Binary Renyi Correlation
A simpler proof of Witsenhausen's result and a tighter upper bound

1 Renyi Correlation

Definition 1. *Let $(X, Y) \sim p(x, y)$ be a pair of dependent RV's. Then the Renyi Correlation $\rho(X, Y)$ is defined as*

$$\begin{aligned}
&\rho(X,Y) = \max_{f(x),g(y)} Ef(X)g(Y) \\
&\textit{subject to} \\
&Ef(X) = Eg(Y) = 0 \\
&Ef(X)^2 = Eg(Y)^2 = 1
\end{aligned}$$

where $f(x)$ and $g(y)$ are functions of x and y respectively.

Lemma 1. *Let $(X, Y) \sim p(x, y)$. Let $f(x), g(y)$ be arbitrary functions. Let $\rho = \rho(X, Y)$ be the Renyi Correlation. Then*

$$E_{XY}f(X)g(Y) \leq E_X f(X) E_Y g(Y) + \rho\sqrt{Var_X f(X)}\sqrt{Var_Y g(Y)} \tag{1}$$

$$E_{XY}f(X)g(Y) \leq \sqrt{E_X f(X)^2}\sqrt{E_Y g(Y)^2} \tag{2}$$

Proof. The first inequality should be clear from the definition of Correlation of 2 real-valued random variables and the fact that Renyi correlation is the maximum correlation.

The second inquality is simply cauchy-schwarts inequality.

In fact, with some work, we can prove

$$\begin{aligned}
E_{XY}f(X)g(Y) &\leq E_X f(X) E_Y g(Y) + \rho\sqrt{\mathrm{Var}_X f(X)}\sqrt{\mathrm{Var}_Y g(Y)} \\
&\leq \sqrt{E_X f(X)^2}\sqrt{E_Y g(Y)^2}
\end{aligned}$$

All three are equal when $f(x), g(y)$ are constants. ∎

2 Witsenhausen's Theorem

Theorem 1. *Let* $(X_1, Y_1) \sim p_1(x_1, y_1)$ *and* $(X_2, Y_2) \sim p_2(x_2, y_2)$ *be independent pairs of RVs. Then,* $\rho(X_1X_2, Y_1Y_2) = \max(\rho(X_1, Y_1), \rho(X_2, Y_2))$.

Proof. Let $\rho(X_1, Y_1) = \rho_1,\ \rho(X_2, Y_2) = \rho_2$.

Let $f(x_1, x_2), g(y_1, y_2)$ be any 2 functions.

Then,

$$\begin{aligned}
Ef(X_1, X_2)g(Y_1, Y_2) &= E_{X_1Y_1}E_{X_2Y_2}f(X_1, X_2)g(Y_1, Y_2) \\
&\leq E_{X_1Y_1}E_{X_2}f(X_1, X_2)E_{Y_2}g(Y_1, Y_2) \\
&\quad + \rho_2 E_{X_1Y_1}\sqrt{\mathrm{Var}_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_2}g(Y_1, Y_2)} \\
&\quad\ \text{(by equn. (1))}
\end{aligned}$$

The first term,

$$\begin{aligned}
&E_{X_1Y_1}E_{X_2}f(X_1, X_2)E_{Y_2}g(Y_1, Y_2) \\
&\leq E_{X_1}E_{X_2}f(X_1, X_2)E_{Y_1}E_{Y_2}g(Y_1, Y_2) \\
&\quad + \rho_1\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1, Y_2)} \\
&\quad\ \text{(by equn. (1))} \\
&= \rho_1\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1, Y_2)} \\
&\quad\ \text{(Due to zero mean constraints on f and g)}
\end{aligned}$$

The second term,

$$\begin{aligned}
&\rho_2 E_{X_1Y_1}\sqrt{\mathrm{Var}_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_2}g(Y_1, Y_2)} \\
&\leq \rho_2\sqrt{E_{X_1}\mathrm{Var}_{X_2}f(X_1, X_2)}\sqrt{E_{Y_1}\mathrm{Var}_{Y_2}g(Y_1, Y_2)} \\
&\quad\ \text{(by equn. (2))}
\end{aligned}$$

Back to the original chain of inequalities,

$$
\begin{aligned}
& Ef(X_1,X_2)g(Y_1,Y_2) \\
& \leq E_{X_1Y_1}E_{X_2}f(X_1,X_2)E_{Y_2}g(Y_1,Y_2) \\
& \quad + \rho_2 E_{X_1Y_1}\sqrt{\mathrm{Var}_{X_2}f(X_1,X_2)}\sqrt{\mathrm{Var}_{Y_2}g(Y_1,Y_2)} \\
& \leq \rho_1\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1,X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1,Y_2)} \\
& \quad + \rho_2\sqrt{E_{X_1}\mathrm{Var}_{X_2}f(X_1,X_2)}\sqrt{E_{Y_1}\mathrm{Var}_{Y_2}g(Y_1,Y_2)} \\
& \leq \max(\rho_1,\rho_2)\left[\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1,X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1,Y_2)}\right. \\
& \quad \left.+\sqrt{E_{X_1}\mathrm{Var}_{X_2}f(X_1,X_2)}\sqrt{E_{Y_1}\mathrm{Var}_{Y_2}g(Y_1,Y_2)}\right] \\
& \leq \max(\rho_1,\rho_2)\left[\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1,X_2)+E_{X_1}\mathrm{Var}_{X_2}f(X_1,X_2)}\right] \\
& \quad \left[\sqrt{E_{Y_1}\mathrm{Var}_{Y_2}g(Y_1,Y_2)+\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1,Y_2)}\right] \\
& \qquad \text{(By Cauchy Schwarts inequality)} \\
& = \max(\rho_1,\rho_2)\sqrt{\mathrm{Var}_{X_1X_2}f(X_1,X_2)}\sqrt{\mathrm{Var}_{Y_1Y_2}g(Y_1,Y_2)} \\
& \qquad \text{(By law of variance)} \\
& = \max(\rho_1,\rho_2)
\end{aligned}
$$

□

3 Binary Renyi Correlation

Definition 2. *Let $(X,Y) \sim p(x,y)$ be a pair of dependent RV's. Then the Binary Renyi Correlation $\rho_b(X,Y)$ is defined as*

$$
\begin{aligned}
& \rho_b(X,Y) = \max_{f(x),g(y)} Ef(X)g(Y) \\
& \textit{subject to} \\
& Ef(X) = Eg(Y) = 0 \\
& Ef(X)^2 = Eg(Y)^2 = 1 \\
& \textit{f(x), g(y) each take atmost 2 distinct values.}
\end{aligned}
$$

where $f(x)$ and $g(y)$ are functions of x and y respectively.

Example: *$f(x) \in \{-2,3\}, g(y) \in \{2.2,-0.86\}$ is permissible in the optimisation.*

Lemma 2. *Let $(X_1,Y_1) \sim p_1(x_1,y_1)$ and $(X_2,Y_2) \sim p_2(x_2,y_2)$ be independent pairs of RVs. Then,*

$$\rho_b(X_1X_2,Y_1Y_2) \leq \max(\rho(X_1,Y_1),\rho_b(X_2,Y_2))$$

Proof. We take a moment to look at the proof of theorem 1. In the above proof, if we were to consider Binary Renyi Correlation $\rho_b(X_1X_2, Y_1Y_2)$, we see that we can readily replace ρ_2 by $\rho_{2b} = \rho_b(X_2, Y_2)$. It is not clear at the moment whether one can replace ρ_1 by ρ_{1b}, so we retain ρ_1. $\square$

Definition 3. Stochastic Binary Renyi Correlation: *The Stochastic Binary Renyi Correlation $\rho_{bs}(X, Y)$ is defined similar to Binary Renyi Correlation $\rho_b(X, Y)$ except that the functions f and g are allowed to be stochastic, i.e. $f(X)$ is replaced by $f(L_X, X)$ where L_X is local randomness at node X. Similarly, $g(Y)$ is replaced by $g(L_Y, Y)$.*

Lemma 3.

$$\rho_{bs}(X, Y) = \rho_b(X, Y)$$

Proof.

$$\begin{aligned}
\rho_{bs}(X, Y) &= \rho_b(L_X X, L_Y Y) \\
&\leq \max(\rho(L_X, L_Y), \rho_b(X, Y)) \\
&= \rho_b(X, Y)
\end{aligned}$$

The last step follows from Witsenhausen's result and the fact that renyi correlation is 0 for independent RVs. The reverse inequality $\rho \leq \rho_s$ is obvious since deterministic functions are a subset of stochastic functions. $\square$

Thus whenever we consider Renyi Correlation, we consider only deterministic functions and not stochastic functions.

Theorem 2. *Let $(X_1, Y_1) \sim p_1(x_1, y_1)$ and $(X_2, Y_2) \sim p_2(x_2, y_2)$ be independent pairs of RVs. Then,*

$$\rho_b(X_1X_2, Y_1Y_2) \leq \max(\rho_b(X_1, Y_1), \rho_b(X_2, Y_2))$$

Proof. In the proof of theorem 1, we proved the inequality chain

$$\begin{aligned}
&E_{X_1Y_1}E_{X_2}f(X_1, X_2)E_{Y_2}g(Y_1, Y_2) \\
&\leq E_{X_1}E_{X_2}f(X_1, X_2)E_{Y_1}E_{Y_2}g(Y_1, Y_2) \\
&\quad + \rho_1\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1, Y_2)} \\
&\qquad \text{(by equn. (1))} \\
&= \rho_1\sqrt{\mathrm{Var}_{X_1}E_{X_2}f(X_1, X_2)}\sqrt{\mathrm{Var}_{Y_1}E_{Y_2}g(Y_1, Y_2)} \\
&\qquad \text{(Due to zero mean constraints on f and g)}
\end{aligned}$$

We note that the first term here depends only on $p(x_1, y_1)p_{X_2}(x_2)p_{Y_2}(y_2)$. Thus the dependence is only w.r.t the marginals of $p(x_2, y_2)$. Therefore, (X_2, Y_2) can be replaced by local randomness. An appeal to Lemma 3 tells us that we can replace ρ_1 by ρ_{1b}. We already saw in Lemma 2 that ρ_2 can be replaced by ρ_{2b}. This completes the proof. $\square$

4 Comments

1. We proved that stochastic binary renyi correlation is no better than deterministic binary renyi correlation. An analogous result can be proved for stochastic renyi correlation vs. deterministic renyi correlation.

2. Witsenhausen's upper bound on error probability of $\frac{1-\rho}{2}$ can be replaced by $\frac{1-\rho_b}{2}$.

3. The above proof carries over for n-ary Renyi Correlation in a straightforward way.