Probabilidad
Parcial II - Ejercicio 2

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Problemas

1. (100 pts.) Sean $X \sim \Gamma(\nu/2, 1/2)$ y $Y \sim \Gamma(\mu/2, 1/2)$ con $\nu, \mu > 0$ no precisamente enteros. Utilizando Probabilidad Total halle la distribución de $F = \frac{\mu X}{\nu Y}$, bajo la hipótesis $X \perp Y$.

   Encontraremos la distribución de F

$$
\begin{aligned}
F_f(t) &= P\left[F \leq t\right] \\
&= \int_0^\infty P\left[F \leq t | Y = y\right] f_Y(y) dy \\
&\text{Por Probabilidad Total sobre Y} \\
&= \int_0^\infty P\left[\frac{\mu X}{\nu y} \leq t\right] f_Y(y) dy \\
&\text{Por independencia de X y Y} \\
&= \int_0^\infty P\left[X \leq t\left(\frac{\nu y}{\mu}\right)\right] f_Y(y) dy \\
&\text{Ya que } P\left[g(X) \leq t\right] = P\left[X \leq g^{-1}(t)\right] \\
&= \int_0^\infty \left(\int_0^t \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}} \left(x\left(\frac{\nu y}{\mu}\right)\right)^{\frac{\nu}{2}-1} e^{-\frac{x\left(\frac{\nu y}{\mu}\right)}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu y}{\mu}\right) dx\right) f_Y(y) dy \\
&\text{Por el Teorema de Cambio de Variable} \\
&= \int_0^\infty \left(\int_0^t \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}} \left(x\left(\frac{\nu y}{\mu}\right)\right)^{\frac{\nu}{2}-1} e^{-\frac{x\left(\frac{\nu y}{\mu}\right)}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu y}{\mu}\right) dx\right) \left(\frac{\left(\frac{1}{2}\right)^{\frac{\mu}{2}} y^{\frac{\mu}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{\mu}{2}\right)}\right) dy \\
&= \int_0^\infty \left(\int_0^t \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}} \left(x\left(\frac{\nu y}{\mu}\right)\right)^{\frac{\nu}{2}-1} e^{-\frac{x\left(\frac{\nu y}{\mu}\right)}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu y}{\mu}\right) \left(\frac{\left(\frac{1}{2}\right)^{\frac{\mu}{2}} y^{\frac{\mu}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{\mu}{2}\right)}\right) dx\right) dy \\
&= \int_0^t \left(\int_0^\infty \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}} \left(x\left(\frac{\nu y}{\mu}\right)\right)^{\frac{\nu}{2}-1} e^{-\frac{x\left(\frac{\nu y}{\mu}\right)}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu y}{\mu}\right) \left(\frac{\left(\frac{1}{2}\right)^{\frac{\mu}{2}} y^{\frac{\mu}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{\mu}{2}\right)}\right) dy\right) dx
\end{aligned}
$$

Por Fubini

$$= \int_0^t \frac{1}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} \left(\frac{\nu}{\mu}\right)^{\frac{\nu}{2}} x^{\frac{\nu}{2}-1} \left(\int_0^\infty \left(\frac{1}{2}\right)^{\frac{\nu}{2}+\frac{\mu}{2}} (y)^{\frac{\nu}{2}+\frac{\mu}{2}-1} e^{-\left(\frac{x\left(\frac{\nu}{\mu}\right)}{2}+\frac{1}{2}\right)y} dy \right) dx$$

$$= \int_0^t \frac{\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} \left(\frac{\nu}{\mu}\right)^{\frac{\nu}{2}} x^{\frac{\nu}{2}-1} \left(1+x\frac{\nu}{\mu}\right)^{-\left(\frac{\nu}{2}+\frac{\mu}{2}\right)}$$

$$\left(\int_0^\infty \left(\left(1+x\frac{\nu}{\mu}\right)\left(\frac{1}{2}\right)\right)^{\frac{\nu}{2}+\frac{\mu}{2}} (y)^{\frac{\nu}{2}+\frac{\mu}{2}-1} e^{-\left(\frac{x\left(\frac{\nu}{\mu}\right)}{2}+\frac{1}{2}\right)y} dy \right) dx$$

$$= \int_0^t \frac{\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} \left(\frac{\nu}{\mu}\right)^{\frac{\nu}{2}} x^{\frac{\nu}{2}-1} \left(1+x\frac{\nu}{\mu}\right)^{-\left(\frac{\nu}{2}+\frac{\mu}{2}\right)} dx$$

Al ser la integral de una Gama con parametros $\left(\frac{\nu}{2}+\frac{\mu}{2}, \left(1+x\frac{\nu}{\mu}\right)\left(\frac{1}{2}\right)\right)$

Por lo tanto

$$f_F(x) = \frac{\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)} \left(\frac{\nu}{\mu}\right)^{\frac{\nu}{2}} x^{\frac{\nu}{2}-1} \left(1+x\frac{\nu}{\mu}\right)^{-\left(\frac{\nu}{2}+\frac{\mu}{2}\right)},$$

con lo que concluimos que

$$F \sim F(\nu, \mu).$$