1. **[More NFA/DFA constructions]** Show that the following languages are regular. You can use NFAs, DFAs or the closure operations of regular languages.
   (a) $L = \{w \mid w$ has equal number of occurrences of 01 and 10$\}$.
   (b) $L = \{w \mid w$ contains an even number of occurrences of 010$\}$.
   The occurrences can overlap here. For instance, the string $01010 \in L$.
   (c) The alphabet $\Sigma = \{a, b, c\}$, and $L = \{w \mid w = a^i b^j c^k, i, j, k \geq 0\}$.
   (d) $L = \{w \mid$ at least one of the last four positions is a 0$\}$.

2. Prove the following two properties of regular languages.
   (a) **[An easy one]** Show that every infinite regular language L contains an infinite proper subset $L' \subset L$ such that L' is also regular.
   (b) **[A slightly harder one]** Show that every infinite regular language L contains a proper subset $L' \subset L$ such that both L' and $L - L'$ are infinite and regular.

3. **[Alternate definition for an NFA]** Let us define a ∀NFA as a 5-tuple $N = (Q, \Sigma, \Delta, q_0, F)$ where $\Delta : Q \times \Sigma \to \mathcal{P}(Q)$, $q_0 \in Q$, and $F \subseteq Q$. The extended transition function is defined exactly as we did in class. The language accepted by N is defined as follows.

$$L(N) = \{w \mid \widehat{\Delta}(q_0, w) \subseteq F\}.$$

   In other words, a string is accepted iff all the non-deterministic choices lead to a final state.

   Show that the languages accepted by ∀NFAs are precisely the class of regular languages.

4. **[More examples of closure properties of regular languages]** These are problems that prove additional closure properties of regular languages, similar to some of the examples we saw in class.
   (a) We saw in class that if L is regular, then $\sqrt{L}$ is also regular. Can you prove a similar statement for $\sqrt[3]{L}$? How will you generalize this to $\sqrt[k]{L}$ for a constant k?
   (b) For a languages $L \subseteq \Sigma^*$, define $L_{\frac{1}{2}-}$ as follows.

$$L_{\frac{1}{2}-} = \{x \mid \exists\, y \text{ such that } |x| = |y| \text{ and } xy \in L\}.$$

   Show that if L is regular, then $L_{\frac{1}{2}-}$ is regular.
   (c) For two languages $L_1, L_2 \subseteq \Sigma^*$ define the *quotient* of L_1 by L_2, denoted as L_1/L_2, as follows.

$$L_1/L_2 = \{x \mid \exists y \in L_2 \text{ such that } xy \in L_1\}.$$

   Show that if L_1 is regular, and L_2 is any language (*not necessarily regular*), then L_1/L_2 is regular.

(d) For a language $L \subseteq \Sigma^*$, define $\mathrm{rot}(L)$, the *rotation* of L, as follows.

$$\mathrm{rot}(L) = \{xy \mid yx \in L\}.$$

Show that if L is regular, then $\mathrm{rot}(L)$ is regular.

5. **[Proving non-regularity of languages]** You can either use the pumping lemma for proving non-regularity, or you can prove it without explicitly using the lemma and directly giving a proof by contradiction.

(a) Consider the following two languages.

$$L_1 = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \geq k, k \geq 1\},$$
$$L_2 = \{1^k x \mid x \in \{0,1\}^* \text{ and } \#1(x) \leq k, k \geq 1\}.$$

One of the languages is regular, and the other is not. Which one is regular and which is non-regular? Justify your answer.

(b) Consider the following two languages.

$$L_1 = \{w \mid \exists\, x, y \in \{0,1\}^* - \varepsilon \text{ such that } w = xyx\},$$
$$L_2 = \{w \mid \exists\, x, y \in \{0,1\}^* - \varepsilon \text{ such that } w = xyx^R\}.$$

One of the languages is regular, and the other is not. Which one is regular and which is non-regular? Justify your answer.

(c) Show that $L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = 1 \Rightarrow j = k\}$ is not regular.

(d) Show that $L = \{w \mid \exists\, w \in \{0,1\}^* \text{ such that } w \neq w^R\}$ is not regular.