Lectures on the Mathematical Foundations of General Relativity

Winter School of Gravity & Light

These are a collection of notes I took while viewing the online lectures delivered by Dr. Frederic Schuller at the Winter School of Gravity and Light in 2015. These lectures are focused on building a self-contained mathematical foundation and motivation towards understanding relativistic gravitational theory.

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THE WE-HERAEUS INTERNATIONAL WINTER SCHOOL ON
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Mathematical Physics Series Notes

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1 Topology / Topological Spaces

Overview of Lectures

The driving topic of this Winter school is the structure of space and time and in particular: Gravity Theory and Relativistic Matter. The Einstein Equations governing General Relativity relates matter content to gravitational effects. While idea that matter gravitates is an old one, the new notion in GR is that the gravitational effect of matter is encoded directly in terms of the curvature of spacetime itself.

To even begin talking about matter or gravity requires a proper notion of spacetime.

We require a mathematical notion that is good enough to properly talk about spacetime.

Physical Key Definition underlying all Modern Physics:

*Spacetime is a four-**dimensional** topological **manifold** with a **smooth atlas** carrying a torsion-free **connection** compatible with a **Lorentzian metric** and a time orientation satisfying the Einstein equations.*

This series of lectures is therefore aimed at unpacking each of these technical definitions so as to cleanly build up a model of spacetime that enables us to talk about GR at an appropriate level of mathematical rigour.

1.1 Topological Spaces

At the coarsest level, spacetime is a set. However, this is not enough to talk about continuity of maps. In classical physics, we have this idea that there are no discontinuous jumps. That is, a particle trajectory will not somehow jump from one point to another in a discontinuous fashion (Such a scenario would be deemed unphysical).

Figure 1.1: “Discontinuous" trajectories such as shown here are considered to be classically unphysical.

The weakest structure that can be established on a set which allows the notion of continuity of maps to be defined is a **topology.**

Definition 1.1: Topology

Let M be a set. A topology $\mathcal{O}$ is a subset $\mathcal{O} \subset \mathcal{P}(M)^a$ satisfying the following:

1. $\emptyset \in \mathcal{O}, M \in \mathcal{O}$
2. *If $U \in \mathcal{O}$ and $V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$.*
3. *Let A be an index set that can be uncountable. Then, if $U_\alpha \in \mathcal{O}\ \forall \alpha \in A$, then $(\cup_{\alpha \in A} U_\alpha) \in \mathcal{O}$.*

[a] $\mathcal{P}(M)$ is the power set of M.

Definition 1.2: Topological Space

A set M together with a topology on M, $\mathcal{O}_M$ is said to be a topological space. It is typically written as the pair $(M, \mathcal{O}_M)$.

Example 1.1: A Simple Finite Set with Topology

Consider the set $M = \{1, 2, 3\}$, equipped with the topology $\mathcal{O}_1 = \{\emptyset, \{1, 2, 3\}\}$. Then $\mathcal{O}_1$ is a topology on M.

> *Proof.* Clearly, the first axiom is satisfied as both $\emptyset$ and M lie in the topology. The intersection between any two choices of elements gives us either the empty set or M, hence second axiom satisfied. We also have the union between any two elements of the topology as either the empty set or M, thereby satisfying the third axiom. □

Consider the same set, but this time equipped with the topology $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$. This is not a topology on M as the union $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{O}_2$ does not lie in the topology.

Definition 1.3: Chaotic Topology

Let M be any set. The chaotic topology on M is defined by

$$\mathcal{O}_{chaotic} = \{\emptyset, M\} \tag{1.1}$$

Definition 1.4: Discrete Topology

Let M be any set. The discrete topology on M is defined by

$$\mathcal{O}_{discrete} = \mathcal{P}(M) \tag{1.2}$$

The chaotic and discrete topologies represent the most extreme possibilities for a topology. The chaotic topology contains the least possible elements for a topology whereas the discrete contains the most possible elements. It's critical to understand that these are utterly useless.

Definition 1.5: Soft Ball / Open Ball

The soft ball / open ball centred at $p \in \mathbb{R}^d$ with radius r is defined as the set

$$B_r(p) := \left\{ (q_1, ..., q_d) \in \mathbb{R}^d \,\middle|\, \sum_{i=1}^{d} (q_i - p_i)^2 < r^2 \right\} \tag{1.3}$$

In this definition, we have used the additional standard metric space structure that $\mathbb{R}^d$ is equipped with.

Definition 1.6: Standard Topology on $\mathbb{R}^d$

The standard topology $\mathcal{O}_{standard} \subset \mathcal{P}(\mathbb{R}^d)$ is the topology defined as a collection of arbitrary unions of soft / open balls. In essence, the standard topology is defined by the following implicit definition:

$$\text{If } U \in \mathcal{O}_{standard}, \text{ then } \forall p \in U : \exists r \in \mathbb{R}^+ \text{ such that } B_r(p) \subset U. \tag{1.4}$$

A natural question that will eventually be asked is on the choice of topology that spacetime should be equipped with.

Terminology / Notation

- *M is a set in the sense of ZFC*
- *$\mathcal{O}$: topology: set of **open** sets*
- *$(M, \mathcal{O})$: topological space*

Definition 1.7: Open Set

*Let $\mathcal{O}$ be a topology. Then $U \in \mathcal{O}$ are called **open sets**.*

Definition 1.8: Closed Set

*Let $(M, \mathcal{O})$ be a topological space. Then for some subset $A \subset M$, if $M \setminus A \in \mathcal{O}$, then A is called a **closed set**.*

Definition 1.9: Maximal Open Set

Let $(X, \mathcal{O})$ be a topological space. A proper nonempty open subset U of X is said to be a maximal open set if any open set which contains U is X or U.

It should be noted that the notion of open and closed sets are not *a priori* related to one another. If a particular set is open, it does not mean that a statement can be made on whether or not it is closed (and vice versa). You may have sets that are both open and closed with respect to a topology. An example of this is the empty set for any topology.

1.2 Continuous Maps

Definition 1.10: Map

A map is defined between two sets. The sets do not require any additional structure for one to define a map over it. We write $f : M \to N$ to indicate that $\forall m \in M$, we associate a unique element $f(m) \in N$. f would therefore be called a map.

Whether or not a map $f : M \to N$ is *continuous* depends on which topologies are chosen on M and N.

Definition 1.11: Preimage

Let $f : M \to N$. Let $U \subset N$, then the preimage of U is defined as

$$\textit{preim}_f(U) = \{m \in M : f(m) \in U\} \tag{1.5}$$

Definition 1.12: Continuous Map

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Then a map

$$f : M \to N \tag{1.6}$$
$$m \mapsto f(m) \tag{1.7}$$

*is called **continuous** (w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_N$) if*

$$\forall V \in \mathcal{O}_N,\ \textit{preim}_f(V) \in \mathcal{O}_M \tag{1.8}$$

Mnemonic Phrase: *A map is continuous iff the preimages of (all) open sets (of the target space) are open sets (of the domain).*

Example 1.2: Continuous Bijective Maps and Their Inverse

Let $M = \{1,2\}$ *and* $N = \{1,2\}$.

a) *Then let us define* $f : M \to N$ *as follows:*

$$f(1) = 2 \text{ and } f(2) = 1 \tag{1.9}$$

We now define the topologies:

$$\mathcal{O}_M := \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \tag{1.10}$$

$$\mathcal{O}_N := \{\emptyset, \{1,2\}\} \tag{1.11}$$

To check if f *is continuous, we enumerate:*

$$\text{preim}_f(\emptyset) = \emptyset \in \mathcal{O}_M \tag{1.12}$$

$$\text{preim}_f(\{1,2\}) = M \in \mathcal{O}_M \tag{1.13}$$

Hence, f *is continuous.*

b) *We now consider a map going in the opposite direction. Let* $g : N \to M$ *by defined as*

$$g(1) = 2 \text{ and } g(2) = 1 \tag{1.14}$$

In essence, $g = f^{-1}$. *We now check whether* g *is also continuous:*

$$\text{preim}_f(\{1\}) = \{2\} \notin \mathcal{O}_N, \tag{1.15}$$

hence, f^{-1} *is not continuous.*

1.3 Composition of Continuous Maps

Definition 1.13: Composition of Maps

Consider the diagram

$$M \xrightarrow{f} N \xrightarrow{g} P \tag{1.16}$$

Then we can define the composition $g \circ f : M \to P$ *as follows:*

$$M \ni m \mapsto (g \circ f)(m) := g(f(m)) \tag{1.17}$$

Lemma 1.1: Decomposition of Composed Preimage

Let $f : M \to N$ *and* $g : N \to P$ *be maps. Let* $V \subset P$. *Then, we have the identity*

$$\text{preim}_{g\circ f}(V) = \text{preim}_f(\text{preim}_g(V)) \tag{1.18}$$

Theorem 1.1: Composition of Continuous Maps is Continuous

Let $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$ and $(P, \mathcal{O}_P)$ be topological spaces. Then, if the maps $f : M \to N$ and $g : N \to P$ are continuous with respect to the topologies, then their composition $g \circ f : M \to P$ is also continuous.

Proof. We are tasked with finding out whether

$$\text{preim}_{g\circ f}(V) \in \mathcal{O}_M \; \forall \, V \in \mathcal{O}_P. \tag{1.19}$$

This is easily accomplished by using the decomposition shown in Lemma 1.1. Since g is continuous, and $V \in \mathcal{O}_P$, then

$$\text{preim}_g(V) \in \mathcal{O}_N. \tag{1.20}$$

Since f is continuous and $\text{preim}_g(V) \in \mathcal{O}_N$, then we have that

$$\text{preim}_f(\text{preim}_g(V)) \in \mathcal{O}_M \tag{1.21}$$

thereby demonstrating continuity of $g \circ f$. □

1.4 Inheriting a Topology

There are many useful ways to inherit a topology from some given topological space(s).

Important to spacetime Physics:

Given a topological space $(M, \mathcal{O}_M)$, can one construct a topology on $S \subset M$ that is induced by the topology on M? **Yes.**

Proposition 1.1: Basic Properties of Sets

Let A, B, C be sets. Some properties of the intersection include

$$A \cap B = B \cap A \text{ (Commutative Law)} \tag{1.22}$$
$$(A \cap B) \cap C = A \cap (B \cap C) \text{ (Associativity)} \tag{1.23}$$
$$(A \cap B) \cap (C \cap B) = (A \cap C) \cap B \tag{1.24}$$

Some properties of the union operation include

$$A \cup B = B \cup A \text{ (Commutative Law)} \tag{1.25}$$
$$(A \cup B) \cup C = A \cup (B \cup C) \text{ (Associativity)} \tag{1.26}$$
$$(A \cap C) \cup (B \cap C) = (A \cup B) \cap C \tag{1.27}$$

Definition 1.14: Subset Topology

Let $(M, \mathcal{O}_M)$ be a topological space. Consider a subset $S \subset M$ whose topology $\mathcal{O}|_S \subset \mathcal{P}(S)$ is defined as

$$\mathcal{O}|_S := \{U \cap S \mid U \in \mathcal{O}_M\}. \tag{1.28}$$

Then, $\mathcal{O}|_S$ is referred to as the subset topology on S induced by M.

Proof. We will demonstrate that this is indeed a topology.

1. Since $\emptyset \in \mathcal{O}_M$, then $\emptyset \cap S = \emptyset \in \mathcal{O}|_S$. Similarly, since $M \in \mathcal{O}_M$, then $M \cap S = S \in \mathcal{O}|_S$. Hence, the first axiom is satisfied.
2. Suppose that $U \in \mathcal{O}|_S$ and $V \in \mathcal{O}|_S$. Then there exists a $\tilde{U}, \tilde{V} \in \mathcal{O}_M$ such that $U = \tilde{U} \cap S$ and $V = \tilde{V} \cap S$. Then we observe that
$$U \cap V = (\tilde{U} \cap S) \cap (\tilde{V} \cap S) = (\tilde{U} \cap \tilde{V}) \cap S \in \mathcal{O}|_S. \tag{1.29}$$
The last statement is true as $\tilde{U} \cap \tilde{V} \in \mathcal{O}_M$, hence $(\tilde{U} \cap \tilde{V}) \cap S \in \mathcal{O}|_S$.
3. Consider an arbitrary index set A, and collection of open sets $U_\alpha \in \mathcal{O}|_S$, where $\alpha \in A$. Then $\forall \alpha \in A$, $\exists \tilde{U}_\alpha \in \mathcal{O}_M$ such that $U_\alpha = \tilde{U}_\alpha \cap S$. We now consider their union:
$$\cup_{\alpha \in A} U_\alpha = \cup_{\alpha \in A} (\tilde{U}_\alpha \cap S) = \big(\cup_{\alpha \in A} \tilde{U}_\alpha \big) \cap S \in \mathcal{O}|_S, \tag{1.30}$$
where we note that $\big(\cup_{\alpha \in A} \tilde{U}_\alpha \big) \in \mathcal{O}_M$ due to the third axiom of a topology.

□

One of the main uses of this specific way to inherit a topology from a superset is to understand maps restricted to this subset. Consider the following map:
$$M \xrightarrow{f} N, \tag{1.31}$$
where $\mathcal{O}_M$ and $\mathcal{O}_N$ are the associated topologies to M and N. Let $S \subset M$, then we can consider the restriction $f|_S : S \to N$.

Definition 1.15: The Restricted Map

Consider the map $f : M \to N$. Let $S \subset M$. Then we define the restriction $f|_S : S \to M$ as the map $f|_S(m) = f(m) \, \forall m \in S$.

Lemma 1.2: Preimage of Restriction Map Identity

Let $f : M \to N$, $V \subset N$ and $S \subset M$. Then,
$$\mathit{preim}_{f|_S}(V) = \mathit{preim}_f(V) \cap S \tag{1.32}$$

Theorem 1.2: Continuous Maps Restricted to Subset Topologies are also Continuous

Let $f : M \to N$ be continuous with respect to the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$. Then consider the topological space $(S, \mathcal{O}|_S)$ where $\mathcal{O}|_S$ is the subset topology induced by $\mathcal{O}_M$. Then the map
$$f|_S : S \to N \tag{1.33}$$
is continuous with respect to $\mathcal{O}|_S$ and $\mathcal{O}_N$.

Proof. Let $V \in \mathcal{O}_N$. Then we note that

$$\text{preim}_{f|_S}(V) = \text{preim}_f(V) \cap S \tag{1.34}$$

Since f is continuous, then $\text{preim}_f(V) \in \mathcal{O}_M$. Then, since we are considering the induced subset topology on S, we must have that

$$\text{preim}_f(V) \cap S \in \mathcal{O}|_S, \tag{1.35}$$

thereby demonstrating $f|_S$ as continuous. □

Definition 1.16: Coarser

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be topologies. A topology $\mathcal{O}_2$ is said to be coarser than a topology $\mathcal{O}_1$ if $\mathcal{O}_2 \subset \mathcal{O}_1$.

1.5 Tutorial

Exercise 1: True or False?

Tick the correct statements, but not the incorrect ones!

a) A topological space

- ○ is defined by a set, a topology, and an atlas.
- ○ is a set without any further structure.
- • defines a notion of open sets.
- ○ always has integer dimension.
- • allows to check the continuity of a map from the underlying set to itself

An *atlas* is additional structure required for topological manifolds, a concept that will be defined in the next lecture. The notion of *dimension* cannot be defined for vanilla topological spaces, but will be defined for topological manifolds.

b) The chaotic topology on a set M.

- ○ cannot be defined on the natural numbers $\mathbb{N}$.
- ○ consists of all subsets of M.
- • contains the empty set.
- • is the coarsest topology on M.
- • makes all maps $f : N \to M$ continuous, where the domain may carry an arbitrary topology.

To see that the last statement is true, we consider both open sets on M. $\text{preim}_f(M) = \{n \in N | f(n) \in M\} = N$ and $\text{preim}_f(\emptyset) = \emptyset$, thereby demonstrating that all such maps are indeed continuous.

c) Consider a map $f : M \to N$ between topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$.

- ○ Continuity can only be defined if $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ for positive integers m and n.
- • For some maps, one can arrange for the topological notion of continuity to coincide with the undergraduate analysis notion of continuity.
- ○ Continuity of a map can only be defined for some topologies.
- ○ Continuity is a property of a map that only depends on the topology $\mathcal{O}_M$.

- Choosing the discrete topology on M makes all maps from M to N continuous.

If we equip $\mathbb{R}^d$ with the standard topology, then one can have maps whose topological notion of continuity coincides with the $\epsilon - \delta$ notion of continuity in analysis. The final statement is true as selecting the topology $\mathcal{O}_m = \mathcal{P}(M)$ means that $\text{preim}_f(V) \subset M \; \forall V \in \mathcal{O}_N$, thereby ensuring that $\text{preim}_f(V) \in \mathcal{P}(M)$.

d) A subset $U \subset M$ of a topological space $(M, \mathcal{O})$

- ○ may be open and not open at the same time.
- may be open, but not closed.
- may be closed, but not open.
- may be open and closed.
- may be not open and not closed.

The first statement is nonsense as the condition of being an open set is Boolean valued [It either is or isn't]. Consider $M = \{1, 2\}$ equipped with the topology $\mathcal{O} = \{\emptyset, M\}$, then clearly $\{1\}$ is neither open or closed.

Exercise 2: Topologies on a Simple Set

Question: Write down the definition of a topology $\mathcal{O}$ on a set M.

Solution: A topology $\mathcal{O} \subset \mathcal{P}(M)$ is a collection of subsets of M that satisfies the following conditions:

1. $\emptyset, M \in \mathcal{O}$.
2. If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$.
3. If $U_\alpha \in \mathcal{O} \; \forall \alpha \in A$, then $\big(\cup_{\alpha \in A} U_\alpha \big) \in \mathcal{O}$.

Question: Let $M = \{1, 2, 3, 4\}$ be a set. Does $\mathcal{O}_1 := \{\emptyset, \{1\}, \{1, 2, 3, 4\}\}$ constitute a topology on M[1]?

Solution: *Yes.* The first axiom is clearly satisfied. We can enumerate the rest. We note that

$$\emptyset \cap U = \emptyset \in \mathcal{O}_1 \text{ for any } U \in \mathcal{O}_1 \tag{1.36}$$
$$\{1\} \cap M = \{1\} \in \mathcal{O}_1 \tag{1.37}$$

thereby demonstrating the second axiom to be satisfied. Finally, taking the union for any combination of the provided 3 open sets gets us an open set.

Question: Consider $M = \{1, 2, 3, 4\}$ once again. Is $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3, 4\}\}$ a topology on M?

Solution: *No.* Observe that

$$\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{O}_2 \tag{1.38}$$

Question: Are there other topologies than the ones recognized so far?

Yes. One can consider $\mathcal{O} = \{\emptyset, \{2\}, M\}$ as one such example. There are also the chaotic and discrete topologies and many more.

Exercise 3: Continuous Functions

Becoming familiar with the striking impact of choosing topologies on the continuity of a map.

Question: What is the definition of a continuous map?

Solution: Let $f : M \to N$ be a map where $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ are topological spaces. Then, the map f is said to be continuous iff

$$\text{preim}_f(V) \in \mathcal{O}_M \; \forall \, V \in \mathcal{O}_N \tag{1.39}$$

[1] In general, consider an arbitrary set M with a topology $\mathcal{O} = \{\emptyset, U, M\}$, where $U \subset M$. Then $\mathcal{O}$ is a topology on M.

Question: Let $M = \{1,2,3,4\}$ and consider the identity map $\mathrm{id}_M : M \to M$ defined by

$$\mathrm{id}_M(1) = 1, \qquad \mathrm{id}_M(2) = 2, \qquad \mathrm{id}_M(3) = 3, \qquad \mathrm{id}_M(4) = 4 \tag{1.40}$$

Is the map id_M continuous if the domain is equipped with the chaotic topology and the target with the topology $\mathcal{O}_{\text{target}} := \{\emptyset, \{1\}, \{1,2,3,4\}\}$?

Solution: *No.* Observe that

$$\mathrm{preim}_{id_M}(\{1\}) = \{1\} \notin \mathcal{O}_{\text{domain}} \tag{1.41}$$

Question: Consider the inverse $\mathrm{id}_M^{-1} : M \to M$ of the identity map id_M such that now the target is equipped with the chaotic topology and the domain with the topology $\{0, \{1\}, \{1,2,3,4\}\}$.

Provide the values of the map id_M^{-1} and decide whether id_M^{-1} is continuous.

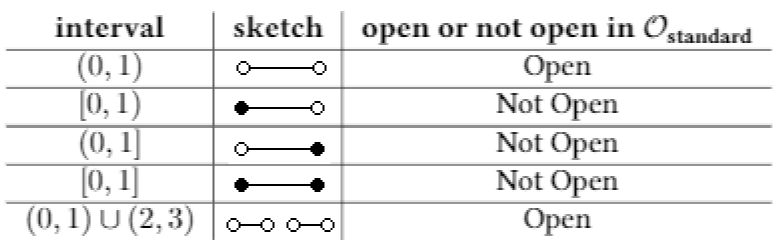
Solution: *Yes.* We note that $id_M^{-1} = id_M$, hence it maps precisely as before. We can enumerate

$$\mathrm{preim}_{id_M^{-1}}(\emptyset) = \emptyset \in \mathcal{O}_{\text{domain}} \tag{1.42}$$
$$\mathrm{preim}_{id_M^{-1}}(M) = M \in \mathcal{O}_{\text{domain}}, \tag{1.43}$$

thereby demonstrating that id_M^{-1} with the associated topologies defined here is a continuous map.

Exercise 4: The Standard Topology on $\mathbb{R}^d$
The topology you always knew, possibly without knowing.

Question: Sketch the real intervals below and decide whether they are open or not in $\mathcal{O}_{\text{standard}}$!



interval	sketch	open or not open in $\mathcal{O}_{\text{standard}}$
$(0,1)$		Open
$[0,1)$		Not Open
$(0,1]$		Not Open
$[0,1]$		Not Open
$(0,1)\cup(2,3)$		Open

Figure 1.2: The openness of several intervals and their sketches.

Question: Which of the following subsets of $\mathbb{R}^2$ are open with respect to the standard topology?

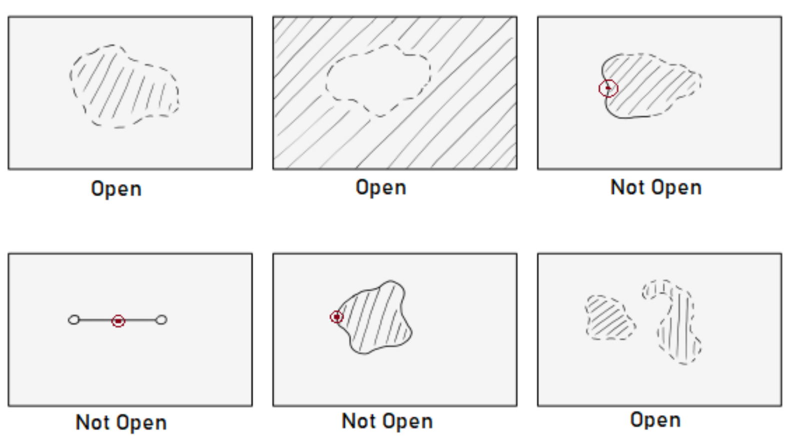


Figure 1.3: Sets in $\mathbb{R}^2$.

Question: Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by the following graph:

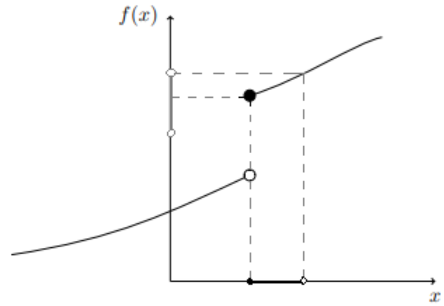


Figure 1.4: A discontinuous map $f : \mathbb{R} \to \mathbb{R}$.

If domain and target of the map are both equipped with the standard topology, is this function continuous?

Solution: f is not continuous. Let (z, b) denote the location of the black dot with the second coordinate corresponding to the image of f. Let (z, a) correspond to the location of the white dot (discontinuity). Then, consider the open set $U = (b - \epsilon, b + \epsilon)$ such that $b - \epsilon > a$. Then, we must have $\text{preim}_f(U) = [z, \delta)$ for some $\delta > z$, which is not an open set in the standard topology of $\mathbb{R}$. Hence, f is not continuous. One can see a visualization above in Figure 1.4.

2 Topological Manifolds

Topological Spaces: Scherrer argues that there are so many topological spaces that they haven't all been classified. However, there are many topological notions such as a Hausdorff space, etc. Imagine you draw out a big chart with every possible topological notion and you go through every possible space and tick off the corresponding boxes (if Boolean value applies) on whether that space satisfies that particular topological notion. There is no such set of topological notions that allows you to define a homeomorphism that preserves all topological properties.

For spacetime physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be charted, analogously to how the surface of the Earth is charted in an atlas.

2.1 Topological Manifolds

Definition 2.1: Topological Manifolds

A topological space $(M, \mathcal{O})$ is called a d−dimensional topological manifold[a] if

$$\forall p \in M : \exists U \in \mathcal{O} \text{ containing } p \in U \text{ such that } \exists x : U \to x(U) \subset \mathbb{R}^d \text{ such that} \tag{2.1}$$

1. *x is invertible (There exists $x^{-1} : x(U) \to U$).*
2. *x is continuous.*
3. *x^{-1} is continuous.*

[a] $\mathbb{R}^d$ is equipped with the standard topology in this definition.

Example 2.1: Simple Topological Manifolds

a) *The doughnut surface (Torus) $M \subset \mathbb{R}^3$.*

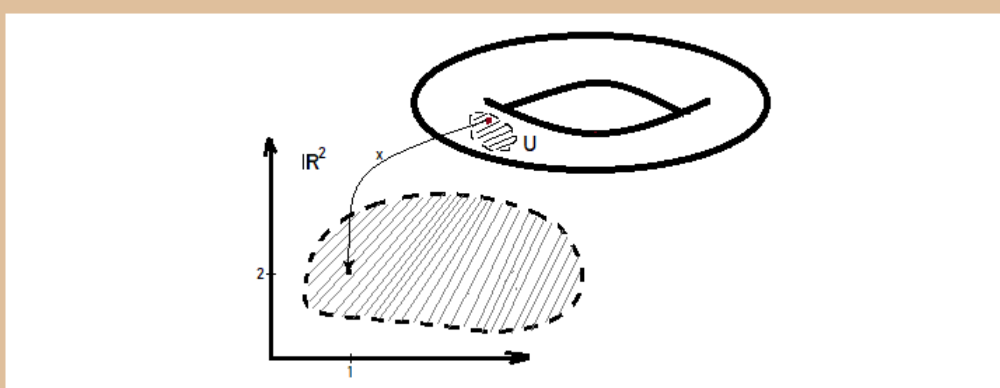


Figure 2.1: A common topological manifold known as the Torus with a chart mapping into $\mathbb{R}^2$.

We claim that this is a 2-dim topological manifold. What is important here is that the original topological manifold M can represent the 'real' world and the coordinates that a point on that surface happens to end up at in $\mathbb{R}^2$ is arbitrary. Suppose that the Earth had a toroidal shape such as shown above and the red dot illustrated in the doughnut represented the foot of the Eiffel tower. When constructing a chart to map out the Earth, imagine that we printed out a sheet of paper and displayed the coordinates of the Eiffel tower to lie at $(1, 2)$ where simply 1 and 2 indicate the amount of distance along their respective axes (perhaps in cm) from the bottom left corner of the published sheet. These coordinates $(1, 2)$ have no physical meaning, they are completely arbitrary.

b) *We now consider a 'one-dimensional' bifurcated diagram.*

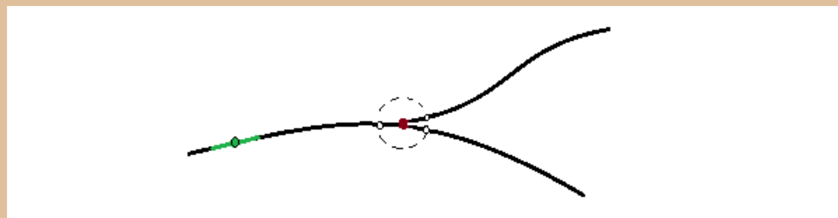


Figure 2.2: A one-dimensional bifurcation.

You can define the induced topology from $(\mathbb{R}^2, \mathcal{O}_{standard})$ onto this space, therefore rendering it to be a honest to god topological space. However, this is not a topological manifold. In particular, the problem arises at the red point shown in Figure 2.2. While you can find an open set in a neighbourhood of the red dot that can possibly be viewed as lying in $\mathbb{R}^2$, there is no such map x that would be continuous and invertible to any open set in $\mathbb{R}$ or $\mathbb{R}^2$.

Definition 2.2: Homeomorphism

*A function $f : X \to Y$ between two topological spaces is said to be a **homeomorphism** if it has the following properties:*

1. *f is a bijection*
2. *f is continuous*
3. *The inverse function f^{-1} is continuous*

*Two topological spaces are said to be **homeomorphic** if there exists a homeomorphism from one space to the other.*

Definition 2.3: Chart

*A **chart** on a topological space $(M, \mathcal{O})$ is a homeomorphism ψ from an open subset $U \subset M$ to an open subset of a Euclidean space. The chart is traditionally written as the ordered pair (U, ψ). ψ is referred to as the chart map.*

Definition 2.4: Atlas

Let $(M, \mathcal{O})$ be a topological space. Let A be some index set and (U_α, x_α) a chart for $(M, \mathcal{O})$ $\forall \alpha \in A$. Then, we say that

$$\mathcal{A} = \{(U_\alpha, x_\alpha) | \alpha \in A\} \tag{2.2}$$

*is an **atlas** of $(M, \mathcal{O})$ if $M = \cup_{\alpha \in A} U_\alpha$.*

Definition 2.5: Maximal Atlas

Let $(M, \mathcal{O}, \mathcal{A})$ be a topological manifold. An atlas $\mathcal{A}$ is said to be a maximal atlas on a topological manifold if there does not exist an atlas $\mathcal{B}$ over the same topological space such that $\mathcal{A} \underset{proper}{\subset} \mathcal{B}$.

Definition 2.6: Coordinate Maps

Suppose that $x : U \to x(U) \subset \mathbb{R}^d$ is a chart map. Since elements of $\mathbb{R}^d$ are simply $d-$tuples, then a chart map can be expressed as

$$x(p) = (x^1(p), x^2(p), ..., x^d(p)), \qquad \text{where } x^i : U \to \mathbb{R},\ i = 1, .., d. \tag{2.3}$$

*The x^i are called the **coordinate maps**. Hence, for some $p \in U$, one could refer to $x^1(p)$ as the first coordinate of point p with respect to the chosen chart (U, x).*

Physics without Telekinesis: The choice of chart is completely irrelevant. It is simply a useful procedure for mapping the 'real world' onto some abstract representation. The way we choose to do this means selecting out a chart. Physics is chart-independent and had it been dependent on how we imagine the world to be, then one would be able to perform physics with telekinesis. Since no experiment has ever revealed this to be the case, we do not need to do chart-dependent physics.

Example 2.2: Chart Examples from Subsets of $\mathbb{R}^2$

Suppose that $M = \mathbb{R}^2$, equipped with $\mathcal{O}_{standard}$. Consider $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Then we can define

$$x : U \to \mathbb{R}^2 \tag{2.4}$$
$$(m, n) \mapsto (-m, -n) \tag{2.5}$$

as one possible chart map. We can define another chart map on $U = \mathbb{R}^2 \setminus \{(a, 0)\}$, where $a \in \mathbb{R}^+_0$.

$$(m, n) \mapsto \left(\sqrt{m^2 + n^2}, \arctan(n/m)\right) \tag{2.6}$$

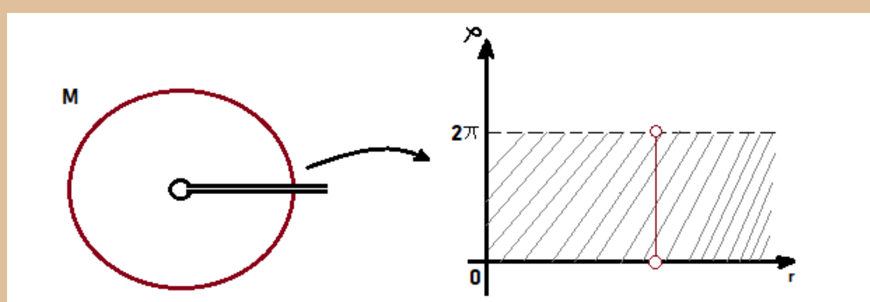


Figure 2.3: Mapping $\mathbb{R}^2 \setminus \{(a, 0)\}$ onto a chart with polar coordinates.

One can see that through the map defined by Equation (2.6), the circle depicted by the red line in M is mapped onto the line segment shown on the (r, ψ) coordinates in $\mathbb{R}^2$.

2.2 Chart Transition Maps

Imagine two charts (U, x) and (V, y) with overlapping region, so that $U \cap V \neq \emptyset$.

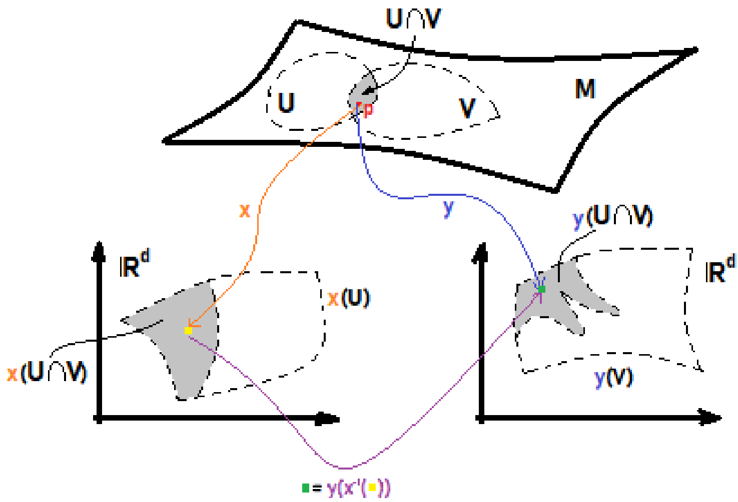


Figure 2.4: Transitioning between two charts with non-empty intersection.

Definition 2.7: Transition Map

Consider a topological manifold M. Let (U, x) and (V, y) be charts such that $U \cap V \neq \emptyset$.

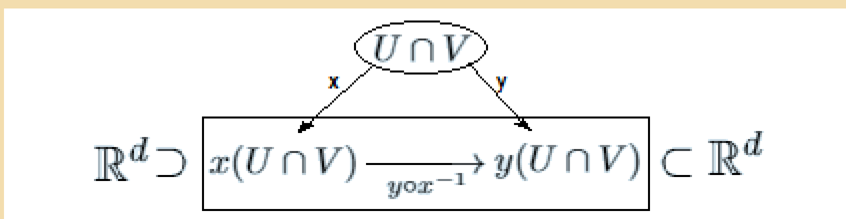


Figure 2.5: The Transition Map.

Then, one can define a map to get from the coordinates chosen by the x chart to those chosen by the y chart. This is of course none other than composing the inverse x^{-1} with y itself:

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V). \tag{2.7}$$

*This is called the **chart transition map**. By the nature of homeomorphisms, the existence of x^{-1} is guaranteed and is also continuous. Similarly, since y is continuous, then $y \circ x^{-1}$ is also continuous. Hence, we have a continuous, invertible map from some open set of $\mathbb{R}^d$ to another open set of $\mathbb{R}^d$.*

In essence, chart transition maps provide a formal correspondence between different ways to chart out the *real world* without ever *touching it*. In Figure 2.5, one can see that $U \cap V$ lives above in the 'real' manifold M, but we are able to go between different chart images that completely lie in $\mathbb{R}^d$.

Informally, the chart transition maps contain the instructions on how to glue together the charts of an atlas.

2.3 Manifold Philosophy

Manifold Philosophy: Suppose that I have the real world M, and I can represent it by drawing charts. You can push the objects you want to study in the 'real world' down to the chart representations and study the objects in the charts themselves. One benefit is that the representations lie in $\mathbb{R}^d$. One can then arrive at the idea that you could define properties of the objects in the 'real world' by properties existing in the chart representations.

Often it is desirable (or indeed the way) to define properties ("continuity") of real-world objects ("$\mathbb{R} \xrightarrow{\gamma} M$") by judging suitable conditions not on the real world object itself but on a chart-representative of the real world object.

Advantages: Consider a curve for some object trajectory. This curve can naturally be parameterized by a real number, hence representing it in the 'real world' could be done by a map $\gamma : \mathbb{R} \to U \subset M$. Now, when we work in $\mathbb{R}^d$, this is done by selecting out a chart $x : U \to x(U) \subset \mathbb{R}^d$ and then considering its image of the curve. In essence, our parameterized curve of the object's image in $\mathbb{R}^d$ can be obtained by the composition $x \circ \gamma : \mathbb{R} \to \mathbb{R}^d$.

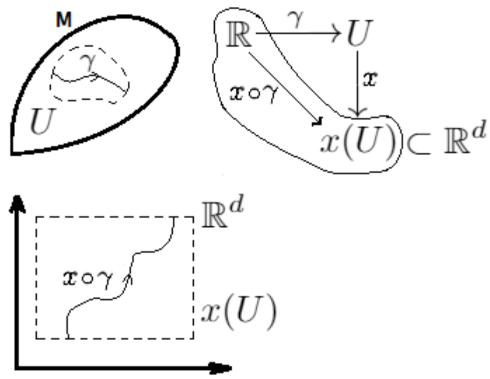


Figure 2.6: An object trajectory in $U \subset M$ and how we work with its image in $\mathbb{R}^d$.

Hence, in undergraduate physics, one would refer to $x \circ \gamma$ as the trajectory of a particle, which is of course just a chart representative. The 'real' curve of the object is given by γ.

Disadvantages: Possible confusion between the chart image and the real world object itself.

A potential key disadvantage is that our definitions may end up being ill-defined. In essence, since we would like to assign a property to γ that depends on properties of $x \circ \gamma$, one potential problem is that our definition may be chart-dependent. Since we have chosen a chart representative, we need to ensure that whatever definitions we assign to γ end up being independent of the charts we chose.

Proposition 2.1: Composed Continuity with Homeomorphism Implies Continuity

Let $(M, \mathcal{O}_m)$, $(N, \mathcal{O}_n)$ and $(P, \mathcal{O}_p)$ be topological spaces. Suppose that we have the maps $x : M \to N$, $y : N \to P$ and $y \circ x : M \to P$. If y is a homeomorphism and $y \circ x$ is continuous, then x is continuous.

Proof. Since y is a homeomorphism, then y is invertible and y^{-1} is continuous. Observe that

$$x = y^{-1} \circ (y \circ x) \tag{2.8}$$

Since y^{-1} and $y \circ x$ is continuous, then so is their composition. We therefore have that x is continuous. □

In Proposition 2.1, we used the fact that y is a homeomorphism to establish that x is also continuous. Was this condition overkill? No. Suppose instead that y and $y \circ x$ were continuous (y is not necessarily a homeomorphism). Then, for x to be continuous requires that

$$\text{preim}_x(V) \in \mathcal{O}_M \;\forall\, V \in \mathcal{O}_N. \tag{2.9}$$

We are given that

$$\text{preim}_x(\text{preim}_y(V)) \in \mathcal{O}_M \;\forall\, V \in \mathcal{O}_P, \tag{2.10}$$

$$\text{preim}_y(V) \in \mathcal{O}_N \; \forall \, V \in \mathcal{O}_P. \tag{2.11}$$

However, by the first and second statement, its key that we have the set equivalence

$$\{\text{preim}_y(V) | V \in \mathcal{O}_P\} = \mathcal{O}_N, \tag{2.12}$$

if we want continuity of x to be true. It's evident that $\{\text{preim}_y(V)|V \in \mathcal{O}_P\} \subset \mathcal{O}_N$, but this might just be a proper subset. Indeed, we can simply give a counter example: Consider $N = P = \{1,2,3,4\}$ where $\mathcal{O}_N = \{\emptyset, \{1\}, \{1,2,3,4\}\}$ and $\mathcal{O}_P = \{\emptyset, \{1,2,3,4\}\}$. Let $y = \text{id}_N$. Then its evident that y is continuous but

$$\{\text{preim}_y(V)|V \in \mathcal{O}_P\} = \{\emptyset, \{1,2,3,4\}\} \neq \mathcal{O}_N \tag{2.13}$$

Therefore, one cannot conclude that $\text{preim}_x(\{1\}) \in \mathcal{O}_M$. Hence, this is why continuity of y alone is insufficient.

Chart-Independence of Continuity

Suppose that we wanted to define continuity of γ. One way to do this is to look at the chart composition $x \circ \gamma$. If $x \circ \gamma$ admits of a property, such as continuity then we want all other charts $y \circ \gamma$ to admit of that same property as well.

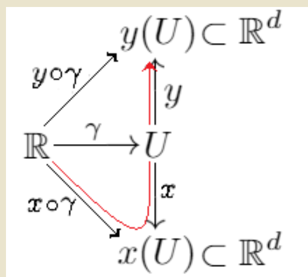


Figure 2.7: Moving between two charts (U, y) and (U, x).

Notice that

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma). \tag{2.14}$$

Suppose that $x \circ \gamma$ is continuous, then $y \circ \gamma$ is also continuous. One can see that $y \circ x^{-1}$ is clearly continuous, as is $x \circ \gamma$, hence by the above composition one has $y \circ \gamma$ as continuous as well.

However, suppose that we wanted to define differentiability of γ by saying that $x \circ \gamma$ is differentiable. One could ask whether this is a chart-independent property and it turns out that for topological manifolds, it is not. Observe that the structure of topological manifolds only contains homeomorphisms - continuous and invertibly continuous maps. Hence, if $x \circ \gamma$ were differentiable, we are not guaranteed that another chart $y \circ \gamma$ is also differentiable since $y \circ x^{-1}$ is not necessarily differentiable (as the composition of a differentiable function and a continuous function is not necessarily differentiable).

2.4 Tutorial

Exercise 1: True or False?

Tick the correct statements, but not the incorrect ones!

a) Topological manifolds

- ● have an integer dimension.
- ○ are equipped with a topology and exactly one chart.
- ● are a special case of topological spaces.
- ○ are the minimal structure required to define continuity of maps.

○ are homeomorphic to $\mathbb{R}^d$

However, we can say that a topological manifold is **locally** homeomorphic to $\mathbb{R}^d$.

b) Which statements about topological manifolds are correct?

● Any topological manifold has a maximal atlas.

● A maximal atlas on a topological manifold may contain infinitely many charts.

● Continuity of a curve on a topological manifold can be checked with respect to an atlas.

○ An atlas of a topological manifold can never contain only one chart.

● If a function is continuous in one chart, it is continuous in every chart of a maximal atlas of a topological manifold.

Given an atlas for a topological manifold, one could always construct a maximal atlas over it.

c) It is correct that

● the real line $(\mathbb{R}, \mathcal{O}_s, \mathcal{A}_\mathbb{R})$ equipped with the standard topology and an atlas whose only chart is the identity map over $(\mathbb{R}, \mathcal{O}_s)$ is a topological manifold of dimension 1.

○ a topological manifold can have a finite set underlying.

○ a topological manifold can never have a topology that is a subset topology.

○ for the topological space $(S^{42}, \mathcal{O}_s|_{S^{42}})$, you cannot build an atlas.

○ a function $f : M \to \mathbb{R}$ can only be said to be continuous if M is a topological manifold.

Exercise 2: An Atlas from a Real World - The Moebius river

How to chart the Moebius strip

Question: Consider the Moebius strip you received as the first "Bastelset" for this tutorial that has a river printed on it. How many charts do you need to cover the Moebius strip?

Draw the image of the river on the Moebius strip under the chart map(s)!

Solution:

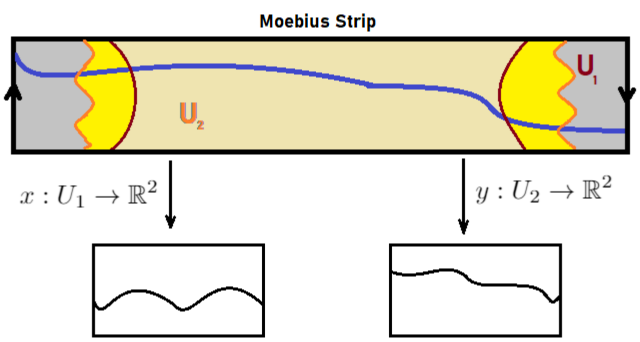


Figure 2.8: Charting Out the Moebius Strip

The Moebius strip requires at least 2 charts to form a topological manifold. We have defined two charts (U_1, x) and (U_2, y). One can construct the Moebius strip as a rectangular strip seen above where we identify the left and right boundaries by a reverse in orientation (We use the arrow directions to indicate this). Formally, viewing the rectangular strip in $\mathbb{R}^2$, setting the origin to lie at the bottom left corner and defining the height of the moebius strip to be h and length as L, we impose the boundary condition

$$(0, y) = (L, h - y)\ \forall\ 0 \leq y \leq 1 \tag{2.15}$$

In Figure 2.8, we have indicated that U_2 lies in between the orange lines, whereas U_1 lies in between the red lines shown. Hence, U_2 covers the inner light yellow and dark yellow section, while U_1 covers the gray and dark yellow section. The dark yellow section represents their intersection. We have also provided an example of how the *river* of the Moebius strip could be mapped into $\mathbb{R}^2$ for the provided charts.

Exercise 3: A Real World from an Atlas

Getting familiar with reconstructing manifolds from its charts.

Question: You received an atlas containing four charts - shown below - of some Real World. Reconstruct the manifold described by this atlas by mentally gluing together the pieces of the second “Bastelset" in the appropriate overlap regions!

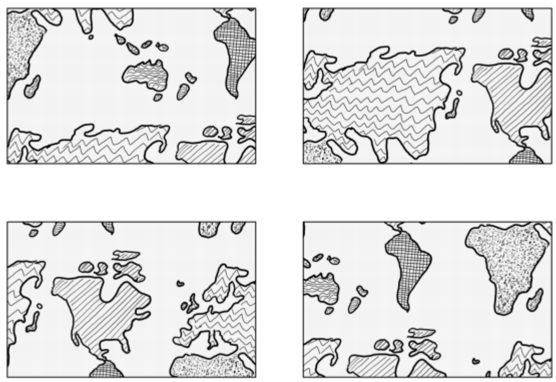


Figure 2.9: Four Charts that glue together to form the *real world.*

It turns out that glueing these charts together gets us a map whose boundaries can be identified with the opposite sides in such a way that we obtain the Torus T^2 in two dimensions.

Exercise 4: Before the Invention of the Wheel

Another one-dimensional topological manifold. Another one?

Consider the set $F^1 := \{(m, n) \in \mathbb{R}^2 | m^4 + n^4 = 1\}$ of pairs of real numbers (m, n). It be equipped with the subset topology $\mathcal{O}_s|_{F^1}$ inherited from the standard topology on $\mathbb{R}^2$.

Question: We look at a map $x : F^1 \to \mathbb{R}$ that maps a pair in F^1 to the first entry of the pair. Write this in formal mathematical terms! Is this map injective?

Solution:

$$x : F^1 \to [-1, 1] \tag{2.16}$$
$$(m, n) \mapsto m \tag{2.17}$$

No, this map is not injective. We note that $m, n \in [-1, 1]$ by the enforced relationship between them. Hence, as a counterexample, one can observe that

$$x[(0, 1)] = 0,\ \ x[(0, -1)] = 0,\ \text{but}\ (0, 1) \neq (0, -1) \tag{2.18}$$

Question: This map may be made injective by restricting its domain to either of two maximal open subsets of F^1. Which ones ? Call them $x_\uparrow$ and $x_\downarrow$!

Solution:

$$x_\uparrow : \{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1 \wedge n > 0\} \to (-1,1) \tag{2.19}$$
$$(m,n) \mapsto m \tag{2.20}$$

$$x_\downarrow : \{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1 \wedge n < 0\} \to (-1,1) \tag{2.21}$$
$$(m,n) \mapsto m \tag{2.22}$$

Question: Now, construct an injective map $y_\uparrow : F^1 \to \mathbb{R}$ that maps every pair in a maximal open subset of F^1 to the *second* entry of the pair.

Solution:

$$y_\uparrow : \{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1 \wedge m > 0\} \to (-1,1) \tag{2.23}$$
$$(m,n) \mapsto n \tag{2.24}$$

Question: Is this map $y_\uparrow$ invertible? If so, construct the inverse $y_\uparrow^{-1}$!

Solution: If we restrict ourselves to $(-1,1)$ as the codomain then $y_\uparrow$ is bijective and therefore invertible.

$$y_\uparrow^{-1} : (-1,1) \to \{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1 \wedge m > 0\} \tag{2.25}$$
$$a \mapsto \left(\sqrt[4]{1-a^4}, a\right) \tag{2.26}$$

Question: Do the domains of the maps $x_\uparrow$ and $y_\uparrow$ overlap? If so, construct the *transition map* $x_\uparrow \circ y_\uparrow^{-1}$ and specify its domain and target!

Solution: There is overlap:

$$\{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1 \wedge m > 0 \wedge n > 0\} \tag{2.27}$$

The transition map is therefore given by

$$x_\uparrow \circ y_\uparrow^{-1} : (0,1) \to (0,1) \tag{2.28}$$
$$s \mapsto x_\uparrow(y_\uparrow^{-1}(s)) = x_\uparrow\left((\sqrt[4]{1-s^4}, s)\right) \tag{2.29}$$
$$= \sqrt[4]{1-s^4} \tag{2.30}$$

Question: How many maps (constructed this way) do you need for their domains to cover the whole set F^1? Does the collection of these domains and maps form an atlas of F^1?

Solution:

$$\text{Domain}(x_\uparrow) \cup \text{Domain}(x_\downarrow) = F^1 \setminus \{(1,0),(-1,0)\} \tag{2.31}$$
$$\text{Domain}(y_\uparrow) \cup \text{Domain}(y_\downarrow) = F^1 \setminus \{(0,1),(0,-1)\} \tag{2.32}$$

These 4 maps $(x_\uparrow, x_\downarrow, y_\uparrow, y_\downarrow)$ cover F^1, therefore forming an atlas for F^1. Hence, F^1 equipped with this atlas is a topological manifold of dimension 1.

3 Multilinear Algebra

We will *not* equip space(time) with a vector space structure.

However, the tangent spaces T_pM (Which will be seen in Lecture 5), which can be defined for smooth manifolds (Lecture 4), do carry a vector space structure.

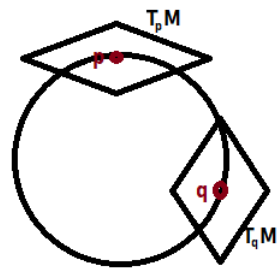


Figure 3.1: Two Tangent Space Diagrams on the Sphere.

It's beneficial to first study vector spaces (and all that comes with it) abstractly for two reasons:

1. for construction of T_pM one needs an intermediate vector space $C^\infty(M)$.
2. *tensor* techniques are most easily understood in an abstract setting.

3.1 Vector Space

Definition 3.1: Vector Space

A $\mathbb{R}$-vector space $(V, +, \cdot)$ is a set V equipped with two operations that we call "addition" $+ : V \times V \to V$ and "s-multiplication" $\cdot : \mathbb{R} \times V \to V$. For all $\lambda, \mu \in \mathbb{R}$ and $u, v, w \in V$, they satisfy the following axioms:

1. *(Commutativity)* $v + w = w + v$.
2. *(Associativity)* $(u + v) + w = u + (v + w)$.
3. *(Identity)* $\exists 0 \in V : \forall\, v \in V : v + 0 = v$.
4. *(Inverse)* $\forall v \in V, \exists(-v) \in V : v + (-v) = 0$.
5. *(Multiplicative Associativity)* $\lambda \odot (\mu \odot x) = (\lambda \cdot \mu) \odot x$.
6. *(Vector Distributivity)* $(\lambda + \mu) \cdot v = \lambda v + \mu v$.
7. *(Scalar Distributivity)* $\lambda \cdot (u + w) = \lambda \cdot u + \lambda \cdot w$.
8. *(Multiplicative Identity)* $1 \cdot v = v$.

Definition 3.2: Vector

An element of a vector space is often referred to, informally as a vector.

Example 3.1: Constructing a Vector Space Structure over the Polynomials

Define a set $P = \{p : (-1, +1) \to \mathbb{R} | p(x) = \sum_{n=0}^{N} p_n \cdot x^n, p_n \in \mathbb{R}\}$. In essence, this is the set of polynomials of (fixed) degree. One can ask whether an object p_s is a vector where $p_s(x) = x^2$. It certainly lies in P, provided that $N \geq 2$, but there is no vector space structure defined on P so p_s cannot be called a vector.

Hence, let us construct a vector space structure on P:

$$+ : P \times P \to P \tag{3.1}$$
$$(p, q) \mapsto p + q \tag{3.2}$$

where

$$(p+q)(x) = p(x) \underset{\mathbb{R}}{+} q(x). \tag{3.3}$$

We'll also define scalar multiplication by

$$\cdot : \mathbb{R} \times P \to P \tag{3.4}$$
$$(\lambda, p) \mapsto \lambda \cdot p \tag{3.5}$$

where $(\lambda \cdot p)(x) := \lambda \underset{\mathbb{R}}{\cdot} p(x)$*. With these definitions, one can confirm that* $(P, +, \cdot)$ *is a vector space. Hence, now one can talk about the notion of* p_s *being a vector as our new object is a vector space. It's recommended to instead only talk about whether something is an element of a vector space rather than a vector. Similarly, one could talk about elements of a tensor space instead of a tensor.*

3.2 Linear Maps

Linear maps are the structure-respecting maps between vector spaces.

Definition 3.3: Linear Map

Let $(V, +_v, \cdot_v)$ *and* $(W, +_w, \cdot_w)$ *be vector spaces. Then a map*

$$\psi : V \to W \tag{3.6}$$

is called ***linear*** *if* $\forall v, \tilde{v} \in V$ *and* $\forall \lambda \in \mathbb{R}$*, the following is true:*

1. $\psi(v +_v \tilde{v}) = \psi(v) +_w \psi(\tilde{v})$
2. $\psi(\lambda \cdot_v v) = \lambda \cdot_w \psi(v)$

Notation: *If* $\psi : V \to W$ *is linear, we will denote this by* $\psi : V \xrightarrow{\sim} W$.

Proposition 3.1: Condition on Linearity

Let $(V, \oplus_v, \odot_v), (W, \oplus_w, \odot_w)$ *be vector spaces over a field* $\mathbb{F}$*. A map* $d : V \to W$ *is linear if and only if*

$$d((\alpha \odot_v x) \oplus_v y) = \alpha \odot_w d(x) \oplus_w d(y) \ \forall x, y \in V \text{ and } \alpha \in \mathbb{F} \tag{3.7}$$

Example 3.2: Differentiation Operator on Polynomials

Let us define a linear map from P *to* P *(as defined in Example 3.1):*

$$\delta : P \to P \tag{3.8}$$
$$p \mapsto \delta(p) := p' \tag{3.9}$$

We can observe that this is linear:

a) $\delta(p +_p q) = (p +_p q)' = p' +_p q' = \delta(p) +_p \delta(q)$
b) $\delta(\lambda p) = (\lambda p)' = \lambda \cdot p' = \lambda \cdot \delta(p)$

Theorem 3.1: Composition of Linear Maps is Linear

If $\psi : V \to W$ and $\phi : W \to U$ are linear maps between vector spaces, then their composition $\phi \circ \psi : V \to U$ is also linear.

As an immediate corollary, the second derivative is linear $\delta \circ \delta : P \xrightarrow{\sim} P$.

3.3 Vector Space of Homomorphisms

Definition 3.4: Vector Space of Homomorphisms Hom(V, W)

Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. Then we define

$$Hom(V, W) := \{\psi : V \xrightarrow{\sim} W\}, \tag{3.10}$$

which is simply just a set at the moment. We define

$$\oplus : Hom(V, W) \times Hom(V, W) \to Hom(V, W) \tag{3.11}$$
$$(\psi, \phi) \mapsto \psi \oplus \phi \tag{3.12}$$

where $(\psi \oplus \phi)(v) := \psi(v) +_w \phi(v)$ and

$$\odot : \mathbb{R} \times Hom(V, W) \to Hom(V, W) \tag{3.13}$$
$$(\lambda, \psi) \mapsto \lambda \odot \psi \tag{3.14}$$

where $(\lambda \odot \psi)(v) = \lambda \cdot \psi(v)$. Hence, $(Hom(V, W), \oplus, \odot)$ is a vector space.

Example 3.3: Building Up Differentiation Elements of Hom(P, P)

We note that Hom(P, P) is a vector space. We can observe that

$$\delta \in Hom(P, P) \tag{3.15}$$
$$\delta \circ \delta \in Hom(P, P) \tag{3.16}$$
$$\vdots \tag{3.17}$$
$$\underbrace{\delta \circ \ldots \circ \delta}_{M} \in Hom(P, P) \tag{3.18}$$

Similarly, we can construct

$$5 \odot \delta \oplus_{Hom(P,P)} \delta \circ \delta \in Hom(P, P) \tag{3.19}$$

3.4 Dual Vector Spaces

Definition 3.5: Dual Vector Space

Let $(V, +, \cdot)$ be a vector space. Then, we define

$$V^* := \{\psi : V \xrightarrow{\sim} \mathbb{R}\} = Hom(V, \mathbb{R}) \tag{3.20}$$

Then, $(V^*, \oplus, \cdot)$[a] is referred to as the ***dual vector space*** *(to V).*

[a] Where $\oplus$ and $\odot$ are the operations defined on $\mathrm{Hom}(V, \mathbb{R})$.

Definition 3.6: Covectors

The terminology that we will sometimes use is that if $\psi \in V^*$, then ψ is called a covector (informally).

Example 3.4: Integration Operator

Let us consider the map

$$I : P \xrightarrow{\sim} \mathbb{R}, \text{ i.e } I \in P^* \tag{3.21}$$

where

$$I(p) := \int_0^1 dx\, p(x) \tag{3.22}$$

It's evident that this map is linear. Hence, I is the integration operator that integrates something from 0 to 1 (i.e $I = \int_0^1 dx$).

3.5 Tensors

Definition 3.7: Multilinear Map

Let $V_1, ..., V_n$ be vector spaces. A map

$$T : V_1 \times ... \times V_n \to \mathbb{R} \tag{3.23}$$

is said to be multilinear if it is a linear map in each argument. For all $\psi_i, \phi_i \in V_i$ and $\forall \alpha, \beta \in \mathbb{R}$, the following is true:

$$T(\psi_1, .., \alpha\psi_i + \beta\phi_i, ..\psi_n) = \alpha T(\psi_1, ..., \psi_i, ..., \psi_n) + \beta T(\psi_1, ..., \phi_i, ..., \psi_n), \qquad 1 \leq i \leq n \tag{3.24}$$

Definition 3.8: (r, s)-Tensor

Let $(V, +, \cdot)$ be a vector space. An (r, s)-tensor T over V is a multi-linear map

$$T : \underbrace{V^* \times ... \times V^*}_{r} \times \underbrace{V \times ... \times V}_{s} \xrightarrow{\sim(r+s)} \mathbb{R} \tag{3.25}$$

Example 3.5: $(1, 1)$-Tensor

If T is a $(1, 1)$ tensor, then

$$T(\psi + \phi, v) = T(\psi, v) + T(\phi, v), \qquad T(\lambda\psi, v) = \lambda \cdot T(\psi, v) \tag{3.26}$$

$$T(\psi, v + w) = T(\psi, v) + T(\psi, w), \qquad T(\psi, \lambda v) = \lambda \cdot T(\psi, v) \tag{3.27}$$

Excursion: In these lectures, we think of a $(1,1)$-tensor as map:

$$T : V^* \times V \xrightarrow{\sim} \mathbb{R}. \tag{3.28}$$

However, some students in the classroom have identified a potential conflict with a definition they were familiar with. They conceived of a $(1,1)$-tensor as a map:

$$\phi_T : V \xrightarrow{\sim} (V^*)^* = V \tag{3.29}$$

$$v \mapsto T(\cdot, v) \tag{3.30}$$

where $T(\cdot, v) : V^* \to \mathbb{R}$. The idea is that these two constructions contain the same data. That is, if I have a map $\phi : V \xrightarrow{\sim} V$, then I can construct

$$T_\phi : V^* \times V \xrightarrow{\sim} \mathbb{R} \tag{3.31}$$

$$(\psi, v) \mapsto \psi(\phi(v)) \tag{3.32}$$

Example 3.6: $(0,2)$-Tensor over P

Let P be the vector space of Polynomials defined earlier.

$$g : P \times P \xrightarrow{\sim} \mathbb{R} \tag{3.33}$$

$$(p, q) \mapsto \int_{-1}^{1} dx\, p(x)q(x) \tag{3.34}$$

Then, this is a $(0,2)$-tensor over P.

Definition 3.9: Tensor Product Space

Suppose that V and W were two vector spaces. We define a new space $V \otimes W$, whose elements are in some sense products of vectors $v \in V$ and $w \in W$. We would denote these products by $v \otimes w$. For all $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$, this product is bilinear:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \tag{3.35}$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \tag{3.36}$$

*If $\{e_i\}_{i=1}^n$ were a basis for V and $\{f_i\}_{i=1}^m$ were a basis for W, then $\{e_i \otimes f_j\}_{i=1,j=1}^{i=n,j=m}$ is a basis for $V \otimes W$. We call $V \otimes W$ the **tensor product** of V and W as the set of all $\mathbb{R}$-valued bilinear functions on $V^* \times W^*$. Given vectors $v \in V$, $w \in W$, we define the tensor product of v and w to be the element of $V \otimes W$ defined as follows*

$$(v \otimes w)(h, g) := v(h)w(g)\ \forall h \in V^*, g \in W^* \tag{3.37}$$

Proposition 3.2: Properties of Tensor Product

1. *Let V, W be vector spaces. Then, we have the following properties on their tensor product:*

$$(V \otimes W)^* \stackrel{\sim}{=} V^* \otimes W^* \tag{3.38}$$

2. *Let V_1, V_2, V_3 be vector spaces. Then, the tensor product is associative:*

$$(V_1 \otimes V_2) \otimes V_3 \stackrel{\sim}{=} V_1 \otimes (V_2 \otimes V_3) \tag{3.39}$$

Proposition 3.3: Tensor Product Space and Set of Tensors

Let V be a vector space. With the tensor product defined in 3.9, we have the following identification:

$$\underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s} = \{\textit{set of all } (r,s)\textit{-tensors over } V\} \tag{3.40}$$

3.6 Vectors and Covectors as Tensors

Theorem 3.2

If $\psi \in V^*$, then $\psi : V \xrightarrow{\sim} \mathbb{R} \iff \psi$ is a $(0,1)-$Tensor. One therefore calls ψ a covector.

Theorem 3.3: Vectors are $(1,0)$-Tensors

For finite-dimensional vector spaces, we have the equivalence $V = (V^*)^*$. Hence, if $v \in V$, then $v : V^* \xrightarrow{\sim} \mathbb{R}$. Hence, one can consider v to be a $(1,0)$-tensor.

3.7 Bases

Definition 3.10: Basis [Lecture]

Let $(V,+,\cdot)$ be a real vector space. A subset $B \subset V$ is called a (Hamel) ***basis*** *if*

$$\forall v \in V,\ \exists \textit{ finite } F = \{f_1, \ldots, f_n\} \subset B : \exists \textit{ unique } v^1, v^2, \ldots, v^n \in \mathbb{R} : \tag{3.41}$$
$$v = v^1 f_1 + \ldots + v^n f_n \tag{3.42}$$

The definition of a basis provided in lecture appears to be wrong. It does not appear to be a very useful definition but nevertheless was what Scherrer defined verbatim. The more appropriate definition encountered in many Linear Algebra texts is the following:

Definition 3.11: Basis

Let $(V,+,\cdot)$ be a real vector space. A finite subset $F = \{f_1, \ldots, f_n\} \subset V$ is called a ***basis*** *of V if*

$$\forall v \in V : \exists \textit{ unique } v^1, v^2, \ldots, v^n \in \mathbb{R} : \tag{3.43}$$
$$v = v^1 f_1 + \ldots + v^n f_n \tag{3.44}$$

In essence, $\{f_1, \ldots, f_n\}$ spans V and $\{f_1, \ldots, f_n\}$ are linearly independent.

Advice: Only choose a basis if you really have to.

Definition 3.12: Dimension

If $\exists$ basis B with finitely many elements, say d many, then we call dim$(V) := d$ the dimension of V.

Definition 3.13: Vector Components with respect to a Basis

Let $(V, +, \cdot)$ be a finite-dimensional vector space. Having chosen a basis $e_1, ..., e_n$ of $(V, +, \cdot)$, we may uniquely associate

$$v \mapsto (v^1, ..., v^n), \tag{3.45}$$

*where $v = v^1 e_1 + ... + v^n e_n$. Then, the v^i are called the **components** of v with respect to the chosen basis.*

3.8 Basis for the Dual Space

Definition 3.14: Dual Basis

Suppose that we choose a basis $e_1, ..., e_n$ for V, then we can choose a basis $\epsilon^1, ..., \epsilon^n$ for V^*. Once $e_1, .., e_n$ on V has been chosen, then we can define

$$\epsilon^a(e_b) = \delta^a_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \tag{3.46}$$

*This uniquely determines choice of $\epsilon^1, ..., \epsilon^n$ from choice of $e_1, ..., e_n$. If a basis $\epsilon^1, ..., \epsilon^n$ of V^* satisfies this, it is called the **dual basis** (of the dual space).*

Example 3.7: Polynomials of Degree at most $N = 3$

Suppose that e_0, e_1, e_2, e_3 is a basis for P where

$$e_0(x) = 1,\ e_1(x) = x,\ e_2(x) = x^2,\ e_3(x) = x^3. \tag{3.47}$$

Compactly, we have that $e_a(x) = x^a$. A dual basis $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$ for P^* is given by

$$\epsilon^a := \frac{1}{a!}\delta^a\big|_{x=0}, \tag{3.48}$$

where $\delta^a\big|_{x=0}$ is the a^{th} derivative evaluated at $x = 0$.

3.9 Components of Tensors

Definition 3.15: Components of Tensor

Let T be an $(r, s)-$tensor over a finite-dimensional vector space V. Let $e_1, ..., e_n$ be the basis for V and $\epsilon^1, ..., \epsilon^n$ be the dual basis of V^*. Then, we define $(r + s)^{dim(V)}$ many real numbers

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} := T\big(\epsilon^{i_1}, ..., \epsilon^{i_r}, e_{j_1}, ..., e_{j_s}\big) \text{ where } i_1, ..i_r, j_1, ..., j_s \in \{1, ..., dim(V)\}. \tag{3.49}$$

*Then, $T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in \mathbb{R}$ are referred to as the **components** of the tensor with respect to the chosen basis.*

This is useful as knowing the components (and basis), one can reconstruct the entire tensor.

Example 3.8: $T : (1,1)$**-Tensor**

Suppose that T is a $(1,1)$-Tensor over n-dimensional vector space V with $e_1, ..., e_n$ being the basis for V and $\epsilon^1, ..., \epsilon^n$ basis for V^*. Then, our components can be expressed as

$$T^i_j := T(\epsilon^i, e_j) \tag{3.50}$$

To reconstruct T, we can expand a $\phi \in V^*$ in the dual basis and $v \in V$ in the basis of V:

$$T(\psi, v) = T\big(\sum_{i=1}^{n} \psi_i \epsilon^i, \sum_{j=1}^{n} v^j e_j\big) \tag{3.51}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_i v^j T(\epsilon^i, e_j) \tag{3.52}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_i v^j T^i_j \tag{3.53}$$

Definition 3.16: Einstein Summation Convention

Let V be a finite-dimensional vector space. Suppose that one takes the convention to express the basis of V with lower indices $e_1, ..., e_n$ and the dual basis of V^* with upper indices $\epsilon^1, ..., \epsilon^n$. Similarly, suppose that we write the components of a vector $v \in V$ with upper indices as $v = \sum_{i=1}^n v^i e_i$ and write components of covectors $\psi \in V^*$ with lower indices $\psi = \sum_{i=1}^n \psi_i \epsilon^i$. Then, whenever we are dealing with tensors over V, one would often expand the vectors / covectors in terms of their basis. For higher rank tensors, one would be required to express many summations and so to reduce this over-saturation of mathematical expressions we define the following convention:

Whenever one sees a term with a lower index and another term (possibly same term) with an upper index in an expression where the index is the same, then it is implicit that one has a hidden summation that is summing over all possible values of that index.

Hence, for a $(1,1)$-Tensor, we would have the convention:

$$\psi_i v^j T^i_j := \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_i v^j T^i_j. \tag{3.54}$$

This convention is referred to as the ***Einstein summation convention****.*

3.10 Tutorial

Exercise 1: True or False?

Tick the correct statements, but not the incorrect ones!

a) What statements on vector spaces are correct?

- ○ Commutativity of multiplication is a vector space axiom.
- ○ Every vector is a matrix with only one column.
- ○ Every linear map between vector spaces can be represented by a unique quadratic matrix.[2]
- ● Every vector space has a corresponding dual vector space.

[2] A Quadratic Matrix is a $n \times n$ matrix for some $n \in \mathbb{N}$. Note that such a quadratic matrix can exist if the dimensions of the vector spaces in the domain and target are the same. In addition, it would not be unique, but rather basis-dependent.

- ○ The set of everywhere positive functions on $\mathbb{R}$ with pointwise addition and $S-$multiplication is a vector space.

The final statement is not true as it doesn't have an additive identity (i.e the zero vector).

b) What is true about tensors and their components?

- • The tensor product of two tensors is a tensor.[3]
- • You can always reconstruct a tensor from its components and the corresponding basis.
- ○ The number of indices of the tensor components depends on dimension.
- ○ The Einstein summation convention does not apply to tensor components.
- ○ A change of basis does not change the tensor components.

c) Given a basis for a d-dimensional vector space V,

- ○ one can find exactly d^2-different dual bases for the corresponding dual vector space V^*.[4]
- • by removing one basis vector of the basis of V, a basis for a $(d-1)$-dimensional vector space V_1 is obtained.
- ○ the continuity of a map $f : V \to W$ depends on the choice of basis for the vector space W.
- • one can extract the components of the elements of the dual vector space V^*.
- • each vector of V can be reconstructed from its components.

Exercise 2: Vector Spaces

Building the vector space $\mathbb{R}^3$ and its dual.

Let $V = \mathbb{R}^3$ be a set of all real triples.

Question: We equip the set V with addition $\oplus : V \times V \to V$ and S- multiplication $\odot : \mathbb{R} \times V \to V$, defined by

$$(a, b, c) \oplus (d, e, f) := (a + d, b + e, c + f) \tag{3.55}$$

and

$$\lambda \odot (a, b, c) := (\lambda \cdot a, \lambda \cdot b, \lambda \cdot c), \tag{3.56}$$

where $+$ and $\cdot$ are the addition and multiplication on $\mathbb{R}$. Check that $(V, \oplus, \cdot)$ is a vector space.

Solution: This is quite trivial to prove but we will nevertheless show that is true.

1. (Associativity) $(x \oplus y) \oplus z = x \oplus (y \oplus z)\ \forall x, y, z \in \mathbb{R}^3$. (Quite obvious)
2. (Abelian) $x \oplus y = y \oplus x\ \forall x, y \in \mathbb{R}^3$. (Quite obvious)
3. $(0, 0, 0)$ is the additive identity as $(0, 0, 0) \oplus (x_1, x_2, x_3) = (x_1, x_2, x_3)\ \forall x_1, x_2, x_3 \in \mathbb{R}$.
4. $\forall x \in \mathbb{R}^3 : \exists -x \in \mathbb{R}^3$ such that $x \oplus -x = 0$.
5. (Multiplicative Distributivity) $\lambda \odot (x \oplus y) = \lambda \odot x \oplus \lambda \odot y\ \forall x, y \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.
6. (Additive Distributivity) $(\lambda + \mu) \odot x = \lambda \odot x + \mu \odot x\ \forall x \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}^3$.[5]
7. (Associativity) $(\lambda \cdot \mu) \odot x = \lambda \odot (\mu \odot x)\ \forall \lambda, \mu \in \mathbb{R}$ and $\forall x \in \mathbb{R}^3$.
8. (Identity) $1 \in \mathbb{R}$ satisfies $1 \odot x = x\ \forall x \in \mathbb{R}^3$.

[3] If T_1 is an (r_1, s_1)-tensor and T_2 is an (r_2, s_2)-tensor over V, then $T_1 \otimes T_2$ is an $(r_1 + r_2, s_1 + s_2)$-tensor over V.

[4] Untrue, as there are unique $\epsilon^a \in V^*$ such that $\epsilon^i(e_j) = \delta^i_j$.

[5] The addition operation $+$ is the canonically defined addition operation over the reals.

Question: Consider the map

$$d : V \to V; (a, b, c) \mapsto d((a, b, c)) := (b, 2c, 0) \tag{3.57}$$

Is d linear?

Solution: We want to check that

$$d(\alpha \odot x \oplus v) = \alpha \odot d(x) \oplus d(y)\ \forall x, y \in V \text{ and } x \in \mathbb{R} \tag{3.58}$$

We can Observe that

$$\begin{aligned} d(\lambda \odot (a, b, c) \oplus (x, y, z)) &= d((\lambda \odot a + x, \lambda \odot b + y, \lambda \odot c + z)) &(3.59)\\ &= (\lambda \odot b + y, 2(\lambda \odot c + z), 0) &(3.60)\\ &= \lambda \odot (b, 2c) \oplus (y, z) &(3.61)\\ &= \lambda \odot d((a, b, c)) + d((x, y, z)) &(3.62)\end{aligned}$$

Question: Show that $d \circ d$ is linear.

Solution: $d \circ d : V \to V$. Since d is linear, then $d \circ d$ is also linear by Theorem 3.1. I won't show this computation explicitly.

Question: Consider the map

$$i : V \to \mathbb{R}; (a, b, c) \mapsto i\big((a, b, c)\big) := a + \frac{1}{2}b + \frac{1}{3}c \tag{3.63}$$

Check linearity. Of what set is i an element?

Solution: We observe that

$$\begin{aligned} i(\lambda \odot (a, b, c) \oplus (x, y, z)) &= i\big((\lambda a + x, \lambda b + y, \lambda c + z)\big) &(3.64)\\ &= \lambda(a + \frac{1}{2}b + \frac{1}{3}c) + (x + \frac{1}{2}y + \frac{1}{3}z) &(3.65)\\ &= \lambda\, i((a, b, c)) + i((x, y, z)) &(3.66)\end{aligned}$$

Hence, i is linear. Consequently i is a linear functional on V; it is an element of the dual space V^*.

Question: Consider the map

$$G : V \times V \to \mathbb{R} \tag{3.67}$$

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto 2 \cdot a_1 \cdot a_2 + \frac{2}{3} \cdot a_1 \cdot c_2 + \frac{2}{3} \cdot b_1 \cdot b_2 + \frac{2}{3} \cdot c_1 \cdot a_2 + \frac{2}{5} \cdot c_1 \cdot c_2 \tag{3.68}$$

Show that G is multilinear.

Solution: We'll show that G is linear in its first entry as an example. That is:

$$G[(\lambda \mathbf{x} \oplus \mathbf{y}, \mathbf{z})] = \lambda G[(\mathbf{x}, \mathbf{z})] + G[(\mathbf{y}, \mathbf{z})] \tag{3.69}$$

We observe that

$$\begin{aligned} &G[(\lambda \odot (u, v, w) \oplus (x, y, z), (a_2, b_2, c_2))] &(3.70)\\ &= 2 \cdot (\lambda u + x) \cdot a_2 + \frac{2}{3} \cdot (\lambda u + x) \cdot c_2 + \frac{2}{3} \cdot (\lambda v + y) \cdot b_2 + \frac{2}{3} \cdot (\lambda w + z) \cdot a_2 + \frac{2}{5} \cdot (\lambda w + z) \cdot c_2 &(3.71)\\ &= \lambda\left[2ua_2 + \frac{2}{3}uc_2 + \frac{2}{3}vb_2 + \frac{2}{3}wa_2 + \frac{2}{5}wc_2\right] + 2xa_2 + \frac{2}{3}xc_2 + \frac{2}{3}yb_2 + \frac{2}{3}za_2 + \frac{2}{5}zc_2 &(3.72)\\ &= \lambda\, G[((u, v, w), (a_2, b_2, c_2))] + G[((x, y, z), (a_2, b_2, c_2))] &(3.73)\end{aligned}$$

By a similar computation, it can be shown that G is linear in its second argument. Hence, G is a $(0,2)$-tensor over V.

Question: Compare the above map $d : V \to V$ with the map $\delta : P_2 \to P_2$ from the lecture and construct a bijective map $j : P_2 \to \mathbb{R}^3$ such that

$$d = j \circ \delta \circ j^{-1} \tag{3.74}$$

Where we recall that $P_2 := \{p : \mathbb{R} \to \mathbb{R} | p(x) = a + bx + cx^2\}$ is the vector space of polynomials of at most degree 2. We also recall that

$$d : \mathbb{R}^3 \to \mathbb{R}^3; d((a,b,c)) = (b, 2c, 0) \tag{3.75}$$

We also have the derivative operator $\delta : P_2 \to P_2$ that acts as $\delta(p) = p' = b + 2cx + 0 \cdot x^2$, where $p = a + bx + cx^2$. Hence, the relation between δ and d appears to be quite similar. We now aim to define

$$j : P_2 \to \mathbb{R}^3; p \mapsto j(p) = (a,b,c), \text{where } p(x) = a + bx + cx^2 \tag{3.76}$$

Hence,

$$j^{-1} : \mathbb{R}^3 \to P_2;\ \big(j^{-1}((a,b,c))\big)(x) = a + bx + cx^2 \tag{3.77}$$

Exercise 3: Indices

Winning the index battle.

Let V be a d-dimensional vector space. Consider two tensors A and B, where

$$A : V^* \times V^* \to \mathbb{R} \tag{3.78}$$

and

$$B : V \times V \to \mathbb{R} \tag{3.79}$$

V has a basis $e_1, e_2, .., e_d$ and V^* has the basis $\epsilon^1, \epsilon^2, ..., \epsilon^d$.

Question: Define the components A^{ab} of A and B_{ab} of B with respect to the given bases.

Solution: This should be

$$A^{ab} = A(\epsilon^a, \epsilon^b), \qquad B_{ab} = B(e_a, e_b) \tag{3.80}$$

Question: We define $A^{[ab]} := \frac{1}{2}(A^{ab} - A^{ba})$. Show that

$$A^{[ab]} = -A^{[ba]} \tag{3.81}$$

and also

$$A^{[ab]}B_{ab} = A^{ab}B_{[ab]} \tag{3.82}$$

Solution: The first relation is trivial. We now consider the second relation, noting that

$$B_{[ab]} = \frac{1}{2}(B_{ab} - B_{ba}) \tag{3.83}$$

Hence, we have that

$$\begin{aligned}
A^{[ab]}B_{ab} &= \frac{1}{2}\big(A^{ab} - A^{ba}\big)B_{ab} &(3.84)\\
&= \frac{1}{2}\big(A^{ab} - A^{ba}\big)(2B_{[ab]} + B_{ba}) &(3.85)\\
&= A^{ab}B_{[ab]} - \frac{1}{2}A^{ba}(B_{ab} - B_{ba}) + \frac{1}{2}A^{ab}B_{ba} - \frac{1}{2}A^{ba}B_{ba} &(3.86)\\
&= A^{ab}B_{[ab]} - \frac{1}{2}\underbrace{(A^{ba}B_{ab} - A^{ab}B_{ba})}_{0} &(3.87)\\
&= A^{ab}B_{[ab]} &(3.88)
\end{aligned}$$

Where the last equality is obtained by the implicit Einstein summation:

$$\sum_{a,b} A^{ba} B_{ab} = \sum_{a,b} A^{ab} B_{ba} \tag{3.89}$$

Question: We additionally define $B_{(ab)} := \frac{1}{2}(B_{ab} + B_{ba})$. Now, show that

$$B_{(ab)} = B_{(ba)} \tag{3.90}$$

and again

$$A^{ab} B_{(ab)} = A^{(ab)} B_{ab} \tag{3.91}$$

Solution: The first relation is trivial. We consider the second relation. We have

$$\begin{aligned} A^{ab} B_{(ab)} &= \frac{1}{2} A^{ab}(B_{ab} + B_{ba}) && (3.92)\\ &= \frac{1}{2} A^{ab} B_{ab} + \frac{1}{2} A^{ab} B_{ba} && (3.93)\\ &= \frac{1}{2} A^{ab} B_{ab} + \frac{1}{2} A^{ba} B_{ab} && (3.94)\\ &= \frac{1}{2}(A^{ab} + A^{ba}) B_{ab} && (3.95)\\ &= A^{(ab)} B_{ab} && (3.96) \end{aligned}$$

Question: Using the results from the previous questions, we can easily show that

$$A^{[ab]} B_{(ab)} = 0 \tag{3.97}$$

i.e., the summation (contraction) of symmetric and anti-symmetric indices yields zero.

Solution:

$$\begin{aligned} A^{[ab]} B_{(ab)} &= \frac{1}{2}(A^{ab} B_{(ba)} - A^{ba} B_{(ba)}) && (3.98)\\ &= \frac{1}{2}(A^{ab} B_{(ba)} - A^{ab} B_{(ab)}) && (3.99)\\ &= \frac{1}{2}(A^{ab} B_{(ba)} - A^{ab} B_{(ba)}) && (3.100)\\ &= 0 && (3.101) \end{aligned}$$

Exercise 4: Linear maps as Tensors

Recognizing that a linear map $V^* \xrightarrow{\sim} V^*$ is a $(1,1)$-tensor over V.

Question: Given a vector space V and linear map $\phi : V^* \xrightarrow{\sim} V^*$, construct a $(1,1)$-tensor T_ϕ.

Solution: We define

$$\begin{aligned} T_\phi : V^* \times V &\to \mathbb{R} && (3.102)\\ (\sigma, v) &\mapsto T_\phi(\sigma, v) := (\phi(\sigma))(v) && (3.103) \end{aligned}$$

Since $\phi(\sigma) \in V^*$, this is well defined. Since $\phi_\sigma := \phi(\sigma)$ is a linear functional, then this ensures that the map is linear in the second entry. In the first entry, ϕ is by design linear. Hence, we have that

$$\begin{aligned} T_\phi(a\sigma + \rho, v) &= (\phi(a\sigma + \rho))(v) = (a\phi(\sigma) + \phi(\rho))(v) && (3.104)\\ &= a(\phi(\sigma))(v) + (\phi(\rho))(v) = aT_\phi(\sigma, v) + T_\phi(\rho, v) && (3.105) \end{aligned}$$

where we have applied the vector space structure of V^* as maps are defined to act as $(f + g)(x) = f(x) + g(x)$.

Question: Given a $(1,1)$-tensor $T : V^* \times V \to \mathbb{R}$, construct a linear map $\phi_T : V^* \xrightarrow{\sim} V^*$.

Solution: We can do this as follows:

$$\phi_T : V^* \xrightarrow{\sim} V^* \tag{3.106}$$
$$\sigma \mapsto \phi_T(\sigma) := \underbrace{T(\sigma, \cdot)}_{\in V^*} \tag{3.107}$$

Hence, one has $T(\sigma, \cdot) : V \xrightarrow{\sim} \mathbb{R}$. It's quite evident that this map is linear.

Question: Show that

a) $T_{\phi_T} = T$

b) $\phi_{T_\phi} = \phi$

Solution:

$$T_{\phi_T} : V^* \times V \xrightarrow{\sim} \mathbb{R} \tag{3.108}$$
$$(\sigma, v) \mapsto T_{\phi_T}(\sigma, v) = (\phi_T(\sigma))(v) = (T(\sigma, \cdot))(v) = T(\sigma, v) \tag{3.109}$$

Hence, we have the equivalence $T_{\phi_T} = T$. Now we check the second equality

$$\phi_{T_\phi} : V^* \xrightarrow{\sim} V^* \tag{3.110}$$
$$\sigma \mapsto \phi_{T_\phi}(\sigma) = T_\phi(\sigma, \cdot) = (\phi(\sigma))(\cdot) = \phi(\sigma) \tag{3.111}$$

Question: Conclude that a linear map $\phi : V^* \xrightarrow{\sim} V^*$ **can be considered** a $(1,1)$-tensor[6].

Solution: Given a linear map ϕ defined in the question, one can always construct a $(1,1)$-tensor $T_\phi : V^* \times V \xrightarrow{\sim} \mathbb{R}$. It should be emphasized that one can **view** ϕ as a $(1,1)$-tensor as a consequence of this but it **is not** itself an actual $(1,1)$-tensor.

[6]One can similarly view another map $\hat{\phi} : V \xrightarrow{\sim} V$ as a $(1,1)$-tensor via a similar construction. It's a bit more sophisticated here as you would have to invoke the isomorphism $(V^*)^* \stackrel{\sim}{=} V$.

4 Differentiable Manifolds

So far, we have looked at topological manifolds which allowed us to talk about curves with continuity so that there are no discontinuous jumps. However, we are also interested in wanting to develop a formal way to talk about the velocity of a particle as well.

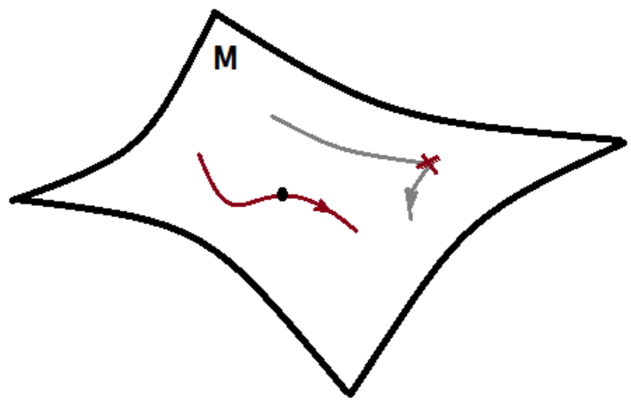


Figure 4.1: A *differentiable* and *non-differentiable* curve on a manifold.

In the above figure, one can see that one of these curves (roughly speaking) is differentiable and the other isn't as you don't have a well-defined velocity at the red cross. For topological manifolds $(M, \mathcal{O})$, one might ask whether its structure is sufficient to talk about differentiable curves. The answer is a resounding **no**. You need to make more *choices* before you can talk about the differentiability of a curve.

It turns out that in dimensions $d = 1, 2, 3$, you have a single 'choice' in how you would go about constructing a notion of differentiation. For manifolds with dimensions $d > 4$, one has a *finite* number of possible ways for each dimension. It turns out that there is a theorem which essentially says that for $d = 4$ manifolds, there is an uncountable number of choices to do this.

Goal: We wish to define a notion of **differentiable**

- curves $\mathbb{R} \to M$
- functions $M \to \mathbb{R}$
- maps $M \to \mathbb{N}$

4.1 Strategy

Choose a chart (U, x) and consider the portion of the curve lying in the chart domain. Hence, considering $\gamma : \mathbb{R} \to U$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \arrow[d, "x"] \\ & x(U) \subset \mathbb{R}^d \end{tikzcd}$$

Figure 4.2: Diagram of curve image in charted region

The idea is to try and *lift* the undergraduate notion of differentiability of a curve in $\mathbb{R}^d$ to a notion of differentiability of a curve on M.

Problem: Can this be well-defined under change of chart?

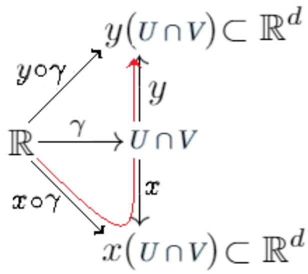


Figure 4.3: Moving Between Charts

Suppose that $x \circ \gamma$ is *undergraduate differentiable* (as a map from $\mathbb{R} \to \mathbb{R}^d$), would $y \circ \gamma$ also be *undergraduate differentiable*? We observe that

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma). \tag{4.1}$$

What you know with certainty is that $y \circ x^{-1}$ is continuous BUT not necessarily differentiable. The composition between a continuous map and differentiable map is NOT necessarily differentiable. Hence, one cannot conclude that $y \circ \gamma$ is also *undergraduate differentiable*.

At first sight, this strategy does not work out. However, there is a remedy for this.

4.2 Compatible Charts

In §4.1, we used any imaginable charts on the topological manifold $(M, \mathcal{O})$. To emphasize this, we may say that we took U and V from the **maximal atlas** $\mathcal{A}$ of $(M, \mathcal{O})$.

From a construction perspective, we can begin with a maximal atlas and throw out every single chart where their transition functions are continuous but not differentiable. What is leftover is a collection of charts such that their transition functions are differentiable.

Definition 4.1: $\mathcal{X}$-Compatible

Two charts (U, x) and (V, y) of a topological manifold are called $\mathcal{X}$-compatible if either
  a) $U \cap V = \emptyset$
  b) $U \cap V \neq \emptyset$, and their chart transition maps

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V), \qquad x \circ y^{-1} : y(U \cap V) \to x(U \cap V), \tag{4.2}$$

both have an undergraduate $\mathcal{X}$-property.

Philosophy: You can assign a variety of different structures to a topological manifold.

Definition 4.2: $\mathcal{X}$-Compatible Atlas

An atlas $\mathcal{A}_{\mathcal{X}}$ is a $\mathcal{X}$-compatible atlas if any two charts in $\mathcal{A}_{\mathcal{X}}$ are $\mathcal{X}$-compatible. A $\mathcal{X}$-manifold is a triple $(\underbrace{M, \mathcal{O}}_{\text{top. mfld}}, A_{\mathcal{X}})$, where $A_{\mathcal{X}} \subset \mathcal{A}_{max}$.

$\mathcal{X}$	Undergraduate $\mathcal{X}$
C^0	$C^(\mathbb{R}^d \to \mathbb{R}^d)$: Continuous maps w.r.t $\mathcal{O}_{st}$
C^1	$C^1(\mathbb{R}^d \to \mathbb{R}^d)$: Differentiable (once) and result is continuous.
C^k	k-times continuously differentiable
D^k	k-times differentiable
C^∞	$C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$: *Smooth* Functions
C^ω	(Analytic) There exists a multi-dimensional Taylor Expansion
$\mathbb{C}^\infty$	$C^\infty(\mathbb{C}^d \to \mathbb{C}^d)$ Complex Smooth

Table 1: Different Types of Manifold Structure

Theorem 4.1

Any $C^{k\geq 1}$-atlas $\mathcal{A}_{C^{k\geq 1}}$ of a topological manifold ***contains*** *a C^∞-atlas.*

Hence, we may WLOG always consider C^∞-manifolds (*smooth manifolds*), unless we wish to define Taylor expandability / complex differentiability.

4.3 Diffeomorphisms

Part of the story thus far has been centred on formalizing a notion of *structure-preserving* maps between two objects. For instance, if M, N are 'naked' sets (without any additional structure), then the structure preserving maps are the bijections (invertible maps).

$$M \xrightarrow{\phi} N \tag{4.3}$$

Two sets M, N are considered to be (set-theoretically) isomorphic $M \cong N$ if $\exists$ bijection $\phi : M \to N$.

Examples: $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{Q}$, $\mathbb{N} \not\cong \mathbb{R}$.

Similarly, for vector spaces, we say that $(V, +_v, \cdot_v) \cong (W, +_w, \cdot_w)$ if there exists $\phi : V \xrightarrow{\sim} W$ that is bijective. We will now want to try and formalize a notion of structure-preserving maps between C^∞ manifolds.

Definition 4.3: Differentiability for Functions of Several Real Variables

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be differentiable at a point x_0 if there exists a linear map $\mathbf{J} : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\mathbf{h}\to 0} \frac{||f(\mathbf{x_0}+\mathbf{h}) - f(\mathbf{x_0}) - \mathbf{J}(\mathbf{h})||_{\mathbb{R}^n}}{||\mathbf{h}||_{\mathbb{R}^m}} = 0 \tag{4.4}$$

If a function is differentiable at $\mathbf{x}_0$, then all of the partial derivatives exist at $\mathbf{x}_0$, and the linear map $\mathbf{J}$ is given by the Jacobian matrix.

Definition 4.4: C^∞-Maps

Consider C^∞-Manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$. Let (U, x) and (V, y) be two charts on M and N respectively. Restricting to these chart domains, we say that $\phi : U \to V$ is a C^∞ map on the charted region if the maps $y \circ \phi \circ x^{-1}$ are undergraduate C^∞ for any choice of charts.

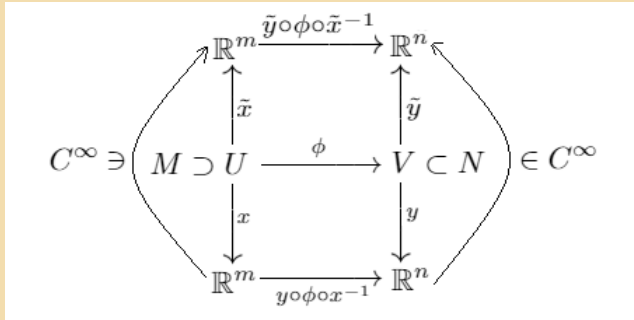


Figure 4.4: Diffeomorphic Map and Transition Charts

One can see a visualization of such a diagram in Figure 4.4.

To ensure that C^∞-maps are well-defined, one would have to show that given a choice of charts $(U, \tilde{x})$ and (U, x) on a region of M and $(V, \tilde{y})$ and (V, y) on region of N, that if $y \circ \phi \circ x^{-1} \in C^\infty$, then $\tilde{y} \circ \phi \circ \tilde{x}^{-1} \in C^\infty$. One can see that

$$\tilde{y} \circ \phi \circ x^{-1} = (\tilde{y} \circ y^{-1}) \circ (y \circ \phi \circ x^{-1}) \circ (x \circ \tilde{x}^{-1}), \tag{4.5}$$

where $\tilde{y} \circ y^{-1}$ and $x \circ \tilde{x}^{-1}$ are C^∞ transition maps. Hence, a composition of C^∞ maps is also C^∞, therefore illustrating that our notion of C^∞-map is well-defined.

Definition 4.5: Diffeomorphic

Two C^∞-Manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ are said to be ***diffeomorphic*** *if there exists a bijection $\phi : M \to N$ such that*

$$\phi : M \to N, \tag{4.6}$$

$$\phi^{-1} : N \to M \tag{4.7}$$

are C^∞-maps.

Theorem 4.2: Unique C^∞ Manifolds (Up to Diffeomorphism)

The number of C^∞ manifolds one can make out of a given C^0-manifold M up to diffeomorphism depends on its dimension. We display the results in a table below.

dim M	Number
1	1
2	1
3	1
4	uncountably infinitely many
5	finite
6	finite
7	finite
$\vdots$	$\vdots$

Table 2: The number of unique C^∞ manifolds one can construct out of a C^0-manifold for varying dimension.

For manifolds of dimensions $d > 4$, some of the results largely arose out of "surgery theory".

4.4 Tutorial

Exercise 1: True or False

a) The function $f : \mathbb{R} \to \mathbb{R}$ ...

- ● ..., defined by $f(x) = x^2$, lies in $C^3(\mathbb{R} \to \mathbb{R})$.
- ○ ..., defined by $f(x) = |x|$, lies in $C^2(\mathbb{R} \to \mathbb{R})$.[7]
- ○ ..., defined by $f(x) = |x^3|$, lies in $C^3(\mathbb{R} \to \mathbb{R})$.[8]
- ○ ..., defined by $f(x) = \exp|x|$, lies in $C^\infty(\mathbb{R} \to \mathbb{R})$.
- ● ..., defined by $f(x) = \ln(x)$, lies in $C^\infty(\mathbb{R} \to \mathbb{R})$.

b) Which statements on differentiable manifolds are true?

- ● Every smooth manifold has a maximal atlas.[9]
- ● Every C^2-manifold is also a C^1-manifold.
- ● Every differentiable manifold is a topological manifold.
- ● In a C^∞-manifold, chart maps are differentiable.
- ● Definition of a C^k-curve requires a $C^{m\geq k}$-manifold.

c) A differentiable manifold

- ● is always a topological space.
- ○ has enough structure to constitute a vectorspace.
- ○ is a generalisation of a set.
- ○ requires the discrete topology.
- ● features at least one set of charts that covers the whole manifold.

Exercise 2: Restricting the Atlas
From topological to differentiable manifolds.

Let $(\mathbb{R}, \mathcal{O}_{st})$ be a topological space. Let it be further equipped with an atlas $\mathcal{A} = \{(\mathbb{R}, x), (\mathbb{R}, y)\}$ where $x : \mathbb{R} \to \mathbb{R}; a \mapsto x(a) = a$ and $y : \mathbb{R} \to \mathbb{R}; a \mapsto y(a) = a^3$.

Question: Construct the chart transition map $y \circ x^{-1} : \mathbb{R} \to \mathbb{R}$ and give its differentiability class.

Solution: Since x is just the identity map, then $x^{-1} : \mathbb{R} \to \mathbb{R}; a \mapsto x^{-1}(a) = a$. Hence, we have that

$$(y \circ x^{-1})(a) = a^3, \tag{4.8}$$

which is $C^\infty(\mathbb{R} \to \mathbb{R})$.

Question: Also construct the chart transition map $x \circ y^{-1} : \mathbb{R} \to \mathbb{R}$. Is $(\mathbb{R}, \mathcal{O}_{st}, \mathcal{A})$ a differentiable manifold?

[7] Clearly the function is not differentiable at $x = 0$.
[8] The second derivative has a kink at $x = 0$, hence third derivative does not exist at $x = 0$.
[9] Any atlas can be made into a maximal atlas.

Solution: We first construct $y^{-1}: \mathbb{R} \to \mathbb{R}; a \mapsto y^{-1}(a) = a^{1/3}$. Hence, we have that

$$(x \circ y^{-1})(a) = a^{1/3} \tag{4.9}$$

Since the function is not continuously differentiable at $a = 0$, then $(\mathbb{R}, \mathcal{O}_{st}, \mathcal{A})$ can't be a differentiable manifold as these transition maps are not C^1-compatible.

Question: Restrict the atlas $\mathcal{A}$ to an atlas $\tilde{\mathcal{A}}$ in order to make $(\mathbb{R}, \mathcal{O}, \tilde{\mathcal{A}})$ into a smooth manifold.

Solution: Remove the chart $(\mathbb{R}, y)$. Hence, $(\mathbb{R}, x)$ covers the entire manifold and is clearly C^∞-compatible with itself. This turns $(\mathbb{R}, \mathcal{O}, \tilde{\mathcal{A}})$ into a smooth manifold.

Exercise 3: Soft Squares on $\mathbb{R} \times \mathbb{R}$

From charts to atlases.

Let $M = \mathbb{R} \times \mathbb{R}$ equipped with the soft square topology $\mathcal{O}_{ssq}$[10] and an atlas $\mathcal{A} = \{(U_n, x_n)\}$, where $U_n = \{(x, y) \in \mathbb{R} \times \mathbb{R} \big| |x| < n, |y| < n, n \in \mathbb{N}^+\}$ and

$$x_n : U_n \to x_n(U_n) \subset \mathbb{R}^2; (x, y) \mapsto x_n((x, y)) := \left(\frac{x+y}{2n}, \frac{x-y}{2n}\right). \tag{4.10}$$

Question: Recall the definition of a chart and show that the (U_n, x_n) are indeed charts.

Solution: We first note that clearly the soft square topology is indeed a topology. Observe that $U_n \subset U_m$ if and only if $n \leq m$. Consider a subset of the integers A as our index set. Then, it's clear that $\cap_{n \in A} U_n = U_{\min(A)} \in \mathcal{O}_{sst}$ and $\cup_{n \in A} U_n = U_{\max(A)} \in \mathcal{O}_{sst}$. We also note that $\mathcal{A}$ is indeed an atlas as U_n's cover all of M: $\cup_{n \in \mathbb{N}^+} U_n = \mathbb{R} \times \mathbb{R}$.

It's obvious that the x_n's are all continuous in the analysis sense but we want to be more rigorous. The first requirement is that charts must be homeomorphisms from an open subset $U \subset M$ to an open subset of Euclidean space. Since U_n are precisely the open sets of $\mathcal{O}_{sst}$, this is good. In addition, we note that

$$x_n(U_n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \big| |x| < 1, |y| < 1\} = U_1 \tag{4.11}$$

We note that U_1 is an open set of the standard topology of $\mathbb{R} \times \mathbb{R}$.[11] We now define $x_n^{-1} : x_n(U_n) \to U_n$ by

$$x_n^{-1}((x, y)) = (n(x+y), n(x-y)), \tag{4.12}$$

which satisfies

$$(x_n^{-1} \circ x_n)((x, y)) = (x, y) \; \forall (x, y) \in U_n, \tag{4.13}$$

verifying that x_n is bijective by existence of x^{-1}. We also note that x_n^{-1} is clearly continuous in the analysis sense, but also one can verify that it is continuous in the topological sense. Normally, one can conclude the x_n's are homeomorphisms and consequently (U_n, x_n) are charts but x_n are NOT continuous in the topological sense[12].

Question: Show that $\mathcal{A}$ is a C^k-atlas by explicitly constructing the chart transition map. What is k?

Solution: We consider $x_n \circ x_m^{-1} : x_m(U_n \cap U_m) \to x_n(U_n \cap U_m)$ which maps

$$(x, y) \mapsto \left(\frac{mx}{n}, \frac{my}{n}\right) = \frac{m}{n}(x, y) \tag{4.14}$$

[10] The open sets are given by the soft squares defined by U_n.

[11] Okay. There might be an issue here as the map $x_n : U_n \to Im(x_n)$ doesn't appear to be continuous in the topological sense. Consider some open ball $B_1(0)$. Clearly, this fits right inside $B_1(0) \subset U_1$, but $\text{preim}_{x_n}(B_1(0)) = \{(x, y) | x_n((x, y)) = \left(\frac{x+y}{2n}, \frac{x-y}{2n}\right) \in B_1(0)\} = B_{2\sqrt{n}}(0) \notin \mathcal{O}_{sst}$. The last equality is obtained because if the point $\left(\frac{x+y}{2n}, \frac{x-y}{2n}\right) \in B_1(0)$, then $\frac{1}{\sqrt{2}n}\sqrt{x^2 + y^2} < 1$.

[12] Above footnote.

which is precisely a scaled identity map, demonstrating that this is a smooth map. Hence, we have "$k = \infty$".

Question: Construct at least one other chart that would lie in the maximal extension of $\mathcal{A}$ and prove that it does.

Solution: Easy example is simply choosing an identity map. For instance, we define

$$\tilde{x} : U_4 \to \tilde{x}(U_4) \subset \mathbb{R}^2 \tag{4.15}$$
$$(x, y) \mapsto \tilde{x}((x, y)) = (x, y) \tag{4.16}$$

Hence, we have that

$$(\tilde{x} \circ x_n^{-1})((x, y)) = (n(a + b), n(a - b)), \qquad (x_n \circ \tilde{x}^{-1})((x, y)) = \left(\frac{x + y}{2n}, \frac{x - y}{2n}\right) \tag{4.17}$$

Short Exercise 4: Undergraduate Multi-Dimensional Analysis

A good notation and basic results for partial differentiation.

For a map $f : \mathbb{R}^d \to \mathbb{R}$ we denote by the map $\partial_i f : \mathbb{R}^d \to \mathbb{R}$ the partial derivative with respect to the i$^{\text{th}}$ entry.

Question: Given a function

$$f : \mathbb{R}^3 \to \mathbb{R}; (\alpha, \beta, \gamma) \mapsto f(\alpha, \beta, \gamma) := \alpha^3\beta^2 + \beta^2\delta + \delta \tag{4.18}$$

calculate the values of the following derivatives:

Solution:

- $(\partial_2 f)(x, y, z) = 2x^3y + 2yz$
- $(\partial_1 f)(\Box, \circ, \star) = 3\Box^2\circ^2$
- $(\partial_1\partial_2 f)(a, b, c) = 6a^2b$
- $(\partial_3^2 f)(299, 1222, 0) = 0$

Exercise 5: Differentiability of a Manifold

How to deal with functions and curves in a chart.

Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth d-dimensional manifold. Consider a chart (U, x) of the atlas $\mathcal{A}$ together with a smooth curve $\gamma : \mathbb{R} \to U$ and a smooth function $f : U \to \mathbb{R}$ on the domain U of the chart.

Question: Draw a commutative diagram containing the chart domain, chart map, function, curve and the respective representatives of the function and the curve in the chart.

Solution:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \arrow[r, "f"] \arrow[d, "x"] & \mathbb{R} \\
& x(U) \arrow[ur, "f \circ x^{-1}"'] & \\
& \mathbb{R}^d \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

Figure 4.5: Commutative Diagram of Smooth Maps

Question: Consider for $d = 2$,

$$(x \circ \gamma)(\lambda) := (\cos(\lambda), \sin(\lambda)), \qquad \text{and } (f \circ x^{-1})((x, y)) := x^2 + y^2. \tag{4.19}$$

Using the chain rule, calculate

$$(f \circ \gamma)'(\lambda) \tag{4.20}$$

Solution: We use the provided chart x to compute this. Observe that

$$f \circ \gamma = (f \circ x^{-1}) \circ (x \circ \gamma) \tag{4.21}$$

Let $F = f \circ x^{-1}$ and $\Gamma = x \circ \gamma$, then we find

$$(f \circ \gamma)'(\lambda) = (\partial_i F)(\Gamma(\lambda)) \cdot (\Gamma^i)'(\lambda) \tag{4.22}$$

$$= 2(\cos(\lambda))(-\sin(\lambda)) + 2(\sin(\lambda))\cos(\lambda) = 0 \tag{4.23}$$

Alternatively, one can directly compute $(f \circ \gamma)(\lambda) = 1$, $(f \circ \gamma)'(\lambda) = 0$.

5 Tangent Spaces

Lead Question: "*What is the velocity of a curve γ at a point p?*

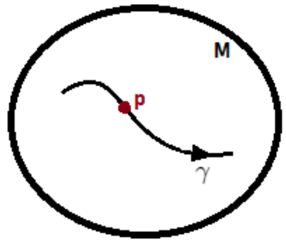


Figure 5.1: Trajectory of Curve.

We're trying to *rediscover* the notion of velocity.

5.1 Velocities

We first define a vector space of smooth functions that we denote $C^\infty(M)$. This will be used in our construction of velocity.

Definition 5.1: $C^\infty(M)$

Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold. Then, we define

$$C^\infty(M) := \{f : M \to \mathbb{R} | f \text{ smooth function}\}, \tag{5.1}$$

equipped with the vector space structure $\oplus, \odot$ operations that are defined as

$$(f \oplus g)(p) := f(p) + g(p), \qquad (\lambda \odot g)(p) := \lambda \cdot g(p). \tag{5.2}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Figure 5.2: Commutative Diagram of $f \circ \gamma$.

Definition 5.2: Smooth Curve

Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold. A curve $\gamma : \mathbb{R} \to M$ is said to be a ***smooth curve*** *if for all $p \in Im(\gamma) \subset M$ with local chart[a] (U_p, x_p) (where $p \in U_p$), the map $x_p \circ \gamma : \mathbb{R} \to \mathbb{R}^d$ is smooth in the undergraduate sense.*

[a]The point $p \in M$ gets hit by our curve γ means that there exists $\lambda \in \mathbb{R}$ such that $\gamma(\lambda) = p$.

Definition 5.3: Velocity

*Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold. Consider a curve $\gamma : \mathbb{R} \to M$ that is at least C^1. Suppose that $\gamma(\lambda_0) = p$. The **velocity** of γ at p is the linear map*

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R} \tag{5.3}$$
$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0) \tag{5.4}$$

Hence, the maps $v_{\gamma,p}$ are $(0,1)$-tensors over $C^\infty(M)$.

In the past, you may have thought about the directional derivative as an object that looks like $\underbrace{v^i}_{\text{vector}}(\partial_i f)$ where we often considered v^i as the vectors. However, also by associativity we have the perspective $\underbrace{v^i}_{\text{vector}}(\partial_i f) = \underbrace{(v^i \partial_i)}_{\text{vector}}(f)$, where one often now views vectors in differential geometry as the $v^i \partial_i$ objects. We'll make this clear in a moment.

5.2 Tangent Vector Space

Definition 5.4: Tangent Vector Space

For every point $p \in M$, we define the tangent space to M at p as the following set:

$$T_pM := \{v_{\gamma,p} | \gamma \text{ smooth curves that pass through } p\}. \tag{5.5}$$

We now want to equip it with a vector space structure. We define $\oplus$ operation as follows:

$$\oplus : T_pM \times T_pM \to T_pM \subset Hom(C^\infty, \mathbb{R}) \tag{5.6}$$
$$(v_{\gamma,p} \oplus v_{\delta,p}) \underbrace{(f)}_{\in C^\infty(M)} := v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f), \tag{5.7}$$

and $\odot$ operation as

$$\odot : \mathbb{R} \times T_pM \to T_pM \subset Hom(C^\infty, \mathbb{R}) \tag{5.8}$$
$$(\lambda \odot v_{\gamma,p})(f) := \lambda \cdot_{\mathbb{R}} v_{\gamma,p}(f) \tag{5.9}$$

Elements of the tangent space T_pM are sometimes referred to as tangent vectors or contravariant vectors.

Proposition 5.1: Closure of Tangent Vector Space Operations

To ensure that T_pM does indeed form a vector space, it remains to show that

1. *$\forall \gamma, \delta$ smooth curves, $\exists$ smooth curve σ such that $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$.*
2. *$\forall \gamma$ smooth curves, $\exists$ smooth curve τ such that $\alpha \odot v_{\gamma,p} = v_{\tau,p}$.*

Proof. We will first prove the second statement: Let us define a curve $\tau : \mathbb{R} \to M$ by

$$\tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) \tag{5.10}$$

where $\gamma(\lambda_0) = p$. Hence, $\tau(0) = p$. We take γ to be smooth and consequently τ is smooth. For convenience, we define a map $\mu_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\mu_\alpha(\lambda) = \alpha\lambda + \lambda_0 \tag{5.11}$$

Hence, one has $\tau = \gamma \circ \mu_\alpha$. We can now compute

$$\begin{align} v_{\tau,p}(f) : &= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \tag{5.12} \\ &= (f \circ \gamma)'(\mu_\alpha(0)) \cdot \frac{\partial \mu}{\partial \lambda}(0) \tag{5.13} \\ &= \alpha \cdot (f \circ \gamma)'(\lambda_0) \tag{5.14} \\ &= \alpha \cdot v_{\gamma,p} \tag{5.15} \end{align}$$

Demonstrating that for some given smooth curve γ that hits $p \in M$, there exists another smooth curve τ that also hits $p \in M$ and satisfies the desired property.

Now, let us prove the first statement. For this purpose, we choose a chart $(U, x) \in \mathcal{A}$ such that $p \in U$. We define the map $\sigma_x : \mathbb{R} \to M$ by the following definition:

$$\sigma_x(\lambda) := x^{-1}\big((x \circ \delta)(\lambda_0 + \lambda) + (x \circ \gamma)(\lambda_1 + \lambda) - (x \circ \delta)(\lambda_0)\big) \tag{5.16}$$

Observe that $\sigma_x(0) = x^{-1}\big(x(\delta(\lambda_0)) + x(\gamma(\lambda_1)) - x(\delta(\lambda_0))\big) = x^{-1}(x(p)) = p$, where we have taken $\delta(\lambda_0) = \gamma(\lambda_1) = p$. For convenience, we define $\mu_i : \mathbb{R} \to \mathbb{R}$ by $\mu_i(\lambda) := \lambda_i + \lambda$ for $i = 0, 1$. In addition, we define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(\lambda) := \lambda_0$. Hence, we have

$$\sigma_x = x^{-1} \circ \big(x \circ \delta \circ \mu_0 + x \circ \gamma \circ \mu_1 - x \circ \gamma \circ \rho\big) \tag{5.17}$$

It is not obvious that the map σ_x itself is a chart-independent quantity. However, what's important is that the velocity associated with σ_x will end up being independent of any chart chosen on σ_x. We'll show this in the next proposition. For now, we compute the velocity of σ_x:

$$\begin{align} v_{\sigma_x,p}(f) : &= (f \circ \sigma_x)'(0) \tag{5.18} \\ &= \big((f \circ x^{-1}) \circ (x \circ \delta \circ \mu_0 + x \circ \gamma \circ \mu_1 - x \circ \gamma \circ \rho)\big)'(0) \tag{5.19} \\ &= \partial_i(f \circ x^{-1})\big(x(p)\big) \cdot \big([x \circ \delta \circ \mu_0 + x \circ \gamma \circ \mu_1 - x \circ \gamma \circ \rho]^i\big)'(0) \tag{5.20} \\ &= \partial_i(f \circ x^{-1})\big(x(p)\big) \cdot \big([x \circ \delta \circ \mu_0 + x \circ \gamma \circ \mu_1]^i\big)'(0) \tag{5.21} \\ &= \partial_i(f \circ x^{-1})\big(x(p)\big) \cdot \big([x \circ \delta \circ \mu_0]^i\big)'(0) + \partial_i(f \circ x^{-1})\big(x(p)\big) \cdot \big([x \circ \gamma \circ \mu_1]^i\big)'(0) \tag{5.22} \\ &= (f \circ x^{-1} \circ x \circ \delta \circ \mu_0)'(0) + (f \circ x^{-1} \circ x \circ \gamma \circ \mu_1)'(0) \tag{5.23} \\ &= (f \circ \delta)'(\mu_0(0)) \cdot \mu_0'(0) + (f \circ \gamma)'(\mu_1(0)) \cdot \mu_1'(0) \tag{5.24} \\ &= (f \circ \delta)'(\lambda_0) + (f \circ \gamma)'(\lambda_1) \tag{5.25} \\ &= v_{\delta,p}(f) + v_{\gamma,p}(f) \tag{5.26} \end{align}$$

Hence, $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$. We note that this is also sufficient to conclude that $v_{\sigma_x,p}$ does not depend on the chosen chart (U, x) as $v_{\delta,p} \oplus v_{\gamma,p}$ bears no dependence on (U, x) and they are equivalent maps. $\square$

5.3 Components of a Vector with respect to a Chart

Definition 5.5: Velocity Components and Basis Notation

Let $(U,x) \in \mathcal{A}_{smooth}$. Let $\gamma : \mathbb{R} \to U$ be a curve and $\gamma(0) = p$. We have that

$$v_{\gamma,p}(f) := (f \circ \gamma)'(0) = (f \circ x^{-1}) \circ (x \circ \gamma))'(0) \tag{5.27}$$

$$= ((x \circ \gamma)^i)'(0) \cdot \partial_i (f \circ x^{-1})\big(x(p)\big) \tag{5.28}$$

The above formula comes up time and again and so we introduce a shorthand notation:

$$\dot{\gamma}_x^i(0) := \big((x \circ \gamma)^i\big)'(0) \tag{5.29}$$

$$\left(\frac{\partial f}{\partial x^i}\right)_p := \partial_i (f \circ x^{-1})\big(x(p)\big) \tag{5.30}$$

We emphasize that since $f : M \to \mathbb{R}$, that the notation $\frac{\partial}{\partial x^i}$ does not signify a partial derivative. Using the above shorthand notation we have

$$v_{\gamma,p}(f) := \dot{\gamma}_x^i(0) \cdot \left(\frac{\partial f}{\partial x^i}\right)_p := \dot{\gamma}_x^i(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p f \tag{5.31}$$

Hence, it appears that one can think of the velocity as the following:

$$v_{\gamma,p} = \underbrace{\dot{\gamma}_x^i(0)}_{\text{components of the velocity } v_{\gamma,p}} \underbrace{\left(\frac{\partial}{\partial x^i}\right)_p}_{\text{basis elements of } T_pM} \tag{5.32}$$

Hence, $(\partial/\partial x^i)_p$ are conceived as basis elements of T_pM with respect to which the components need to be understood. It is the "chart induced basis of T_pM".

Proposition 5.2: Chart Independence of Tangent Vector $v_{\sigma_x,p}$

Let $(M, \mathcal{O}, \mathcal{A})$ be an n-dimensional smooth manifold. Let $\delta, \gamma : \mathbb{R} \to M$ be smooth curves that intersect at the point of interest $p \in M$. Let $\delta(\lambda_0) = \gamma(\lambda_1) = p$ and let (U,x) be a local chart around p. Then, as we had previously found, the smooth curve

$$\sigma_x : \mathbb{R} \to M \tag{5.33}$$

$$\sigma_x(\lambda) := x^{-1}\big((x \circ \delta)(\lambda_0 + \lambda) + (x \circ \gamma)(\lambda_1 + \lambda) - (x \circ \delta)(\lambda_0)\big), \tag{5.34}$$

is a curve that satisfies

$$v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma_x,p}. \tag{5.35}$$

Given another chart (V,y) where $p \in V$, we note that $\sigma_y : \mathbb{R} \to M$ defined analogously to σ_x also satisfies the above relation. However, we will show that $v_{\sigma_x,p}$ bears no dependence on the chosen chart x. Explicitly, consider a local chart (W,z) around p. Then, we can expand in the chart-basis induced by z:

$$v_{\sigma_x,p} = \dot{\sigma_x}^i_{(z)} \left(\frac{\partial}{\partial z^i}\right)_p, \tag{5.36}$$

where $\dot{\sigma_x}^i_{(z)}$ denotes the velocity components with respect to this chart-induced basis z. We want to show that $v_{\sigma_x,p}$ is indeed chart-independent (with respect to the chart x in σ_x), which amounts to show that given a chosen chart basis (W,z), that all the components are equal irregardless of what chart (U,x) we chose to write σ_x. We want to demonstrate that

$$\dot{\sigma_x}^i_{(z)} = \dot{\sigma_y}^i_{(z)}, \ \forall i = 1, ..., n \tag{5.37}$$

Proof. We recall that $\sigma_x(0) = p$. In addition, for convenience we define $\mu_i : \mathbb{R} \to \mathbb{R}$ by $\mu_i(\lambda) := \lambda_i + \lambda$ for $i = 0, 1$, as well as $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(\lambda) := \lambda_0$.

$$\dot{\sigma_x}^i_{(z)} := (z \circ \sigma_x)^{i'}(0) = (z^i \circ \sigma_x)'(0) \tag{5.38}$$
$$:= \left(z^i \circ x^{-1} \circ (x \circ \delta \circ \mu_0 + x \circ \gamma \circ \mu_1 - x \circ \delta \circ \rho)\right)'(0) \tag{5.39}$$
$$= \partial_j (z^i \circ x^{-1})[x(p)] \cdot \left(x^j \circ \delta \circ \mu_0 + x^j \circ \gamma \circ \mu_1 - x^j \circ \delta \circ \rho)\right)'(0) \tag{5.40}$$
$$:= \left(\frac{\partial z^i}{\partial x^j}\right)_p \cdot \left[(x^j \circ y^{-1} \circ y \circ \delta \circ \mu_0)'(0) + (x^j \circ y^{-1} \circ y \circ \gamma \circ \mu_1)'(0) - (x^j \circ y^{-1} \circ y \circ \delta \circ \rho)\right)'(0)\right] \tag{5.41}$$
$$= \left(\frac{\partial z^i}{\partial x^j}\right)_p \left[\partial_k (x^j \circ y^{-1})[y(p)] \left((y^k \circ \delta \circ \mu_0)'(0) + \cdot (y^k \circ \gamma \circ \mu_1)'(0) - \cdot (y^k \circ \delta \circ \rho)'(0)\right)\right] \tag{5.42}$$
$$= \left(\frac{\partial z^i}{\partial x^j}\right)_p \left(\frac{\partial x^j}{\partial y^k}\right)_p \left(y^k \circ \delta \circ \mu_0 + y^k \circ \gamma \circ \mu_1 - y^k \circ \delta \circ \rho\right)'(0) \tag{5.43}$$
$$= \left(\frac{\partial z^i}{\partial y^k}\right)_p \left(y^k \circ \delta \circ \mu_0 + y^k \circ \gamma \circ \mu_1 - y^k \circ \delta \circ \rho\right)'(0) \tag{5.44}$$
$$:= \partial_k (z^i \circ y^{-1})[y(p)] \cdot \left(y \circ \delta \circ \mu_0 + y \circ \gamma \circ \mu_1 - y \circ \delta \circ \rho\right)^{k'}(0) \tag{5.45}$$
$$= \left(z^i \circ y^{-1} \circ (y \circ \delta \circ \mu_0 + y \circ \gamma \circ \mu_1 - y \circ \delta \circ \rho)\right)'(0) \tag{5.46}$$
$$:= \left(z^i \circ \sigma_y\right)'(0) \tag{5.47}$$
$$:= \dot{\sigma_y}^i_{(z)} \tag{5.48}$$

□

5.4 Chart-Induced Basis

Theorem 5.1: Basis of T_pM

Let $(M, \mathcal{O}, \mathcal{A}_{smooth})$ be a smooth d-dimensional manifold. Let $(U, x) \in \mathcal{A}_{smooth}$, then $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_pM$ constitutes a basis of T_pU.

Proof. It is sufficient to show that these are linearly independent. We want to show that

$$\sum_{i=1}^{d} \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p = 0 \tag{5.49}$$

if and only if $\lambda^i = 0$ for all i. We choose the function $x^j : U \to \mathbb{R}$, hence we have

$$\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^j) = \lambda^i \cdot \partial_i (x^j \circ x^{-1})'(x(p)) \tag{5.50}$$
$$= \lambda^i \delta^j_i = \lambda^j \tag{5.51}$$

Hence, this requires that $\lambda^j = 0$ for all j for consistency, thereby demonstrating linear-independence. □

Corollary 5.1: Dimension of T_pM

Hence, one has dim $T_pM = d =$ dim M.

Terminology: If $X \in T_pM$, then $\exists\gamma : \mathbb{R} \to M$ such that $X = v_{\gamma,p}$. We also know that $\exists X^1, ..., X^d \in \mathbb{R}$ such that $X = X^i\big(\frac{\partial}{\partial x^i}\big)_p$.

5.5 Change of Vector Components Under a Change of Chart

Proposition 5.3: Multi-dimensional Chain Rule

Let $w : \mathbb{R}^m \to \mathbb{R}$ and $x : \mathbb{R}^n \to \mathbb{R}^m$ with coordinate maps $x_i : \mathbb{R}^n \to \mathbb{R}$, where $x_i = x_i(t_1, ..., t_n)$ of n independent variables. With this, we let $w = f(x_1, ..., x_m)$. Then the following is true

$$\frac{\partial w}{\partial t_j} = \sum_{i=1}^{m} \frac{\partial w}{\partial x_i}\frac{\partial x_i}{\partial t_j} \tag{5.52}$$

In a more coordinate-free way, one could write

$$\partial_j(w \circ x)[t] = \partial_i(w)[x(t)]\,\partial_j(x^i)[t], \tag{5.53}$$

where the notation $f[t]$ denotes the function f being evaluated at t.

It should be emphasized that a vector does **not** change under a change of chart, but its components may.

Let (U, x) and (V, y) be overlapping charts and $p \in U \cap V$. Let $X \in T_pM$. Then,

$$X^i_{(y)}\left(\frac{\partial}{\partial y^i}\right)_p \underbrace{=}_{(V,y)} X \underbrace{=}_{(U,x)} X^i_{(x)}\left(\frac{\partial}{\partial x^i}\right)_p \tag{5.54}$$

We want to see what happens under a change of chart and so we evaluate:

$$\begin{aligned}
\left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i(f \circ x^{-1})(x(p)) && (5.55)\\
&= \partial_i\big((f \circ y^{-1}) \circ (y \circ x^{-1})\big)(x(p)) && (5.56)\\
&= \big(\partial_i(y \circ x^{-1})\big)^j(x(p)) \cdot \big(\partial_j(f \circ y^{-1})\big)(y(p)) && (5.57)\\
&= \big(\partial_i(y^j \circ x^{-1})\big)(x(p)) \cdot \big(\partial_j(f \circ y^{-1})\big)(y(p)) && (5.58)\\
&= \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial f}{\partial y^j}\right)_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial f}{\partial y^j}\right)_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p f && (5.59)
\end{aligned}$$

Hence, we can conclude that

$$X^i_{(x)}\left(\frac{\partial y^j}{\partial x^i}\right)_p\left(\frac{\partial}{\partial y^j}\right)_p = X^j_{(y)}\left(\frac{\partial}{\partial y^j}\right)_p \longrightarrow \left(X^i_{(x)}\left(\frac{\partial y^j}{\partial x^i}\right)_p - X^j_{(y)}\right)\left(\frac{\partial}{\partial y^j}\right)_p = 0 \tag{5.60}$$

Since $\big(\frac{\partial}{\partial y^j}\big)_p$ form a basis, we can finally conclude that under a change of chart, the components transform as

$$X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)} \tag{5.61}$$

5.6 Cotangent Spaces

Definition 5.6: Gradient

Let $f \in C^\infty(M)$. We define

$$(df)_p : T_pM \xrightarrow{\sim} \mathbb{R} \tag{5.62}$$

$$X \mapsto (df)_p(X) := Xf \tag{5.63}$$

We call $(df)_p$ the gradient of f at $p \in M$. Hence, the gradient is a $(0,1)$-tensor over T_pM.

The components of the gradient with respect to a chart-induced basis (U, x) is given by

$$\big((df)_p\big)_j := (df)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \left(\frac{\partial f}{\partial x^j}\right)_p \tag{5.64}$$

Definition 5.7: Cotangent Space

Let T_pM be a Tangent space to $p \in M$. Then, we define the cotangent space $(T_pM)^*$ as follows:

$$(T_pM)^* := \{\psi : T_pM \xrightarrow{\sim} \mathbb{R}\} \tag{5.65}$$

We will often employ the notation $T_p^*M := (T_pM)^*$. Elements of T_p^*M are sometimes referred to as cotangent vectors or covariant vectors.

Hence, objects such as the gradient $(df)_p \in (T_pM)^*$ lie in the cotangent space.

Theorem 5.2: Dual Basis of Cotangent Space

Let T_pM be the tangent space to $p \in M$. Consider a chart (U, x) where $p \in U$ with coordinate maps $x^i : U \to \mathbb{R}$. Then,

$$(dx^1)_p, \ldots, (dx^d)_p \tag{5.66}$$

*constitutes a basis for T_p^*M. In particular, it forms the **dual** basis as*

$$(dx^a)_p\left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial x^a}{\partial x^b}\right)_p = \delta^a_b \tag{5.67}$$

5.7 Change of Components of a Covector under Change of Chart

Suppose that $w \in T_p^*M$. Then, let (U, x) and (V, y) be overlapping charts around p. With these charts, one can choose to represent w in terms of these charts. Hence, we have

$$w_{(x)i}(dx^i)_p \underbrace{=}_{(U,x)} w \underbrace{=}_{(V,y)} w_{(y)i}(dy^i)_p \tag{5.68}$$

Let's act on some arbitrary vector $X \in T_pM$, and we choose another chart (W, z) simply for representing X intermediately. Hence, we write $X = X^i_{(z)}\big(\frac{\partial}{\partial z^i}\big)_p$. By the above equivalence, we will want to connect the components of w under a change of chart. Hence,

$$w_{(y)i}(dy^i)_p X = w_{(x)i}(dx^i)_p X \tag{5.69}$$
$$= w_{(x)i} X^j_{(z)} \left(\frac{\partial x^i}{\partial z^j}\right)_p \tag{5.70}$$
$$= w_{(x)i} X^j_{(z)} \partial_j \left(x^i \circ z^{-1}\right)[z(p)] \tag{5.71}$$
$$= w_{(x)i} X^j_{(z)} \partial_j \left((x^i \circ y^{-1}) \circ (y \circ z^{-1})\right)[z(p)] \tag{5.72}$$
$$= w_{(x)i} X^j_{(z)} \partial_k \left((x^i \circ y^{-1})\right)[y(p)]\, \partial_j (y^k \circ z^{-1})[z(p)] \tag{5.73}$$
$$= w_{(x)i} X^j_{(z)} \left(\frac{\partial x^i}{\partial y^k}\right)_p \left(\frac{\partial y^k}{\partial z^j}\right)_p \tag{5.74}$$
$$= w_{(x)i} \left(\frac{\partial x^i}{\partial y^k}\right)_p X^j_{(z)} \left(\frac{\partial}{\partial z^j}\right)_p y^k \tag{5.75}$$
$$= w_{(x)i} \left(\frac{\partial x^i}{\partial y^k}\right)_p (dy^k)_p X \tag{5.76}$$

Hence, we can conclude that under a change of charts, the components of the gradient change as follows:

$$w_{(y)i} = w_{(x)j} \left(\frac{\partial x^j}{\partial y^i}\right)_p . \tag{5.77}$$

5.8 Tutorial

Exercise 1: True or False?

Tick the correct statements, but not the incorrect ones!

a) A tangent vector to a differentiable manifold
   - ○ has a length and a direction.
   - ○ must not be zero.
   - ● maps a function on the manifold to the real numbers
   - ○ is of the same dimension as the manifold.
   - ● arises as the velocity to some curve through the vector's base point.

b) The tangent space T_pM to a d-dimensional differentiable manifold
   - ○ is of the dimension $2 \cdot \dim M$.
   - ○ is no real vector space, because its elements are only tangent vectors.
   - ● can be defined at *every* point p of M.
   - ● has no tangent vectors in common with a tangent space T_qM for $q \neq p$.
   - ● admits a linear bijection to the vector space $(\mathbb{R}^d, \oplus, \odot)$.[13]

c) If (U, x) is a chart for a d-dimensional differentiable manifold, then
   - ○ the coordinate maps $x^i : U \to \mathbb{R}$ with $i = 1, ..., d$ are only continuous, not differentiable.
   - ○ for the basis $\left(\frac{\partial}{\partial x^1}\right)_p, ..., \left(\frac{\partial}{\partial x^d}\right)_p$ of T_pM, there is no dual basis in the dual space.
   - ● $(dx^1)_p, ...(dx^d)_p$ constitute a basis of T_p^*M.

[13]Every finite d-dimensional vector space over $\mathbb{R}$ is isomorphic to the vector space $\mathbb{R}^d$, equipped with its standard operations.

- the components of the vector X with respect to the chart-induced basis are $(dx^i)_p(X)$.
- the expression $(dx^a)_p(\left(\frac{\partial}{\partial x^a}\right)_p)$ yields the dimension of the manifold.

Exercise 2: Virtuoso Use of the Symbol $\left(\frac{\partial}{\partial x^i}\right)_p$

Translating the symbol into undergraduate analysis symbols and vice versa.

Question: For a smooth function f and a chart (U, x), provide the definition of the expression

$$\left(\frac{\partial f}{\partial x^i}\right)_p \tag{5.78}$$

Solution:

$$\left(\frac{\partial f}{\partial x^i}\right)_p := \partial_i(f \circ x^{-1})[x(p)] \tag{5.79}$$

Question: Show that, for overlapping charts (U, x) and (V, y), one has

$$\left(\frac{\partial x^a}{\partial y^m}\right)_p \left(\frac{\partial y^m}{\partial x^b}\right)_p = \delta^a_b \tag{5.80}$$

for any $p \in U \cap V$.

Solution:

$$\begin{aligned}
\delta^a_b = \left(\frac{\partial}{\partial x^b}\right)_p (x^a) &= \partial_b(x^a \circ x^{-1})[x(p)] && (5.81)\\
&= \partial_b((x^a \circ y^{-1}) \circ (y \circ x^{-1}))[x(p)] && (5.82)\\
&= \partial_m((x^a \circ y^{-1})[y(p)]\ \partial_b(y^m \circ x^{-1}))[x(p)] && (5.83)\\
&= \left(\frac{\partial x^a}{\partial y^m}\right)_p \left(\frac{\partial y^m}{\partial x^b}\right)_p && (5.84)
\end{aligned}$$

Question: After inserting $y^{-1} \circ y$, where y is another chart map on the same chart domain U, at an appropriate position in the definition of the left hand side of

$$\left(\frac{\partial f}{\partial x^i}\right)_p = \left(\frac{\partial y^m}{\partial x^i}\right)_p \left(\frac{\partial f}{\partial y^m}\right)_p, \tag{5.85}$$

use the undergraduate multi-dimensional chain rule to show that it equals the right hand side.

Solution:

$$\begin{aligned}
\left(\frac{\partial f}{\partial x^i}\right)_p &= \partial_i(f \circ x^{-1})[x(p)] && (5.86)\\
&= \partial_i((f \circ y^{-1}) \circ (y \circ x^{-1}))[x(p)] && (5.87)\\
&= \partial_j(f \circ y^{-1})[y(p)]\ \partial_i(y^j \circ x^{-1})[x(p)] && (5.88)\\
&= \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial f}{\partial y^j}\right)_p && (5.89)
\end{aligned}$$

Question: Do the dim M many quantities defined by the left hand side of the above expression constitute the components of a tensor? If so, what are the valence and the rank of the tensor?

Solution: Yes, we can see that letting $p_{i(x)}$ denote the left hand side, that one obtains under a change of chart the following relation

$$p_{i(x)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p p_{j(y)} \tag{5.90}$$

Hence, we can see that the quantities $p_{i(x)}$ transform as a covector: a $(0,1)$-tensor. Given the form for $p_{i(x)}$, we can indeed verify that

$$\left(\frac{\partial f}{\partial x^i}\right)_p \tag{5.91}$$

constitutes the components of the gradient $(df)_p$ in the chosen basis $(dx^i)_p$. Let $f_{i(x)}$ denote the components of $(df)_p$ with respect to the basis $(dx^i)_p$, then

$$(df)_p(\left(\frac{\partial}{\partial x^j}\right)_p) = f_{i(x)}(dx^i)_p(\left(\frac{\partial}{\partial x^j}\right)_p) = f_{j(x)} \tag{5.92}$$

Hence, we can compute the components by the very left hand-side of the above expression. We thus have

$$f_{j(x)} = (df)_p(\left(\frac{\partial}{\partial x^j}\right)_p) = \left(\frac{\partial f}{\partial x^j}\right)_p, \tag{5.93}$$

verifying that $\left(\frac{\partial f}{\partial x^j}\right)_p$ constitute the components of $(df)_p$ in the chosen chart (U,x). This is a rank 1 tensor.

Exercise 3: Transformation of Vector Components

Understanding the vector component transformation law the pedestrian way.

Let the topological manifold $(\mathbb{R}^2, \mathcal{O}_{st})$ be equipped with the atlas $\mathcal{A} = \{(\mathbb{R}^2, x), (\mathbb{R}^2, y)\}$, where

$$x : \mathbb{R}^2 \to \mathbb{R}^2, \qquad y: \mathbb{R}^2 \to \mathbb{R}^2 \tag{5.94}$$

$$(a,b) \mapsto (a,b) \qquad (a,b) \mapsto (a, b+a^3) \tag{5.95}$$

Question: Calculate the objects $\left(\frac{\partial x^i}{\partial y^j}\right)_p$!

Solution: Given these maps, one can verify that the inverse map y^{-1} is defined as follows:

$$y^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5.96}$$

$$(a,b) \mapsto (a, b-a^3) \tag{5.97}$$

We now aim to compute

$$\left(\frac{\partial x^i}{\partial y^j}\right)_p = \partial_j(x^i \circ y^{-1})[y(p)] \tag{5.98}$$

We note that we're evaluating these derivatives at $y(p)$. Our points $p \in \mathbb{R}^2$ are expressed as (a,b), hence $y((a,b)) = (a, b+a^3)$.

$$\left(\frac{\partial x^1}{\partial y^1}\right)_p [(a, b+a^3)] = 1 \tag{5.99}$$

$$\left(\frac{\partial x^1}{\partial y^2}\right)_p [(a, b+a^3)] = 0 \tag{5.100}$$

$$\left(\frac{\partial x^2}{\partial y^1}\right)_p [(a, b+a^3)] = -3a^2 \tag{5.101}$$

$$\left(\frac{\partial x^2}{\partial y^2}\right)_p [(a, b+a^3)] = 1 \tag{5.102}$$

$$\tag{5.103}$$

In the lectures, the velocity $v_{\gamma,p}$ of the curve at a point $p = \gamma(\lambda_0)$ has been defined by its action on a smooth function f

$$v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0). \tag{5.104}$$

By choosing a chart (U, x), inserting $x^{-1} \circ x$ at the appropriate place in this definition and employing the chain rule, you found the components of the velocity with respect to the chart

$$\dot{\gamma}_x^i(\lambda_0) := (x \circ \gamma)^{i'}(\lambda_0). \tag{5.105}$$

Now consider the curve

$$\gamma : \mathbb{R}^2 \to \mathbb{R}^2; \qquad \lambda \mapsto (\lambda, -\lambda) \tag{5.106}$$

Question: Calculate the components $\dot{\gamma_x}^i(\lambda_0)$ and $\dot{\gamma_y}^i(\lambda_0)$! [Note that we are still using the same x and y chart maps defined in Exercise 3's main statement].

Solution:

$$\dot{\gamma}_x(\lambda_0) = (1, -1) \;\mapsto\; \dot{\gamma}_x^1(\lambda_0) = 1, \dot{\gamma}_x^2(\lambda_0) = -1 \tag{5.107}$$

For the next part, we note that $(y \circ \gamma)(\lambda) = (\lambda, -\lambda + \lambda^3)$.

$$\dot{\gamma}_y(\lambda_0) = (1, -1 + 3\lambda_0^2) \;\mapsto\; \dot{\gamma}_y^1(\lambda_0) = 1, \dot{\gamma}_y^2(\lambda_0) = -1 + 3\lambda_0^2 \tag{5.108}$$

Question: With the result of the first question, how could you have obtained the components $\dot{\gamma}_x^i(\lambda_0)$ from the $\dot{\gamma}_y^i(\lambda_0)$?

Solution: We can compute that under a change of charts $(U, x) \to (V, y)$, we obtain

$$\dot{\gamma}_x^i(\lambda_0) = \left(\frac{\partial x^i}{\partial y^j}\right)_{\gamma(\lambda_0)} \dot{\gamma}_y^j(\lambda_0) \tag{5.109}$$

Hence, to verify this holds for our case of interest, we can see that

$$\dot{\gamma}_x^1(\lambda_0) = (1)(1) + 0 = 1, \tag{5.110}$$

$$\dot{\gamma}_x^2(\lambda_0) = (-3\lambda_0^2)(1) + (1)(-1 + 3\lambda_0^2) = -1, \tag{5.111}$$

which indeed verifies our findings with the previous question.

Exercise 4: The Gradient

Not the only covector undergoing an identity crisis.

Given a function f on a manifold M, the level sets of f for a constant $c \in \mathbb{R}$ are defined as

$$N_c(f) := \{p \in M | f(p) = c\} \tag{5.112}$$

Question: Formulate the condition for a curve $\gamma : \mathbb{R} \to M$ to take values solely in one of the level sets of a function f!

Solution: We can notice that

$$\forall \lambda \in \mathbb{R} : f(\gamma(\lambda)) = c, \longrightarrow f \circ \gamma \equiv c. \tag{5.113}$$

Hence, $f \circ \gamma$ must be the constant function that maps $\lambda \mapsto c \; \forall \lambda \in \mathbb{R}$.

Question: Now show that the gradient of a function annihilates the velocity vector $v_{\gamma,p}$ for any such curve γ through a point p in $N_c(f)$. In other words, show that

$$(df)_p(v_{\gamma,p}) = 0 \tag{5.114}$$

Solution: We compute

$$(df)_p(v_{\gamma,p}) := v_{\gamma,p}(f) = (f \circ \gamma)'(\lambda_0), \tag{5.115}$$

where λ satisfies $\gamma(\lambda_0) = p$. For such curves γ, we found that $f \circ \gamma$ must be a constant function. Hence,

$$(f \circ \gamma)'(\lambda_0) = 0, \tag{5.116}$$

thereby illustrating that $(df)_p(v_{\gamma,p}) = 0$.

Exercise 5: Is there a well-defined sum of curves?
On the dangers of defining concepts by use of charts.

Let the topological manifold $(\mathbb{R}^2, \mathcal{O}_{st})$ be equipped with the atlas $\mathcal{A} = \{(\mathbb{R}^2, x), (\mathbb{R}^2, y)\}$, where

$$x : \mathbb{R}^2 \to \mathbb{R}^2, \qquad y : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5.117}$$
$$(a,b) \mapsto (a,b) \qquad (a,b) \mapsto (a, b \cdot e^a) \tag{5.118}$$

Question: Is $\mathcal{A}$ a C^∞-atlas?

Solution: We first compute their inverses:

$$x^{-1} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad y^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5.119}$$
$$(a,b) \mapsto (a,b) \qquad (a,b) \mapsto (a, b \cdot e^{-a}) \tag{5.120}$$

We can now examine their transition maps:

$$1.\ x \circ y^{-1} : (a,b) \mapsto (a, b \cdot e^{-a}), \tag{5.121}$$
$$2.\ y \circ x^{-1} : (a,b) \mapsto (a, b \cdot e^{a}), \tag{5.122}$$

which are clearly both $C^\infty(\mathbb{R}^2)$ as all partial derivatives to any arbitrary order exist and are continuous over $\mathbb{R}^2$.

Question: On M, consider the curves $\gamma : \mathbb{R} \to M$ and $\delta : \mathbb{R} \to M$, mapping

$$\gamma : \lambda \mapsto (\lambda, 1) \text{ and} \tag{5.123}$$
$$\delta : \lambda \mapsto (1, \lambda) \tag{5.124}$$

Without referring to any chart, can you give the sum $\gamma + \delta$ of these curves?

Solution: One natural way to define the sum is to say that $\gamma + \delta$ is the function that acts as $(\gamma+\delta)(\lambda) = \gamma(\lambda) + \delta(\lambda)$. However, $\gamma(\lambda), \delta(\lambda) \in M$ but M does not have a vector space structure to allow us to talk about the addition of two elements. Hence, the answer is **no**. We note that $M = \mathbb{R}^2$, but $\mathbb{R}^2$ is taken to be a topological manifold. There is currently no vectorspace structure attached to it.

Question: Calculate the representatives of both curves with respect to both charts! Illustrate the results! Where do the curves of the charts intersect?

Solution: The chart representatives $x \circ \gamma, y \circ \gamma, x \circ \delta, y \circ \delta : \mathbb{R} \to \mathbb{R}^2$ are all computed:

1. $(x \circ \gamma)(\lambda_0) = (\lambda_0, 1)$
2. $(y \circ \gamma)(\lambda_0) = (\lambda_0, e^{\lambda_0})$
3. $(x \circ \delta)(\lambda_1) = (1, \lambda_1)$
4. $(y \circ \delta)(\lambda_1) = (1, \lambda_1 e)$

We note that $(x \circ \gamma)$ and $(x \circ \delta)$ intersect at the single point $(1,1)$ which occurs for parameters $\lambda_0 = \lambda_1 = 1$. For the curves $(y \circ \gamma)$ and $(y \circ \delta)$, the curves intersect at the single point $(1, e)$, which occurs for parameters $\lambda_0 = \lambda_1 = 1$.

Recall the definition of the sum of curves with respect to a chart (U, x) from the lectures. There, for curves γ and δ in the common point $\gamma(\lambda_0) = \delta(\lambda_1)$, we defined

$$\sigma_x : \mathbb{R} \to U, \qquad \lambda \mapsto x^{-1}\big(x(\gamma(\lambda + \lambda_0)) + x(\delta(\lambda + \lambda_1)) - x(\gamma(\lambda_0))\big) \tag{5.125}$$

as the sum of γ and δ in the real world.

Question: Implement this construction with our chart $(\mathbb{R}^2, x)$ in order to determine the sum σ_x of our curves

γ and δ! Draw the result in the real world.

Solution: The appearance of x in this definition might seem at first glance to be a bit redundant but we emphasize that it is not. The main reason is that x and x^{-1} are not necessarily linear; more appropriately, a notion of linearity does not even make sense here as we are dealing with charts (U, x). You would have to equip the manifold with a vector space structure to begin to talk about addition. What this formulation allows us to do is map into $\mathbb{R}^d$, add together quantities how we normally would and then go back into the manifold M.

$$\sigma_x(\lambda) = (\lambda+1, \lambda+\lambda_1) = (\lambda+1, \lambda+1), \tag{5.126}$$

as we take the point where $\gamma(\lambda_0) = \delta(\lambda_1)$, which was found to occur at $\lambda_0 = \lambda_1 = 1$.

Question: Repeat the construction, but now using the chart $(\mathbb{R}^2, y)$ to obtain the curve σ_y. Do you get the same curve in the real world?

Solution: For the second curve, we have

$$\sigma_y(\lambda) = (\lambda+1, [e^{\lambda_0}(e^{\lambda}-1) + (\lambda_1+\lambda)e]e^{-(1+\lambda)}) = (\lambda+1, 1+\lambda e^{-\lambda}), \tag{5.127}$$

where we similarly chose $\lambda_0 = \lambda_1 = 1$ as that is the set of parameters where the curves intersect. Evidently, the curves σ_x and σ_y differ from one another.

Question: Show that - despite the above result - the velocity of σ_x equals the velocity of σ_y at the intersection point. In essence, we want to show that

$$\dot{\sigma_x}_{(x)} = \dot{\sigma_y}_{(x)} \qquad \text{and} \qquad \dot{\sigma_x}_{(y)} = \dot{\sigma_y}_{(y)} \tag{5.128}$$

Solution: We enumerate over every possible case. We note that there are only two charts to choose from[14], hence simplifying this process.

$$\dot{\sigma_x}_{(x)} := (x \circ \sigma_x)'(0) = (\lambda+1, \lambda+1)'(0) = (1, 1), \tag{5.129}$$

$$\dot{\sigma_y}_{(x)} := (x \circ \sigma_y)'(0) = (\lambda+1, 1+\lambda e^{-\lambda})'(0) = (1, e^{-\lambda}(1-\lambda))|_{\lambda=0} = (1, 1), \tag{5.130}$$

verifying that the same velocity is obtained for the chart induced basis x. We now repeat this procedure for the other chart-induced basis y:

$$\dot{\sigma_x}_{(y)} := (y \circ \sigma_x)'(0) = (\lambda+1, (\lambda+1)e^{\lambda+1})'(0) = (1, e^{\lambda+1}(2+\lambda))|_{\lambda=0} = (1, 2e), \tag{5.131}$$

$$\dot{\sigma_y}_{(y)} := (y \circ \sigma_y)'(0) = (\lambda+1, (1+\lambda e^{-\lambda})e^{\lambda+1})'(0) = (1, e^{\lambda+1}+e)|_{\lambda=0} = (1, 2e), \tag{5.132}$$

verifying that the same velocity is obtained for the chart induced basis y.

[14] Given a collection of n charts that have overlapping intersection at the point p of interest, we would require n computations in a fixed chart-induced basis. Showing this for all chart-induced bases would therefore require n^2 computations in total. If one were feeling masochistic, they could feel free to take on such an endeavour, but since we have proven that any chart works for computing velocity components, we (the sane) will stick to choosing whichever chart is the most convenient.

6 Fields

6.1 Bundles

Definition 6.1: Bundles

*A **bundle** is a triple* (E, M, π)

$$\underbrace{E}_{\text{smooth mfld}} \xrightarrow{\pi} \underbrace{M}_{\text{smooth mfld}}, \tag{6.1}$$

where we refer to E *as the total space,* M *as the base space and* π *is a smooth surjective map that is referred to as the projection map.*

Example 6.1: A Cylindrical, Circle Bundle

We can take E *to be a cylinder equipped with the standard topology etc, and similarly choose our base space* M *to be a circle.*

Definition 6.2: Fiber

Let $E \xrightarrow{\pi} M$ *be a bundle and let* $p \in M$*. Then, we define the **fiber** over* p *by*

$$\pi^{-1}(p) := \textit{preim}_\pi(\{p\}), \tag{6.2}$$

with $\pi^{-1}(p)$ *denoting this fiber.*

Definition 6.3: Section

*A **section*** σ *of a bundle* $E \xrightarrow{\pi} M$ *is a map* $\sigma : M \to E$ *such that* $\pi \circ \sigma = id_M$*. In the physics jargon, sections are **fields** and what type of field depends on the total space (ex. vector, tensor etc.).*

Side Remark: In quantum mechanics, the wave functions $\psi : M \to \mathbb{C}$ are not 'functions' but sections.

6.2 Tangent Bundle of Smooth Manifold

Definition 6.4: Disjoint Union

Let $\{A_i : i \in I\}$ *be a family of sets indexed by* I*. The **disjoint union** of this family is the set*

$$\sqcup_{i\in I} A_i := \cup_{i\in I}\{(x,i) : x \in A_i\} \tag{6.3}$$

Definition 6.5: Tangent Bundle

Let $(M, \mathcal{O}, \mathcal{A})$ *be a smooth manifold. We want to collect all of our tangent spaces into one big bag but still manage to keep them separate from one another.*

a) We define the tangent bundle (at the set level) TM *to be the disjoint union of tangent spaces* T_pM

$$TM := \sqcup_{p\in M} T_pM := \cup_{p\in M}\{(x,p) : x \in T_pM\} \tag{6.4}$$

b) *We now want to define the projection map*

$$\pi : TM \to M \tag{6.5}$$
$$(X, p) \mapsto p, \tag{6.6}$$

where $p \in M$ is the unique point such that $X \in T_pM$. Hence, π is a surjective map. The current situation at this moment is that we have the following:

$$\underbrace{TM}_{\text{set}} \overset{\pi}{\underset{\text{surjective}}{\longrightarrow}} \underbrace{M}_{\text{smooth mfld.}} \tag{6.7}$$

c) *We now construct a topology on TM that is the coarsest topology such that π is (just) continuous. ("initial topology w.r.t π"). We will see that this is explicitly defined as*

$$\mathcal{O}_{TM} := \{preim_\pi(U) \mid U \in \mathcal{O}\} \tag{6.8}$$

Construction of C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M:
We now want to construct a smooth atlas over TM. To do this, we induce a collection of charts from the atlas $\mathcal{A}$ over M. What we will aim to do is use the available data to slot in all the relevant coordinate components. The first d many will correspond to the coordinates components of p in the (U, x) basis, and the last d many will be the coordinates of X in the (U, x) basis. We now define

$$\mathcal{A}_{TM} := \{(TU, \xi_x) \mid (U, x) \in \mathcal{A}\}, \tag{6.9}$$

where

$$\xi_x : TU \to \mathbb{R}^{2\cdot dim(M)} \tag{6.10}$$
$$\mathbf{X} \mapsto \big(\underbrace{(x^1 \circ \pi)(\mathbf{X}), ..., (x^d \circ \pi)(\mathbf{X})}_{(U,x)-\text{coords of } \pi(\mathbf{X})}, (dx^1)_{\pi(\mathbf{X})}(\mathbf{X}), ..., (dx^d)_{\pi(\mathbf{X})}(\mathbf{X})\big). \tag{6.11}$$

We note that one can always write $\mathbf{X} = (X, p)$, where $X \in T_{\pi(\mathbf{X})}U$ and $p = \pi(\mathbf{X})$.

Lemma 6.1: Intersection of Preimages

Let $f : A \to B$ be a function. Let $U, V \subset B$. Then,

$$preim_f(U \cap V) = preim_f(U) \cap preim_f(V) \tag{6.12}$$

Lemma 6.2: Union of Preimages

Let $f : A \to B$ be a function. Let $U_\alpha \subset B\ \forall \alpha \in I$, where I is some index set. Then,

$$\cup_{\alpha \in I}\, preim_f(U_\alpha) = preim_f\big(\cup_{\alpha \in I} U_\alpha\big) \tag{6.13}$$

Proposition 6.1: Tangent Bundle Topology

We will show that

$$\mathcal{O}_{TM} := \{preim_\pi(U) \mid U \in \mathcal{O}\} \tag{6.14}$$

is the coarsest topology on TM such that π is continuous.

Proof. We will first show that $\mathcal{O}_{TM}$ is a topology on TM.

1. Since $\emptyset \in \mathcal{O}$, then $\emptyset \in \mathcal{O}_{TM}$. Since $\pi : TM \to M$, then $\text{preim}_\pi(M) = TM$. Hence, as $M \in \mathcal{O}$, then $TM \in \mathcal{O}_{TM}$.
2. Let $U, V \in \mathcal{O}_{TM}$, then there exist $\tilde{U}, \tilde{V} \in \mathcal{O}$ such that
$$U = \text{preim}_\pi(\tilde{U}), \qquad V = \text{preim}_\pi(\tilde{V}). \tag{6.15}$$
Hence, their intersection is given by
$$U \cap V = \text{preim}_\pi(\tilde{U}) \cap \text{preim}_\pi(\tilde{V}) = \text{preim}_\pi(\tilde{U} \cap \tilde{V}) \tag{6.16}$$
Since $\tilde{U} \cap \tilde{V} \in \mathcal{O}$, then $\text{preim}_\pi(\tilde{U} \cap \tilde{V}) = U \cap V \in \mathcal{O}_{TM}$.
3. Let $U_\alpha \in \mathcal{O}_{TM}$, indexed by some set $\alpha \in I$. Then, $\forall U_\alpha$, there exists $\tilde{U}_\alpha \in \mathcal{O}$ such that
$$U_\alpha = \text{preim}_\pi(\tilde{U}_\alpha) \tag{6.17}$$
Their union is therefore given by
$$\cup_{\alpha \in I}\, U_\alpha = \cup_{\alpha \in I}\text{preim}_\pi(\tilde{U}_\alpha) = \text{preim}_\pi(\cup_{\alpha \in I}\tilde{U}_\alpha). \tag{6.18}$$
Since $\cup_{\alpha \in I}\tilde{U}_\alpha \in \mathcal{O}$, then $\text{preim}_\pi(\cup_{\alpha \in I}\tilde{U}_\alpha) = \cup_{\alpha \in I}U_\alpha \in \mathcal{O}_{TM}$.

We now address continuity: π is said to be continuous if
$$\text{preim}_\pi(U) \in \mathcal{O}_{TM} \forall\, U \in \mathcal{O} \tag{6.19}$$
Suppose that TM was equipped with a topology $\mathcal{O}_t$ that ensured π is continuous. Then it's quite evident that $\mathcal{O}_{TM} \subset \mathcal{O}_t$. Observe that if $\tau \in \mathcal{O}_{TM}$, then there exists $U \in \mathcal{O}$ such that $\tau = \text{preim}_\pi(U)$. Hence, by this construction, it's easy to see that $\tau \in \mathcal{O}_t$. Hence, $\mathcal{O}_{TM}$ is the coarsest topology on TM that ensures continuity of π. $\square$