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An information dimension of weighted complex networks

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HIGHLIGHTS

- This work presents a new information dimension of complex networks.
- In the proposed method, the detailed influence of the nodes is considered in each box.
- The information dimension can reveal the properties of complex networks.
- Some experiments are applied to show the effectiveness of the proposed method.

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ABSTRACT

The fractal and self-similarity are important properties in complex networks. Information dimension is a useful dimension for complex networks to reveal these properties. In this paper, an information dimension is proposed for weighted complex networks. Based on the box-covering algorithm for weighted complex networks (BCANw), the proposed method can deal with the weighted complex networks which appear frequently in the real-world, and it can get the influence of the number of nodes in each box on the information dimension. To show the wide scope of information dimension, some applications are illustrated, indicating that the proposed method is effective and feasible.

1. Introduction

Recently, nature science methodology has made great progress due to the development of complexity science. With its outstanding complexity and nonlinearity, complex network is developing rapidly and becomes a very important subject in the field of nature science over the years. Because complex network is an excellent mathematical model for studying complexity, it has been used in various scientific fields, including complexity science, nonlinear science, circuits and systems, computer science, control theory, theoretical physics, mathematics, biology. All of these prominently pay attention to topological features of system structure [1,2], the efficient spreading strategies of networks [3–8], and the importance of the node [9–12]. Through the study of complex networks, the fuzzy world can be quantified and predicted. Based on the research result of complex network, it can predict the development and operation of the system in a certain range [13], meanwhile it is able to predict the network crash [14]. It has produced a large number of practical models, and these models have been applied in a lot practical productions and have achieved a lot of practical results, like detecting the edge of image [15,16], solving the overload failures in real-world networks [17], analyzing the balance of the whole public traffic roads networks [18–21], analyzing the spread of the disease [22], studying the progress of society [23–25]. It has been applied in some important topics like the brain [26], cell differentiation [27], and multilayer networks [28]. It also has been improved to a sandbox algorithm to investigate the multifractality of complex networks [29,30].

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Complex networks [31] have some properties like self-organization [32], self-similarity [33,34], attractor, small-world phenomenon [35], and scale-free degree. These properties have been revealed in many real-world networks, like citation network [36], human mobility on social networks [37–39], protein networks [40–43], and some fractal networks [44], specifically the Sierpinski networks [45–47], new Koch networks [48]. A large number of works [49–54] have been investigated for the fractal and self-similarity which can show the structure characteristic of complex networks. In order to depict the properties of complex networks better, lots of dimensions are proposed like fractal dimension [55], information dimension [56], volume dimension [57] and correlation dimension [58], and so on. More specific progress is that F. Hausdorff [59] proposed fractal dimension. Because it cannot describe the fractal and self-similarity accurately, Renyi [56] has proposed an information dimension which is more effective than previous work. Meanwhile, the box-covering algorithm was used into some real complex networks by Song et al. [55,60]. It has been proved to be an effective and accurate method. After that, the box-covering algorithm has been improved [61] and modified in weighted complex networks by Wei et al. [62,63] to accommodate the real-world network. Then, a new method which is called maximal entropy minimal coverings [64] enhances the ability to compute the information dimension in some cases. With the improving of information dimension, it can be applied to obtain a more reasonable evaluation of vulnerability [65]. In these previous researches, information dimension only can be obtained from some real unweighted complex networks [66,67]. Although it can reveal that boxes which have different number of nodes have different influences to the information dimension, it cannot be applied in the weighted complex networks because it cannot deal with the complex networks whose whole edge-weight is not equal to one.

Entropy can describe the degree of chaos of the system, which has been used in some important topics like probability theory, number theory and control theory [68]. Shannon entropy [69,70] is an efficient measure of information volume of a process or system in information theory, which can measure the uncertainty of the result of random experiment. It is the mostly used method of representing uncertainty relationships. Because of its excellent features, it has been used in complex networks [71]. Then the information dimension [56] of complex networks is proposed on the basis of Shannon entropy. So the information dimension can clearly give out how much information a complex network contains, and reveal the fractal and self-similarity properties of complex networks.

In the real application, most of the real-world networks are the weighted complex networks. To the previous researches cannot solve these problems well. If more detailed properties about these real networks want to be obtained, the information dimension should be modified in order to adapt to the real weighted complex networks. In this paper, by improving the previous work, an information dimension combining with box-covering algorithm for weighted complex networks (BCANw) and information dimension based on Shannon entropy is presented. This proposed algorithm is broad enough to accommodate various real weighted complex networks. In addition, it can deal with the unweighted networks when the complex network's whole edge-weight is equal to one.

The remainder of this paper is organized as follows. In Section 2, some preliminaries of this work are introduced. Section 3 proposed an information dimension which can be used in weighted complex networks. The applications of this proposed method are illustrated in Section 4. Finally, some conclusions are given in Section 5.

2. Preliminaries

2.1. The shortest path between any two nodes

The shortest path between any two nodes is an important factor to the complex networks. In any unweighted complex networks $\tilde{G}(\mathrm{N},\mathrm{V})$, $N = (1, 2, \ldots, n)$ is a set of nodes and $V = (1, 2, \ldots, v)$ is a set of edges. An adjacency matrix of $\tilde{G}$ can be created by the relationship between nodes and edges, which can be represented by $X = (\mathrm{x}_{ij})$. When $\mathrm{x}_{ij} = 1$, it indicates that there is an edge between node i and node j, and $\mathrm{x}_{ij} = 0$ in opposite case. The shortest path between node i and node j in unweighted networks is defined as follows,

Definition 2.1. The length of the shortest path between node i and node j is denoted by $\tilde{\mathrm{d}}_{ij}$, which satisfies

$$\tilde{\mathrm{d}}_{ij} = \{x_{ih_1} + x_{h_1h_2} + \cdots + x_{h_kj}\} \tag{1}$$

where $i, h_1, h_2, \ldots, h_k, j$ represent the IDs of nodes.

In real world, most of complex networks are weighted networks, so the previous definition is not suit for the most real-world networks. Compared with unweighted complex networks, weighted complex networks $G = (N, V, W)$ have $W = (1, 2, \ldots, w)$ which is a set of edge-weight and denoted by w_{ij}, and w_{ij} is any real number. When all of the edge-weight $w_{ij}(i, j = 1, 2, \ldots, n)$ equal to one, weighted complex networks G generate to unweighted complex networks $\tilde{G}$. The shortest path in weighted complex networks is defined as follows,

Definition 2.2. The length of the shortest path between node i and node j in weighted complex networks is denoted by d_{ij}, which satisfies

$$\mathrm{d}_{ij} = \{w_{ij_1} + w_{j_1j_2} + \cdots + w_{j_kj}\} \tag{2}$$

where $i, j_1, j_2, \ldots, j_k, j$ represent the IDs of nodes.

2.2. Box-covering algorithm for fractal dimension of weighted networks (BCANw)

The previous box-covering algorithm for unweighted networks is that every covering-box with a box-size s can cover nodes whose shortest distance d_{ij} is less than the box size s. When the whole complex network is under covering with minimum number of boxes, the fractal dimension can be obtained as follows,

$$\mathrm{d}_f = -\lim_{s\to 0} \frac{\ln N(s)}{\ln s} \tag{3}$$

where the $N(s)$ is the number of boxes with the box size s.

Because this method cannot deal with the problem that the distance between two nodes is non-integers and the maximum value of the shortest distance $\tilde{d}_{ij}^{\max}$ from the center node i is less than one, Wei et al. have proposed a new box-covering algorithm which can be used in weighted complex networks. In this method, the initial value box size and the box size increased method has a different definition. In order to make the information dimension established in weighted complex network, the initial box size no longer equals to one and it does not increase by one. Instead its initial box size is equal to the shortest distance between any two nodes in the network. Then it can be obtained by accumulating the value of the distance expanding from small to big until the value of the box size is more than the values of $\max(d_{ij})(i, j = 1, 2, \ldots, n)$. When setting a specific value of box size l_B, it is time to use graph-coloring algorithm to obtain the minimum value of the number of boxes. And the order of graph-coloring algorithm is obtained by descending order of the node-strength in this method [72].

Then there is an example to demonstrate how BCANw is applied in network in detail. For a random network W_1 shown in Fig. 1, the values of each edge can be obtained from Fig. 1. The value of box size l_B can be obtained by adding the value of the distance from small to big. Facing different networks, there are different numbers of boxes with different values of box size l_B. Some detailed steps with $l_B = 0.85$ are shown in Fig. 1. Node i would be connected to node j when $d_{\mathrm{ij}} \geq 0.85$ in W_1, and it is transformed to a new weighted network W_2. Those nodes which are connected directly have different color, and the rest of nodes have the same color with the node which has the maximum node strength. Finally, the network is divided into many boxes by different colors. The minimum number of box $N_B = 3$ is obtained. From the result, this method is effective to obtain the minimum number of boxes and the number of nodes in each box.

2.3. Information dimension of complex networks

Based on the classic information dimension, Wei et al. proposed a novel information dimension. By using this novel information dimension to real complex networks, different sizes of boxes cover the whole real network. Because of the different values of the box size, the whole network can be divided into many boxes, each of which has their own number of node.

Definition 2.3. The information dimension is given as follows:

$$I(\lambda) = -\sum_{i=1}^{N_b} p_i(\lambda) \ln p_i(\lambda) \tag{4}$$

where N_b represents the number of the box, and the probability of nodes in the ith box $p_i(\lambda)$ is obtained by Eq. (5).

$$p_i(\lambda) = \frac{n_i(\lambda)}{n}, i = 1, 2, \ldots, N_b \tag{5}$$

where $n_i(\lambda)$ represents the number of the nodes which are contained in the ith box, n is the total number of the nodes in the whole complex network.

Definition 2.4. The value of this information dimension d_I can be obtained by Eq. (4) and the Eq. (5)

$$d_I = -\lim_{l\to 0} \frac{I(\lambda)}{\ln(\lambda)} = \lim_{l\to 0} \frac{\sum_{i=1}^{N_b} p_i(\lambda) \ln p_i(\lambda)}{\ln(\lambda)}. \tag{6}$$

In the real unweighted complex networks, the box size cannot be very small because the distance between any two nodes is bigger than one. Meanwhile, most of the real-world complex networks are weighted complex networks. Using this method, more details like the fractal and self-similarity of real weighted networks cannot be obtained because of the limited edge-weight.

3. An information dimension of real weighted complex networks

3.1. Basic method

In this section, an information dimension of weighted complex networks is proposed. The new method is based on BCANw and the information dimension of complex networks. When setting different values of the box size, most boxes have different

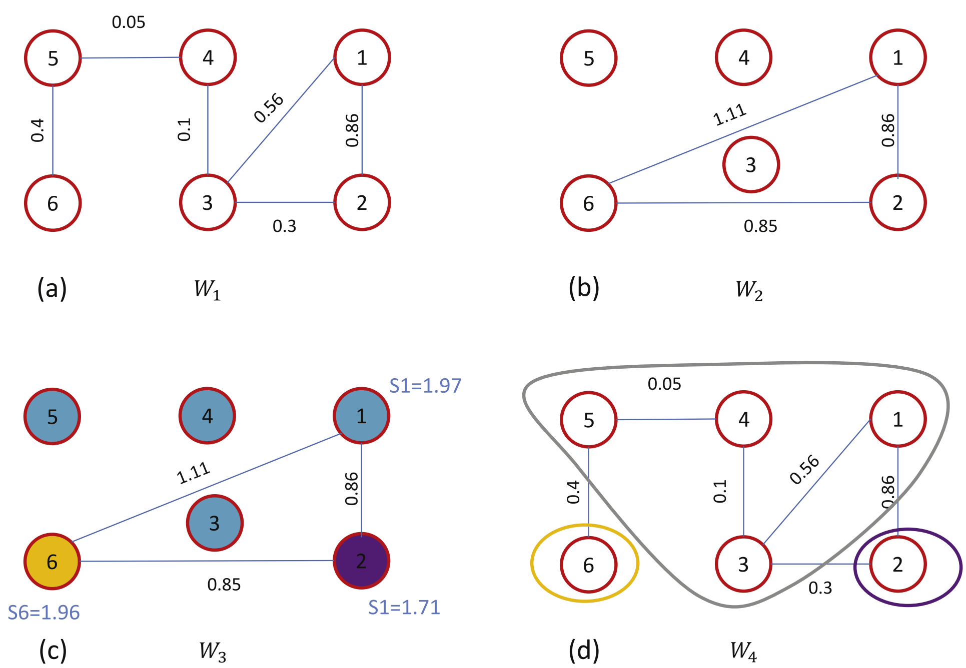

Fig. 1. BCANw method. (a) There is a random weighted network called W_1 which has six nodes and six weighted edges. Observing from the figure, the path between any two nodes can be easily obtained. Then the shortest path in W_1 can be calculated by Eq. (2). (b) Node *i* can be connected to node *j* when $d_{ij} \geq 0.85$ where $0.85 = 0.05 + 0.1 + 0.3 + 0.4$. A new weighted network is established and it is called W_2. (c) These nodes (1, 2, 6) connected directly have different colors, and the rest of the nodes (3, 4, 5) have the same color with node 1 because node 1 has the maximum value of node-strength. (d) Finally, as one color for one box, the minimum number of box of W_1 can be obtained, and it is three, as shown in W_4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

numbers of nodes by using BCANw. The number of nodes in each box is under consideration, when a box size l is given, the shortest path d_{ij} between any two nodes can be obtained by Eq. (2), and the probability of nodes in the *ith* box $p_i(l)$ is obtained as follows:

$$p_i(l) = \frac{n_i(l)}{n}, i = 1, 2, \ldots, N_b \tag{7}$$

where $n_i(l)$ represents the number of nodes in the *ith* box, n is the total number of the whole complex network, and N_b is the number of the boxes. Information dimension is obtained as follows:

$$I(l) = -\sum_{i=1}^{N_b} p_i(l) \ln p_i(l). \tag{8}$$

Definition 3.1. The information dimension of real weighted complex networks d_w can be obtained as follows:

$$d_w = -\lim_{l \to 0} \frac{I(l)}{\ln(l)}. \tag{9}$$

By substituting Eqs. (7) and (8) into Eq. (9), the following formula can be obtained:

$$d_w = \lim_{l \to 0} \frac{\sum_{i=1}^{N_b} \frac{n_i(l)}{n} \ln \frac{n_i(l)}{n}}{\ln(l)}, i = 1, 2, \ldots, N_b \tag{10}$$

where $n_i(l)$ is the number of node in ith box, n is the total number of node in this complex network, N_b is the number of box when the box size is equal to l.

In real weighted complex networks, because the edge-weight w_{ij} can be any real number excluding zero, d_{ij} has a different method to obtain, and it can be obtained by Eq. (2). Due to the unknown edge-weight of real weighted complex networks, the size of box is no longer limited to greater than 1 or just an integer. It can equal to any real number excluding zero by using BCANw. With different values of box size, the complex network can be divided into different number of boxes and each box contains different number of nodes. The ratio of the number of nodes in each nodes to the total number of nodes plays a important role to the information dimension and reveals the uncertainty of the complex network. Information dimension $I(l)$ gives a measure of information, and the relation between $I(l)$ and $ln(l)$ is a linear relationship in the case of log–log plot

Table 1
The probability of nodes in each box ($l = 0.0624$).

Box	1	2	3	···	64	65
Number of nodes	37	17	19	···	1	4
Probability	$p_1(l) = \frac{37}{332}$	$p_2(l) = \frac{17}{332}$	$p_3(l) = \frac{19}{332}$	···	$p_{64}(l) = \frac{1}{332}$	$p_{65}(l) = \frac{4}{332}$

Table 2
Information dimension for USAir network.

Properties	l_1	l_2	l_3	···	l_{153}	l_{154}
l	0.0009	0.0019	0.0030	···	0.9484	0.9582
$\ln l$	−7.0131	−6.2659	−5.8091	···	−0.0530	−0.0427
$I(l)$	5.8051	5.8051	5.8010	···	0.0368	0.0205

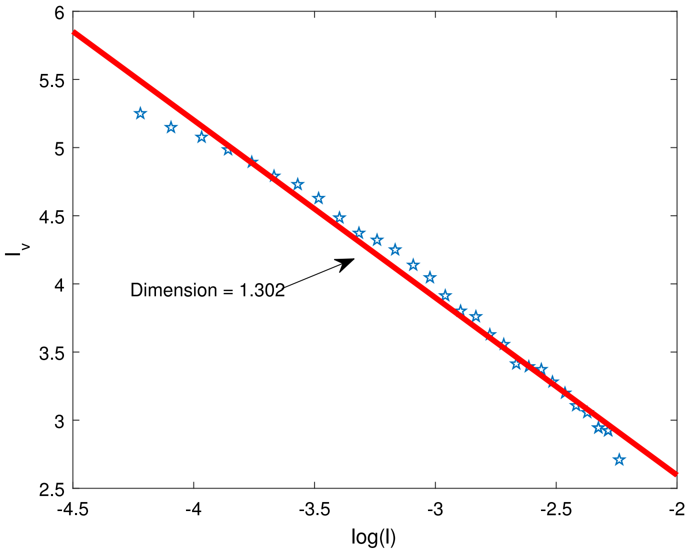

Fig. 2. The information dimension of the USAir network.

which can be studied in this paper. The information dimension of complex networks d_w is the slope of the straight line which can reveal the fractal and self-similarity of complex networks.

3.2. Example explanation

A complex network which is called the US-airlines weighted network (http://vlado.fmf.uni-lj.si/pub/networks/data/) is used to explain this proposed method more specific. This complex network has enough points to do the linear fit because it is sufficiently complex and has 332 nodes and 2126 edges.

Step 1: The size of box is increased from the shortest distance between any two nodes ($l_{\min} = 0.0009$) in the network. In this example, the size of box is selected as 0.0624 based on BCANw algorithm. The whole network would be divided in 65 boxes and each box has different number of nodes. The probability of nodes in each box can be obtained by Eq. (7), and part of the data is shown in Table 1.

Step 2: Then the box size is increasing, and the information dimension $I(l)$ in different box sizes l can be obtained until the box size is greater than the diameter of the complex network ($l = 0.9680 > d = 0.9632$).

Step 3: Part of the information dimension in the different box sizes is shown in Table 2. The horizontal coordinate is the logarithm of the box size $\ln l$, and the vertical coordinate is the information dimension $I(l)$. Using the data whose linearization degree is good in Table 2, the information dimension of US-airlines weighted complex network d_w can be calculated by the relation between $\ln l$ and $I(l)$. The slope of the straight line in Fig. 2 is the information dimension d_w and it is 1.302.

4. Experimental study

4.1. Fractal property of theoretical weighted fractal network

To show this method's effectiveness and feasibility, "Sierpinski" weighted fractal networks have been used to obtain the fractal properties by this method. The theoretical weighted fractal networks are generated by iterated function system. To

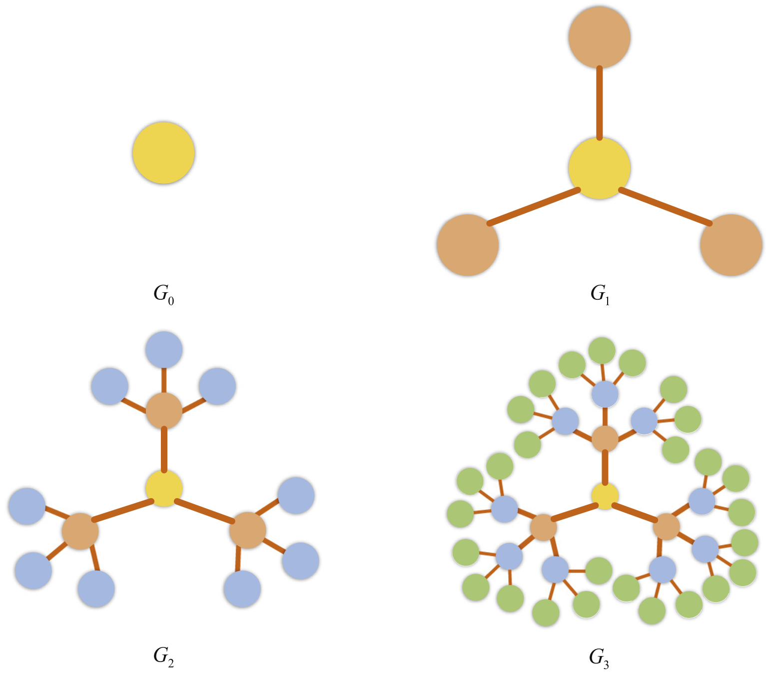

Fig. 3. Sierpinski weighted fractal network.

generate the "Sierpinski" fractal network, the initial network G_0 is a single node. Then, each iteration makes three copies of the original graph and reduces it by 1/2, G_n with nth generation can be obtained and the networks G_0, G_1, G_2, G_3 are shown in Fig. 3. Considering the size of network, "Sierpinski" network G_7 with 7th generation which has 3280 nodes and 3279 edges is used in this method and the edge-weights are 1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64 respectively. The largest value of distance in G_7 is less than 4.

The information dimension of "Sierpinski" weighted fractal network obtained by this proposed method is shown in Fig. 4. By means of the least square fitting, the slope of the fitting line is -1.4170, so the value of information dimension for "Sierpinski" weighted fractal network is 1.4170.

To verify this proposed method's validity, a method based on classic box covering algorithm in Ref. [55] which is called as the greedy coloring algorithm is used. The greedy coloring algorithm is used to obtain the number of nodes $n_i(l)$ in ith box and the number of boxes N_b, and the total number of nodes n equals to 3280 in G_7 network. So the probability of nodes in ith box is obtained by Eq. (7), and information dimension $I(l)$ is obtained by Eq. (8). The information dimension of "Sierpinski" weighted fractal network can be obtained by the relationship between $I(l)$ and $\ln(l)$ in Eq. (9). The box size growth mode in this comparative experiment is same as this proposed method. The relationship between $I(l)$ and $\ln(l)$ is shown in Fig. 4 and the information dimension based on this classic method is 1.3738.

Comparing the results, the information dimension obtained by different methods are close to each other. The fitted line obtained by this proposed method is more accurate than the greedy coloring algorithm because these points obtained by this proposed method are more close to the fitting line. So this proposed method is more effective, and the information dimension can describe the fractal and self-similarity properties of complex networks.

4.2. Fractal properties of real-world weighted complex networks

To demonstrate the effectiveness of this model, it is applied to the US-airlines weighted network, the collaboration scientists working on network theory weighted network, the coappearances of characters weighted network, the authors collaboration weighted network in computational geometry, the collaboration weighted network of scientists posting preprints on the high-energy theory (http://vlado.fmf.uni-lj.si/pub/networks/data/), and some conclusions can be listed from the results in this section.

In "Sierpinski" fractal network, the edge-weight represents no physical meaning and it is only a number. In real-world complex networks, there are two opposite situations. The first case is that the distance will be larger if the edge-weight w_{ij} is higher, just like the edge-weight represents the Euclidean distance between any two cities in the real city network. The second case is just the opposite: the higher edge-weight w_{ij} is, the less the distance will be. For example, the edge-weight

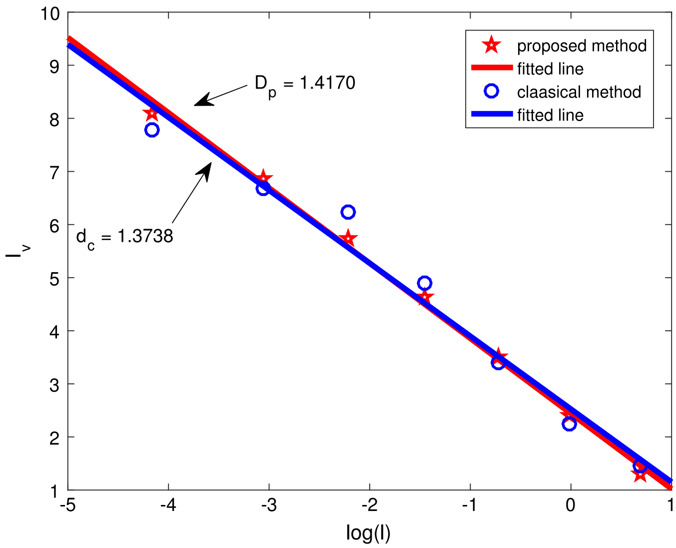

Fig. 4. The information dimension of Sierpinski weighted fractal network.

Table 3
Information dimension of real networks.

Name of network	Nodes	Edges	d_w	RMSE	R-square
USAir	332	2 126	1.302	0.09251	0.9866
Netscience	1589	2 742	0.6338	0.05578	0.9889
Characters	77	254	1.17	0.07997	0.9529
Geom	7343	11 898	2.4185	0.1209	0.9874
High	8360	15 750	3.2532	0.07831	0.9955

represents the number of the seats available, and the real complex networks is the scheduled flights in the airline networks. So the edge-weight represents different meanings in different real-world complex networks.

For edge-weight of collaboration weighted networks, the edge-weights represent the degree of scientists' collaboration which is defined as follows,

$$w_{ij} = \sum_k \frac{\delta_i^k \delta_j^k}{n_k - 1} \tag{11}$$

where n_k is the number of authors for kth paper, δ_i^k would be equal to 1 if the ith scientist is one of the author for kth paper, and it would equal to 0 in the opposite case. When two scientists have collaborated on many articles, the edge-weight would be low and the distance would be small. For edge-weight of USAir weighted complex network, the edge-weight represents how many seats are offered on the scheduled flights. When the number of seats is large, the edge-weight would be low, and this air line can afford more traffic.

The first column in Table 3 is the name of real-world weighted complex networks, the second and the third columns show the number of nodes and edges. The dot sign indicates the correlation between $I(l)$ and $ln(l)$, and the slope of the solid straight line is obtained in the log–log plot by means of the least squares fit. The points which match the linear relationship are selected to do the linear fit and the information dimension is the slope of the fitted line. Some indicators are used to analyze the fitting lines' error, just like sum square error (SSE) of line in Ref. [73]. In this method, root mean squared error (RMSE) and coefficient of determination (R-square) are used. These indicators are very important to the error analysis.

RMSE is defined as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n} w_i (y_i - \widehat{y}_i)^2} \tag{12}$$

where n is the value of the total number, y_i is the value of discrete point obtained by this method, and $\widehat{y}_i$ is the function of the line by fitting. This indicator is based on the error between the predicted value and the original value, which describes the relation between different points.

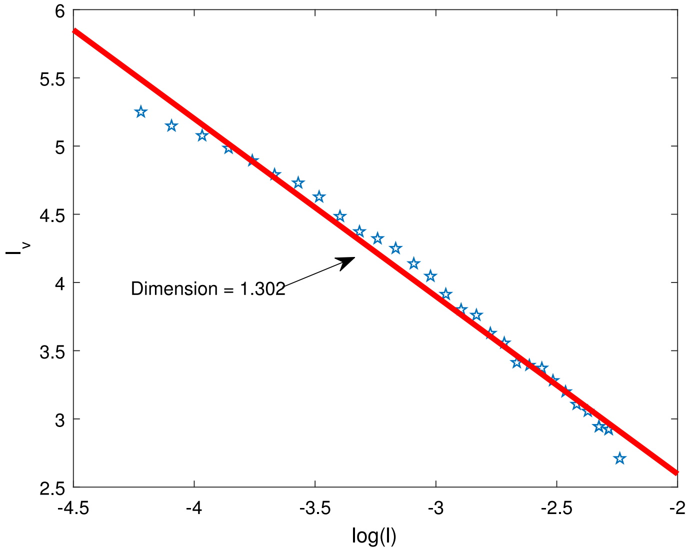

Fig. 5. The US-airlines weighted network (USAir).

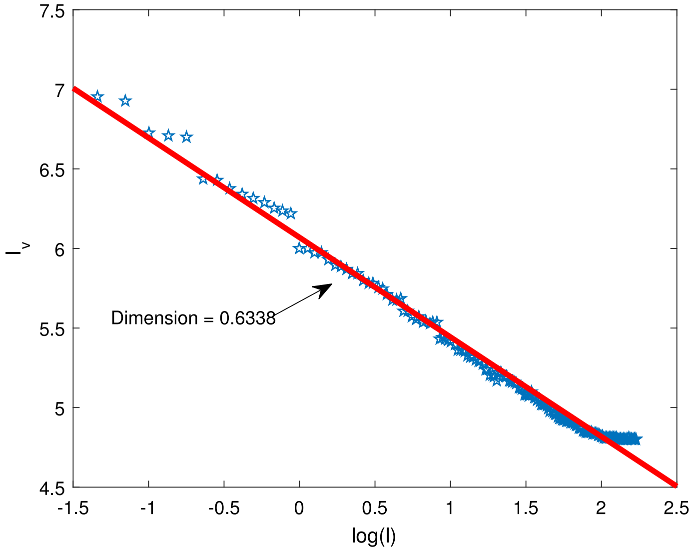

Fig. 6. The collaboration scientists working on network theory weighted network (Netscience).

R-square is different, which is based on the predicted value and the average of original value. It describes a point relative to all of other points, and has a definition as follows:

$$R\text{-}square = 1 - \frac{\sum_{i=1}^{n} w_i(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n} w_i(y_i - \bar{y}_i)^2} \tag{13}$$

where y_i is the value of discrete point obtained by this method, $\bar{y}_i$ is the mean value of y_i, and $\widehat{y}_i$ is the function of the line by fitting.

As shown in Eq. (12), the closer to zero the value of *RMSE* is, the more successful the date prediction is, which shows that the model selection and fitting is better. The value of *R-square* can indicate the fitting degree through the changing of the date. From the Eq. (13) can show the range of *R-square* is [0,1]. The closer to one the value of *R-square* is, the stronger the explanatory power of the equation to the dependent variable *y*.

From the date in Table 3 and Figs. 5–9, not only is the value of *RMSE* enough small and close to zero, but also the value of *R-square* is close to one. These calculating values indicate that the fitting degree is accurate and the fitting line is close to

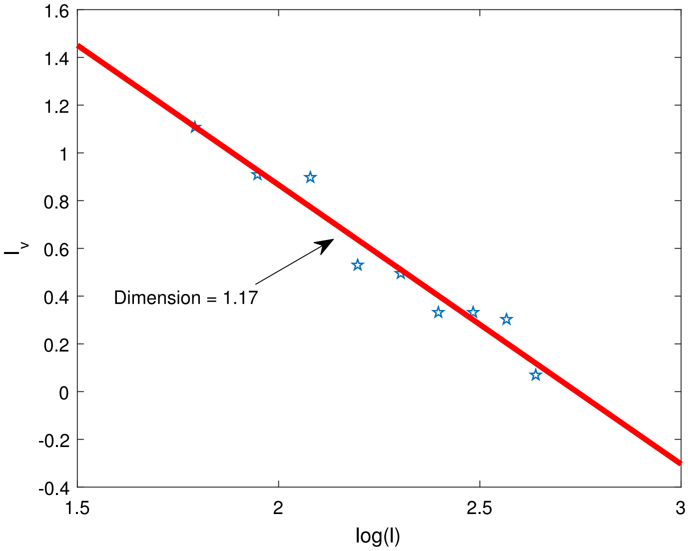

Fig. 7. The coappearances of characters weighted networks (Characters).

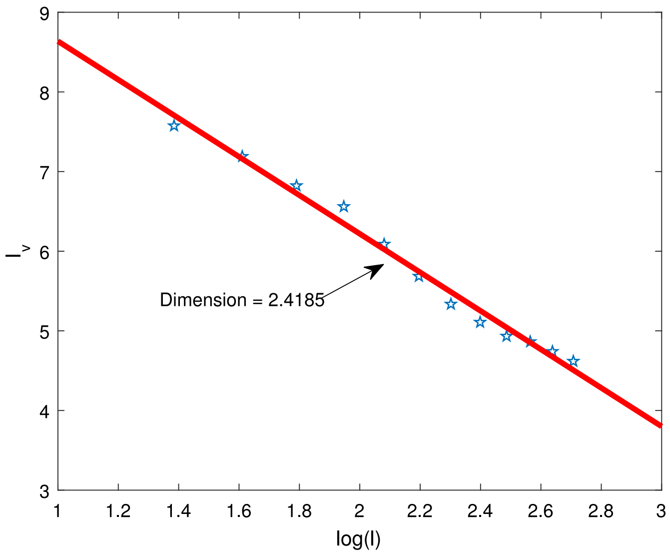

Fig. 8. The collaboration scientists in computational geometry weighted networks (Geom).

this series of points. All of these indicators also represent the information dimension with this method is more precise to describe the fractal and self-similarity properties of complex networks.

5. Conclusion

The information dimension of real weighted complex networks is a form of Shannon entropy of complex networks, so the information dimension is a tool to measure the uncertainty degree of complex networks. Based on this method, the uncertainty of the complex networks is mainly the ratio of the number of nodes in each box to the total number of nodes. When using in the real-world networks, it can effectively represent some properties in the real network. When the whole edge-weight equal to 1, the weighted networks degenerate to unweighted networks.

Information dimension can precisely illustrate self-similarity properties and fractal in the weighted complex networks. In this thesis, a method based on BCANw and information jo/ndimension is proposed to calculate the information dimension of real weighted complex networks. Since this model can use the edge-weight which equals to any real number excluding zero,

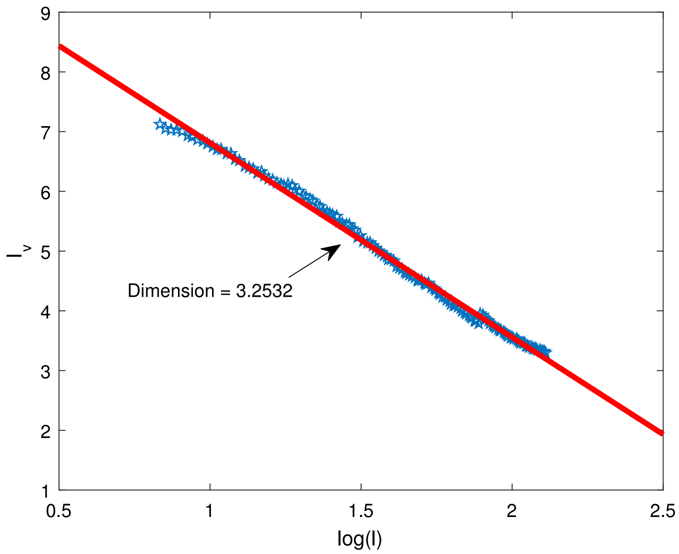

Fig. 9. The collaboration network of scientists posting preprints on the high-energy theory (High).

it can not only be widely applied to real weighted complex networks, but also can be applied to the unweighted networks. Also, being used in the real networks, the simulation results reveal the effectiveness and flexibility of this method to obtain the information dimension.

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References

[1] J.L. Liu, Z.G. Yu, V. Anh, Topological properties and fractal analysis of a recurrence network constructed from fractional Brownian motions, Phys. Rev. E 89 (3) (2014) 12.
[2] F. Alcalde Cuesta, P. Gonzalez Sequeiros, A. Lozano Rojo, Exploring the topological sources of robustness against invasion in biological and technological networks, Sci. Rep. 6 (2016) 20666.
[3] S. Pei, H.A. Makse, Spreading dynamics in complex networks, J. Stat. Mech. Theory Exp. (2013) 21.
[4] L.K. Gallos, N.H. Fefferman, The effect of disease-induced mortality on structural network properties, PLoS One 10 (8) (2015) 17.
[5] X. Zhang, L. Feng, Y. Berman, N. Hu, H.E. Stanley, Exacerbated vulnerability of coupled socio-economic risk in complex networks, Europhys. Lett. EPL 116 (1) (2016) 6.
[6] F. Morone, H.A. Makse, Influence maximization in complex networks through optimal percolation, Nature 524 (7563) (2015) 65–U122.
[7] X. Teng, S. Pei, F. Morone, H.A. Makse, Collective influence of multiple spreaders evaluated by tracing real information flow in large-scale social networks, Sci. Rep. 6 (2016) 11.
[8] D.W. Huang, Z.G. Yu, Dynamic-sensitive centrality of nodes in temporal networks, Sci. Rep. 7 (2017) 11.
[9] L. Lu, T. Zhou, Q.M. Zhang, H.E. Stanley, The H-index of a network node and its relation to degree and coreness, Nature Commun. 7 (2016) 7.
[10] L.K. Gallos, N.H. Fefferman, Simple and efficient self-healing strategy for damaged complex networks, Phys. Rev. E 92 (5) (2015).
[11] X.G. Yan, C. Xie, G.J. Wang, Stock market network's topological stability: Evidence from planar maximally filtered graph and minimal spanning tree, Internat. J. Modern Phys. B 29 (22) (2015) 19.
[12] L. Yin, Y. Deng, Measuring transferring similarity via local information, Physica A (2018). http://dx.doi.org/10.1016/j.physa.2017.12.144. Published on line.
[13] J.X. Gao, B. Barzel, A.L. Barabasi, Universal resilience patterns in complex networks, Nature 530 (7590) (2016) 307–312.
[14] Y. Yu, G.X. Xiao, J. Zhou, Y.B. Wang, Z. Wang, J. Kurths, H.J. Schellnhuber, System crash as dynamics of complex networks, Proc. Natl. Acad. Sci. USA 113 (42) (2016) 11726–11731.
[15] H. Zheng, Y. Deng, Evaluation method based on fuzzy relations between Dempster-Shafer belief structure, Int. J. Intell. Syst. (2017), Article ID INT21956.
[16] V.A. Paun, V.P. Paun, Fracture surface evaluation of zircaloy-4, Mater. Plastice 53 (2) (2016) 326–331.
[17] S. Mizutaka, K. Yakubo, Robustness of scale-free networks to cascading failures induced by fluctuating loads, Phys. Rev. E 92 (1) (2015) 8.
[18] X.L. An, L. Zhang, Y.Z. Li, J.G. Zhang, Synchronization analysis of complex networks with multi-weights and its application in public traffic network, Physica A 412 (2014) 149–156.
[19] X.L. An, L. Zhang, J.G. Zhang, Research on urban public traffic network with multi-weights based on single bus transfer junction, Physica A 436 (2015) 748–755.

[20] W.J. Du, J.G. Zhang, X.L. An, S. Qin, J.N. Yu, Outer synchronization between two coupled complex networks and its application in public traffic supernetwork, Discrete Dyn. Nat. Soc. (2016) 8.
[21] W.J. Du, J.G. Zhang, Y.Z. Li, S. Qin, Synchronization between different networks with time-varying delay and its application in bilayer coupled public traffic network, Math. Probl. Eng. (2016) 11.
[22] Z. Wang, C.T. Bauch, S. Bhattacharyya, A. d'Onofrio, P. Manfredi, M. Perc, N. Perra, M. Salathe, D.W. Zhao, Statistical physics of vaccination, Phys. Rep. 664 (2016) 1–113.
[23] W. Jiang, B. Wei, X. Liu, X. Li, H. Zheng, Intuitionistic fuzzy power aggregation operator based on entropy and its application in decision making, Int. J. Intell. Syst. 33 (1) (2018) 49–67.
[24] Z. Wang, A. Szolnoki, M. Perc, Rewarding evolutionary fitness with links between populations promotes cooperation, J. Theoret. Biol. 349 (2014) 50–56.
[25] M. Schich, C.M. Song, Y.Y. Ahn, A. Mirsky, M. Martino, A.L. Barabasi, D. Helbing, A network framework of cultural history, Science 345 (6196) (2014) 558–562.
[26] L.K. Gallos, M. Sigman, H.A. Makse, The conundrum of functional brain networks: small-world efficiency or fractal modularity, Front. Physiol. 3 (2012) 9.
[27] V. Galvao, J.G.V. Miranda, R.F.S. Andrade, J.S. Andrade, L.K. Gallos, H.A. Makse, Modularity map of the network of human cell differentiation, Proc. Natl. Acad. Sci. USA 107 (13) (2010) 5750–5755.
[28] L.K. Gallos, D. Rybski, F. Liljeros, S. Havlin, H.A. Makse, How people interact in evolving online affiliation networks, Phys. Rev. X 2 (3) (2012) 11.
[29] J.L. Liu, Z.G. Yu, V. Anh, Determination of multifractal dimensions of complex networks by means of the sandbox algorithm, Chaos 25 (2) (2015) 9.
[30] Y.Q. Song, J.L. Liu, Z.G. Yu, B.G. Li, Multifractal analysis of weighted networks by a modified sandbox algorithm, Sci. Rep. 5 (2015) 10.
[31] M.E.J. Newman, The structure and function of complex networks, SIAM Rev. 45 (2) (2003) 167–256.
[32] A. Watanabe, S. Mizutaka, K. Yakubo, Fractal and small-world networks formed by self-organized critical dynamics, J. Phys. Soc. Japan 84 (11) (2015) 10.
[33] L.K. Gallos, F.Q. Potiguar, J.S. Andrade, H.A. Makse, IMDB network revisited: Unveiling fractal and modular properties from a typical small-world network, PLoS One 8 (6) (2013) 8.
[34] C.M. Song, S. Havlin, H.A. Makse, Self-similarity of complex networks, Nature 433 (7024) (2005) 392–395.
[35] L.K. Gallos, H.A. Makse, M. Sigman, A small world of weak ties provides optimal global integration of self-similar modules in functional brain networks, Proc. Natl. Acad. Sci. USA 109 (8) (2012) 2825–2830.
[36] J.R. Clough, T.S. Evans, What is the dimension of citation space? Physica A 448 (2016) 235–247.
[37] P. Deville, C.M. Song, N. Eagle, V.D. Blondel, A.L. Barabasi, D.S. Wang, Scaling identity connects human mobility and social interactions, Proc. Natl. Acad. Sci. USA 113 (26) (2016) 7047–7052.
[38] L.K. Gallos, P. Barttfeld, S. Havlin, M. Sigman, H.A. Makse, Collective behavior in the spatial spreading of obesity, Sci. Rep. 2 (2012) 9.
[39] D.S. Wang, C.M. Song, Impact of human mobility on social networks, J. Commun. Netw. 17 (2) (2015) 100–109.
[40] Y.W. Zhou, J.L. Liu, Z.G. Yu, N.Q. Zhao, V. Anh, Fractal and complex network analyses of protein molecular dynamics, Physica A 416 (2014) 21–32.
[41] R.O. Akinola, G.K. Mazandu, N.J. Mulder, A quantitative approach to analyzing genome reductive evolution using protein-protein interaction networks: A case study of mycobacterium leprae, Front. Genet. 7 (2016) 12.
[42] W. Jiang, B. Wei, X. Liu, X. Li, H. Zheng, Intuitionistic fuzzy evidential power aggregation operator and its application in multiple criteria decision-making, Int. J. Syst. Sci. (2018). http://dx.doi.org/10.1002/int.21939. Published on line.
[43] K. Hahn, P.R. Massopust, S. Prigarin, A new method to measure complexity in binary or weighted networks and applications to functional connectivity in the human brain, BMC Bioinform. 17 (2016) 18.
[44] X. Deng, W. Jiang, An evidential axiomatic design approach for decision making using the evaluation of belief structure satisfaction to uncertain target values, Int. J. Intell. Syst. 33 (1) (2018) 15–32.
[45] J. Chen, A. Le, Q. Wang, L.F. Xi, A small-world and scale-free network generated by sierpinski pentagon, Physica A 449 (2016) 126–135.
[46] S.J. Wang, L.F. Xi, H. Xu, L.H. Wang, Scale-free and small-world properties of sierpinski networks, Physica A 465 (2017) 690–700.
[47] A.B. Le, F. Gao, L.F. Xi, S.H. Yin, Complex networks modeled on the sierpinski gasket, Physica A 436 (2015) 646–657.
[48] D.W. Huang, Z.G. Yu, V. Anh, Multifractal analysis and topological properties of a new family of weighted Koch networks, Physica A 469 (2017) 695–705.
[49] Z.W. Wei, B.H. Wang, Emergence of fractal scaling in complex networks, Phys. Rev. E 94 (3) (2016) 6.
[50] L.K. Gallos, N.H. Fefferman, Revealing effective classifiers through network comparison, Europhys. Lett. 108 (3) (2014) 38001.
[51] M. Locci, G. Concas, R. Tonelli, I. Turnu, Three algorithms for analyzing fractal software networks, WSEAS Trans. Inf. Sci. Appl. 7 (3) (2010) 371–380.
[52] T. Emmerich, A. Bunde, S. Havlin, G.L. Li, D.Q. Li, Complex networks embedded in space: Dimension and scaling relations between mass, topological distance, and Euclidean distance, Phys. Rev. E 87 (3) (2013) 8.
[53] T. Hasegawa, T. Nogawa, K. Nemoto, Profile and scaling of the fractal exponent of percolations in complex networks, Europhys. Lett. EPL 104 (1) (2013) 6.
[54] P. Chelminiak, Emergence of fractal scale-free networks from stochastic evolution on the cayley tree, Phys. Lett. A 377 (40) (2013) 2846–2850.
[55] C.M. Song, L.K. Gallos, S. Havlin, H.A. Makse, How to calculate the fractal dimension of a complex network: the box covering algorithm, J. Stat. Mech. Theory Exp. (2007) 16.
[56] A. Rényi, Dimension, entropy and information, in: Trans. 2nd Prague Conf. Information Theory, 1960, pp. 545–556.
[57] O. Shanker, Defining dimension of a complex network, Modern Phys. Lett. B 21 (6) (2007) 321–326.
[58] L. Lacasa, J. Gomez-Gardenes, Correlation dimension of complex networks, Phys. Rev. Lett. 110 (16) (2013) 5.
[59] F. Hausdorff, Dimension and outer dimension, Math. Ann. 79 (1919) 157–179.
[60] C.M. Song, S. Havlin, H.A. Makse, Origins of fractality in the growth of complex networks, Nat. Phys. 2 (4) (2006) 275–281.
[61] Y.Y. Sun, Y.J. Zhao, Overlapping-box-covering method for the fractal dimension of complex networks, Phys. Rev. E 89 (4) (2014) 7.
[62] R. Zhang, B. Ashuri, Y. Deng, A novel method for forecasting time series based on fuzzy logic and visibility graph, Adv. Data Anal. Classif. 11 (4) (2017) 759–783.
[63] T. Liu, Y. Deng, F. Chan, Evidential supplier selection based on DEMATEL and game theory, Int. J. Fuzzy Syst. (2017). http://dx.doi.org/10.1007/s40815-017-0400-4.
[64] E. Rosenberg, Maximal entropy coverings and the information dimension of a complex network, Phys. Lett. A 381 (6) (2017) 574–580.
[65] B. Kang, G. Chhipi-Shrestha, Y. Deng, K. Hewage, R. Sadiq, Stable strategies analysis based on the utility of Z-number in the evolutionary games, Appl. Math. Comput. (2017). http://dx.doi.org/10.1016/j.amc.2017.12.006. Published on line.
[66] W. Jiang, Y. Chang, S. Wang, A method to identify the incomplete framework of discernment in evidence theory, Math. Probl. Eng. 2017 (2017) Article ID 7635972.
[67] X. Deng, D. Han, J. Dezert, Y. Deng, Y. Shyr, Evidence combination from an evolutionary game theory perspective, IEEE Trans. Cybern. 46 (9) (2016) 2070–2082.
[68] W. Jiang, S. Wang, An uncertainty measure for interval-valued evidences, Int. J. Comput. Commun. Control 12 (5) (2017) 631–644.
[69] C.E. Shannon, A mathematical theory of communication, Bell Syst. Tech. J. 27 (4) (1948) 623–656.
[70] S.J. Cao, M. Dehmer, Degree-based entropies of networks revisited, Appl. Math. Comput. 261 (2015) 141–147.

[71] P. Riera-Fernandez, C.R. Munteanu, M. Escobar, F. Prado-Prado, R. Martin-Romalde, D. Pereira, K. Villalba, A. Duardo-Sanchez, H. Gonzalez-Diaz, New Markov-Shannon entropy models to assess connectivity quality in complex networks: From molecular to cellular pathway, parasite-host, neural, industry, and legal-social networks, J. Theoret. Biol. 293 (2012) 174–188.
[72] Y.A.O. Can-zhong, Y. Jian-mei, Improved box dimension calculation algorithm for fractality of complex networks, Comput. Eng. Appl. 46 (8) (2010) 5.
[73] W. Pedrycz, A. Bargiela, Fuzzy fractal dimensions and fuzzy modeling, Inform. Sci. 153 (2003) 199–216.