Ex 3.2

a) $z^n = 1$

$$\Rightarrow z^n = 1 = e^{i2\pi \cdot k} = [\cos(2\pi k) + i\sin(2\pi k)]$$

$$\Rightarrow z = 1^{1/n} = [\cos(2\pi k) + i\sin(2\pi k)]^{\frac{1}{n}}$$

$$\Rightarrow z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \quad [\because \text{Using De Moivre's Thm}]$$

(1) When $n=5$

n^{th} roots of unity for $n=5$, will be given for $k=0,1,2,3,4$ as shown below:

- For $k=0$:

$z_0 = \cos 0 + i\sin 0$

$\Rightarrow z_0 = 1 + 0i$

- For $k=1$:

$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$

$$\Rightarrow z_1 \approx 0.309 + 0.951i$$

- For $k=2$:

$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$\Rightarrow z_2 \approx -0.809 + 0.588i$$

- For $k=3$:

$$z_3 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$$

$$\Rightarrow z_3 = \cos\left(\frac{6\pi}{5} - 2\pi\right) + i\sin\left(\frac{6\pi}{5} - 2\pi\right) \quad [\because \text{To maintain Principal Argument}]$$

$$\Rightarrow z_3 = \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$$

$$\Rightarrow z_3 \approx -0.809 - 0.588i$$

- For $k=4$:

$$z_4 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$$

$$\Rightarrow z_4 = \cos\left(\frac{8\pi}{5} - 2\pi\right) + i\sin\left(\frac{8\pi}{5} - 2\pi\right) \quad [\because \text{To maintain Principal Argument}]$$

$$\Rightarrow z_4 = \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

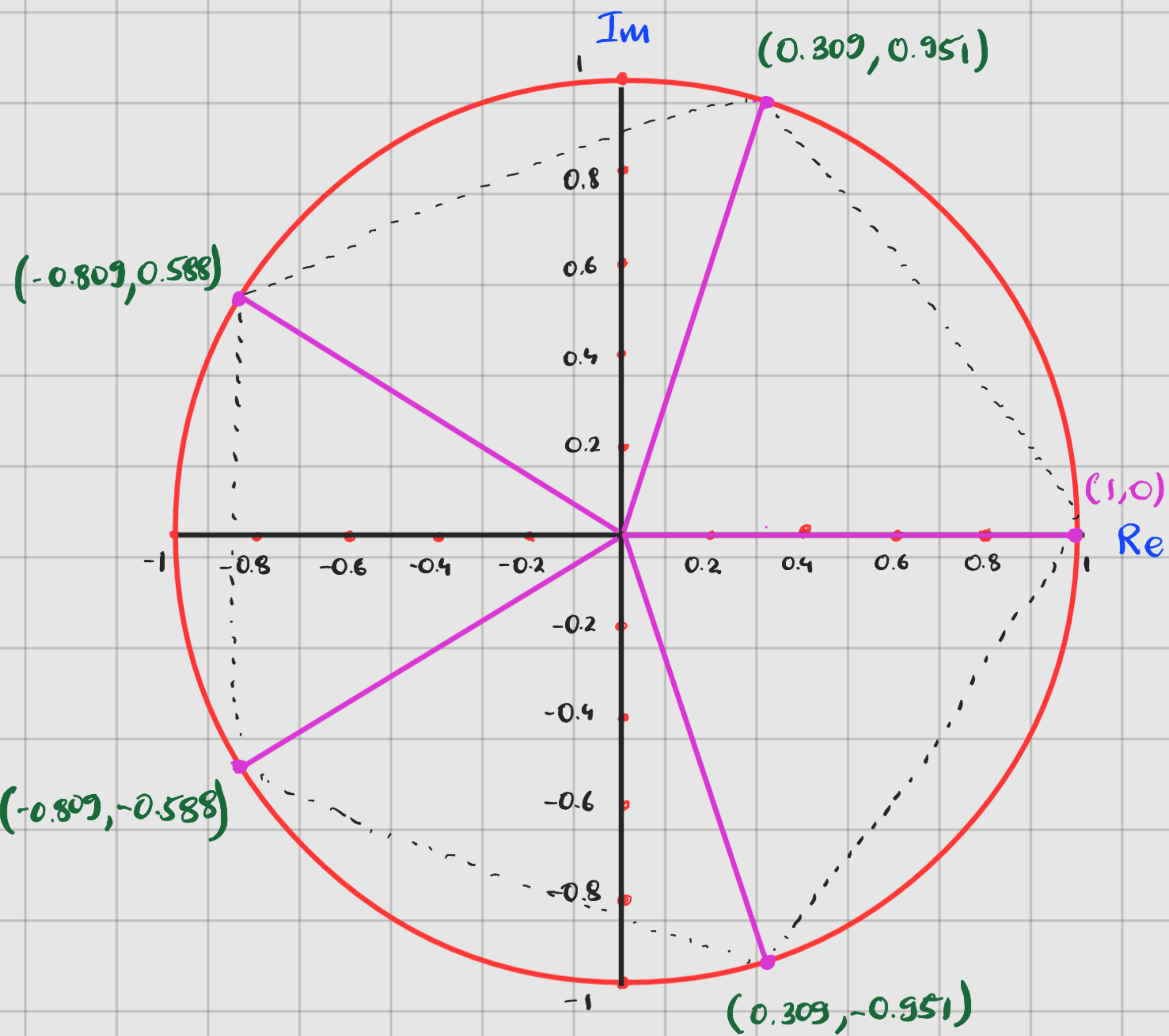
$$\Rightarrow z_4 \approx 0.309 - 0.951i$$

$\therefore$ So, the n^{th} roots of unity for $n=5$ are:

$$1,\ e^{\frac{2\pi}{5}i},\ e^{\frac{4\pi}{5}i},\ e^{-\frac{4\pi}{5}i},\ e^{-\frac{2\pi}{5}i}$$

$\therefore$ Similarly, the primitive n^{th} roots of unity for $n=5$ would be:

$$e^{\frac{2\pi}{5}i},\ e^{\frac{4\pi}{5}i},\ e^{-\frac{4\pi}{5}i},\ e^{-\frac{2\pi}{5}i}$$



n^{th} roots of unity for $n=5$

- indicates primitive n^{th} roots of unity for $n=5$

② When $n=7$

n^{th} roots of unity for $n=7$, will be given for $K=0,1,2,3,4,5,6$ as shown below:

- For $K=0$:

$z_0 = \cos 0 + i\sin 0$

$\Rightarrow z_0 = 1 + 0i$

- For $K=1$:

$z_1 = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$\Rightarrow z_1 \approx 0.623 + 0.782i$

- For $K=2$:

$z_2 = \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$

$\Rightarrow z_2 \approx -0.222 + 0.975i$

- For $K=3$:

$z_3 = \cos\left(\frac{6\pi}{7}\right) + i\sin\left(\frac{6\pi}{7}\right)$

$\Rightarrow z_3 \approx -0.901 + 0.434i$

- For $K=4$:

$$z_4 = \cos\left(\frac{8\pi}{7}\right) + i\sin\left(\frac{8\pi}{7}\right)$$

$$\Rightarrow z_4 = \cos\left(\frac{8\pi}{7} - 2\pi\right) + i\sin\left(\frac{8\pi}{7} - 2\pi\right)$$ [∵ To maintain Principal Argument]

$$\Rightarrow z_4 \approx \cos\left(\frac{6\pi}{7}\right) - i\sin\left(\frac{6\pi}{7}\right)$$

$$\Rightarrow z_4 \approx -0.901 - 0.434i$$

- For $K=5$:

$$z_5 = \cos\left(\frac{10\pi}{7}\right) + i\sin\left(\frac{10\pi}{7}\right)$$

$$\Rightarrow z_5 = \cos\left(\frac{10\pi}{7} - 2\pi\right) + i\sin\left(\frac{10\pi}{7} - 2\pi\right)$$ [∵ To maintain Principal Argument]

$$\Rightarrow z_5 \approx \cos\left(\frac{4\pi}{7}\right) - i\sin\left(\frac{4\pi}{7}\right)$$

$$\Rightarrow z_5 \approx -0.222 - 0.975i$$

- For $K=6$:

$$z_6 = \cos\left(\frac{12\pi}{7}\right) + i\sin\left(\frac{12\pi}{7}\right)$$

$$\Rightarrow z_6 = \cos\left(\frac{12\pi}{7} - 2\pi\right) + i\sin\left(\frac{12\pi}{7} - 2\pi\right)$$ [∵ To maintain Principal Argument]

$$\Rightarrow z_6 = \cos\left(\frac{2\pi}{7}\right) - i\sin\left(\frac{2\pi}{7}\right)$$

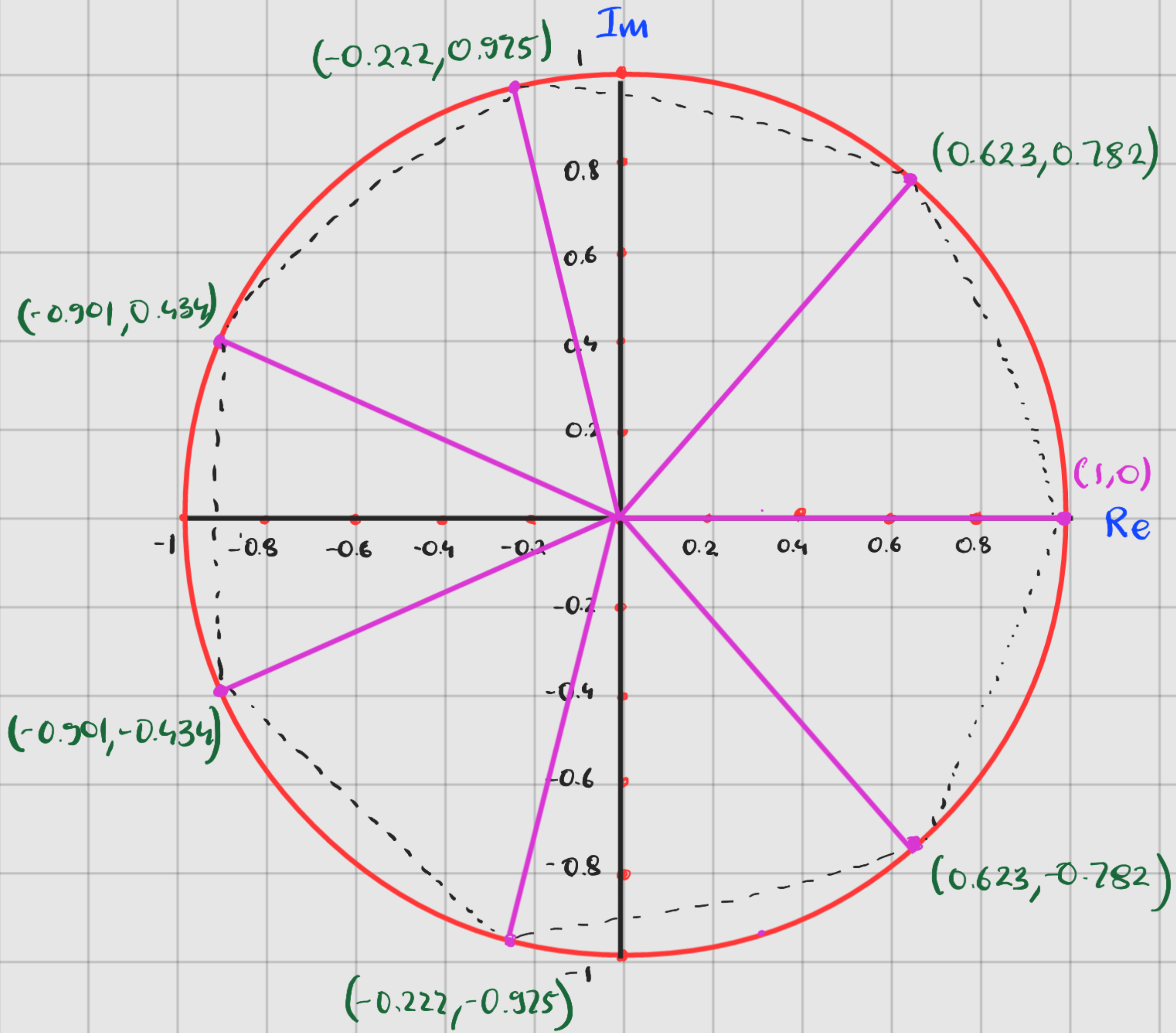
$$\Rightarrow z_6 \approx 0.623 - 0.782i$$

∴ So, the n^{th} roots of unity for $n=7$ are:

$$1,\ e^{\frac{2\pi}{7}i},\ e^{\frac{4\pi}{7}i},\ e^{\frac{6\pi}{7}i},\ e^{\frac{-6\pi}{7}i},\ e^{\frac{-4\pi}{7}i},\ e^{\frac{-2\pi}{7}i}$$

∴ Similarly, the primitive n^{th} roots of unity for $n=7$ would be:

$$e^{\frac{2\pi}{7}i},\ e^{\frac{4\pi}{7}i},\ e^{\frac{6\pi}{7}i},\ e^{\frac{-6\pi}{7}i},\ e^{\frac{-4\pi}{7}i},\ e^{\frac{-2\pi}{7}i}$$



n^{th} roots of unity for $n=7$

– indicates primitive n^{th} roots of unity for $n=7$

b) & c) To Prove: $\sum_{k=0}^{n-1} q^k = \dfrac{1-q^n}{1-q}$, $\forall\, q \in \mathbb{C},\ q \neq 1$

Proof: Let $q^n = 1$

$$\Rightarrow q = \sqrt[n]{1} = 1^{1/n}$$

$$\Rightarrow q = (\cos 0^\circ + i \sin 0^\circ)^{1/n}$$

$$\Rightarrow q = \left(\cos(2\pi k + 0^\circ) + i \sin(2\pi k + 0^\circ)\right)^{\frac{1}{n}} \quad — \text{(1)}$$

[∵ General Polar Form]

where our $k = 0, 1, 2, \ldots\ldots, n-1$

Using De Moivre's Theorem on (1):

$$\Rightarrow q = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$$

$$\Rightarrow q = e^{i\frac{2\pi k}{n}} \qquad [\because \text{Euler's Form}]$$

- For $k=0$, 1^{st} Root $= e^{i0} = e^0 = 1$
- For $k=1$, 2^{nd} Root $= e^{i\frac{2\pi}{n}} = \alpha$
- For $k=2$, 3^{rd} Root $= e^{i\frac{4\pi}{n}} = \alpha^2$

$\vdots$

- For $k=(n-1)$, n^{th} Root $= e^{\frac{i2\pi(n-1)}{n}} = \alpha^{n-1}$

n^{th} roots of unity are: $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ where $\alpha = e^{i\frac{2\pi}{n}}$

$\Rightarrow \quad 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}$ [Sum of n^{th} roots of unity]

This is a Geometric Progression!

$$S_n = \frac{a(1-r^n)}{1-r}$$ [$\because$ Sum of n-terms in G.P]

Here a = first term ; r = common ratio $= \alpha$

$$\Rightarrow S_n = \frac{1(1-\alpha^n)}{1-\alpha}$$

$$\Rightarrow S_n = \frac{1-\left(e^{i\frac{2\pi}{n}}\right)^n}{1-e^{i\frac{2\pi}{n}}}$$

$$\Rightarrow S_n = \frac{1-\left(e^{i\frac{2\pi k}{n}}\right)^n}{1-e^{i\frac{2\pi k}{n}}}$$ [$\because$ Since, cos & sin has a period of 2π] — 2

$$\Rightarrow \boxed{\sum_{k=1}^{n-1} q^k = \frac{1-q^n}{1-q}}$$ [$\because$ Substituting the value of q back]

Hence, Proved

To Prove : sum of all n^{th} roots of unity equals to 0, for $n>1$

Proof :- The Exact same steps as above. We will now just expand a bit further on (2) :

$$S_n = \frac{1-\left(e^{\frac{i2\pi k}{\cancel{n}}}\right)^{\cancel{n}}}{1-e^{\frac{i2\pi k}{n}}} \quad , \quad n>1$$

$$\Rightarrow \quad S_n = \frac{1-\{\cancelto{1}{\cos(2\pi k)} + i\cancelto{0}{\sin(2\pi k)}\}}{1-e^{\frac{i2\pi k}{n}}} \quad [\because \text{Euler's Formula}]$$

$$\Rightarrow \quad S_n = \frac{\cancel{1}-\cancel{1}}{1-e^{\frac{i2\pi k}{n}}} = \boxed{0}$$

Hence, Proved