计算方法及MATLAB实现

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第五节　弦截法和抛物线法
Secant Method and Parabola Method

- 5.1. **弦截法**(Secant Method)

【定义 1】 弦截法 (Secant Method) 将非线性方程 $f(x)=0$ 的牛顿迭代法

$$x_{k+1} := \varphi(x_k) = x_k - \frac{f(x_k)}{f'(x_k)} \tag{5.1}$$

其中的导数替换为 *差商* **(Divided Difference)**

$$f[x_{k-1}, x_k] = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \tag{5.2}$$

迭代法

$$x_{k+1} := \varphi(x_k) = x_k - \frac{f(x_k)}{f[x_{k-1}, x_k]} = x_k - \frac{f(x_k)}{\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}} \tag{5.3}$$

即

$$x_{k+1} := \varphi(x_k) = x_k - \frac{f(x_k)}{f(x_k) - f(x_{k-1})}(x_k - x_{k-1}) \tag{5.4}$$

称 $x_{k+1} = \varphi(x_k)$ 为 **弦截法 (Secant Method)** ，

【 弦截法的几何意义】

注意到求根函数曲线上经过两个近似点 $(x_{k-1}, f(x_{k-1})), (x_k, f(x_k))$ 的 割线 方程为

$$y = f(x_k) + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_k) \tag{5.5}$$

故弦截法 下级迭代值x_{k+1} 即为弦线与 x 横轴的交点 $y = 0$. 换言之，弦截法的几何意义是 将弦线外推到 0. 因此弦截法亦称 割线法, 也是一种 线性化方法.

【定理 1】 弦截法的局部收敛定理 求根函数 $f(x)$ 在不动点 $x^*: f(x^*) = 0, f'(x^*) \neq 0$ 附近邻域

$$\triangle : [x^* - \delta, x^* + \delta]$$

内至少二阶连续可微，且 $\forall x \in \triangle, f'(x) \neq 0$. 对于迭代初值 $x_0, x_1 \in \triangle$, 当邻域足够小时，比如满足

$$\delta < 1/M, \quad M = \frac{\max |f''(x)|}{2 \min |f'(x)|} \tag{5.6}$$

则弦截法迭代序列 $x_{k+1} = \varphi(x_k)$超线性收敛，

收敛阶为 $p=\dfrac{1+\sqrt{5}}{2}\approx 1.618.$ 即

$$\lim_{k\to+\infty}|\frac{e_{k+1}}{e_k^p}|=\lim_{k\to+\infty}|\frac{x_{k+1}-x^*}{(x_k-x^*)^p}|=Const=\widetilde{M}^{p-1},$$
$$\widetilde{M}:=|\frac{f^{''}(x^*)}{2f^{'}(x^*)}| \tag{5.7}$$

【 弦截法与牛顿法的比较】

弦截法与牛顿法均为 *线性化方法* 即以直线拟合曲线，而且都具有 *超线性收敛阶* ；但两者有显著区别：

(1) 弦截法以弦线拟合曲线，牛顿法以切线拟合曲线；

(2) 弦截法以差商替换导数；不需要导数信息，却要具备两个迭代初值 (有得有失).

(3) 弦截法的收敛速度稍慢于牛顿法，收敛阶为
$$\frac{1+\sqrt{5}}{2} \approx 1.618 < 2.$$

【例 1 弦截法求解超越方程的单根】 非线性方程初值问题

$$f(x) = xe^x - 1 = 0, x_0 = 0.5, x_1 = 0.6,$$

即求指数曲线 $y = e^x$ 与双曲线 $y = \dfrac{1}{x}$ 的交点，试用弦截法计算近似根.

【解】

用弦截法：

$$f[x_0, x_1] = \frac{f(0.6) - f(0.5)}{0.6 - 0.5}$$
$$= \frac{0.09327 + 0.17564}{0.1}$$
$$= 2.6891$$

故

$$
\begin{aligned}
x_{k+1} &:= \varphi(x_k) = x_k - \frac{f(x_k)}{f[x_{k-1}, x_k]} \\
&\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f[x_0, x_1]} \\
&= 0.6 - \frac{0.09327}{2.6891} \\
&= 0.6 - 0.0346847 = 0.5653153
\end{aligned}
$$

即 $x_2 \approx 0.565315$，如是递推即得各次弦截法迭代近似根.

根据迭代函数利用迭代递推公式作出近似根表如下:

k	x_k	
0	0.5	
1	0.6	
2	0.565315	
3	0.56709	
4	0.56714	

• 5.2. 抛物线法 (Müller-Parabola Method)

【定义 2】 **抛物线法** (Müller-Parabola Method) 考虑差商系数的二次抛物插值多项式或即抛物线函数

$$\begin{aligned} f(x) &= f(x_k) + f[x_{k-1}, x_k](x - x_k) \\ &+ f[x_{k-2}, x_{k-1}, x_k](x - x_{k-1})(x - x_k) \end{aligned} \tag{5.8}$$

将之外推到 0 求交点 x 满足 $f(x) = 0$，即

$$\begin{aligned} 0 &= f(x_k) + f[x_{k-1}, x_k](x - x_k) \\ &+ f[x_{k-2}, x_{k-1}, x_k](x - x_{k-1})(x - x_k) \end{aligned} \tag{5.9}$$

根据弦截法与牛顿法的迭代形式，下级迭代值是上级迭代值与微增量的差，如：

牛顿法 $x_{k+1} := \varphi(x_k) = x_k - \dfrac{f(x_k)}{f'(x_k)}$；

弦截法 $x_{k+1} := \varphi(x_k) = x_k - \dfrac{f(x_k)}{f[x_{k-1}, x_k]}$.

故可形式假设抛物线法的下级迭代值 $x = x_{k+1}$ 亦是上级迭代值 x_k 与微增量 (步长) 的差

$$x = x_{k+1} = x_k - \theta \tag{5.10}$$

代入 (5.8) 式得

$$
\begin{aligned}
&0 = f(x_k) + f[x_{k-1}, x_k](x - x_k) \\
&+f[x_{k-2}, x_{k-1}, x_k](x - x_{k-1})(x - x_k) \\
&= f(x_k) + f[x_{k-1}, x_k](x_k - \theta - x_k) \\
&+f[x_{k-2}, x_{k-1}, x_k](x_k - \theta - x_{k-1})(x_k - \theta - x_k) \\
&= f(x_k) - f[x_{k-1}, x_k]\theta + f[x_{k-2}, x_{k-1}, x_k](\theta - (x_k - x_{k-1}))\theta \\
&= f[x_{k-2}, x_{k-1}, x_k]\theta^2 \\
&-(f[x_{k-1}, x_k] + (x_k - x_{k-1})f[x_{k-2}, x_{k-1}, x_k])\theta + f(x_k)
\end{aligned} \tag{5.11}
$$

引入记号

$$
\begin{aligned}
&h = x_k - x_{k-1} \\
&a = f[x_{k-2}, x_{k-1}, x_k] = f_3 \\
&\omega = f[x_{k-1}, x_k] + (x_k - x_{k-1}) f[x_{k-2}, x_{k-1}, x_k] = f_2 + h f_3 \\
&c = f(x_k) = f_1
\end{aligned}
\tag{5.12}
$$

即获得关于微增量 θ 的二次方程

$$a\theta^2 - \omega\theta + c = 0 \tag{5.13}$$

于是由二次方程的求根公式有

$$\begin{aligned}\theta &= \frac{\omega \pm \sqrt{\omega^2 - 4ac}}{2a} \\ &= \frac{\omega \pm \sqrt{\omega^2 - 4f_3 f_1}}{2f_3} \\ &= \frac{\omega \pm \sqrt{\omega^2 - 4f[x_{k-2}, x_{k-1}, x_k]f(x_k)}}{2f[x_{k-2}, x_{k-1}, x_k]}\end{aligned} \tag{5.14}$$

故迭代法为

$$
\begin{aligned}
x_{k+1} &:= \varphi(x_k) = x_k - \theta \\
&= x_k - \frac{\omega \pm \sqrt{\omega^2 - 4f[x_{k-2}, x_{k-1}, x_k]f(x_k)}}{2f[x_{k-2}, x_{k-1}, x_k]}
\end{aligned} \tag{5.15}
$$

称 $x_{k+1} = \varphi(x_k)$ 为 抛物线法或缪勒法 (Müller-Parabola Method).

关于正负号，具体运算时就选取 ω 与根式 $\sqrt{\omega^2-4f[x_{k-2},x_{k-1},x_k]f(x_k)}$同号，以确保误差微增量绝对值 $|\theta|$ 较小．即有

$$\theta_{+,-}=\frac{2f_1}{\omega\pm\sqrt{\omega^2-4f_3f_1}}=\begin{cases}\dfrac{2|f_1|}{\omega+\sqrt{\omega^2-4f_3f_1}}, & \omega>0;\\ \dfrac{2|f_1|}{-\omega-\sqrt{\omega^2-4f_3f_1}}, & \omega<0.\end{cases} \tag{5.19}$$

【 抛物线法的几何意义】

抛物线法 下级迭代值x_{k+1} 即为抛物线

$$\begin{aligned}p_2(x) := f(x) &= f(x_k) + f[x_{k-1}, x_k](x - x_k)\\&+f[x_{k-2}, x_{k-1}, x_k](x - x_{k-1})(x - x_k)\end{aligned}$$

与 x 横轴的交点 $y = 0$. 换言之，抛物线法的几何意义是 将抛物线外推到 0. 因此抛物线法是一种 非线性化方法.

【**定理 2**】 **抛物线法的局部收敛定理** 求根函数 $f(x)$ 在不动点 $x^*: f(x^*) = 0, f'(x^*) \neq 0$ 附近邻域

$$\triangle : [x^* - \delta, x^* + \delta]$$

内至少三阶连续可微，且 $\forall x \in \triangle, f'(x) \neq 0$. 对于迭代初值 $x_0, x_1 \in \triangle$, 当邻域足够小时，则抛物线法迭代序列 $x_{k+1} = \varphi(x_k)$超线性收敛，收敛阶为 $p \approx 1.840$. 即
$\lim\limits_{k\to+\infty} |\frac{e_{k+1}}{e_k^p}| = \lim\limits_{k\to+\infty} |\frac{x_{k+1} - x^*}{(x_k - x^*)^p}| = Const = \widetilde{M}^{0.42}$,

$\widetilde{M} := |\frac{f'''(x^*)}{6f'(x^*)}|$

【 抛物线法 (缪勒法) 与牛顿法和弦截法的比较】

抛物线法和牛顿法、弦截法均为具有 超线性收敛阶 ；但三者有显著区别：

(1) 抛物线法为 非线性化方法 ， 而牛顿法和弦截法均为 线性化方法 ；

(2) 抛物线法要具备三个迭代初值 $x_0, x_1, , x_2$.

(3) 抛物线法和牛顿法、弦截法的收敛速度不同，牛顿法 $p = 2$, 抛物线法 $p \approx 1.840$, 弦截法收敛阶为 $\dfrac{1+\sqrt{5}}{2} \approx 1.618$.

【例 1 抛物线法求解超越方程的单根】 非线性方程初值问题

$$f(x) = xe^x - 1 = 0, x_0 = 0.5, x_1 = 0.6, x_2 = 0.56532$$

即求指数曲线 $y = e^x$ 与双曲线 $y = \dfrac{1}{x}$ 的交点，试用抛物线法计算近似根.

【解】

用抛物线法：一阶差商

$$
\begin{aligned}
f[x_0, x_1] &= \frac{f(0.6) - f(0.5)}{0.6 - 0.5} \\
&= \frac{0.0933 + 0.1756}{0.1} \\
&= 2.6891
\end{aligned}
$$

且

$$f[x_1, x_2] = \frac{f(0.5653) - f(0.6)}{0.565315 - 0.6}$$

$$= \frac{-0.0050 - 0.0933}{-0.0347}$$

$$= \frac{-0.0983}{-0.0347}$$

$$= 2.8345$$

故二阶差商满足

$$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$=\frac{2.8345-2.6891}{0.5653-0.5}$$

$$=\frac{0.1454}{0.0653}$$

$$=2.2267$$

从而

$$
\begin{aligned}
&\omega = f[x_{k-1}, x_k] + (x_k - x_{k-1}) f[x_{k-2}, x_{k-1}, x_k] = f_2 + h f_3 \\
&\Rightarrow \omega = f[x_1, x_2] + (x_2 - x_1) f[x_0, x_1, x_2] \\
&= 2.8345 - 0.0347 \cdot 2.2267 \\
&= 2.7573
\end{aligned}
$$

于是

$$
\begin{aligned}
x_{k+1} &:= \varphi(x_k) = x_k - \theta \\
&= x_k - \frac{2f(x_k)}{\omega \pm \sqrt{\omega^2 - 4f[x_{k-2}, x_{k-1}, x_k]f(x_k)}} \\
\Rightarrow x_3 &= x_2 - \frac{2f(x_2)}{\omega + \sqrt{\omega^2 - 4f[x_0, x_1, x_2]f(x_2)}} \\
&= 0.5653 - \frac{-0.0050 \cdot 2}{2.7573 + \sqrt{2.7573^2 + 4 \cdot 2.2267 \cdot 0.0050}} \\
&= 0.5671
\end{aligned}
$$

即 $x_3 \approx 0.5671$. 可见抛物线法比弦截法收敛更快.