计算方法及MATLAB实现

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第六节　　非线性方程组的牛顿法

Newton Method for Non-linear System

- 6.1. 非线性方程组的牛顿法

【定义 1】 **二阶非线性方程组的牛顿法**　设二元方程组

$$\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases} \tag{6.1}$$

其求根函数 $f_1(x,y), f_2(x,y)$ 中至少有一个是变量 x,y 的非线性函数，则称为 二阶非线性方程组. 以矩阵 - 向量方程形式可写为

$$F(x)=0, \qquad x=(x,y)^T, F=(f_1,f_2)^T \tag{6.2}$$

定义向量函数 $F(x)$ 的 导函数矩阵

$$J = F^{'}(x) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} \tag{6.3}$$

为非线性方程组的 雅可比矩阵 (Jacobi Matrix). 设它是非奇异阵 (可逆阵)，即其行列式

$$\mid J \mid = Det(F^{'}(x)) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \neq 0 \tag{6.4}$$

则存在逆矩阵

$$J^{-1} = (F^{'}(x))^{-1} = \frac{1}{\mid J \mid} \begin{pmatrix} \dfrac{\partial f_2}{\partial y} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_1}{\partial x} \end{pmatrix} \tag{6.5}$$

令向量迭代函数

$$\varphi(\overrightarrow{x}) = \overrightarrow{x} - (F^{'}(\overrightarrow{x}))^{-1}F(\overrightarrow{x}) \tag{6.6}$$

由此构造向量迭代序列

$$\overrightarrow{x}_{k+1} = \overrightarrow{x}_k - (F^{'}(\overrightarrow{x}_k))^{-1}F(\overrightarrow{x}_k) \tag{6.7}$$

即

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{|J|} \begin{pmatrix} \dfrac{\partial f_2}{\partial y} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_1}{\partial x} \end{pmatrix} \begin{pmatrix} f_1(x_k, y_k) \\ f_2(x_k, y_k) \end{pmatrix} \tag{6.8}$$

称之为 二阶非线性方程组的牛顿迭代法.

【定义 2】 **n 阶非线性方程组的牛顿法** 设 n 元方程组

$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases} \tag{6.9}$$

其求根函数 $f_1, f_2, \cdots, f_n$ 中至少有一个是变量 $x_1, x_1, \cdots, x_n$ 的非线性函数，则称为 **n 阶非线性方程组**. 以矩阵 - 向量方程形式可写为

$$F(x) = 0, \qquad x = (x_1, x_2, \cdots, x_n)^T, F = (f_1, f_2, \cdots, f_n)^T \tag{6.10}$$

定义向量函数 $F(x)$ 的 **导函数矩阵**

$$J = F^{'}(x) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{6.11}$$

为非线性方程组的 **雅可比矩阵** (Jacobi Matrix). 设它是非奇异阵 (可逆阵)，即其行列式

$$
|J| = Det(F^{'}(x)) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \cdots \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \cdots \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} \cdots \dfrac{\partial f_n}{\partial x_n} \end{vmatrix} \neq 0 \qquad (6.12)
$$

则存在逆矩阵

$$J^{-1} = (F^{'}(x))^{-1} \tag{6.13}$$

令向量迭代函数

$$\varphi(\overrightarrow{x}) = \overrightarrow{x} - (F^{'}(\overrightarrow{x}))^{-1}F(\overrightarrow{x}) \tag{6.14}$$

由此构造向量迭代序列

$$\overrightarrow{x}^{(k+1)} = \overrightarrow{x}^{(k)} - (F^{'}(\overrightarrow{x}^{(k)}))^{-1}F(\overrightarrow{x}^{(k)}) \tag{6.15}$$

即

$$
\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} -
$$

$$
(F^{'}(\overrightarrow{x}^{(k)}))^{-1} \begin{pmatrix} f_1(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}) \\ \vdots \\ f_n(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}) \end{pmatrix}
$$

称之为 n 阶非线性方程组的牛顿迭代法.

- 6.2. **例题选讲**

【例 1 二阶非线性方程组的牛顿迭代法】

设二元非线性方程组初值问题

$$\begin{cases} f_1(x,y) = x^2 + y^2 - 4 = 0 \\ f_2(x,y) = x^2 - y^2 - 1 = 0 \\ x_0 = 1.6, y_0 = 1.2 \end{cases}$$

试用牛顿迭代法计算近似根.

【解】

向量函数 $F(x)$ 的 导函数矩阵 即非线性方程组的 雅可比矩阵 (Jacobi Matrix) 为

$$J = F^{'}(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

它是非奇异阵 (可逆阵)，其行列式

$$|J| = Det(F^{'}(x)) = \begin{vmatrix} 2x & 2y \\ -2x & -2y \end{vmatrix} = -8xy \neq 0$$

故存在逆矩阵

$$J^{-1} = (F^{'}(x))^{-1} = \frac{1}{\mid J \mid} \begin{pmatrix} \dfrac{\partial f_2}{\partial y} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_1}{\partial x} \end{pmatrix}$$

$$= -\frac{1}{8xy} \begin{pmatrix} -2y & -2y \\ -2x & 2x \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \dfrac{1}{x} & \dfrac{1}{x} \\ \dfrac{1}{y} & -\dfrac{1}{y} \end{pmatrix}$$

由此构造迭代序列

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{|J|} \begin{pmatrix} \frac{\partial f_2}{\partial y} & -\frac{\partial f_1}{\partial y} \\ -\frac{\partial f_2}{\partial x} & \frac{\partial f_1}{\partial x} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$
= \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{1}{x_k} & \frac{1}{x_k} \\ \frac{1}{y_k} & -\frac{1}{y_k} \end{pmatrix} \begin{pmatrix} x_k^2 + y_k^2 - 4 \\ x_k^2 - y_k^2 - 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{2x_k^2 - 5}{x_k} \\ \frac{2y_k^2 - 3}{y_k} \end{pmatrix}
$$

$$
= \frac{1}{4} \begin{pmatrix} \frac{2x_k^2 + 5}{x_k} \\ \frac{2y_k^2 + 3}{y_k} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2x_k + \frac{5}{x_k} \\ 2y_k + \frac{3}{y_k} \end{pmatrix}
$$

即是

$$\begin{cases} x_{k+1} = \frac{1}{4}(2x_k + \frac{5}{x_k}) \\ y_{k+1} = \frac{1}{4}(2y_k + \frac{3}{y_k}) \\ x_0 = 1.6, y_0 = 1.2 \end{cases}$$

由此迭代即得近似根．如

$$\begin{cases} x_1 = \frac{1}{4}(2 \cdot 1.6 + \frac{5}{1.6}) = \frac{1}{4}(3.2 + 3.125) = 1.581250 \\ y_1 = \frac{1}{4}(2 \cdot 1.2 + \frac{3}{1.2}) = \frac{1}{4}(2.4 + 2.5) = 1.225000 \end{cases}$$

【注记】显然，本二元非线性方程组求根问题的几何意义是要我们求圆 $x^2+y^2=4$ 与等轴双曲线 $x^2-y^2=1$ 的交点，直接求解或作图易知共有 4 个交点

$$(\sqrt{5/2},\sqrt{3/2}),(-\sqrt{5/2},\sqrt{3/2}),$$

$$(-\sqrt{5/2},-\sqrt{3/2}),(\sqrt{5/2},-\sqrt{3/2})$$

对于所给初值 $x_0=1.6,y_0=1.2$，即是求在第一象限内的交点 $(\sqrt{5/2},\sqrt{3/2})$.