Assignment 3

Foundations of Audio Signal Processing

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Exercise 3.1

a)

$$\begin{aligned}
cos(\alpha) &= \frac{1}{2}(2\cos\alpha) = \frac{1}{2}(2\cos\alpha + i\sin\alpha - i\sin\alpha) \\
&= \frac{1}{2}(cos\alpha + i\sin\alpha + \cos(-\alpha) + i\sin(-\alpha)) \\
&= \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})
\end{aligned}$$

b)

$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{i\sin(\alpha+\beta)}{i} = \frac{i\sin(\alpha+\beta) + \cos(\alpha+\beta) - \cos(\alpha+\beta)}{i} \\
&= \frac{e^{\alpha+\beta} - \cos(\alpha+\beta)}{i} = \frac{e^{\alpha}e^{\beta} - \cos(\alpha+\beta)}{i} \\
&= \frac{1}{i}((\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) - \cos(\alpha+\beta)) \\
&= \frac{1}{i}(\cos\alpha\cos\beta + \cos\alpha\ i\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta - \cos(\alpha+\beta)) \\
&= \cos\alpha\sin\beta + \cos\beta\sin\alpha + \frac{1}{i}(\cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos(\alpha+\beta)) \\
&= \cos\alpha\sin\beta + \cos\beta\sin\alpha + \frac{1}{i}(\cos(\alpha+\beta) - \cos(\alpha+\beta)) \\
&= \cos\alpha\sin\beta + \cos\beta\sin\alpha
\end{aligned}$$

c)

$$\begin{aligned}
\sin(\alpha)^2 + \cos(\alpha)^2 &= \sin(\alpha)^2 + i\sin(\alpha)\cos(\alpha) - i\sin(\alpha)\cos(\alpha) + \cos(\alpha)^2 \\
&= (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha) \\
&= e^{i\alpha}e^{-i\alpha} = e^{i\alpha - i\alpha} = e^0 = 1
\end{aligned}$$

Ex 3.2

a) $z^n = 1$

$\Rightarrow z^n = 1 = e^{i2\pi \cdot k} = [\cos(2\pi k) + i\sin(2\pi k)]$

$\Rightarrow z = 1^{1/n} = [\cos(2\pi k) + i\sin(2\pi k)]^{\frac{1}{n}}$

$\Rightarrow z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$ [$\because$ Using De Moivre's Thm]

① When $n=5$

n^{th} roots of unity for $n=5$, will be given for $k=0,1,2,3,4$ as shown below:

- For $k=0$:

$z_0 = \cos 0 + i\sin 0$

$\Rightarrow z_0 = 1 + 0i$

- For $K=1$:

$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$

$$\Rightarrow z_1 \approx 0.309 + 0.951i$$

- For $K=2$:

$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$\Rightarrow z_2 \approx -0.809 + 0.588i$$

- For $K=3$:

$$z_3 = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$$

$$\Rightarrow z_3 = \cos\left(\frac{6\pi}{5} - 2\pi\right) + i\sin\left(\frac{6\pi}{5} - 2\pi\right)$$ [$\because$ To maintain Principal Argument]

$$\Rightarrow z_3 = \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$$

$$\Rightarrow z_3 \approx -0.809 - 0.588i$$

- For $K=4$:

$$z_4 = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$$

$$\Rightarrow z_4 = \cos\left(\frac{8\pi}{5} - 2\pi\right) + i\sin\left(\frac{8\pi}{5} - 2\pi\right)$$ [$\because$ To maintain Principal Argument]

$$\Rightarrow z_4 = \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

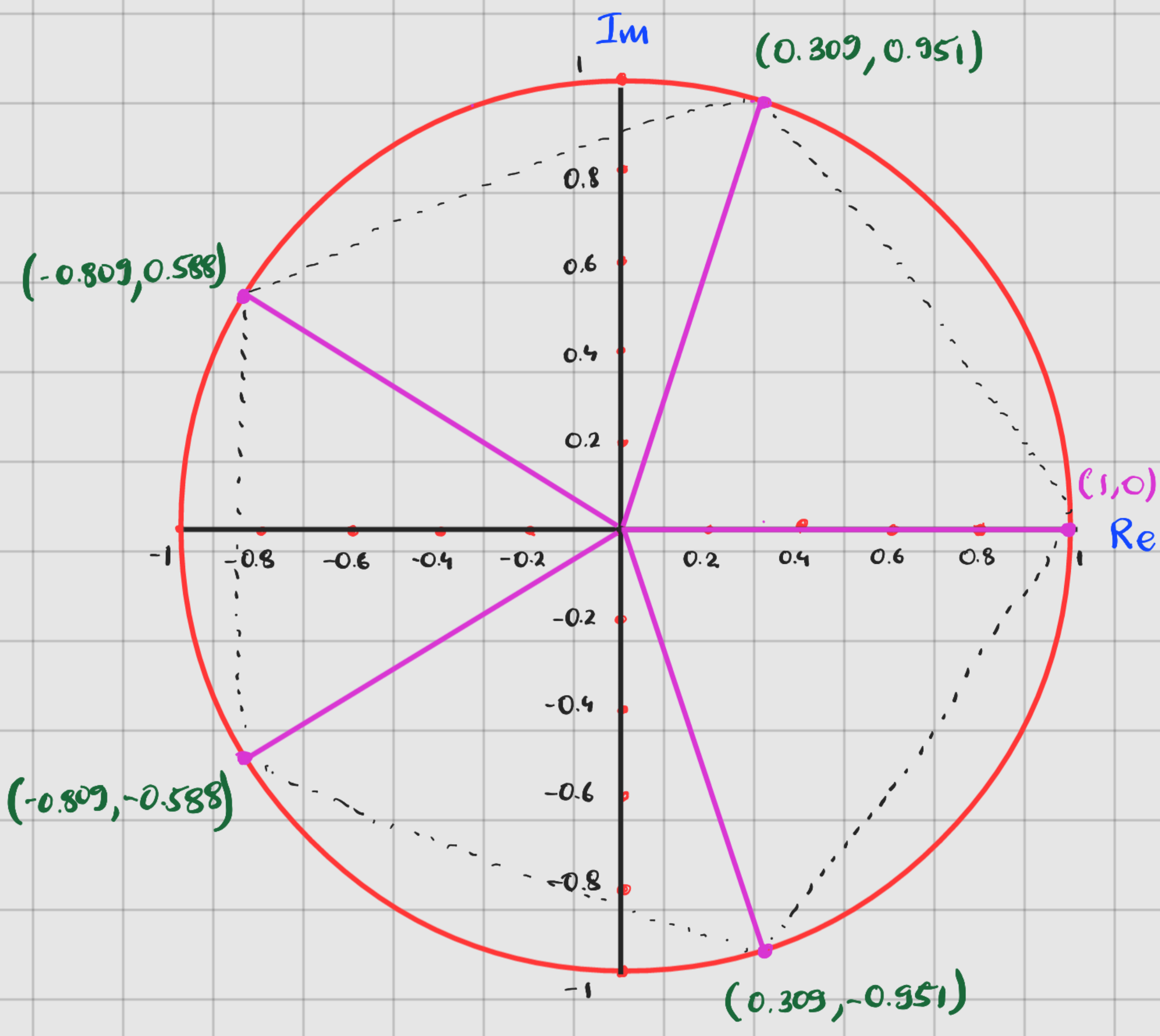
$$\Rightarrow z_4 \approx 0.309 - 0.951i$$

∴ So, the n^{th} roots of unity for $n=5$ are:

$$1,\ e^{\frac{2\pi}{5}i},\ e^{\frac{4\pi}{5}i},\ e^{-\frac{4\pi}{5}i},\ e^{-\frac{2\pi}{5}i}$$

∴ Similarly, the primitive n^{th} roots of unity for $n=5$ would be:

$$e^{\frac{2\pi}{5}i},\ e^{\frac{4\pi}{5}i},\ e^{-\frac{4\pi}{5}i},\ e^{-\frac{2\pi}{5}i}$$



n^{th} roots of unity for $n=5$

▨ - indicates primitive n^{th} roots of unity for $n=5$

② When $n=7$

n^{th} roots of unity for $n=7$, will be given for $K=0,1,2,3,4,5,6$ as shown below:

- For $K=0$:

$z_0 = \cos 0 + i \sin 0$

$\Rightarrow z_0 = 1 + 0i$

- For $K=1$:

$z_1 = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$

$\Rightarrow z_1 \approx 0.623 + 0.782i$

- For $K=2$:

$z_2 = \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$

$\Rightarrow z_2 \approx -0.222 + 0.975i$

- For $K=3$:

$z_3 = \cos\left(\frac{6\pi}{7}\right) + i\sin\left(\frac{6\pi}{7}\right)$

$\Rightarrow z_3 \approx -0.901 + 0.434i$

• For $k=4$:

$$z_4 = \cos\left(\frac{8\pi}{7}\right) + i\sin\left(\frac{8\pi}{7}\right)$$

$$\Rightarrow z_4 = \cos\left(\frac{8\pi}{7} - 2\pi\right) + i\sin\left(\frac{8\pi}{7} - 2\pi\right)$$ [$\because$ To maintain Principal Argument]

$$\Rightarrow z_4 \approx \cos\left(\frac{6\pi}{7}\right) - i\sin\left(\frac{6\pi}{7}\right)$$

$$\Rightarrow z_4 \approx -0.901 - 0.434i$$

• For $k=5$:

$$z_5 = \cos\left(\frac{10\pi}{7}\right) + i\sin\left(\frac{10\pi}{7}\right)$$

$$\Rightarrow z_5 = \cos\left(\frac{10\pi}{7} - 2\pi\right) + i\sin\left(\frac{10\pi}{7} - 2\pi\right)$$ [$\because$ To maintain Principal Argument]

$$\Rightarrow z_5 \approx \cos\left(\frac{4\pi}{7}\right) - i\sin\left(\frac{4\pi}{7}\right)$$

$$\Rightarrow z_5 \approx -0.222 - 0.975i$$

• For $k=6$:

$$z_6 = \cos\left(\frac{12\pi}{7}\right) + i\sin\left(\frac{12\pi}{7}\right)$$

$$\Rightarrow z_6 = \cos\left(\frac{12\pi}{7} - 2\pi\right) + i\sin\left(\frac{12\pi}{7} - 2\pi\right)$$ [$\because$ To maintain Principal Argument]

$$\Rightarrow z_6 = \cos\left(\frac{2\pi}{7}\right) - i\sin\left(\frac{2\pi}{7}\right)$$

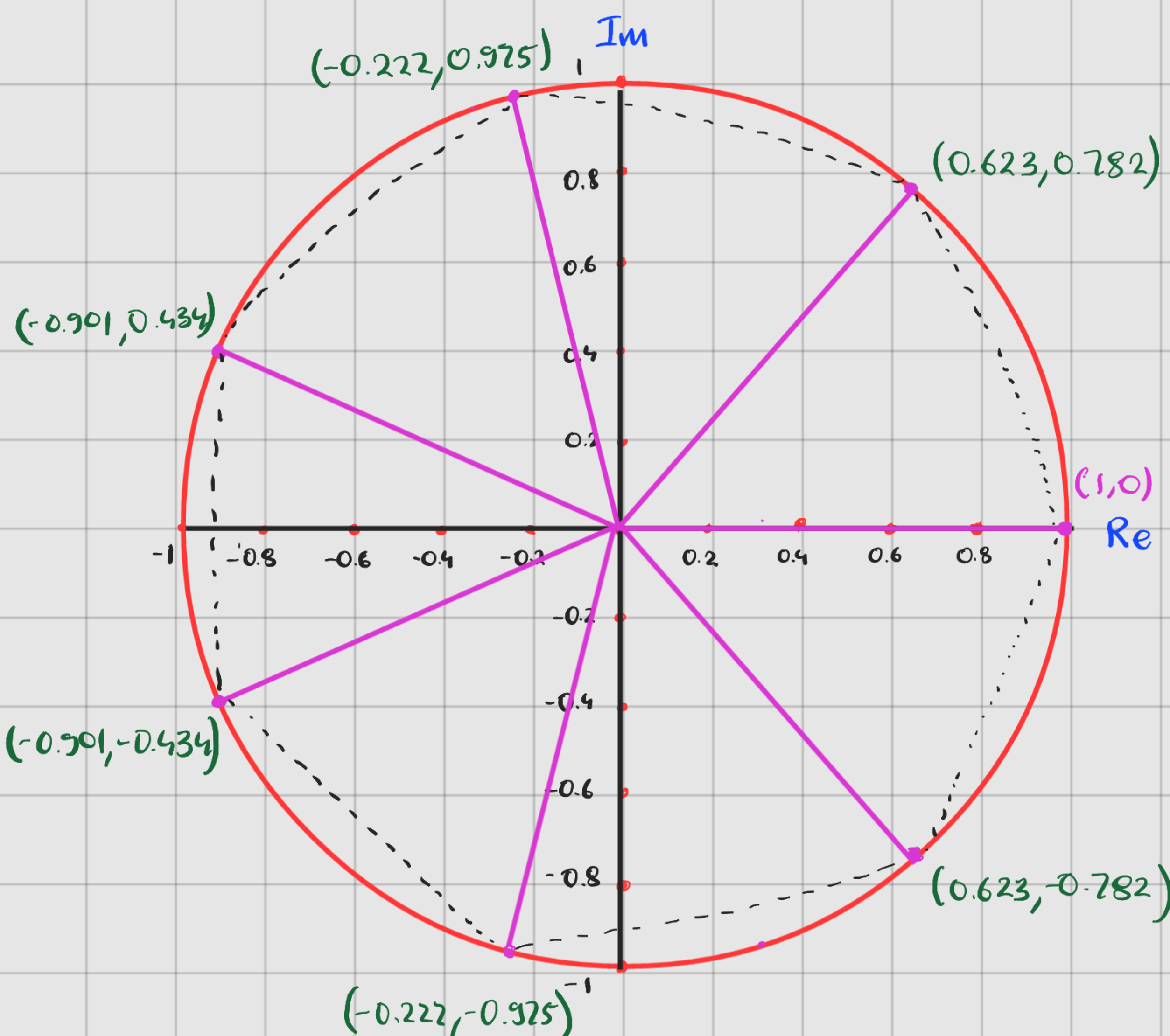
$$\Rightarrow z_6 \approx 0.623 - 0.782i$$

$\therefore$ So, the n^{th} roots of unity for $n=7$ are:

$$1, e^{\frac{2\pi}{7}i}, e^{\frac{4\pi}{7}i}, e^{\frac{6\pi}{7}i}, e^{\frac{-6\pi}{7}i}, e^{\frac{-4\pi}{7}i}, e^{\frac{-2\pi}{7}i}$$

$\therefore$ Similarly, the primitive n^{th} roots of unity for $n=7$ would be:

$$e^{\frac{2\pi}{7}i}, e^{\frac{4\pi}{7}i}, e^{\frac{6\pi}{7}i}, e^{\frac{-6\pi}{7}i}, e^{\frac{-4\pi}{7}i}, e^{\frac{-2\pi}{7}i}$$



n^{th} roots of unity for $n=7$

- indicates primitive n^{th} roots of unity for $n=7$

b) & c) To Prove : $\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$, $\forall q \in \mathbb{C}$, $q \neq 1$

Proof : Let $q^n = 1$

$\Rightarrow q = \sqrt[n]{1} = 1^{1/n}$

$\Rightarrow q = (\cos 0^\circ + i \sin 0^\circ)^{1/n}$

$\Rightarrow q = \left(\cos(2\pi k + 0^\circ) + i \sin(2\pi k + 0^\circ)\right)^{\frac{1}{n}}$ — ①

[$\because$ General Polar Form]

where our $k = 0, 1, 2, \ldots\ldots n-1$

Using De Moivre's Theorem on ① :

$\Rightarrow q = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$

$\Rightarrow q = e^{i\frac{2\pi k}{n}}$ [$\because$ Euler's Form]

- For $k=0$, 1^{st} Root $= e^{i0} = e^0 = 1$
- For $k=1$, 2^{nd} Root $= e^{i\frac{2\pi}{n}} = \alpha$
- For $k=2$, 3^{rd} Root $= e^{i\frac{4\pi}{n}} = \alpha^2$

$\vdots$

- For $k=(n-1)$, n^{th} Root $= e^{\frac{i2\pi(n-1)}{n}} = \alpha^{n-1}$

n^{th} roots of unity are: $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ where $\alpha = e^{i\frac{2\pi}{n}}$

$\Rightarrow$ $1+\alpha+\alpha^2+\ldots+\alpha^{n-1}$ [Sum of n^{th} roots of unity]

This is a Geometric Progression !

$$S_n = \frac{a(1-r^n)}{1-r}$$ [$\because$ Sum of n-terms in G.P]

Here a = first term ; r = common ratio = α

$$\Rightarrow S_n = \frac{1(1-\alpha^n)}{1-\alpha}$$

$$\Rightarrow S_n = \frac{1-\left(e^{i\frac{2\pi}{n}}\right)^n}{1-e^{i\frac{2\pi}{n}}}$$

$$\Rightarrow S_n = \frac{1-\left(e^{\frac{i2\pi k}{n}}\right)^n}{1-e^{\frac{i2\pi k}{n}}}$$ [$\because$ Since, cos & sin has a period of 2π] —— 2

$$\Rightarrow \boxed{\sum_{k=1}^{n-1} q^k = \frac{1-q^n}{1-q}}$$ [$\because$ Substituting the value of q back]

Hence, Proved

To Prove : sum of all n^{th} roots of unity equals to 0, for $n>1$

Proof :- The Exact same steps as above. We will now just expand a bit further on (2) :

$$S_n = \frac{1-\left(e^{\frac{i2\pi k}{\cancel{n}}}\right)^{\cancel{n}}}{1-e^{\frac{i2\pi k}{n}}} \quad , \quad n>1$$

$$\Rightarrow \quad S_n = \frac{1-\{\cancelto{1}{\cos(2\pi k)} + i\cancelto{0}{\sin(2\pi k)}\}}{1-e^{\frac{i2\pi k}{n}}} \quad [\because \text{Euler's Formula}]$$

$$\Rightarrow \quad S_n = \frac{\cancel{1}-\cancel{1}}{1-e^{\frac{i2\pi k}{n}}} = \boxed{0}$$

Hence, Proved