorphic graph H (in which she also does not know a Hamiltonian cycle). Victor could simulate the protocol by himself (without Peggy) because she knows what he will ask to see. Therefore, Victor gains no information about the Hamiltonian cycle in G from the information revealed in each round.

Why Is This ZKP: Soundness

If Peggy does not know the information, she can guess which question Victor will ask and generate either a graph isomorphic to G or a Hamiltonian cycle for an unrelated graph, but since she cannot do both without this guesswork, her chance of fooling Victor is 2^{-n}, where n is the number of rounds. For all realistic purposes, it is infeasibly difficult to defeat a zero-knowledge proof with a reasonable number of rounds this way,

Graph Coloring Example

Peggy wants to convince Victor that a particular graph G, known to both of them, is 3-colorable and that Peggy knows such a coloring, without revealing to Victor any information about how the coloring looks.



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Peggy then removes colors from the vertices, labels the i^{th} vertex of G with the cryptotext $y_i = e_i(c_i)$, and designs the following table.

1	red	e_1	$e_1(red) = y_1$
2	green	e_2	$e_2(green) = y_2$
3	blue	e_3	$e_3(blue) = y_3$
4	red	e_4	$e_4(red) = y_4$
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6	green	e_6	$e_6(green) = y_6$

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Peggy finally shows Victor the graph with vertices labeled by cryptotexts.

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Victor performs encryptions to verify that the nodes have the colors as shown.

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Protocol 1: Alice sends Bob messages *head* and *tail* encrypted by a one-way function f. Bob guesses which one of them is the encryption of *head*. Alice tells Bob whether his guess was correct. If Bob does not believe her, Alice sends f to Bob.

Coin Flipping By Phone

Protocol 2: Alice chooses two large primes p, q and sends $n = pq$ to Bob, keeping p and q secret. Bob chooses randomly an integer $x \in \{1, 2, \ldots, \frac{n-1}{2}\}$, sends Alice $y \equiv x^2 \pmod{n}$ and tells Alice 'If you guess x correctly, you get the car'.

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Alice then guesses $x = x_1'$ or $x = x_2'$ and tells Bob of her choice (for example by reporting the position and value of the leftmost bit in which x_1' and x_2' differ).

Coin Flipping By Phone

Protocol 2: Alice chooses two large primes p, q and sends $n = pq$ to Bob, keeping p and q secret. Bob chooses randomly an integer $x \in \{1, 2, \ldots, \frac{n-1}{2}\}$, sends Alice $y \equiv x^2 \pmod{n}$ and tells Alice 'If you guess x correctly, you get the car'.

Alice computes four square roots $(x_1, n - x_1)$ and $(x_2, n - x_2)$ of x.

Let

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Since $x \in \{1, \ldots, \frac{n-1}{2}\}$, either $x = x_1'$ or $x = x_2'$.

Alice then guesses $x = x_1'$ or $x = x_2'$ and tells Bob of her choice (for example by reporting the position and value of the leftmost bit in which x_1' and x_2' differ).

Bob tells Alice whether her guess was correct, and then later if necessary, Alice reveals p and q and Bob reveals x.

A More Mathematical Example

Let $n = pq$ for large primes p and q. Assume $p, q \equiv 3 \pmod 4$.

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Suppose Peggy knows the factorization $n = pq$.

- Victor chooses a random integer x and sends $x^4 \pmod n$ to Peggy.
- Peggy computes the principal square root

$$y_1 = (x^4)^{\frac{p+1}{2}}$$

of $x^4 \pmod p$, and principal square root

$$y_2 = (x^4)^{\frac{q+1}{2}}$$

of $x^4 \pmod q$, and uses the Chinese Remainder Theorem (and the Euclidean Algorithm) to compute y so that $y \equiv y_1 \pmod p$ and $y \equiv y_2 \pmod q$. Peggy sends this value back to Victor.

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This formula for square root modulo primes congruent to $3 \pmod 4$ returns the principal square root, which is itself a square. For a prime $p \equiv 3 \pmod 4$ and for $a \equiv b^2 \pmod 4$, the two square roots are $\pm b$, and exactly one of $\pm b$ is itself a square since -1 is a nonsquare and $\mathbb{Z}/p^*$ is cyclic.

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Since Victor can already compute x^2, Peggy has certainly imported no information to Victor.

A More Mathematical Example

Victor should be convinced that there is no other way for Peggy to have found the square root than by knowing the factors p and q because in any case, being able to take a square roots $\pmod{n}$ gives a probabilistic algorithm for factoring n (when n is in the special form $n = pq$ with distinct primes p and q) as following:

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Victor should be convinced that there is no other way for Peggy to have found the square root than by knowing the factors p and q because in any case, being able to take a square roots $\pmod{n}$ gives a probabilistic algorithm for factoring n (when n is in the special form $n = pq$ with distinct primes p and q) as following:

If we have an <u>oracle</u> (meaning, some otherwise unexplained mechanism) which computes square roots modulo n), we repeatedly do the following: pick a random number x, compute $x^2 \pmod{n}$ and feed the result to the oracle, which returns a square root of $x^2 \pmod{n}$. Since these are exactly two square roots of any nonzero square modulo a prime (by the Chinese Remainder Theorem), there are exactly 4 square roots of any square modulo $n = pq$ and $\pm x$ is just two of them. Let the other two be $\pm x'$.

A More Mathematical Example

Assuming that the original x was really chosen randomly, the probability is $\frac{1}{2}$ that the oracle will return x' as y. If so, then n does not divide either of $x \pm y$ but nevertheless n divides $x^2 = y^2$ (since $x^2 \equiv y^2 \pmod{n}$). So, p divides one of $x \pm y$ and q divides the other one. therefore, $((x - y), n)$ is either p or q which could easily be computed. Since the oracle can be called repeatedly, at each invocation there is probability $\frac{1}{2}$ that a factorization will be obtained. So the probability that after l invocations we don't obtain s factorization is $\left(\frac{1}{2}\right)^l$. This goes to 0 quickly as l goes to ∞, which we construe as an indication that we will obtain a factorization in reasonable time.

Fiat and Shamir

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The central entity can calculate k different integers $v_j = f(I, c_j)$, where c_j is an integer and v_j has integer value for which $u_j^2 \equiv v_j \pmod{n}$ with u_j between 0 and $n-1$.

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The function $f(.,.)$, which is public, can be realized with, for example, the triple-DES. The k value can be obtained by repeatedly calculating different values of $f(I, c_j)$, until k values for v_j have been found, which satisfy the above condition.

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The central entity will then calculate the smallest roots of $v_j^{-1} \pmod{n}$, for each of the k values of v_j. These are denoted by s_j.

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We should note that the calculation of these roots requires information about the factors p and q. Since we may assume the factorization of a large number is computationally infeasible, as in RSA, no one other than the central entity can calculate the values of s_j. Thus, these values are used as the secret values with which others can ascertain another person's identity. The actual values of s_j will provide no information with respect to p and q. Therefore there is nothing to prevent n being shared by more than one member.

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After this initialization phase, the ZKP can commence.

Using This ZKP

Let us suppose Bob wishes to ascertain the identity of Alice. Alice must therefore prove in some manner that she has access to the secret values $s_1, \ldots, s_k$ without actually revealing any of these values. The protocol requires the following:

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Bob generates a binary random vector $(t_{i1}, \ldots, t_{ik})$ and sends this to Alice.

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Steps 3-6 are repeated for $i = 1 \ldots, t$.

Bob will accept that the person claiming to be Alice is really Alice if all t checks out successfully.

Using This ZKP

If all proceeds according to plan, then for all i, $z_i = x_i$. By combining the equations in steps 3,4 and 6, it follows that

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Suppose Charles wishes to pretend he is Alice. Since he knows neither the values of s_j, not the values of r_i, he cannot calculate y_i in step 5.

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Furthermore, Charles will also find it computationally infeasible to deduce the values of y_i from the equation in step 6; Charles knows the values of $z_i s_i$ since these must be equal to those of x_i, and the values of v_j and the elements t_{ij} of the random vector, but he will still be defeated by the problem of finding a square root of a modulo value.

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Consider the simplest case where $k = t = 1$. Alice has one secret key s and must demonstrate this to Bob.

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If he returns a zero, then Alice will respond by sending r. If he returns a 1, Alice will respond by sending the product rs.

Bob can verify that Alice has responded correctly by using his knowledge of the values of I and x. If Alice returns r, Bob will not learn anything of s, since r is a random number. And if Alice returns rs, he will still learn nothing of s since in this scenario, r is not known.