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Activator-Inhibitor System with Delay and Pattern Formation

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Abstract—In the present paper, a description and mathematical analysis of a simple model of nonlinear pattern formation is given. The model is based on the so-called activator-inhibitor system proposed by Thomas. We introduce time delay into the reaction term and focus on its influence on morphogenesis and pattern formation. Numerical simulations are presented and compared for both cases without and with delay.

Keywords—Reaction-diffusion equation, Equilibrium state, Stability, Turing instability, Pattern formation, Morphogenesis.

1. INTRODUCTION

Pattern formation is one of the processes that were modeled very often and on the background of different mechanisms (see e.g., [1] and references therein or [2,3]), starting from the paper of Turing [4] where a linear model was proposed. One of the most popular nonlinear models is the model based on the so-called Thomas mechanism, [1,4,5]. This model was used to explain the mechanisms governing pattern formation in embryos growth and mammalian coat formation.

The main idea of the Thomas mechanism consists in considering a type of pattern formation arising from the interactions of two chemicals. One of them, with the concentration $u(x, y, t)$, is activator and the second one, with the concentration $v(x, y, t)$, is inhibitor. Basically, both substances are supplied at constant rates a and αb, degrade linearly proportional to their concentrations and both are used up in the reaction at a rate $\rho h(u, v)$. The function $h(u, v)$ describes the reaction term. It is assumed that this function is proportional to v but the dependence on u is more complicated. For a given v and small u it is nearly bilinear, i.e., we have $h \approx uv$. For larger u, the reaction is slower. In this case, it can be described by the term, $uv/(1 + u)$. Finally, large values of u stop the reaction, i.e., h decreases with u and we assume $h \approx uv/(1 + u^2)$.

Summarizing, the following dimensionless system of equations is obtained, [1,5],

$$\partial_t u = \gamma f(u, v) + \nabla^2 u, \qquad \partial_t v = \gamma g(u, v) + d\nabla^2 v, \tag{1}$$

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where

$$f(u,v) = a - u - h(u,v),$$
$$g(u,v) = \alpha(b - v) - h(u,v), \tag{2}$$
$$h(u,v) = \frac{\rho uv}{1 + u + Ku^2},$$

and positive coefficients a, b, d, K, α, γ, ρ; $t \in \mathbb{R}^+$, $(x,y) \in \Omega$, $\Omega = [0,1]^2$ with no-flux boundary conditions,

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad \text{at } \partial\Omega,$$

where n is the unit outward normal to the boundary $\partial\Omega$ and the initial data $u(x,y,0) > 0$ and $v(x,y,0) > 0$.

Turing [4] observed that if the homogeneous equilibrium state is stable to small perturbations in the absence of diffusion but unstable to small spatial perturbations when diffusion is present, then the chemicals u and v can react and diffuse in such a way as to produce spatial patterns of chemical concentration.

In the paper, we introduce time delay into the reaction term $h(u,v)$ and study its influence on the behaviour of solutions. We focus on the similarities and differences between the patterns that arising in the models with and without delay. Similar ideas appeared e.g., in [6] where the distributed delay is introduced into linear terms.

2. PROPERTIES OF THE MODEL WITHOUT DELAY

We begin our analysis with the system without diffusion, i.e.,

$$\partial_t u = \gamma f(u,v), \qquad \partial_t v = \gamma g(u,v), \tag{3}$$

with the functions f and g defined by equation (2) and positive initial conditions u_0, v_0. It is easily seen that there exists unique solution to equations (3),(2) with $u_0, v_0 > 0$. Moreover, for positive initial data the solution is also positive that implies at most exponential growth of both variables u and v. Therefore, every solution with positive initial data exists for all $t \geq 0$.

The next step of the analysis is to find equilibrium states and study their stability. Now, we present some results known in the literature (e.g., [1]) which are useful for the analysis developed in the paper. From the biological point of view we are only interested in the case with one positive equilibrium state, $(\bar{u}, \bar{v})$. This is the case when Turing instability can occur and stable patterns can be formed. Conditions guaranteeing the stability of unique positive equilibrium state $(\bar{u}, \bar{v})$ are the following,

$$-\frac{\alpha\rho\bar{v}(K\bar{u}^2 - 1)}{(1 + \bar{u} + K\bar{u}^2)^2} + \frac{\rho\bar{u}}{1 + \bar{u} + K\bar{u}^2} + \alpha > 0, \qquad \frac{\rho\bar{v}(K\bar{u}^2 - 1)}{(1 + \bar{u} + K\bar{u}^2)^2} - \frac{\rho\bar{u}}{1 + \bar{u} + K\bar{u}^2} - \alpha - 1 < 0.$$

Now, we study equations (1),(2), i.e., the system with diffusion. Applying the existence theorem, (see [7,8]), we can prove the existence of unique solutions for all $t \geq 0$ and initial functions near the equilibrium state. In this case, we want the equilibrium state to be unstable. It can be shown (see e.g., [1]) that the coefficient d must be greater than one to obtain instability. Moreover, there must exist coefficient γ such that the following inequality,

$$\gamma L < \pi^2(n^2 + m^2) < \gamma M,$$

for some $n, m \in \mathbb{N}$ is satisfied. In the above inequality constants L and M are defined as

$$L = \frac{(df_u + g_v) - \sqrt{(df_u + g_v)^2 - 4d\det A}}{2d}, \qquad M = \frac{(df_u + g_v) + \sqrt{(df_u + g_v)^2 - 4d\det A}}{2d},$$

where A is the Jacobi matrix for equation (3) at $(\bar{u}, \bar{v})$ and $f_u = \frac{\partial f}{\partial u}(\bar{u}, \bar{v})$, $f_v = \frac{\partial f}{\partial v}(\bar{u}, \bar{v})$, $g_u = \frac{\partial g}{\partial u}(\bar{u}, \bar{v})$, and $g_v = \frac{\partial g}{\partial v}(\bar{u}, \bar{v})$.

3. THE MODEL WITH TIME DELAY

We first introduce time delay into equation (3), namely, we consider the functions f, g, and h defined as follows,

$$\begin{aligned} f(u,v)(t) &= a - u(t) - h(u,v)(t-\tau), \\ g(u,v)(t) &= \alpha(b - v(t)) - h(u,v)(t-\tau), \\ h(u,v)(s) &= \frac{\rho u(s)v(s)}{1+u(s)+Ku^2(s)}, \end{aligned} \tag{4}$$

where τ denotes the constant delay of activator-inhibitor reaction and the other constants are the same as in equations (1),(2). Let $\Phi : [-\tau, 0] \to (\mathbb{R}^+)^2$ be a continuous function, $\Phi = (\Phi_1, \Phi_2)$, and

$$u(t) = \Phi_1(t) > 0, \quad v(t) = \Phi_2(t) > 0, \qquad \text{for } t \in [-\tau, 0],$$

be initial data for equations (3),(4).

Using, e.g., the step method (cf., [9]) it is easy to see that for every initial function Φ there exists a unique solution to equations (3),(4). Applying Theorem 1.2 from [10], we conclude that the solution can become negative in a finite time interval. On the other hand, if the solution is positive, then its growth is at most exponential (as in the case without delay) and therefore, such a solution exists for every $t \geq 0$. Assuming that the solution is positive for every $t > 0$, we study its asymptotic behaviour.

The characteristic equation of equations (3),(4) has the following form,

$$D(\lambda) = \lambda^2 + (\gamma + \alpha\gamma + \gamma\kappa e^{-\lambda\tau} + \gamma\mu e^{-\lambda\tau})\lambda + \alpha\gamma^2 + \alpha\gamma^2\kappa e^{-\lambda\tau} + \gamma^2\mu e^{-\lambda\tau} = 0, \tag{5}$$

where

$$\kappa = \frac{\rho\bar{v}(1-K\bar{u}^2)}{(1+\bar{u}+K\bar{u}^2)^2}, \qquad \mu = \frac{\rho\bar{u}}{1+\bar{u}+K\bar{u}^2},$$

and $(\bar{u}, \bar{v})$ is the equilibrium state that is the same for equations (3),(2), equations (3),(4), and equations (1),(2), obviously.

LEMMA 3.1. *Suppose equation (5) has no purely imaginary roots.*

(i) *If $\alpha + \alpha\kappa + \mu < 0$, then at least one root of equation (5) has positive real part.*
(ii) *If $\alpha + \alpha\kappa + \mu > 0$, $\alpha + 1 > |\kappa + \mu|$ and $\tau < ((\alpha+1) - |\kappa+\mu|)/(\gamma|\alpha\kappa+\mu|)$, then all roots of equation (5) have strictly negative real parts.*

PROOF. We use the Mikhailov criterion (see e.g., [12,13] for details). It is enough to investigate when the change of the argument of vector $D(i\omega)$ with ω increasing from 0 to $+\infty$ (i.e., $\Delta \arg D(i\omega)$) is equal to $\pi/2$ multiply by degree of $P(\lambda)$ where $D(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau}$. We can write down this condition in the following form,

$$\mathop{\Delta}_{\omega\in[0,+\infty)} \arg D(i\omega) = \frac{\pi}{2} \cdot \deg P(i\omega).$$

Equation (5) implies

$$P(\lambda) = \lambda^2 + \gamma(\alpha+1)\lambda + \alpha\gamma^2, \qquad Q(\lambda) = \gamma(\kappa+\mu)\lambda + \gamma^2(\alpha\kappa+\mu).$$

Thus, the roots of equation (5) have strictly negative real parts iff $\Delta_{\omega\in[0,+\infty)} \arg D(i\omega) = \pi$. We have

$$\begin{aligned} \Re(D(i\omega)) &= -\omega^2 + \alpha\gamma^2 + \gamma(\kappa+\mu)\omega\sin(\omega\tau) + \gamma^2(\alpha\kappa+\mu)\cos(\omega\tau), \\ \Im(D(i\omega)) &= \gamma(\alpha+1)\omega + \gamma(\kappa+\mu)\omega\cos(\omega\tau) - \gamma^2(\alpha\kappa+\mu)\sin(\omega\tau). \end{aligned} \tag{6}$$

Moreover, $\Re(D(0)) = \gamma^2(\alpha + \alpha\kappa + \mu)$ and $\Im(D(0)) = 0$. If $\alpha + \alpha\kappa + \mu < 0$, then $\Re(D(0)) < 0$ and $\Im(D(0)) = 0$. It is clear that $\lim_{\omega\to+\infty} \Re(D(i\omega)) = -\infty$. Hence, the change of the argument of vector $D(i\omega)$ is equal to $2k\pi$, $k \in \mathbb{Z}$ when ω increases from 0 to $+\infty$. This proves (i).

Suppose $\alpha + \alpha\kappa + \mu > 0$. In this case, we have $\Re(D(0)) > 0$ and $\Im(D(0)) = 0$. Hence, if $\Im(D(i\omega)) > 0$, for all $\omega > 0$, then the change of the argument of $D(i\omega)$ is equal to π.

Substituting $x = \omega\tau$ in equation (6) and multiplying the result by τ, we obtain

$$\tau\Im(D(x)) = \gamma(\alpha+1)x + \gamma(\kappa+\mu)x\cos(x) - \gamma^2\tau(\alpha\kappa+\mu)\sin(x).$$

For every $x \geq 0$, we have

$$\gamma(\kappa+\mu)x\cos(x) \geq -\gamma x|\kappa+\mu|,$$
$$\gamma^2\tau(\alpha\kappa+\mu)\sin(x) \leq \gamma^2\tau|\alpha\kappa+\mu|x.$$

Therefore,

$$\tau\Im(D(x)) \geq \gamma x\left((\alpha+1) - |\kappa+\mu| - \gamma\tau|\alpha\kappa+\mu|\right).$$

Applying the above inequality, we see that if

$$\alpha + 1 > |\kappa+\mu| \qquad \text{and} \qquad \tau < \frac{(\alpha+1) - |\kappa+\mu|}{\gamma|\alpha\kappa+\mu|},$$

then all roots of equation (5) have strictly negative real parts. This completes the proof. ∎

LEMMA 3.2. *Equation (5) has purely imaginary roots $\lambda_j(\tau_j) = i\omega_j$, for some $\tau_j = \tau_0 + 2j\pi$ $(j = 0,1,2,\dots)$, $\tau_0 > 0$, iff*

$$\alpha^2 > (\alpha\kappa+\mu)^2,$$
$$\alpha^2 + 1 < (\kappa+\mu)^2,$$
$$\left(\alpha^2 + 1 - (\kappa+\mu)^2\right)^2 = 4\left(\alpha - (\alpha\kappa+\mu)^2\right),$$

or

$$\alpha^2 + 1 < (\kappa+\mu)^2,$$
$$\left(\alpha^2 + 1 - (\kappa+\mu)^2\right)^2 > 4\left(\alpha - (\alpha\kappa+\mu)^2\right),$$

or

$$\alpha^2 < (\alpha\kappa+\mu)^2.$$

PROOF. Assume that equation (5) has purely imaginary roots $\lambda_j(\tau_j) = i\omega_j$, for some $\tau_j = \tau_0 + 2j\pi$ $(j = 0,1,2,\dots)$. We obtain

$$|P(i\omega_j)|^2 = |Q(i\omega_j)|^2.$$

Following [14], we define the auxiliary function $S(\omega_j)$,

$$S(\omega_j) = |P(i\omega_j)|^2 - |Q(i\omega_j)|^2,$$

and look for stability switches that can occur when $S(\omega_j) = 0$. Substituting $y = \omega_j^2$, we obtain the equation,

$$y^2 + \gamma^2\left[\alpha^2 + 1 - (\kappa+\mu)^2\right]y + \gamma^4\left[\alpha^2 - (\alpha\kappa+\mu)^2\right] = 0. \tag{7}$$

We look for positive roots of equation (7). Let $\alpha^2 + 1 - (\kappa+\mu)^2 = b$ and $\alpha^2 - (\alpha\kappa+\mu)^2 = c$. Viete's formula implies that equation (7) has positive roots iff

$$\begin{matrix} c > 0, & & c > 0, & & c = 0, & & c < 0, \\ b < 0, & \text{or} & b < 0, & \text{or} & b < 0, & \text{or} & b \in \mathbb{R}, \\ \Delta = 0, & & \Delta > 0, & & \Delta > 0, & & \Delta > 0. \end{matrix}$$

The above conditions can be rewritten in the following form,

$$\begin{array}{ccccc} c > 0, & & c \in \mathbb{R}, & & c < 0, \\ b < 0, & \text{or} & b < 0, & \text{or} & b \in \mathbb{R}, \\ \Delta = 0, & & \Delta > 0, & & \Delta > 0. \end{array}$$

Moreover, we can see that the inequality $c < 0$ implies the condition $\Delta > 0$. Hence, the lemma follows. ∎

LEMMA 3.3. *If $\tau_j = \tau_0 + 2j\pi$ ($j = 0, 1, 2, \ldots$), $\tau_0 > 0$, and $\lambda_j(\tau_j)$ is a root of equation (5) satisfying $\Re(\lambda_j(\tau_j)) = 0$ and $\Im(\lambda_j(\tau_j)) = \omega_j$, then*

$$\left.\frac{d\Re(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} > 0,$$

iff

$$-2\omega_j^4 + \gamma^2\omega_j^2\left((\kappa+\mu)^2 - (\alpha^2+1)\right) < 0. \tag{8}$$

PROOF. For $\lambda(\tau) = x + iy$, we have

$$\begin{aligned} \Re(D(x+iy)) &= x^2 - y^2 + \alpha\gamma^2 + \gamma(\alpha+1)x + \left(\gamma^2(\alpha\kappa+\mu) + \gamma(\kappa+\mu)x\right)e^{-x\tau}\cos(y\tau) \\ &\quad + \gamma(\kappa+\mu)ye^{-x\tau}\sin(y\tau) = 0, \\ \Im(D(x+iy)) &= (\gamma(\alpha+1)+2x)\,y - \left(\gamma^2(\alpha\kappa+\mu) + \gamma(\kappa+\mu)x\right)e^{-x\tau}\sin(y\tau) \\ &\quad + \gamma(\kappa+\mu)ye^{-x\tau}\cos(y\tau) = 0. \end{aligned}$$

Using the theorem of implicit function and differentiating with respect to τ for $\tau = \tau_j$, we obtain the following system of equations,

$$\begin{aligned} &\left.\frac{\partial}{\partial x}\Re(D(x+iy))\right|_{\tau=\tau_j}\left.\frac{d\Re(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} + \left.\frac{\partial}{\partial y}\Re(D(x+iy))\right|_{\tau=\tau_j}\left.\frac{d\Im(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} + \left.\frac{\partial}{\partial \tau}\Re(D(x+iy))\right|_{\tau=\tau_j} = 0, \\ &\left.\frac{\partial}{\partial x}\Im(D(x+iy))\right|_{\tau=\tau_j}\left.\frac{d\Re(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} + \left.\frac{\partial}{\partial y}\Im(D(x+iy))\right|_{\tau=\tau_j}\left.\frac{d\Im(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} + \left.\frac{\partial}{\partial \tau}\Im(D(x+iy))\right|_{\tau=\tau_j} = 0. \end{aligned}$$

Applying the equalities $\lambda(\tau_j) = \lambda_j(\tau_j) = i\omega_j$, we calculate

$$\begin{aligned} \left.\frac{\partial}{\partial x}\Re(D(x+iy))\right|_{\tau=\tau_j} &= \left.\frac{\partial}{\partial y}\Im(D(x+iy))\right|_{\tau=\tau_j} = A, \\ \left.\frac{\partial}{\partial y}\Re(D(x+iy))\right|_{\tau=\tau_j} &= -\left.\frac{\partial}{\partial x}\Im(D(x+iy))\right|_{\tau=\tau_j} = -B, \\ &\left.\frac{\partial}{\partial \tau}\Re(D(x+iy))\right|_{\tau=\tau_j} = C, \\ &\left.\frac{\partial}{\partial \tau}\Im(D(x+iy))\right|_{\tau=\tau_j} = D, \end{aligned}$$

where

$$\begin{aligned} A &= \gamma(\alpha+1) + \gamma(\kappa+\mu)\cos(\omega_j\tau_j) - \gamma^2(\alpha\kappa+\mu)\tau_j\cos(\omega_j\tau_j) - \gamma(\kappa+\mu)\omega_j\tau_j\sin(\omega_j\tau_j), \\ B &= -2\omega_j + \gamma(\kappa+\mu)\sin(\omega_j\tau_j) - \gamma^2(\alpha\kappa+\mu)\tau_j\sin(\omega_j\tau_j) + \gamma(\kappa+\mu)\omega_j\tau_j\cos(\omega_j\tau_j), \\ C &= \gamma(\kappa+\mu)\omega_j^2\cos(\omega_j\tau_j) - \gamma^2(\alpha\kappa+\mu)\omega_j\sin(\omega_j\tau_j), \\ D &= -\gamma(\kappa+\mu)\omega_j^2\sin(\omega_j\tau_j) - \gamma^2(\alpha\kappa+\mu)\omega_j\cos(\omega_j\tau_j). \end{aligned}$$

The first and the fourth expressions are the same, we denote them by A. The second and the third have the opposite signs, we denote them by $-B$ and B, respectively. Finally, let denote the last two expressions by C and D. If $A^2 + B^2 \neq 0$, then

$$\left.\frac{d\Re(\lambda_j(\tau))}{d\tau}\right|_{\tau=\tau_j} = -\frac{AC+BD}{A^2+B^2}.$$

Looking for the condition guarantying positivity of this derivative and applying formulas calculated above, we obtain

$$-2\omega_j^4 + \gamma^2\omega_j^2[(\kappa+\mu)^2 - (\alpha^2+1)] < 0.$$

We can show that $A = B = 0$ implies $C = D = 0$ and this condition contradicts positivity of γ. This proves the lemma. ∎

Lemma 3.3 gives conditions for stability switches. If the equilibrium state is stable for $\tau = 0$, then it looses stability at $\tau_0 > 0$ when inequality (8) is fulfilled. Moreover, for all $\tau > \tau_0$, the equilibrium state stays in the region of instability.

Applying all above lemmas to equations (3),(4), we have the following results about the local stability of the equilibrium state $(\bar{u}, \bar{v})$.

THEOREM 3.1. *For equations (3),(4), the following statements hold,*

(i) *if equation (5) has no purely imaginary roots and* $\alpha + \alpha\kappa + \mu > 0$ *and* $\alpha + 1 > |\kappa + \mu|$ *and*

$$\tau < \frac{(\alpha+1) - |\kappa+\mu|}{\gamma\,|\alpha\kappa+\mu|},$$

then the equilibrium state $(\bar{u}, \bar{v})$ *is stable;*

(ii) *if equation (5) has no purely imaginary roots and* $\alpha + \alpha\kappa + \mu < 0$, *then* $(\bar{u}, \bar{v})$ *is unstable;*

(iii) *if equation (5) has purely imaginary roots* $\lambda_j(\tau_j) = i\omega_j$, *for some* $\tau_j = \tau_0 + 2j\pi$ $(j = 0, 1, 2, \dots)$, $\tau_0 > 0$ *and*

$$-2\omega_j^4 + \gamma^2\omega_j^2\left((\kappa+\mu)^2 - (\alpha^2+1)\right) < 0,$$

then

1. *the equilibrium state* $(\bar{u}, \bar{v})$, *stable for* $\tau = 0$, *is stable for* $\tau < \tau_0$ *and becomes unstable for* $\tau > \tau_0$,
2. *the equilibrium state* $(\bar{u}, \bar{v})$, *unstable for* $\tau = 0$, *is unstable, for every* $\tau > 0$.

REMARK 3.1. Suppose

$$\alpha^2 < (\alpha\kappa+\mu)^2 \qquad \text{and} \qquad (\kappa+\mu)^2 \leq (\alpha^2+1).$$

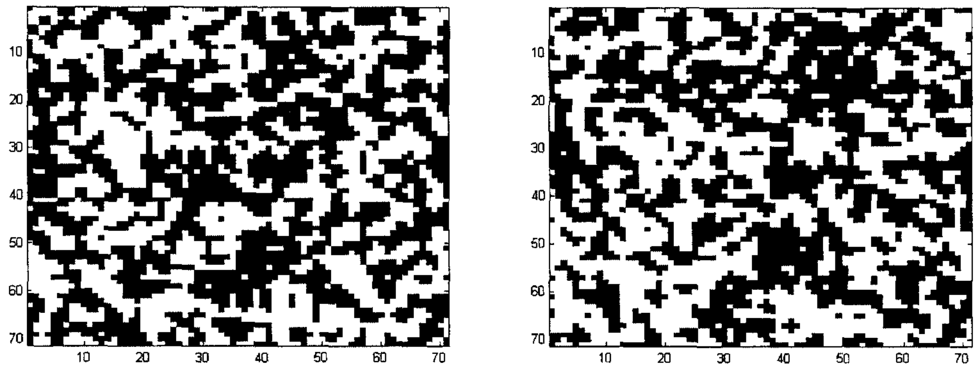
(i) If the equilibrium state $(\bar{u}, \bar{v})$ is stable for equations (3),(2), then it is stable for equations (3),(4) with small delays and there exists some $\tau_0 > 0$, such that $(\bar{u}, \bar{v})$ is stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$.

(ii) If the equilibrium state $(\bar{u}, \bar{v})$ is unstable for equations (3),(2), then it is also unstable for equations (3),(4) with any positive τ.

Now, we turn to equations (1)–(4), i.e., the model with nonzero diffusion and delay. Using the theory presented in [7], we can show that the unique solution to equations (1)–(4) with the initial functions near the equilibrium state exist locally, however, it can become negative due to the properties of equations (3),(4). In the next section, we assume the global existence (for every $t \geq 0$) and nonnegativity of solutions and study influence of time delay into the pattern formation. We perform this study numerically assuming the pattern formation in the model without delay (i.e., equations (1),(2)).

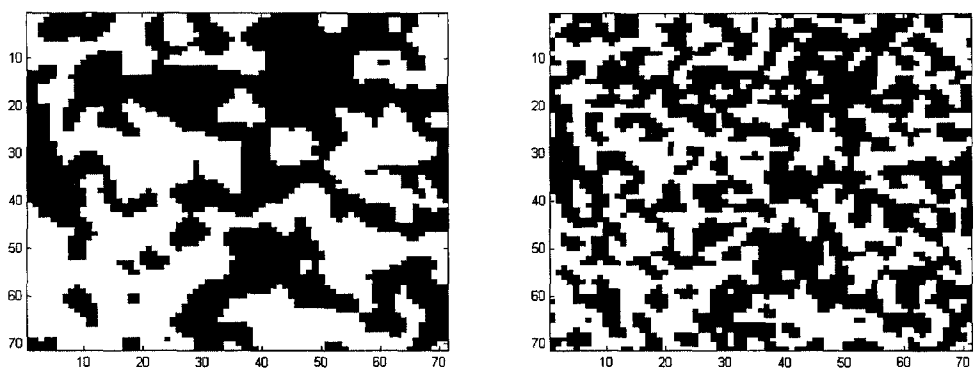
4. NUMERICAL SIMULATIONS

In this section, we present some numerical simulations of the model with and without delay. We focus on the influence of delay on generating animals coat patterns. The solutions to equations (1)–(4) and (1),(2) with zero flux boundary conditions and the same initial functions are computed. We use the initial conditions defined in the following way. First, two arrays for $t = 0$ are taken as a small random perturbations of the equilibrium state. For equations (1)–(4), we need to define $2N$ arrays with $\tau = Nk$ and k is the time step.



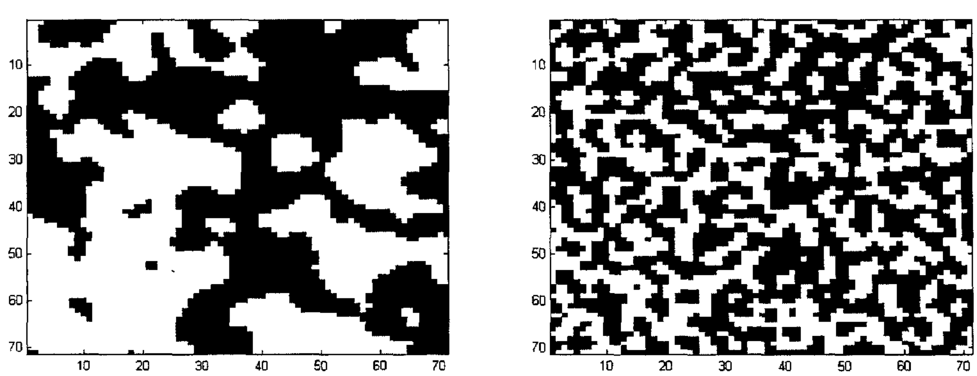
(a) (b)

Figure 1. Numerical simulations for $\tau = 0$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after one step.



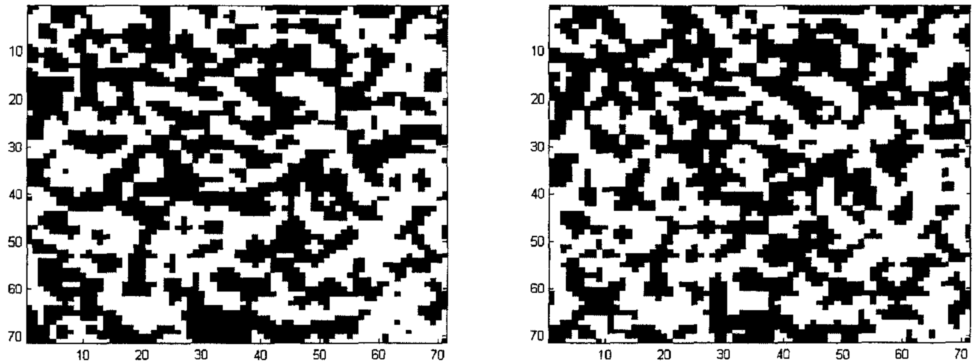
(a) (b)

Figure 2. Numerical simulations for $\tau = 0$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 50 steps.



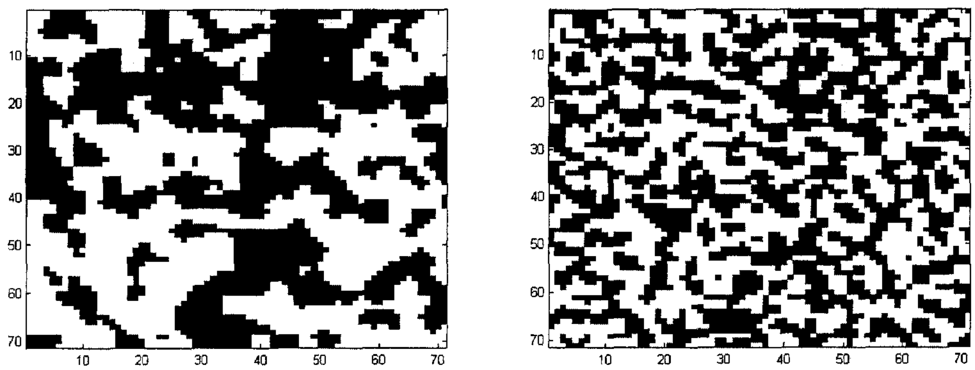
(a) (b)

Figure 3. Numerical simulations for $\tau = 0$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 100 steps.



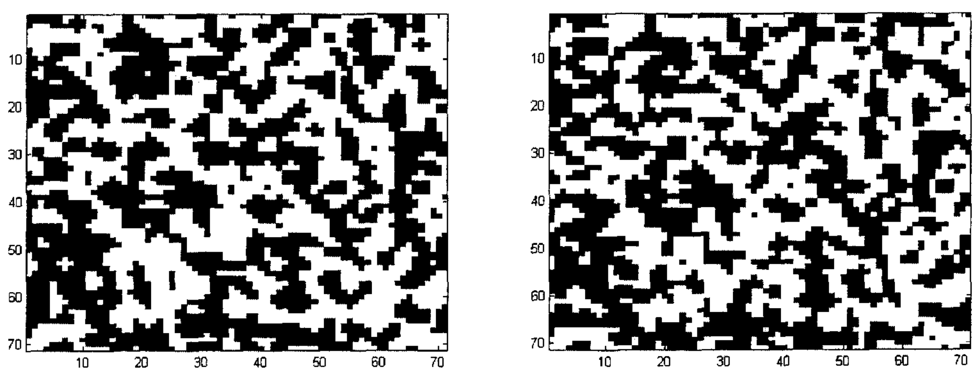
(a) (b)

Figure 4. Numerical simulations for $N = 50$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 50 steps.



(a) (b)

Figure 5. Numerical simulations for $N = 50$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 100 steps.



(a) (b)

Figure 6. Numerical simulations for $N = 80$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 50 steps.

Hence, we generate every tenth array for u and v and then, we interpolate the rest of arrays. We choose the same initial conditions for all numerical simulations because the patterns depend strongly on initial data. We let the black regions represent concentrations of u above the equilibrium state $\bar{u}$ and white regions—under $\bar{u}$. We present numerical results for fixed parameter values: $\alpha = 1.5$, $K = 0.125$, $\rho = 13$, $a = 103$, $b = 77$, and $d = 7$. We only change the coefficient γ and delay τ.

Figures 1–3 show simulation results for equations (1),(2). In the left column, we see figures for $\gamma = 0.1$ and in the right column for $\gamma = 5000$. We can observe that after 50 steps in both cases

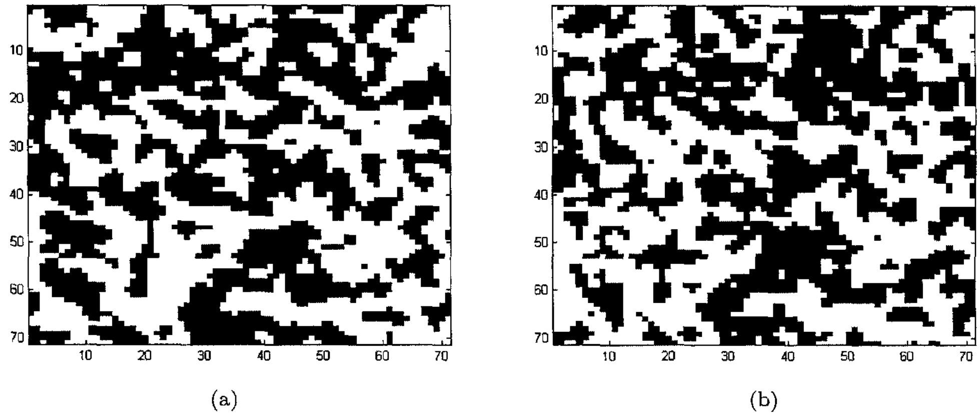


Figure 7. Numerical simulations for $N = 80$ and $\gamma = 0.1$ (a) and $\gamma = 5000$ (b) after 100 steps.

the pattern is completely different. The dramatic effect of scale is clearly demonstrated in these figures as the different values of γ were taken, see [15,16].

Figures 4 and 5 and Figures 6 and 7 show differences between results for different γ (values as above) and $N = 50$ and $N = 80$, respectively.

The results of numerical simulations suggest that the introduced delay changes the generated patterns. Moreover, when the delay increases the coefficient γ looses its crucial role in the mechanism of pattern formation comparing to no delay case.

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