

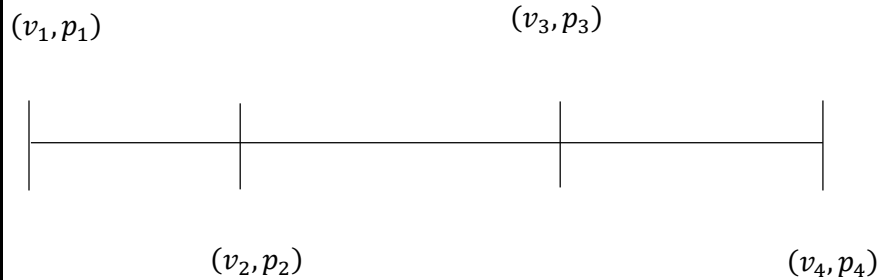
Método de los Elementos Finitos para el problema de dinámica de fluidos en una dimensión con funciones de forma lineales y con peso de Galerkin

$$Ec. 1 \quad f = \bar{v} \frac{d}{dx} v_x - \eta \frac{d}{dx} \left(\frac{d}{dx} v_x \right) + \frac{1}{\rho} \frac{d}{dx} p$$

$$Ec. 2 \quad \frac{d}{dx} \bar{v} = 0$$

Mallado:

Asumimos que las incógnitas están en los nodos



Interpolación:

Utilizaremos funciones de forma lineales para una dimensión.

$$N_i = \frac{x_{i+1} - x}{x_{i+1} - x_i}$$
$$N_{i+1} = \frac{x - x_i}{x_{i+1} - x_i}$$

Discretización:

$$\mathbf{v} \approx [N_i \quad N_{i+1}] \begin{bmatrix} v_i \\ v_{i+1} \end{bmatrix} = \mathbf{N}\mathbf{v}, \quad N(x)$$
$$\mathbf{p} \approx [N_i \quad N_{i+1}] \begin{bmatrix} P_i \\ P_{i+1} \end{bmatrix} = \mathbf{N}\mathbf{p}, \quad N(x)$$

Combinando Ecuaciones:

$$\bar{v} \frac{d}{dx} \mathbf{N}\mathbf{v} - \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N}\mathbf{v} \right) + \frac{1}{\rho} \frac{d}{dx} \mathbf{N}\mathbf{p} - f + \frac{d}{dx} \mathbf{N}\mathbf{v} \approx 0$$

Calculo Residual:

$$\bar{v} \frac{d}{dx} \mathbf{N}\mathbf{v} - \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N}\mathbf{v} \right) + \frac{1}{\rho} \frac{d}{dx} \mathbf{N}\mathbf{p} - f + \frac{d}{dx} \mathbf{N}\mathbf{v} = \xi$$

Método de los residuos ponderados:

$$\int_{\Omega} \xi_i w_i d\Omega = 0$$

$$\mathbf{W} = \begin{bmatrix} w_{x_i} \\ w_{x_{i+1}} \end{bmatrix}$$

$$\int_{\Omega} \mathbf{W} \left[\bar{v} \frac{d}{dx} \mathbf{N} \mathbf{v} - \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N} \mathbf{v} \right) + \frac{1}{\rho} \frac{d}{dx} \mathbf{N} \mathbf{p} - f + \frac{d}{dx} \mathbf{N} \mathbf{v} \right] d\Omega = 0$$

Método de Galerkin:

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_{i+1}} \mathbf{W} \left[\bar{v} \frac{d}{dx} \mathbf{N} \mathbf{v} - \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N} \mathbf{v} \right) + \frac{1}{\rho} \frac{d}{dx} \mathbf{N} \mathbf{p} - f + \frac{d}{dx} \mathbf{N} \mathbf{v} \right] dx = 0$$

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^T \bar{v} \frac{d}{dx} \mathbf{N} \mathbf{v} dx - \int_{x_i}^{x_{i+1}} \mathbf{N}^T \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N} \mathbf{v} \right) dx + \int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{1}{\rho} \frac{d}{dx} \mathbf{N} \mathbf{p} dx - \int_{x_i}^{x_{i+1}} \mathbf{N}^T f dx + \int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{d}{dx} \mathbf{N} \mathbf{v} dx = 0$$

Integrando:

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^T \bar{v} \frac{d}{dx} \mathbf{N} v dx$$

$$= \bar{v} \int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{d}{dx} \mathbf{N} dx v$$

$$= \bar{v} \int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{x_{i+1} - x}{x_{i+1} - x_i} \\ \frac{x - x_i}{x_{i+1} - x_i} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ x_{i+1} - x_i & x_{i+1} - x_i \end{bmatrix} dx v$$

$$= \frac{\bar{v}}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} v$$

$$= \mathbf{A} v$$

Integración por partes:

$$- \int_{x_i}^{x_{i+1}} \mathbf{N}^T \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N} \mathbf{v} \right) dx$$

$$u = \mathbf{N}^T$$

$$du = \frac{d}{dx} \mathbf{N}^T dx$$

$$dv = \eta \frac{d}{dx} \left(\frac{d}{dx} \mathbf{N} \mathbf{v} \right) dx$$

$$v = \eta \frac{d}{dx} \mathbf{N} \mathbf{v}$$

$$= - \mathbf{N}^T \eta \frac{d}{dx} \mathbf{N} \mathbf{v} \Big|_{\Gamma} + \eta \int_{x_i}^{x_{i+1}} \frac{d}{dx} \mathbf{N}^T \frac{d}{dx} \mathbf{N} \mathbf{v} dx$$

$$= \frac{\eta}{x_{i+1} - x_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{v}$$

$$= \mathbf{B} \mathbf{v}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{1}{\rho} \frac{d}{dx} \mathbf{N} \mathbf{p} dx$$

$$= \frac{1}{\rho} \int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{d}{dx} \mathbf{N} \mathbf{p} dx$$

$$= \frac{1}{2\rho} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{p}$$

$$= \mathbf{C} \mathbf{p}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^T f dx = f \int_{x_i}^{x_{i+1}} \mathbf{N}^T dx = \frac{-f(x_{i+1} - x_i)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\mathbf{b}$$

Integración:

$$\int_{x_i}^{x_{i+1}} \mathbf{N}^T \frac{d}{dx} \mathbf{N} \mathbf{v} dx = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{v} = \mathbf{D} \mathbf{v}$$

Sistema de Ecuaciones Lineales:

$$Ec. 1 \mathbf{A} \mathbf{v} + \mathbf{B} \mathbf{v} + \mathbf{C} \mathbf{p} = \mathbf{b} + \mathbf{N}^T \eta \frac{d}{dx} \mathbf{N} \mathbf{v} \Big|_{\Gamma}$$

$$Ec. 2 \mathbf{D} \mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{A} + \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} + \mathbf{N}^T \eta \frac{d}{dx} \mathbf{N} \mathbf{v} \Big|_{\Gamma}$$

$$\mathbf{K} \mathbf{X} = \mathbf{W}$$

Para cada elemento i :

$$k = \begin{bmatrix} (A+B)_{i11} & (A+B)_{i12} & C_{i11} & C_{i12} \\ (A+B)_{i21} & (A+B)_{i22} & C_{i21} & C_{i22} \\ D_{i11} & D_{i12} & 0 & 0 \\ D_{i21} & D_{i22} & 0 & 0 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} v_{i1} \\ v_{i2} \\ p_{i1} \\ p_{i2} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_{i1} \\ b_{i2} \\ 0 \\ 0 \end{bmatrix}$$

Así por ejemplo:

$$(A+B)_{i11} = \frac{-v}{2} + \frac{\eta}{x_{i+1} - x_i}$$

Ensamblaje:

Elemento	<i>i</i>	<i>i</i> + 1
1	1	2
2	2	3
3	3	4

$(A + B)_{111}$	$(A + B)_{112}$	0	0	C_{111}	C_{112}	0	0
$(A + B)_{121}$	$(A + B)_{122} + (A + B)_{211}$	$(A + B)_{212}$	0	C_{121}	$C_{122} + C_{211}$	C_{212}	0
0	$(A + B)_{221}$	$(A + B)_{222} + (A + B)_{311}$	$(A + B)_{312}$	0	C_{221}	$C_{222} + C_{311}$	C_{312}
0	0	$(A + B)_{321}$	$(A + B)_{322}$	0	0	C_{321}	C_{322}
D_{111}	D_{112}	0	0	0	0	0	0
D_{121}	$D_{122} + D_{211}$	D_{212}	0	0	0	0	0
0	D_{221}	$D_{222} + D_{311}$	D_{312}	0	0	0	0
0	0	D_{321}	D_{322}	0	0	0	0

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} + b_{21} \\ b_{22} + b_{31} \\ b_{32} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$