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The fractal dimension of complex networks: A review

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ABSTRACT

The fractal property is one of the most important properties in complex networks. It describes the power law relationship between characteristics of the box and the box size. There are numerous research studies focusing on the fractal property in networks through different dimensions. In order to study the problems across various disciplines, fractal dimension and local dimension are proposed to study network and node properties respectively. In this review paper, various network covering algorithms, which form the basis for obtaining fractal dimension are being reviewed. The different dimensions used to describe the fractal property of networks and their applications are then discussed. Through these studies, we emphasize that the fractal property is an important tool for understanding network characteristics. In the last section, we give our conclusion and discuss possible future directions for fractal dimension research.

1. Introduction

A complex system consists of many components which may interact with one another. Complex systems include social and economic organizations, urban cities, the human brain, infrastructure and machinery, an ecosystem, and ultimately the entire universe. They can be modeled by complex networks where the individual is represented as a node and the relationship between individuals is represented as an edge in the network. Complex networks have been studied extensively to solve practical problems [1,2], for example, large-scale expert social networks in which experts give their opinions on certain issues [3–5]; curbing the spread of diseases or rumors using network modeling [6–8]; studying the synchronization phenomenon of fireflies flashes where the firefly colony is regarded as the connected network [9–11]; analyzing the predatory–prey dynamics in a network [12–14]. Other interdisciplinary theories, such as the state estimation [15,16], information fusion [17–19], game theory [20–23], and machine learning [24–27] have also been deeply studied using complex networks. However, a fundamental aspect in network science is its structure characteristic exploration [28,29]. The three most important structural properties are (a) small-world [30], (b) scale-free [31], and (c) self-similarity property [32]. The self-similarity property refers to the self-repeating pattern for the network structure, that is, invariant distribution of degree under renormalization. The minimal number of boxes (needed to cover the network) and the box size follow the power law function $n \sim l^{-d}$, and the fractal property of networks refers to the finite dimension d in the network [32,33]. In contrast, non-fractal networks are compact systems, which means that hub nodes are closely connected with hub nodes, showing the sharp decay of n with increasing l (infinite fractal dimension) [34,35].

Since the discovery of fractal property, research in this field has mainly focused on three aspects. The first aspect is how to cover the network, which is the foundation of fractal dimension. Numerous algorithms have been proposed, for instance, Song et al. proposed the greedy coloring algorithm [36], but this algorithm requires repeated experiments due to the random covering sequence, resulting in a higher time complexity. In order to address this problem, Wei et al. proposed an improved algorithm with a pre-determined covering sequence based on nodes' degree [37]. There are many other algorithms that weigh between the accuracy and time complexity to cover the network, such as the multi-objective algorithm [38], sampling-based method [39], minimal partition covering method [40], and central-node covering method [41]. The second aspect is to explore the fractal property by suitable dimensions when the focus of study is on the network or node [42,43]. The fractal dimension [36,44] describes the fractal property of the network, where the number and the size of boxes have the power law relation. The fuzzy fractal dimension [45] addresses the NP hard problem (network covering problem) via the box covering ability, and the power law relation between the box covering ability and the box size still exists in the network. The local dimension [46] is different because it reveals the fractal property around the central node rather than the network, hence the box will only be used to cover the central node. The other aspect is how to apply the fractal dimension

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in real-world complex systems. The fractal dimension has been used to measure the vulnerability of the airline network when the network structure is adjusted every year [47]. The local dimension and fractal dimension can be combined to measure similar nodes, resulting in recommender systems [48]. Several improved local dimensions [49,50] have also been put forward to identify influential spreaders to curb epidemic in complex networks [51,52]. Readers can refer to Rosenberg [53,54] for more background material and further study on the topic. The basic definitions and fractal dimension are introduced in Section 2. Due to the importance of the network covering method, we first introduce some covering algorithms in Section 3, which are available for unweighted and weighted networks when the value of the edge is the same or different. In order to introduce these fractal dimensions in an organized manner, we have three subsections in Section 4. The fractal dimension which describes the fractal property of the network is discussed in Section 4.1, and other dimensions which consider different information volumes in the box are also discussed. In Section 4.2, another dimension describing the fractal property for the central node is introduced. The local dimension only considers the information around the central node (unique for each node), hence it can be used to identify the importance of nodes. This will be discussed in Section 5.2.1. In Section 4.3, we discuss the multi-fractal property which can shed light on the fractal property of networks in different scales. The multi-fractal dimension can degenerate to other dimensions when $q = 0, 1, 2$. The applications of the fractal property are illustrated in Section 5, including the research in network property and node property discussed in Sections 5.1 and 5.2 respectively. The conclusions and further outlook are discussed in Section 6. All figures depicted in this review paper are self-generated based on the algorithms in the primary papers.

2. Fractal property of networks

The basic introduction of complex network parameters is first given in this section. Given a network $G(N, E)$, where $N = \{1, 2, \dots, n\}$ and $E = \{1, 2, \dots, m\}$ are the set of nodes and edges, respectively. The number of nodes and edges are $|N|$ and $|E|$, respectively. The topological structure is represented by the adjacency matrix A, whose size is $|N| \times |N|$. $a_{ij} = 1$ in A means node i is connected with node j, and 0 means there is no connected edge. The shortest distance d_{ij} is obtained by the Floyd algorithm or Dijkstra algorithm. The maximum value of the shortest distance d_{ij} is the diameter D of the network.

The fractal property [55] was first discovered by Benoit B. Mandelbrot to measure the length of the coast of Britain. He found that objects not only have integer dimension, but also fractal dimension. The length of the coast will greatly exceed under incorrect dimension measurement: the length is different under different scale measurement, and approaches positive infinity when the scale approaches zero. Therefore, its dimension is 1.25, and not of integer dimension 1. Inspired by this work and Hausdorff dimension [56], the fractal dimension of the network [32] is proposed to describe the relationship between the minimum number of boxes needed to cover the network n_b and the box size l,

$$n_b \sim l^{-d_f}, \tag{1}$$

where d_f is the fractal dimension of network G, which can be re-written as:

$$d_f = -\lim_{l \to 0} \frac{\ln n_b}{\ln l}. \tag{2}$$

In Euclidean space, $l \to 0$ is suitable for regular fractal objects because the object can be covered by increasingly smaller sizes of box, and we have the power law function between the size and number of boxes when size tends to 0. However, the shortest distance between nodes cannot be less than 1 in unweighted networks, and the Euclidean metric is not relevant for graphs [57]. Hence, the box size cannot tend to 0, and the graph distance metric is applied in the unweighted network. The box size should be in a given range (larger than 1 but less than the diameter of the network), although $l \to 0$ is shown in Eq. (2) and following formulas. Apart from the box size $l \to 0$ in the network, tending to infinity is also a solution in growing networks [57]. In Ref [32], Song et al. found that most real-world networks show self-similarity property and have fractal dimension. For example, the world-wide web network's dimension is 4.1, protein–protein interaction network's dimension is 2.3, and actors network's dimension is 6.3. In general, the power law function is suitable for values x in the distribution greater than the minimum value of x_{min}; Clauset et al. [58] has shown that the tail of the distribution follows the power law function. Finally, there should be a lower bound to the power law phenomenon, which means that l should be larger than l_{min} in the network. More information about estimating the lower bound and scaling parameter can be found in Ref. [58]. The Sierpinski triangle network shown in Fig. 1 is a classical fractal object, and its fractal dimension is the ratio of the number of new copies to the scaling of new copies. Its dimension is $\log 3 / \log 2 = 1.585$, and the dimension obtained from network perspective is 1.456, which is close to its dimension. The network covering algorithm in Ref. [37] is applied to obtain the fractal dimension of Sierpinski network.

3. Network covering

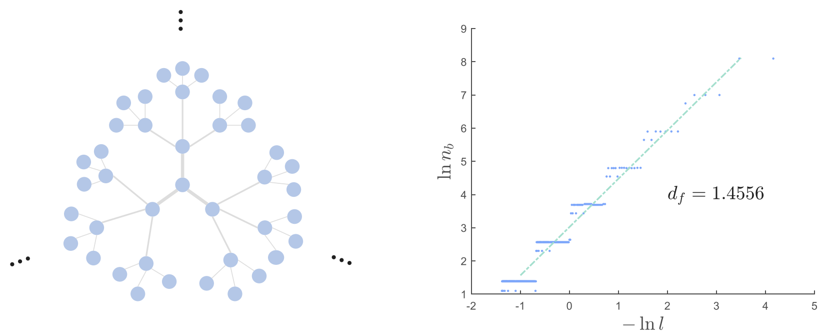
According to the definition of fractal dimension (shown in Eq. (2)), one of the key issues is to cover the network with the least number of boxes. This is an NP-hard problem, and a solution cannot be obtained within a short period of time. Therefore, it remains an open problem. Currently, there are two kinds of algorithms to cover the network, and the difference is whether a central node is needed to be selected for each box in advance. The shortest distance among all nodes should be obtained first because it will determine the assignment of box. When the shortest distance between a pair of nodes is greater than the box size, they will not be selected into a box. For the nodes that do not belong to the same box, the box allocation sequence can be predetermined or random, depending on the algorithm. The central node growth method needs to select central nodes in advance, then use these nodes as the center to cover the network with boxes. We discuss below some of these commonly used methods.

3.1. Greedy coloring algorithm

The greedy coloring algorithm [36] is used to cover the network. As the algorithm does not have a pre-determined color assignment sequence, it needs numerous repeated experiments to ensure the accuracy of the result, thereby leading to a higher time complexity over other methods. The detailed steps are shown in Algorithm 1.

Algorithm 1 Greedy coloring algorithm.

Each node in the network should be assigned an unique ID number $(1, 2, \cdots, n)$;
for Box size $l = 1$ **to** l^{max} **do**
  Given a new network G';
  for Node $i, j = 1$ **to** $|N|$
    if $d_{ij} > l$ in G **then do**
      Node i and j are connected in G';
    end if
  end for
  The color of nodes at both ends of one edge in G' cannot be the same;
  Nodes with the same color ($d_{ij} < l$ in G) will be in one box;
end for



(A) Sierpinski Triangle Network (B) Fractal dimension

Fig. 1. The structure of Sierpinski triangle network and the fractal dimension obtained from network perspective. (A) There are three new copies in each generation step. The line width refers to the scaling and it is 0.5 in each step. (B) The fractal dimension of Sierpinski triangle network is obtained from network perspective.
Source: Adapted from Ref. [37].

3.2. Box-covering algorithm for weighted networks

By building on the previous algorithm, Wei et al. proposed a method whereby the box-covering sequence is determined by the strength of the node [37]. This effectively helps to reduce the time complexity of the box-covering process. The steps of this method are shown in Algorithm 2.

Algorithm 2 Box-covering algorithm for weighted networks.

Each node in the network should be assigned an unique ID number $(1, 2, \cdots, n)$;
The size of box l should increase by accumulating the edge length;
for $l = l^{min}$ **to** D **do**
  Given a new network G';
  for Node $i, j = 1$ **to** $|N|$ **do**
    if $d_{ij} > l$ in G **then**
      Node i and j are connected in G';
    end if
  end for
  Calculate the strength of node in G';
  Nodes connected in G' should be in different colors. The color of nodes with high strength should be assigned first, and nodes without any connections will be assigned a random color;
  Each color corresponds to one box in the network;
end for

3.3. Burning algorithm

Song et al. applied the burning algorithm, a traditional breadth-first algorithm to cover the network [36]. In this method, each box will have a central node, and the largest box size l equals to the largest distance in this box plus one ($d_{ij}^{max_k} + 1$ in box k). The relationship between the box size l and radius r is $l = 2r + 1$. They defined two properties in the box: (1) Compact: each box covers the maximum number of nodes; (2) Connected: the nodes in the box are mutually reachable. The detailed steps of this algorithm are shown in Algorithm 3.

This algorithm takes a fairly long time to compute ($\mathcal{O}(N^2)$). Song et al.et al. [36] then modified this algorithm to achieve the same result in a shorter amount of time ($\mathcal{O}(N)$ - $\mathcal{O}(N^2)$) [33,59], as shown in Algorithm 4.

Algorithm 3 Burning algorithm (Burning with the diameter l).

for Box size $l = 1$ **to** l^{max} **do**
  while There are still uncovered nodes in the network **do**
    One initial uncovered node is chosen as the seed;
    for Node $i = 1$ **to** $|N|$ **do**
      if Node i is not covered by other boxes and its d_{ij} from other nodes in this box is less than l **then**
        Node i will be added to this box;
      end if
    end for
  end while
end for

Algorithm 4 Compact-box-burning algorithm.

for Box size $l = 1$ **to** l^{max} **do**
  while There are still uncovered nodes in the network **do**
    One initial uncovered node is chosen as the seed;
    All nodes whose d_{ij} from this seed that is less than l, will be added to this box;
    for Node $i = 1$ **to** Number of nodes in this box **do**
      The nodes in this box whose d_{ij} from node i is larger than l, will be removed from this box;
    end for
    $d_{ij} < l$ is true for every pair of nodes in this box;
  end while
end for

3.4. Sandbox algorithm

The sandbox algorithm [60] has also been proposed to cover the network, and it has been used to obtain the multi-fractal dimension [61]. This algorithm has the central node for each box, similar to the burning algorithm. However, the box's assigned sequence is random. Therefore, numerous repeated experiments have to be taken to ensure that the result is accurate. This method will find a random series of nodes, and in turn, set them as central nodes. Through this process, the network will eventually be covered by boxes generated by these central nodes.

3.5. Other methods

There are many other network covering algorithms. For example, Sun et al. proposed an overlapping-box-covering method [62]. Its characteristic is different from the previous method in that a node is covered by several overlapping boxes rather than one box. Experimental results show that this method is available for deterministic networks and real-world networks with smaller errors [62]. Niu et al. proposed an outside-in box-covering algorithm for artificially generated networks [63]. The theoretical analysis and experiments have proven the validity of this covering method. Wu et al. proposed a multi-objective optimization algorithm to cover the network [38]. The two

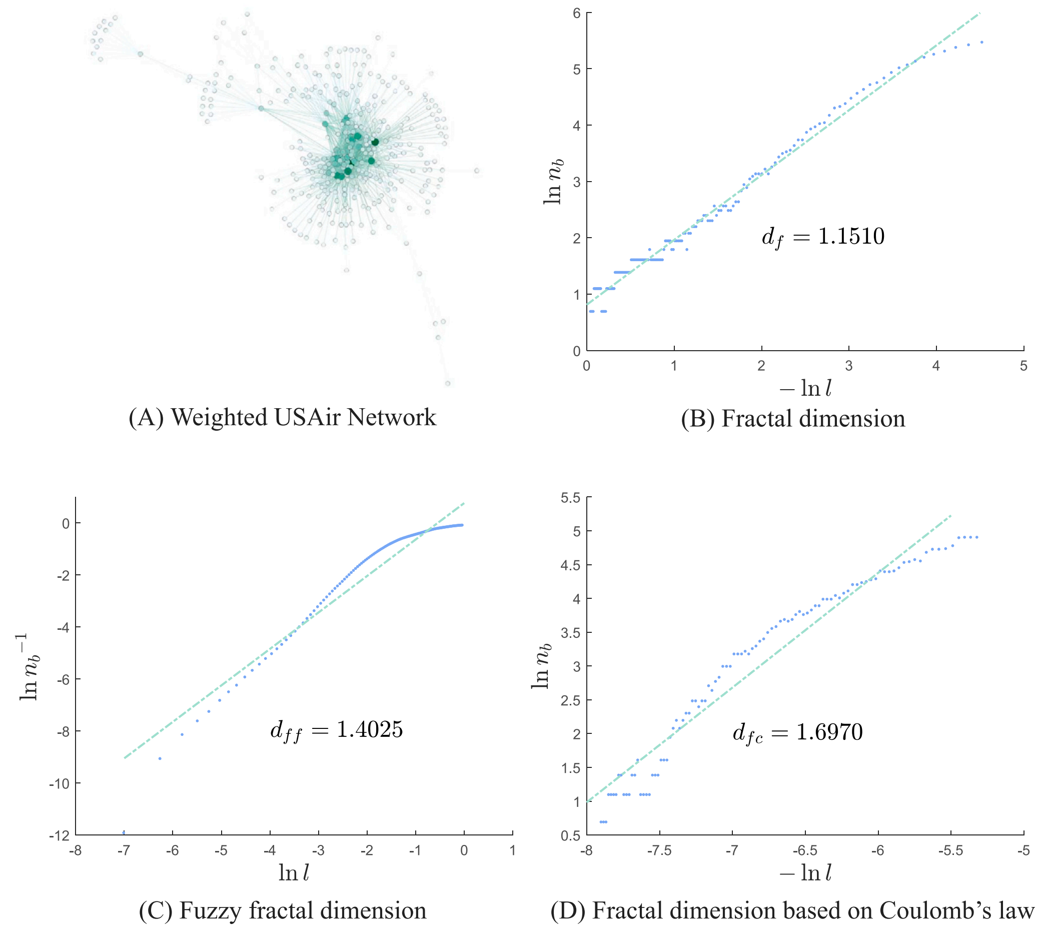


Fig. 2. The structure of weighted USAir network with three fractal dimensions.
Source: Adapted from Ref. [37,45,64].

objective functions in this algorithm are the minimization of ratio cut and the maximization of ratio association.

4. Dimension

After the network has been covered by different algorithms, there are many different dimensions [53,54] to describe the fractal property which concentrates on different aspects of the network (like the fractal property of the network or node).

4.1. Network property

Considering different information in the box, such as the number of boxes, the number of nodes in each box, and the volume in each box, all characteristics of the box and the box size follow the power law function. The network property usually needs network covering in advance, which is different from the node property.

4.1.1. Fractal dimension

As discussed in the introduction of Section 2, the classical dimension is the fractal dimension d_f [32] shown below:

$$d_f = -\lim_{l \to 0} \frac{\ln n_b}{\ln l}. \tag{3}$$

After this dimension was proposed, researchers began to study whether other dimensions could better show the fractal property of the network. Zhang et al. modeled the self-similarity of the network based on Coulomb's law [64]. Due to the key role of hub nodes, Coulomb's law is applied to describe the repulsive force among nodes. The repulsive force R_{ij} of edge is obtained by the degree of connected nodes shown below:

$$R_{ij} = k_i \times k_j, \tag{4}$$

where k_i is the degree of node i. There is no R_{ij} when node i and node j are not connected in network G, and the length of the edge in the new network G' equals to R_{ij}. The box-covering algorithm is applied in network G' to obtain the fractal dimension d_{fc}. Since this method concentrates on the repulsion between nodes, the nodes with weak repulsive force have a high probability of being selected into one box.

However, all these methods need repeated experiments when the network is covered by several boxes, leading to a higher time complexity. Therefore, the fuzzy fractal dimension which does not need repeated experiments is proposed [45]. This method is inspired by the dimension in time series [65]. In the fuzzy fractal dimension, each node does not have the same contribution to the box, because nodes closer to the center of the box have greater weight. The fuzzy fractal dimension d_{ff} is shown below:

$$d_{ff} \approx -\frac{\ln n_b(l)}{\ln l} \approx \frac{\ln n_b(l)^{-1}}{\ln l}, \tag{5}$$

where $n_b(l)$ is the number of boxes, l is the box size, and $n_b(l)^{-1}$ is the covering ability of boxes. The covering ability is higher when more nodes can be covered by one box. The covering ability of each box can be obtained by the fuzzy theory,

$$n_b(l)^{-1} = \frac{1}{|N|(|N|-1)} \sum_{i \in N} \sum_{j \in N, j \neq i} \Omega_{ij}(l) F_{ij}(l), \tag{6}$$

where $\Omega_{ij}(l)$ and $F_{ij}(l)$ are selection function and membership function respectively shown below:

$$\Omega_{ij}(l) = \begin{cases} 1, & d_{ij} \le l \\ 0, & d_{ij} > l \end{cases},$$
$$F_{ij}(l) = \exp(-\frac{d_{ij}^2}{l^2}),$$

where $\Omega_{ij}(l)$ sets the node inside the box to 1. Otherwise set the node outside the box to 0. $F_{ij}(l)$ will assign different weights to nodes in the box (nodes closer to the center node have greater weight). Each node has different contribution to the box, like how each individual contributes differently to the community. This method does not need to cover the network by any box-covering algorithm as the covering ability can be directly obtained.

The weighted USAir network [1] shown in Fig. 2 (A) is applied to show the fractal property via three fractal dimensions. The three fractal dimensions d_f [37], d_{ff} [45], and d_{fc} [64] are shown in Fig. 2 (B), (C), and (D). The network is covered by the algorithm shown in Section 3.2, and all three fractal dimensions follow the power law function.

There are still other dimensions that can be used to describe the fractal property in networks. For example, Guo et al. found that d_{ij} and the average density $\langle \rho(d_{ij}) \rangle$ also follow the power law function [66]. Ramirez et al. was motivated by the geometric measure theory and proposed a D-summable fractal dimension that could reduce the error caused by fitting [67]. The experiments also reveal the rich structural diversity in the network.

4.1.2. Information dimension

These fractal dimensions mainly focus on the relationship between the number and the size of boxes, but they do not consider the information within the box. Inspired by the Shannon entropy, Wei et al. applied the ratio of the number of nodes in the box to the number of nodes in the network, as the probability to explore the information, and proposed the information dimension [68]. The probability $p_i(l)$ is shown below:

$$p_i(l) = \frac{n_i(l)}{|N|},$$

where $n_i(l)$ is the number of nodes in the ith box. The information volume $I(l)$ in the network is then obtained by:

$$I(l) = -\sum_{i=1}^{N_b} p_i(l) \ln p_i(l),$$

where N_b is the number of boxes needed to cover the network. The information dimension of the network is obtained by:

$$d_I = -\lim_{l\to 0} \frac{I(l)}{\ln l} = \lim_{l\to 0} \frac{\sum_{i=1}^{N_b} p_i(l) \ln p_i(l)}{\ln l}. \tag{7}$$

In this method, an in-depth analysis of the information in the box can be conducted. In addition, the information volume and the box size also follow the power law relation.

Wen et al. extended the information dimension to weighted networks [69], thereby enhancing its applicability as weighted network is ubiquitous in real-world problems. For example, the distance between airports in an aviation network is different, relationship between friends in a social network is different, similarly, the product similarity in a recommender system is also different. Clearly, these phenomena have shown that the dimension of weighted network has many real-world applications. Rosenberg proposed the maximal entropy minimal covering algorithm and a novel definition of information dimension [70]. This leads to an enhanced calculation ability with a positive influence on the linear fitting of data.

[1] http://vlado.fmf.uni-lj.si/pub/networks/data/.

4.1.3. Tsallis information dimension

The information dimension discussed in the previous section takes into account the information in each box, but it cannot decide which box has greater effect on the fractal property. In some cases, the box with the small number of nodes is important in the network. In order to consider this situation, Zhang et al. proposed the Tsallis information dimension based on the nonextensive statistical mechanics, Tsallis entropy [71]. They introduced a non-extensive coefficient q to obtain the information volume in the box:

$$I_T(l) = k\frac{1 - \sum_{i=1}^{N_b} p_i(l)^q}{q-1},$$

where $k = 1$ is a constant. The Tsallis information dimension is obtained by:

$$d_T = \lim_{l\to 0} \frac{I_T(l)}{\ln l} = \lim_{l\to 0} \frac{\frac{1-\sum_{i=1}^{N_b} p_i(l)^q}{q-1}}{\ln l}. \tag{8}$$

It is similar to the information dimension because of the power law relation between the information volume $I_T(l)$ and box size l. However, it can give different weights to boxes with different information volume by the non-extensive coefficient q:

(a) When $q \to -\infty$, the box with less nodes has greater effect on the dimension;
(b) When $q \to 0$, the boxes with different information volume have the same effect on the dimension as $p_i(l)^0 = 1$;
(c) When $q \to 1$, the dimension degenerates to the information dimension;
(d) When $q \to \infty$, the box with more nodes has greater effect on the dimension, and the dimension approaches 0 as q increases.

Different dimensions can be obtained by different entropy [72] because entropy can focus on different information in the box. Rényi entropy is also the nonextensive statistical mechanics, and closely related to Tsallis entropy [73]. Therefore, Duan et al. generalized the dimension to Rényi information dimension [74] as shown below:

$$d_R = \lim_{l\to 0} \frac{\frac{1}{1-q} \ln \sum_{i=1}^{N_b} p_i(l)^q}{\ln l}. \tag{9}$$

The positive weight factor w_p decided by q and self weight factor w_s decided by the network structure are defined as:

$$w_p = \frac{1}{1-q},$$
$$w_s = \sum_{i=1}^{N_b} p_i(l)^q.$$

w_p and w_s are defined in such a way to simplify the expression of the information volume of Tsallis information dimension and Rényi information dimension to:

$$I_R = w_p \ln w_s,$$
$$I_T = w_p(w_s - 1).$$

The non-extensive coefficient q is the focus of Tsallis information dimension and Rényi information dimension, because it decides which box to focus on. Meanwhile, these two dimensions will degenerate to information dimension and fractal dimension when $q = 1$ and $q = 0$.

Ramirez et al. applied the Tsallis information dimension to show rich diversity in the structure of network [75]. They found that most networks are sub-extensive, and there are almost no super-extensive network. These findings reveal that the dimension is suitable for measuring the information in the network.

4.1.4. Volume dimension

When two boxes have the same number of nodes, how can dimension be used to distinguish their effect based on the structure? Due to the existence of the hub nodes, nodes with large degree will be

Table 1
Summary of fractal dimension and its improved forms in complex networks.

Study	Year	Dimension	Power law function
Song et al. [32]	2005	Fractal dimension	The number of boxes and the box size
Zhang et al. [64]	2016	Fractal dimension based on Coulomb's law	The number of boxes and the box size based on Coulomb's law
Zhang et al. [45]	2014	Fuzzy fractal dimension	The covering ability of boxes and the box size
Wei et al. [68]	2014	Information dimension	The information volume and the box size
Zhang et al. [71]	2015	Tsallis information dimension	The information volume based on the Tsallis entropy and the box size
Wei et al. [76]	2014	Volume dimension	The average volume and the box size
Lacasa et al. [77]	2013	Correlation dimension	The correlation sum and the box size
Wei et al. [78]	2019	Cluster-growing dimension	The number of nodes and the box size based on the closeness sequence
Shanker [79]	2007	Zeta dimension	The number of nodes in the border of box and the box size based on the Zeta function
Grassberger et al. [80]	1983	Generalized dimension	The information volume based on coefficient q and the box size
Rozenfeld et al. [81]	2007	Transfinite fractal dimension	The mass and diameter in self-similar but non-fractal networks (infinite dimension, but finite transfinite dimension)
Komjáthy et al. [57]	2019	Transfinite Cesaro fractal dimension	The Cesaro-sum and the box size in graph sequences

important in the network. Therefore, Wei et al. proposed the volume dimension d_v based on nodes' degree [76]. The average volume of the network $\langle V(l)\rangle$ and the box size follow the power law function as shown:

$$\langle V(l)\rangle \sim l^{d_v},$$

where the average volume $\langle V(l)\rangle$ is obtained by $\sum_{i\in N} V_i(l)/|N|$, $V_i(l)$ is the sum of the degree in the ith box with the size l. Therefore, the volume dimension is obtained by:

$$d_v = -\lim_{l\to 0}\frac{\ln\langle V(l)\rangle}{\ln l}. \tag{10}$$

4.1.5. Correlation dimension

The correlation dimension was initially applied to describe geometric object. Lacasa et al. applied it to investigate the property of the urban network [77]. The correlation sum $C(l)$ and the box size l follow the power law function $C(l) \sim l^{d_c}$. The number of nodes $C_i(l)$ around node i ($d_{ij} < l$) is shown as:

$$C_i(l) = \sum_{j\in N, j\neq i} I(l - d_{ij}),$$

where $I(\chi) \equiv 1$ when $\chi > 0$. Otherwise $I(\chi)$ equals to 0. The ratio of nodes in the box around node i to the number of nodes in the network (apart from node i itself), is then defined as $C_i(l)/(|N|-1)$, and the correlation sum $C(l)$ is obtained by:

$$C(l) = \frac{1}{|N|}\sum_{i\in N}\frac{C_i(l)}{|N|-1}.$$

$C(2) = \bar{k}/(|N|-1)$, where $\bar{k}$ is the average degree, because $C_i(2) = k_i$. The correlation dimension is obtained by:

$$d_c = \lim_{l\to 0}\frac{\ln C(l)}{\ln l} = \lim_{l\to 0}\frac{\ln\left[\frac{1}{|N|}\sum_{i\in N}\frac{C_i(l)}{|N|-1}\right]}{\ln l}. \tag{11}$$

There are still other dimensions, such as the transfinite fractal dimension [82], cluster-growing dimension [78] and Zeta dimension [79]. The fractal dimensions and its improved forms are summarized in Table 1. These dimensions consider different information in the box and show the fractal property of the network.

4.2. Node property

All the dimensions discussed above are used to show the structural properties from network scale, hence box-covering is required as the foundation. In this subsection, we introduce a dimension at a different scale, called the local dimension. This kind of dimension only focuses on the central node and shows the fractal property in the network. Since local dimensions reveal the local structure around the central node (which is related to node importance identification), several experiments of importance will be discussed in Section 5.2.1 to show their real-world applications.

4.2.1. Local dimension

The local dimension d_{l_i} was proposed by Silva et al. to reveal the topological structure of geographic network [46]. They found the power law relation shown below:

$$B_i(r) \sim r^{d_{l_i}}, \tag{12}$$

where $B_i(r)$ is the number of nodes within radius r (include r) for central node i. d_{l_i} can be obtained by the double logarithmic fitting shown below:

$$d_{l_i}(r) = \frac{d}{d\ln r}\ln B_i(r). \tag{13}$$

The radius r will increase from 1 to $\max d_{ij}$. Meanwhile, d_{l_i} varies for the choice of central node. Eq. (13) can be rewritten based on the discrete nature of the network:

$$d_{l_i}(r) = \frac{r}{B_i(r)}\frac{d}{dr}B_i(r) \simeq r\frac{b_i(r)}{B_i(r)},$$

where $b_i(r)$ is the number of nodes whose d_{ij} equals to r, and $d_{l_i}(r)$ varies for the choice of radius. In general, the local structure of different central nodes is different; the local dimension is thus improved to consider the varied locality for different nodes [49]. For node i, the scale of the locality r_i is unique and it is $\max_{j\in N} d_{ij}$. The improved local dimension is defined as:

$$d_{l_i} = \frac{d}{d\ln r_i}\ln B_i(r_i) \simeq r\frac{b_i(r_i)}{B_i(r_i)}. \tag{14}$$

The setting of r_i allows the radius to increase according to the structural characteristics of the selected central node.

4.2.2. Fuzzy Local dimension

Similar to the fuzzy fractal dimension, nodes have different contributions to the box due to their different positions. The node has a greater contribution when it is closer to the central node; the fuzzy local dimension d_{fl_i} [84] is defined as:

$$d_{fl_i} = \frac{d}{d\ln r_i}\ln\beta_i(r_i), \tag{15}$$

where r_i is the locality of node i which has already been discussed above, and $\beta_i(r_i)$ is the fuzzy number of nodes in the box. Due to the discrete nature, it can be rewritten as:

$$d_{fl_i} \simeq r_i\frac{\alpha_i(r_i)}{\beta_i(r_i)},$$

where $\alpha_i(r_i)$ is the fuzzy number of nodes whose d_{ij} equals to r_i. $\beta_i(r_i)$ is defined by the fuzzy theory shown below:

$$\beta_i(r_i) = \frac{\sum_{j\in N_i} D_{ij}(r_i)}{n_i(r_i)},$$

$$D_{ij}(r_i) = \exp\left(-\frac{{d_{ij}}^2}{{r_i}^2}\right),$$

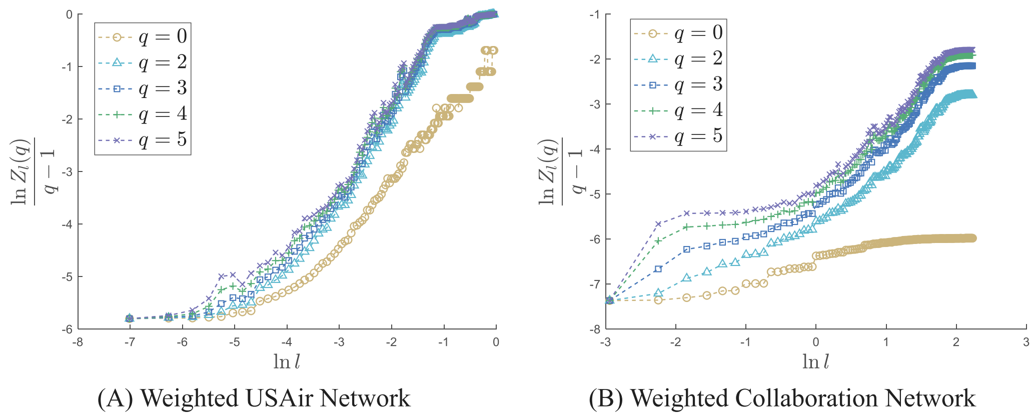


Fig. 3. The multi-fractal dimension of weighted USAir network and collaboration network.
Source: Adapted from Ref. [83].

where $n_i(r_i)$ is the number of nodes within the box, node j represents the other nodes in the box, and $D_{ij}(r_i)$ is the Gaussian membership function. Compared with the local dimension, each node in the box will have a unique contribution ($D_{ij} \in [0,1]$) based on the distance.

4.2.3. Local information dimension

Since d_{l_i} and d_{fl_i} directly consider the number of nodes in the box as the information, the local information dimension d_{li_i} is then proposed [85] when the information is measured by Shannon entropy. This dimension is a quasilocal method due to its different box size growth method. The information $I(r_i)$ in the box around central node i is shown below:

$$I(r_i) = p_i(r_i) \ln p_i(r_i),$$

where $p_i(r_i) = n_i(r_i)/|N|$ is the probability in the box when the radius equals to r_i. The local information dimension can be obtained based on the information $I(r_i)$:

$$d_{li_i} = \frac{d}{d \ln r_i} I(r_i) = \frac{d}{d \ln r_i}\left[p_i(r_i) \ln p_i(r_i)\right] \simeq \frac{r_i}{1 + \ln \frac{n_i(r_i)}{|N|}} \frac{\eta_i(r_i)}{|N|}, \tag{16}$$

where $\eta_i(r_i)$ is the number of nodes whose $d_{ij} = r_i$, and r_i increases from 1 to $\tau_i = 0.5 \max_{j \in N} d_{ij}$. This box size growth method is different from the previous local dimensions and has a lower time complexity. The locality τ_i of the local information dimension varies with the choice of central node.

4.2.4. Multi-local dimension

d_{l_i} can be extended to multi-scale to account for the local property. With different values of coefficient q, the structural property can be explored from different perspectives by the multi-local dimension [50]. The information in the box is obtained by $p_i(r_i) = n_i(r_i)/|N|$. The partition consideration $Z_i(q, r_i)$ of the box is then defined as:

$$Z_i(q, r_i) = \begin{cases} \left[p_i(r_i)\right]^q, & q \neq 0, q \neq 1 \\ 1/p_i(r_i), & q = 0 \\ p_i(r_i) \ln p_i(r_i), & q = 1 \end{cases},$$

where q can be adjusted based on the different scales. $Z_i(0, r_i)$ keeps this dimension larger than 0 when $q-1 < 0$, $Z_i(1, r_i)$ considers the information in the box via Shannon entropy, similar with the local information dimension. Therefore, the multi-local dimension is defined as:

$$d_{ml_i}(q) = \begin{cases} \frac{1}{q-1} \lim_{r_i \to 0} \frac{\ln Z_i(q, r_i)}{\ln r_i}, & q \neq 1 \\ \lim_{r_i \to 0} \frac{Z_i(q, r_i)}{\ln r_i}, & q = 1 \end{cases}, \tag{17}$$

where r_i increases from 1 to $\max_{j \in N} d_{ij}$. $q-1$ equals to 0 when $q = 1$, thus the multi-local dimension has a different expression in this case. $d_{ml_i}(1)$ is obtained by the slope of $Z_i(q, r_i)$ and $\ln r_i$, and $d_{ml_i}(q)$ when $q \neq 1$ is obtained from the relationship between $\ln Z_i(q, r_i)/(q-1)$ and $\ln r_i$. Under different values of q, $d_{ml_i}(q)$ can degenerate to other dimensions:

(a) $q = 1$, $d_{ml_i}(1)$ is d_{li_i}, except for different box-size growth method;
(b) $q = 0$, $d_{ml_i}(0)$ is the variant of d_{l_i}, because of the existence of $|N|$ in $p_i(r_i)$.

4.3. Multi property

The fractal dimension and local dimension describe the property of network and node respectively. Apart from these two kinds of dimension, there is also the multi-fractal property of the network [86–88] which can be influenced by the coefficient q, like the multi-local dimension in Section 4.2.4. This is the generalized form of fractal dimension used to describe the fractal property.

4.3.1. Multi-fractal dimension

After the network is covered by network covering algorithm, the partition sum $Z_l(q)$ can be obtained by the information $\mu_i(l)$ shown below:

$$Z_l(q) = \sum_i \left[\mu_i(l)\right]^q,$$

where $q \in R$ is the coefficient, $\mu_i(l) = n_i(l)/|N| \in [0,1]$ is the probability in the ith box. When the box size l increases, the network covering will be different, thereby causing different $Z_l(q)$. The partition sum and the box size follow the power law function $Z_l(q) \sim l^{\tau(q)}$, the mass exponent $\tau(q)$ is thus defined by:

$$\tau(q) = \lim_{l \to 0} \frac{\ln Z_l(q)}{\ln l}.$$

The multi-fractal dimension $d_{mf}(q)$ based on the coefficient q is obtained by:

$$d_{mf}(q) = \begin{cases} \frac{\tau(q)}{q-1}, & q \neq 1 \\ \lim_{l \to 0} \frac{Z_l(1)}{\ln l}, & q = 1 \end{cases}, \tag{18}$$

where $Z_l(1) = \sum_i \mu_i(l) \ln \mu_i(l)$. $Z_l(1)$ is different from the definition of $Z_l(q)$ when $q \neq 1$, preventing the inadmissible definition of $d_{mf}(1)$. Rosenberg further found the minimal covering method [40] and non-monotonicity property [89] of multi-fractal dimension with different values of q. The multi-fractal will degenerate to other dimensions with different coefficient q:

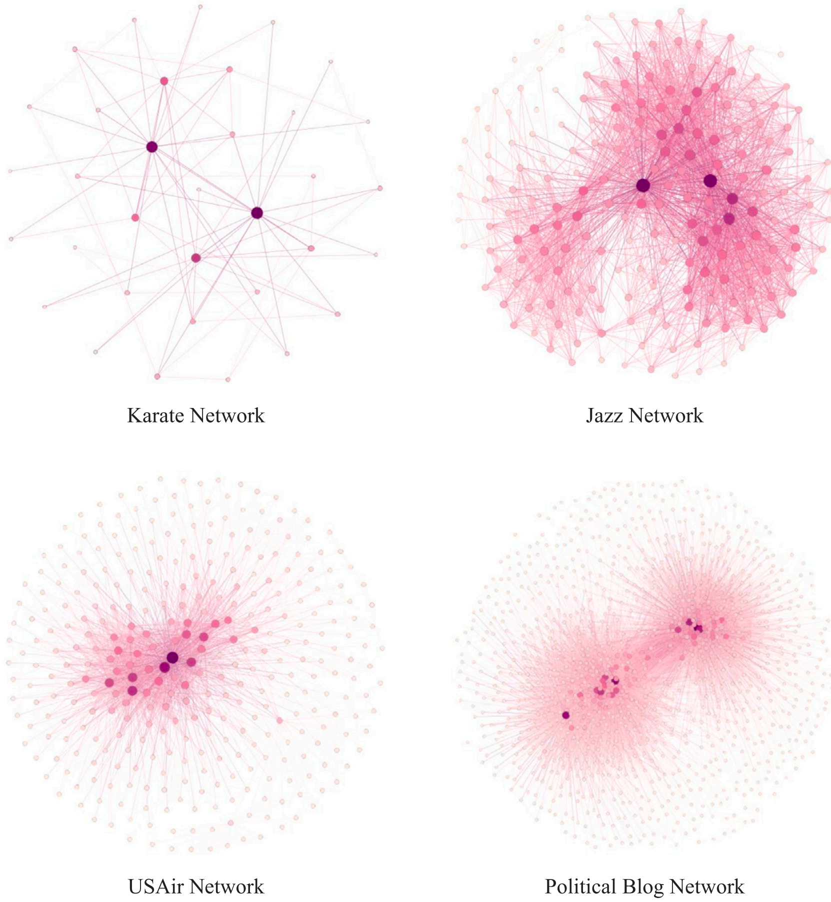


Fig. 4. The structure of four real-world complex networks.

(a) $q = 0$, $d_{mf}(0) = d_f$;
(b) $q = 1$, $d_{mf}(1) = d_I$;
(c) $q = 2$, $d_{mf}(2) = d_c$.

The multi-fractal dimensions of weighted USAir and collaboration networks [2] are shown in Fig. 3. As observed from the results, both of them follow the power law function under different values of q. Therefore, these two real-world networks have the multi-fractal property. Yu et al. [90] proposed new methods to generate different families of fractal networks. They also analyzed the fractal and multi-fractal properties in these networks. Rosenberg [91] studied the property of generalized Hausdorff dimension in real-world networks, and compared it with generalized dimension. Zheng et al. [92] proposed a new box covering algorithm (based on the degree volume dimension) to analyze multi-fractal property in network, and found the relationship between multi-fractal property and the distribution of node degree. The reliable multi-fractal property of networks is further studied by Xue et al. [93].

5. Real-world applications

Complex network can be a useful tool for modeling real-world complex systems with complicated topological structure [94–96], thus attracting researchers from different fields such as power networks [97], social networks [98–100], sensor networks [101–104], and stock-market networks [105–107]. In these networks, most applications can be classified under two main types of problems. The first application is about exploring the network property [108–110]. For example, how to measure the vulnerability of the airline network when the network structure is adjusted every year [111]; how to assess the pipe resilience to failures in water supply network [112]; how to identify the underlying structure of large scale economic network [113]; and how to determine the average weighted receiving times of the weighted hierarchical graph on random walk [114]. These problems can be solved using the property of complex networks, such as vulnerability and robustness of the network [115,116].

The other application is in measuring the property of each node, namely the individual in the complex system. For instance, how to find the influential spreaders in a social network [117,118]; how to find these key nodes to avoid the dismantling of networks [119]; and how to construct the recommender system in the online shopping network [48,120]. The applications of fractal property are summarized in Table 2. Different dimensions can be used to address a variety of problems across different disciplines: the fractal dimensions are applied to explore the global information of the complex system, and the local dimensions are applied to study the property of individuals (local information) in the network. The networks [3] and their structural properties are shown in Table 3, and their topological graphs are shown in Fig. 4. $|N|$ and $|E|$ represent the number of nodes and edges in the network, $\langle k\rangle$ is the average degree of node, $\langle C\rangle$ and ρ are the average clustering coefficient and density of the network, $\langle l\rangle$ and D are the average path length and diameter of the network.

[2] http://vlado.fmf.uni-lj.si/pub/networks/data/.

[3] http://www-personal.umich.edu/~mejn/netdata/.

Table 2
Summary of the applications of fractal property in complex networks.

Application	Methods	Reference
Average weighted receiving time determination	Box dimension	Ref. [114]
Network robustness measurement	Fractal dimension	Ref. [115]
Shape classification	Fractal dimension	Ref. [121]
Disease spread study	Fractal dimension	Ref. [122]
Digital imaging processing	Fractal dimension	Ref. [123]
Urban Development Research	Fractal dimension	Ref. [124]
Power system operation	Fractal dimension	Ref. [125]
Network vulnerability evaluation	Fractal dimension and improved forms	Ref. [47,111,126]
Degradation state recognition of planetary gear	Information dimension	Ref. [127]
Structural description of financial networks	Generalized volume-based dimension	Ref. [128]
Changes of soil properties	Multi-fractal dimension	Ref. [129]
Underlying structure of economic network identification	Fractal dimension and multi-fractal dimension	Ref. [113]
Urban bus-transport network exploration	Fractal dimension and multi-fractal dimension	Ref. [130]
Image edge detection	Local dimension	Ref. [131]
Influential nodes identification	Local dimension and improved forms	Ref. [49,50,84,85]
Similar nodes measurement	Combine fractal dimension and local dimension	Ref. [48]

Table 3
Topological properties of four real-world networks.

Network	$\|N\|$	$\|E\|$	$\langle k\rangle$	$\langle C\rangle$	ρ	$\langle l\rangle$	D
Karate Network	34	78	4.588	0.588	0.139	2.408	5
Jazz Network	198	2742	27.697	0.633	0.141	2.235	6
USAir Network	332	2126	12.807	0.749	0.039	2.738	6
Political Blog Network	1222	16714	27.355	0.360	0.022	2.738	8

5.1. Network property

In this subsection, we mainly introduce the vulnerability and robustness of networks via two examples to show the applications of fractal property.

5.1.1. Network vulnerability

The airline networks downloaded from Bureau of Transportation Statistics[4] are used to measure the vulnerability. The node in the network represents the airport, and the edge represents the existence of airline. The self-loops are removed, and only the largest connected subgraph is retained. The network will become less vulnerable over time, because this is the purpose of the manager to change the network structure. There are two methods to measure the vulnerability of networks. Gou et al. proposed the approach based on fractal dimension [47],

$$V_{d_f} = \left(\frac{1}{|E|} \sum_{e\in E} {b_e}^{d_f} \right)^{\frac{1}{|d_f|}}. \tag{19}$$

Wen et al. then modified the key coefficient to the fuzzy fractal dimension, and proposed an improved method [126],

$$V_{d_{ff}} = \left(\frac{1}{|E|} \sum_{e\in E} {b_e}^{d_{ff}} \right)^{\frac{1}{|d_{ff}|}}, \tag{20}$$

where b_e is the betweenness of edge e, d_f and d_{ff} are the fractal dimension and fuzzy fractal dimension, respectively. The comparison methods include the average inverse geodesic length l^{-1}, the size of the largest component LGS, and the average edge betweenness b_{nor}. The normalized results of these five methods in airline networks are shown in Fig. 5.

As observed from Fig. 5, V_{d_f} and $V_{d_{ff}}$ give a more suitable result because the vulnerability of the airline network gradually decreases over the years. These comparison methods give the wrong order for some networks. For example, the airline network in 2007 is identified as the most vulnerable network by l^{-1}, LGS, and b_{nor}; the airline network in 2003 is identified as the most stable network by b_{nor}. These results are inconsistent with the development of airline network. In order to close this critical gap, the fractal property can be used to measure the vulnerability of real-world complex networks.

[4] https://www.transtats.bts.gov/DL_SelectFields.asp?.

5.1.2. Network robustness

Wu et al. applied the fractal dimension to quantify the robustness of networks [115]. As way of example, three typical complex networks (small-world network [30] , scale-free network [31], and random network [132]) are used, with their topological graphs shown in Fig. 6 (A) – (C). We observe that nodes are closely connected with each other in the small-world network, and most nodes are connected to a few hub nodes in the scale-free network. In the scale-free network, the efficiency becomes lower when the size of network increases, thereby causing the network to become less robust. The fractal dimension d_f monotonously decreases with the increase in network size, and other methods also reveal a monotonous change in this situation. The same situation can be seen in the other two networks. Therefore, the fractal dimension is a viable method to measure the robustness of the network.

5.2. Node property

Apart from the network property, fractal property can be also used to measure the property of node (individual) in the network, which will be discussed in this subsection.

5.2.1. Node importance

In Section 4.2, the local dimension (LD) [49], fuzzy local dimension (FLD) [84], local information dimension (LID) [85], and multi-local dimension (MLD) [50] have been introduced. The importance of nodes can be directly identified by these dimensions due to the unique dimension of each node. The weighting coefficient q equals to 2 in MLD to depict a general situation because it can degenerate to other dimensions when $q = 0$ and $q = 1$. Firstly, the importance of nodes in four networks are identified by (a) the four dimensions, (b) betweenness centrality (BC), (c) closeness centrality (CC), and (d) degree centrality (DC), where BC, CC, and DC are the three classical node importance identification algorithms (introduced in [133]). The list of top-10 node identified by dimensions and classical methods are shown in Table 4. The nodes in bold mean that they are identified by dimensions but not identified by classical methods. In Karate network, node 1 and 34 are identified as the top-2 node by most dimensions, and this is similar with the classical methods. The same situation can be shown in other networks, such as node 136 and 60 in Jazz network, node 118 in USAir network, node 12 and 28 in Political blogs network. From the overall results, the list of top-10 node is also similar among these methods.

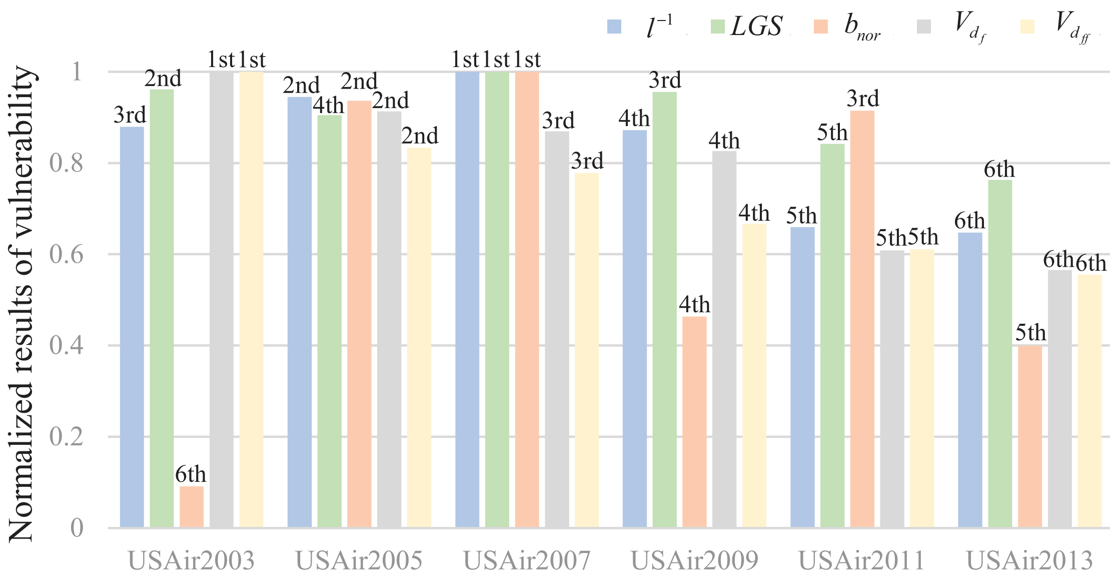
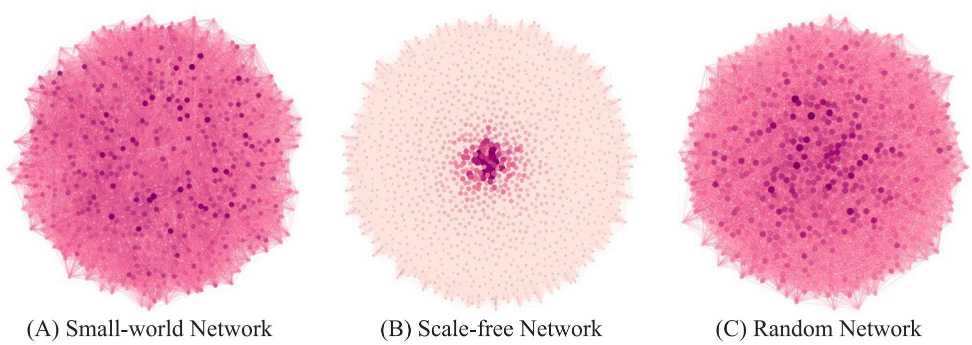


Fig. 5. Normalized results of vulnerabilities obtained by different methods in airline networks.
Source: Adapted from Ref. [126] after normalization.



Number of Node	Small-world Network			Scale-free Network			Random Network		
	b_G	l^{-1}	d_f	b_G	l^{-1}	d_f	b_G	l^{-1}	d_f
300	28.4137	0.5612	3.1511	15.1841	0.6164	4.2420	27.9413	0.5664	3.6684
600	63.8520	0.5003	2.5193	30.0585	0.5622	4.0388	59.6943	0.5234	2.9049
900	104.0939	0.4622	2.2637	45.1777	0.5412	3.7202	94.9720	0.4960	2.5185
1200	145.5859	0.4397	2.1821	60.5962	0.5289	3.5041	133.0558	0.4745	2.3020

(D) Robustness result of three kinds of networks

Fig. 6. Structure of the three networks. Robustness result due to the different networks with different number of nodes.
Source: Adapted from Ref. [115] under three typical networks.

The importance of each node is ranked by the score obtained by the different methods, and the nodes with the same score cannot be ranked properly. Therefore, the frequency ω (the number of nodes with the same score) for different methods is shown in Fig. 7. In Karate network (Fig. 7 (A)), there are around 20 nodes with the same score in BC, which means the importance of up to half the nodes (56%) cannot be reasonably assessed by BC. As for CC and DC, there is also a lot of nodes with the same score between 5th and 15th. In contrast, dimensions have better performance to distinguish the node importance. More than two nodes with the same score only appear in the 12th and 15th rank, which is much better than the classical methods. In other networks, BC results in the same situation: numerous nodes have the same score in the last rank. The rank obtained by DC is the shortest, measuring half of the length obtained by other methods. It is consistent with the scale-free property, because a lot of nodes is only connected with the hub node and their degrees equal to 1. Therefore, these dimensions based on the fractal property can be applied to identify the node importance, and have better performance than the classical methods.

5.2.2. Node similarity

Other than identifying influential nodes, the local information can also be used to measure similar nodes after being combined with fractal dimension. The approach [48] based on the modified relative entropy (Kullback–Leibler divergence) is shown below:

$$\xi_{ij} = \sum_{k=1}^{m'} \frac{\left(\frac{p_i(k)}{p_j(k)}\right)^{d_f} - \left(\frac{p_i(k)}{p_j(k)}\right)}{1 - d_f}, \tag{21}$$

where $m' = \min\{k_i, k_j\} + 1$, d_f is the fractal dimension of the network, and $p_i(k)$ is based on the local dimension shown as:

$$p_i(k) = \begin{cases} \frac{d_{l_i}}{\Sigma_{i\in N} d_{l_i}}, & k \le k_i + 1 \\ 0, & k > k_i + 1 \end{cases}, \tag{22}$$

where d_{l_i} is the local dimension of node i. $p_i(k)$ should be in decreasing order, to ensure accurate result. The relevance matrix describes the

Table 4
The list of top-10 node identified by dimensions and the classical methods. (The nodes in bold mean that they are identified by dimensions but not identified by classical methods.)

Rank	Karate Network							Jazz Network						
	BC	CC	DC	LD	FLD	LID	MLD	BC	CC	DC	LD	FLD	LID	MLD
1	1	1	34	34	1	32	34	136	136	136	60	136	136	60
2	3	3	1	1	34	3	1	60	60	60	136	168	168	136
3	34	34	33	33	33	14	33	153	168	132	132	70	70	132
4	33	32	3	24	3	9	24	5	70	168	83	122	122	83
5	32	33	2	3	2	20	3	149	83	70	168	83	**178**	168
6	6	14	32	2	32	33	2	189	132	108	99	194	83	99
7	2	9	4	**30**	24	1	**30**	167	194	99	108	174	**18**	108
8	28	20	24	6	28	2	6	96	122	158	158	149	153	158
9	24	2	14	**7**	**31**	34	**7**	115	174	83	194	**178**	**118**	194
10	9	4	9	28	**30**	4	28	83	158	7	7	**53**	132	7
Rank	USAir Network							Political Blogs Network						
	BC	CC	DC	LD	FLD	LID	MLD	BC	CC	DC	LD	FLD	LID	MLD
1	118	118	118	118	118	118	118	12	28	12	12	28	12	12
2	8	261	261	261	261	67	261	304	12	28	28	36	28	28
3	261	67	255	152	67	261	152	94	16	304	304	**449**	16	304
4	47	255	182	230	255	201	230	28	14	14	14	**931**	14	14
5	201	201	152	255	201	47	255	145	36	16	16	**146**	304	16
6	67	182	230	182	182	255	182	6	67	94	94	**26**	94	94
7	313	47	166	112	166	166	112	16	94	6	6	16	67	6
8	13	248	67	**147**	47	248	**147**	300	35	67	67	67	36	67
9	182	166	112	166	248	182	166	163	145	35	35	**129**	35	35
10	152	112	201	**293**	112	112	**293**	25	304	145	36	12	6	36

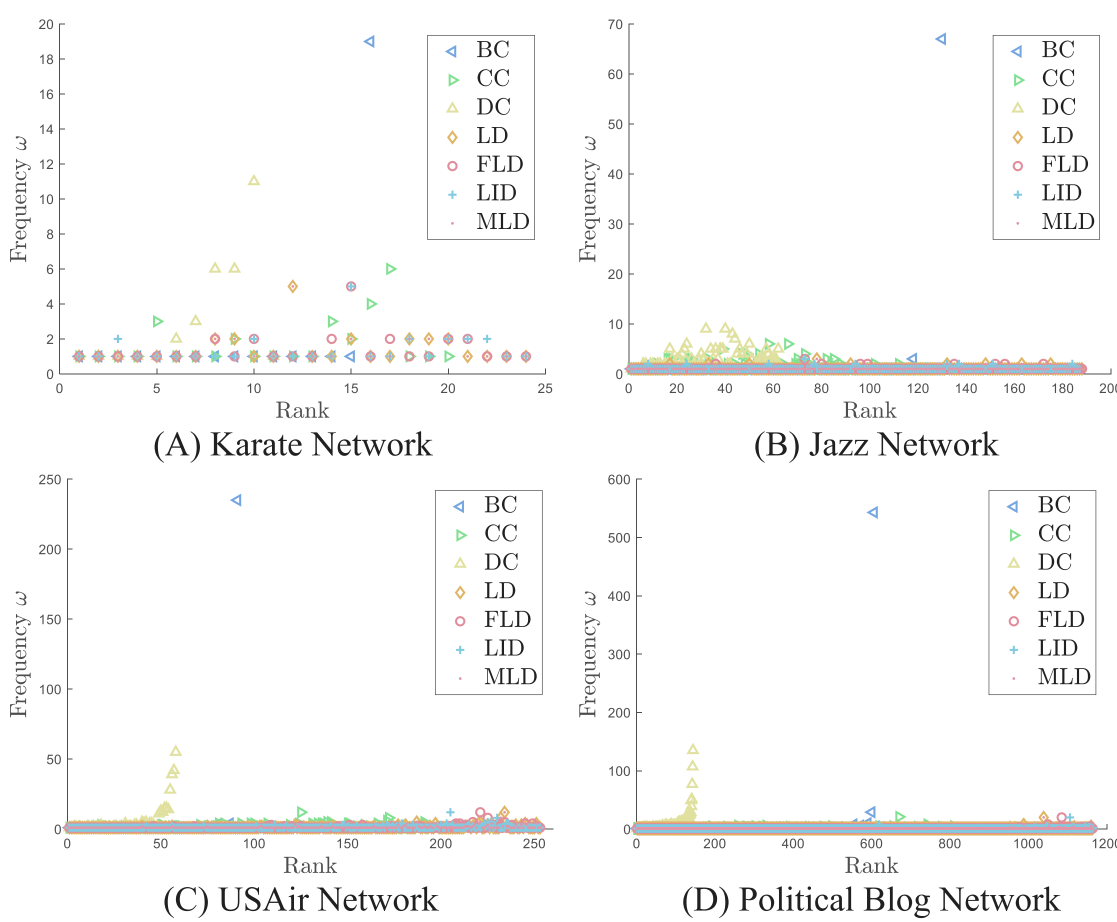


Fig. 7. The frequency of nodes in the four networks obtained by different methods.
Source: Adapted from Ref. [50] under different methods.

difference, and the larger value in the matrix means a larger difference. The relevance matrix (after applying the logarithm) is given in Fig. 8 to show nodes' difference in the Karate network. This matrix is symmetrical, which means a node and its most similar node is one pair of mutual similar node. The method which can identify more pairs of mutual similar node can be considered to be more effective. The ratio

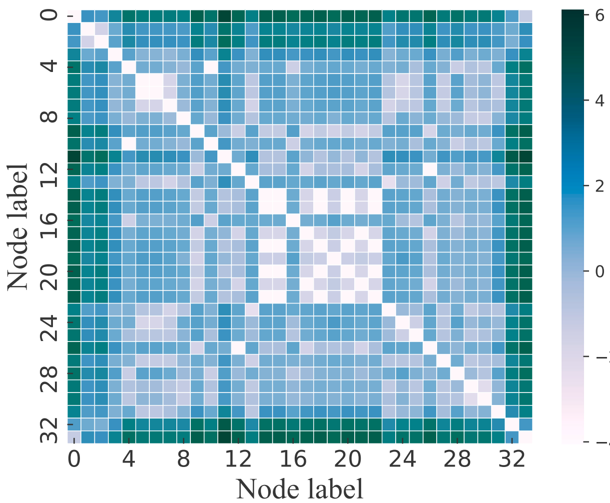


Fig. 8. The relevance matrix of Karate network obtained by the modified relative entropy (after applying the logarithm).
Source: Image adapted from Ref. [135] under different methods.

of mutual similar node of the four networks is shown in Fig. 9. These comparison methods (CN, Katz, and LP) are introduced in Ref. [134]. As observed from the results, the modified relative entropy (RE) is the most viable for using to identify similar nodes compared to the other three comparison methods, because of its largest ratio of mutual similar node.

In general, if one pair of nodes is more similar, then their influential ability will be closer. The susceptible–infected model is useful for measuring the influential ability of nodes. The influential ability difference $\mu(t)$ is obtained by:

$$\mu(t)=\frac{1}{|N|}\sum_{i=1}^{|N|}\left|\frac{y_i(t)-y_i{}'(t)}{|N|}\right|,$$

where $y_i(t)$ and $y_i{}'(t)$ are the number of nodes infected by node i and its most similar node. The results based on the different networks are shown in Fig. 10 by taking the average of 100 independent repeated experiments. The modified relative entropy obtains the lowest $\mu(t)$ in the whole infection process (time t), which suggests that the modified relative entropy can measure the similarity among the nodes, and at the same time, have better performance.

6. Conclusion and outlook

Fractal property describes the self-similarity of the object and is an important property for complex networks and real-world complex system. In this review paper, the fractal property of the network is being discussed. To obtain the fractal dimension, the network should be covered properly in advance. A variety of network covering algorithms have been introduced in Section 3. Numerous dimensions for describing the fractal property are then introduced in Section 4. Due to their different emphasis on the network structure, the dimension can show the fractal property of the network and central node. Most real-world complex systems can be modeled by network using the individual as a node and their connection via the edge. In this manner, the fractal property can be used to deal with these real-world problems (Section 5) involving the network vulnerability measurement and node importance identification.

We also discuss various future direction in this field:

(1) Dynamic and multi-layer network: Existing researches are concentrated in single-layer static networks, although various researches have also been undertaken on weighted and unweighted networks. Does the multi-layer network exhibit fractal property, and will the fractal property of the dynamic network change? These are some of the open problems related to this topic and warrant a more detailed study.
(2) Meso-scale application: Due to the characteristics of fractal dimension and local dimension, network (macro-scale) and node (micro-scale) property have been studied extensively, but there is limited research on the fractal property of the community (meso-scale). Does the community structure have fractal property, and what applications do they have for the community?

CRediT authorship contribution statement

Tao Wen: Software, Methodology, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Writing - review & editing, Visualization. **Kang Hao Cheong:** Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing - original draft, Writing - review & editing, Visualization, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Some useful tools with links are introduced in this Appendix.

(1) The basic introduction about complex networks can be found on the website.[5]
(2) Clauset [58] found that few empirical phenomena follow the power law function, and proposed a method to estimate the minimum value x_{max} and scaling parameter α. The code can be found on Aaron Clauste's website.[6]
(3) Some real-world networks data and useful links (code and data) can be found on Hernan Makse's website,[7] SNAP website,[8] Mark Newman's data sets,[9] Pajek data sets,[10] and Arenas's data sets.[11]
(4) The box covering algorithms, including the maximum-excluded-mass-burning, the compact-box-burning, and random covering in Song et al. paper [36], can be found on the website[12] and website.[13]
(5) More box covering algorithms, including greedy coloring, overlapping box covering algorithm, fuzzy box-covering, max-excluded mass burning, particle swarm optimization box-covering algorithm, ratio of excluded mass to closeness centrality, merge algorithm, simulated annealing algorithm, differential

5 http://networksciencebook.com/.
6 https://aaronclauset.github.io/powerlaws/.
7 https://hmakse.ccny.cuny.edu/software-and-data/.
8 http://snap.stanford.edu/data/.
9 http://www-personal.umich.edu/~mejn/netdata/.
10 http://vlado.fmf.uni-lj.si/pub/networks/data/default.htm.
11 https://deim.urv.cat/~alexandre.arenas/data/welcome.htm.
12 http://www-levich.engr.ccny.cuny.edu/~hernanlab/uploads/modules.py.
13 http://www-levich.engr.ccny.cuny.edu/~hmakse/MEMB.zip.

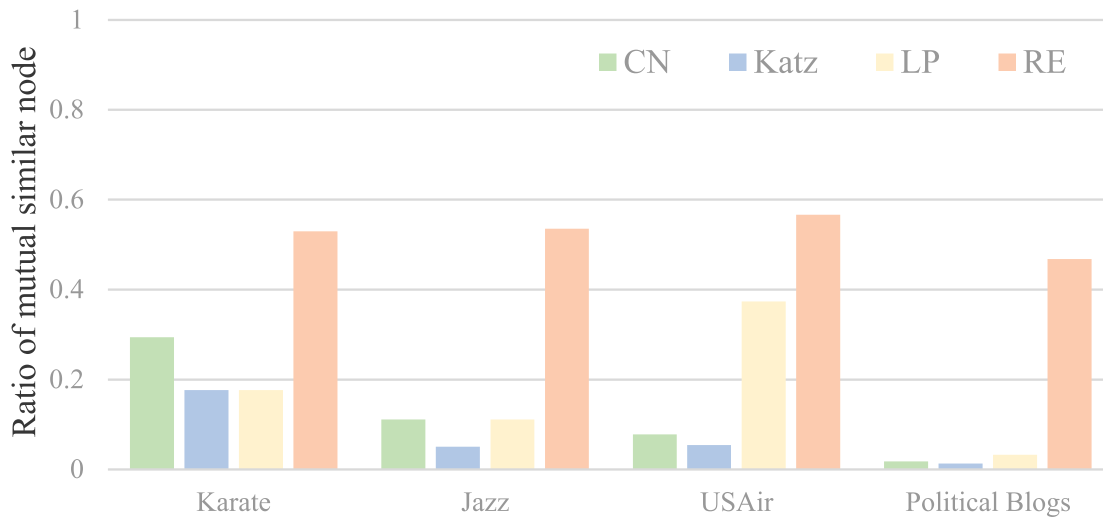

Fig. 9. The ratio of mutual similar node of the four networks.
Source: Adapted from Ref. [48].

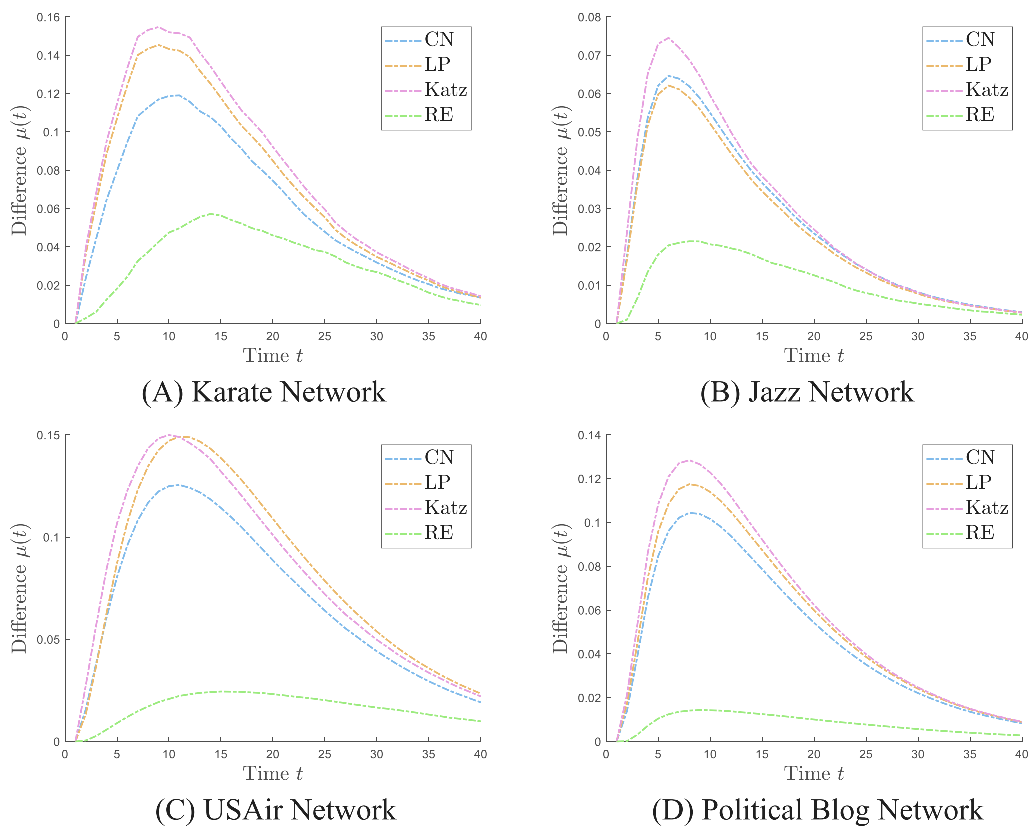

Fig. 10. Difference in the infection ability between each pair of mutual similar node.
Source: Adapted from Ref. [48] under the evaluation of $\mu(t)$.

evolution, random covering, and compact-box-burning, are implemented in the boxes package by Marcell Nagy and Péter Kovács, and they can be found on the website.[14]

(6) For applications, the fractal property can be used to identify influential nodes and similar nodes. To validate the applicability of models in these problems, the susceptible–infected model can be a useful tool. The code and further information about the epidemic in network, including SIS, SIR, SEIR, and multi-layer spreading, can be found on the website.[15]

(7) Since the fractal property of the community (meso-scale) is one of the future directions, the community detection algorithm is

[14] https://github.com/PeterTKovacs/boxes/tree/bd0cd5797f8afacff5754f802e9e458af7a0be31.

[15] https://www.ece.k-state.edu/netse/files/GEMF_Tool_Tutorial.pdf.

important in this regard. The code "Fast Modularity" community detection can be found on the website,[16] and the code of numerous algorithms can be found on the website.[17]

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