1. Consider the following constrained programming paradigm. You are allowed to access an input stream via a function next() that returns the next character of the input. If the input stream has ended the function returns a special character $\bot$. Essentially, you can read the input once from left to right. Furthermore, you are allowed to use variables that can only store a finite set of values that is independent of the actual length of the input. For instance, you cannot have a counter len that increments whenever you see a new character other than $\bot$. On the other hand, you can have a variable var that can store a number between 1 and 1000000000000.

   You also have access to a function write(c) that writes the character c to an output tape. Once again, characters once written cannot be overwritten, and neither can you go back and insert characters in-between ones already written. The goal of this problem is to understand what can be done by such a constrained programming paradigm.

   (a) Write pseudocode to check if the input stream starts with the symbols int.

   (b) Suppose that the input stream is an alphanumeric string. Write pseudocode to check if this string contains the substring 2024.

   (c) Suppose that the input stream is a number in decimal representation starting from the most-significant digit. Write pseudocode to check if the number is divisible by 11.

   (d) The input stream consists of tuples of digits $(0-9)$ that denote two non-negative integers from their least significant digit. For instance 12 and 123 will be represented by the input stream as $(2, 3), (1, 2), (0, 1), \bot$. Write pseudocode to output the sum of the two numbers starting from the least significant digit. In this example, you should output 5 3 1. *Do you think you can also compute the product of the two numbers?*

   (e) The input stream consists of an alphanumeric string. Write pseudocode to check if every occurence of the character i has an occurrence of e after it. For instance aaaiiiie and aaiieeeee are valid. But, aaaaiiieiabb is not valid.

   Think of the string as a program and the characters i and e as corresponding to if and else. In this constrained programming paradigm, do you think you can check if there are dangling else statements?

   For all the problems given above, check the number of variables that you used and the range of values that the variables can hold. Can you optimize these quantities?

2. Let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of natural numbers, rational numbers, and real numbers respectively. Let $\mathcal{P}(\mathbb{N})$ denote the set of all subsets of $\mathbb{N}$, and let $\mathcal{P}_f(\mathbb{N})$ denote the set of all *finite* subsets of $\mathbb{N}$.

   (a) Show that there is a bijection $f : \mathbb{Q} \to \mathbb{N}$.

   (b) Show that there exists a bijection $g : \mathbb{N} \to \mathcal{P}_f(\mathbb{N})$.

   (c) Show that there exists a bijection $h : \mathbb{R} \to \mathcal{P}(\mathbb{N})$.

3. There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that the others are also highly logical and devout (and they all know that they all know that the others are highly logical and devout, and so forth).

For the purposes of this logic puzzle, "highly logical" means that any conclusion that can logically deduced from the information and observations available to an islander, will automatically be known to that islander.

Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

One day, a blue-eyed foreigner visits the island and wins the complete trust of the tribe.

One evening, he addresses the entire tribe to thank them for their hospitality.

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What effect, if anything, does this faux pas have on the tribe?

Outcome 1: The foreigner's words have no effect, because his comments do not tell the tribe anything that they do not already know (everyone in the tribe can already see that there are several blue-eyed people in their tribe).

Outcome 2: 100 days after the address, all the blue eyed people commit suicide together.

Which of the two outcomes will happen, and why?

Hint: Use induction. What are the base cases, and how will you reason about them?