
Linear Algebra

Linear Algebra Notes

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Prelude

This set of notes covers Linear Algebra from the textbook *Linear Algebra Done Right* by *Sheldon Axler*. These notes are one of many that I have decided to make available for anyone interested. One of the benefits of this series are having a compendium of definitions, examples and personal thoughts that I can always refer back to if I need a reminder on a particular topic. In addition, I find that this medium reduces the search time for specific definitions, theorems, examples etc and thus aids in reinforcing my own knowledge when frequented.

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Chapter 1

Vector Spaces

Linear Algebra is the study of linear maps on finite-dimensional vector spaces. This chapter aims to define vector spaces and discuss their elementary properties.

1.1 \mathbb{R}^n and \mathbb{C}^n

1.1.1 Complex Numbers

Definition 1.1: Complex Numbers & their Structure

A **complex number** is an ordered pair (a, b) , where $a, b \in \mathbb{R}$. We will typically write (a, b) as $a + bi$, with i satisfying $i^2 = -1$ (i is typically referred to as the imaginary unit). The set of all complex numbers is denoted by \mathbb{C} :

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \quad (1.1)$$

We define addition and multiplication on \mathbb{C} by

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \quad (1.2)$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \quad (1.3)$$

$\forall a, b, c, d \in \mathbb{R}$. If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a . In this sense, one can consider \mathbb{R} as a subset of \mathbb{C} . Similarly, we will usually write $0 + bi$ as just bi and $0 + 1i$ as just i .

Proposition 1.1: Properties of Complex Arithmetic

1. **Commutativity:**

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \quad \forall \alpha, \beta \in \mathbb{C} \quad (1.4)$$

2. **Associativity:**

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\lambda = \alpha(\beta\lambda) \quad \forall \alpha, \beta, \lambda \in \mathbb{C} \quad (1.5)$$

3. **Identities:**

$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda \quad \forall \lambda \in \mathbb{C} \quad (1.6)$$

4. **Additive Inverse:**

$$\text{For every } \alpha \in \mathbb{C}, \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha + \beta = 0 \quad (1.7)$$

5. **Multiplicative Inverse:**

$$\text{For every } \alpha \in \mathbb{C} \text{ where } \alpha \neq 0, \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha\beta = 1 \quad (1.8)$$

6. Distributive Property:

$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \text{ for all } \lambda, \alpha, \beta \in \mathbb{C} \quad (1.9)$$

Definition 1.2: Subtraction and Division on \mathbb{C}

Let $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the additive inverse of α , so that it is the unique number satisfying $\alpha + (-\alpha) = 0$. Then, we define *subtraction* on \mathbb{C} by

$$\beta - \alpha := \beta + (-\alpha) \quad (1.10)$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α . Thus, $1/\alpha$ is the unique number satisfying $\alpha(1/\alpha) = 1$. Then, we define *division* on \mathbb{C} by

$$\beta/\alpha := \beta(1/\alpha) \quad (1.11)$$

Notation: \mathbb{F}

Throughout these notes, \mathbb{F} stands for either \mathbb{R} or \mathbb{C} . The chosen notation \mathbb{F} is used because both \mathbb{R} and \mathbb{C} are examples of *fields*. Hence, if we prove a theorem involving \mathbb{F} , it holds for both \mathbb{R} and \mathbb{C} .

Definition 1.3: Scalars

Let \mathbb{F} be a field. Then, we refer to elements of \mathbb{F} as **scalars**.

Definition 1.4: Integral Exponentiation

Let $\alpha \in \mathbb{F}$ and m be a positive integer. We define α^m as the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}} \quad (1.12)$$

1.1.2 Lists**Definition 1.5: List, Length**

Let n be a nonnegative integer. A **list of length** n is an ordered collection of n elements (which can be numbers, lists or other abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like

$$(x_1, \dots, x_n). \quad (1.13)$$

Two lists are considered equal if and only if they have the same length and the same elements in the same order. In the mathematical literature, a list of length n is also often referred to as an **n -tuple**. A list of length 0 is written as $()$.

1.1.3 \mathbb{F}^n

Definition 1.6: \mathbb{F}^n and Coordinates

We define \mathbb{F}^n as the set of all lists of length n of elements of \mathbb{F} :

$$\mathbb{F}^n := \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n\} \quad (1.14)$$

For $(x_1, \dots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \dots, n\}$, we say that x_j is the j^{th} **coordinate** of (x_1, \dots, x_n) .

Definition 1.7: Addition on \mathbb{F}^n

We define **addition** in \mathbb{F}^n by adding the corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) \quad (1.15)$$

Proposition 1.2: Commutativity of addition on \mathbb{F}^n

Let $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Definition 1.8: Zero Vector 0

Let 0 denote the list of length n whose coordinates are all 0 :

$$0 = (0, \dots, 0) \quad (1.16)$$

Definition 1.9: Additive Inverse in \mathbb{F}^n

Let $x \in \mathbb{F}^n$. Then, the **additive inverse** of x , denoted by $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0. \quad (1.17)$$

In essence, if $x = (x_1, \dots, x_n)$, then $-x = (-x_1, \dots, -x_n)$.

Definition 1.10: Scalar Multiplication in \mathbb{F}^n

The **product** of a number λ and a vector in \mathbb{F}^n is computed by multiplying each coordinate of the vector by λ :

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n), \quad (1.18)$$

where $\lambda \in \mathbb{F}$ and $(x_1, \dots, x_n) \in \mathbb{F}^n$.

1.1.4 Digression on Fields

Definition 1.11: Field

A **field** is a set containing at least two distinct elements called 0 and 1 (representing the additive inverse and multiplicative inverses respectively) along with operations of addition and multiplication satisfying all the properties listed in Proposition 1.1.

1.2 Definition of Vector Space

Definition 1.12: Addition Operator

An **addition** on a set V is a function $+: V \times V \rightarrow V$ that assigns an element $u + v \in V$ to every pair $u, v \in V$.

Definition 1.13: Scalar Multiplication

A **scalar multiplication** on a set V is a function $*: V \times \mathbb{F} \rightarrow V$ that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Definition 1.14: Vector Space

A **vector space** is a set V , along with an addition on V and a scalar multiplication on V such that the following properties hold:

1. **Commutativity:**

$$u + v = v + u \quad \forall u, v \in V \quad (1.19)$$

2. **Associativity:**

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv) \quad \forall u, v, w \in V \text{ and } \forall a, b \in \mathbb{F} \quad (1.20)$$

3. **Additive Identity:**

$$\text{There exists an element } 0 \in V \text{ such that } v + 0 = v \quad \forall v \in V \quad (1.21)$$

4. **Additive Inverse:**

$$\text{For every } u \in V, \text{ there exists } w \in V \text{ such that } v + w = 0 \quad (1.22)$$

5. **Multiplicative Identity:**

$$1v = v \quad \forall v \in V \quad (1.23)$$

6. **Distributive Properties:**

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv \quad \forall a, b \in \mathbb{F} \text{ and } \forall u, v \in V \quad (1.24)$$

The scalar multiplication in a vector space depends on \mathbb{F} . Hence, to be precise, we will say that V is a **vector space over \mathbb{F}** .

Definition 1.15: Vector / Point

Elements of a vector space are called **vectors** or **points**.

Definition 1.16: Real Vector Space, Complex Vector Space

- A vector space over \mathbb{R} is called a **real vector space**.
- A vector space over \mathbb{C} is called a **complex vector space**.

Definition 1.17: Notation: \mathbb{F}^S

Let S be a set. Then, \mathbb{F}^S denotes the set of all functions from S to \mathbb{F} .

- Let $f, g \in \mathbb{F}^S$, the **sum** $f + g \in \mathbb{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x) \quad (1.25)$$

$\forall x \in S$.

- Let $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x) \quad (1.26)$$

$\forall x \in S$.

If S is a nonempty set, then \mathbb{F}^S , with the operations of addition and scalar multiplication defined above is a vector space over \mathbb{F} . For the rest of these notes, V denotes a vector space over \mathbb{F} .

Proposition 1.3: Unique Additive Identity

A vector space has a unique additive identity.

Proposition 1.4: Unique Additive Inverse

A vector space has a unique additive inverse.

Definition 1.18: Notation: $-v, w - v$

Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v
- $w - v$ is defined to be $w + (-v)$

Proposition 1.5

Let V be a vector space. Then, $0v = 0 \forall v \in V$.

Proposition 1.6

Let \mathbb{F} be a field. Then, $a0 = 0 \forall a \in \mathbb{F}$.

Corollary 1.1

Let V be a vector space. Then, $(-1)v = -v \forall v \in V$.

1.3 Subspaces

Definition 1.19: Subspace

A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V). Some mathematicians use the term **linear subspace**, which means the same as subspace.

Theorem 1.1: Conditions for a Subspace

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. **Additive Identity**

$$0 \in U \quad (1.27)$$

2. **Closed Under Addition**

$$u, w \in U \text{ implies } u + w \in U \quad (1.28)$$

3. **Closed under Scalar Multiplication**

$$a \in \mathbb{F} \text{ and } u \in U \text{ implies } au \in U \quad (1.29)$$

1.3.1 Sums of Subspaces

Definition 1.20: Sum of Subsets

Suppose that U_1, \dots, U_m are subsets of V . The **sum** of U_1, \dots, U_m , denoted as $U_1 + \dots + U_m$, is the set of all possible sums of elements of U_1, \dots, U_m . More precisely,

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\} \quad (1.30)$$

Proposition 1.7: Sum of Subspaces is the smallest containing Subspace

Suppose that U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is the smallest subspace of V containing U_1, \dots, U_m .

1.3.2 Direct Sums

Definition 1.21: Direct Sum

Suppose that U_1, \dots, U_m are subspaces of V .

- The sum $U_1 + \dots + U_m$ is called a **direct sum** if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $u_1 + \dots + u_m$, where each u_j is in U_j ^a.
- If $U_1 + \dots + U_m$ is a direct sum, then $U_1 \oplus \dots \oplus U_m$ denotes $U_1 + \dots + U_m$, with the \oplus notation serving as an indication that this is a direct sum.

The definition of direct sum requires that every vector in the sum have a unique representation as an appropriate sum.

^aTypically, suppose that A, B were two Abelian groups or objects with an abelian structure. Then, we can define the direct

sum as the set of ordered pairs $A \oplus B = \{(a, b) | a \in A, b \in B\}$ with the new addition structure defined coordinate-wise.

^bThe uniqueness condition in writing the summation is sufficient so that we may identify it with the usual notion of direct sum.

Proposition 1.8: Condition for a Direct Sum

Suppose that U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \dots + u_m$, where each $u_j \in U_j$, is by taking each u_j equal to 0.

Proposition 1.9: Direct Sum of Two Subspaces

Suppose U and W are subspaces of V . Then, $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

1.4 Exercises

Definition 1.22: Even Function

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **even** if

$$f(-x) = f(x) \quad (1.31)$$

for all $x \in \mathbb{R}$.

Definition 1.23: Odd Function

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **odd** if

$$f(-x) = -f(x) \quad (1.32)$$

for all $x \in \mathbb{R}$.

Proposition 1.10

Let U_e denote the set of real-valued even functions on \mathbb{R} and let U_o denote the set of real-valued odd functions on \mathbb{R} . Then,

$$\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o \quad (1.33)$$

Proof. First, we note that any function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be expressed as a sum of an even and odd function. We define $f_e(x) = \frac{f(x) + f(-x)}{2}$ and $f_o(x) = \frac{f(x) - f(-x)}{2}$. Then $f = f_e + f_o$ where f_e and f_o are even and odd functions respectively. We will first observe that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$. Let $f \in U_e$ and $g \in U_o$, then $f + g : \mathbb{R} \rightarrow \mathbb{R}$. Hence, we necessarily have $f + g \in \mathbb{R}^{\mathbb{R}}$. Now, let $h \in \mathbb{R}^{\mathbb{R}}$. Then, we define h_e and h_o as the even and odd components of h respectively. Since $h_e \in U_e$ and $h_o \in U_o$, then $h \in U_e + U_o$. Hence, we have established that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.

We now want to show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. We observe that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$ as they satisfy all the properties of a subspace. It's clear that aside from the zero function, a function cannot simultaneously be both even and odd^a. Since U_e, U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$ and $U_e \cap U_o = \{0\}$, by Proposition 1.9, $U_e + U_o$ is a direct sum, thereby establishing

$$\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o \quad \square$$

^aSuppose that a function f was both even and odd. Then, we can conclude that $f(x) = -f(x) \forall x \in \mathbb{R}$. Hence, $f(x) = 0 \forall x \in \mathbb{R}$, signifying that $f = 0$.

Chapter 2

Finite-Dimensional Vector Spaces

Definition 2.1: Notation: \mathbb{F} , V

- \mathbb{F} denotes \mathbb{R} or \mathbb{C}
- V denotes a vector space over \mathbb{F}

2.1 Span and Linear Independence

Definition 2.2: List of Vectors

We will usually write lists of vectors without surrounding parentheses^a.

^aFor instance $(4, 1, 6), (9, 5, 7)$ is a list of length 2 of vectors in \mathbb{R}^3 .

2.1.1 Linear Combinations and Span

Definition 2.3: Linear Combination

A **linear combination** of a list v_1, \dots, v_m of vectors in V is a vector of the form

$$a_1 v_1 + \dots + a_m v_m, \tag{2.1}$$

where $a_1, \dots, a_m \in \mathbb{F}$.

Definition 2.4: Span

The set of all linear combinations of a list of vectors $v_1, \dots, v_m \in V$ is called the **span** of v_1, \dots, v_m , denoted as $\text{span}(v_1, \dots, v_m)$. In essence,

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\} \tag{2.2}$$

The span of the empty list $()$ is defined to be $\{0\}$. Some mathematicians use the term **linear span**, which means the same as span.

Proposition 2.1: Span is the Smallest Containing Subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Definition 2.5: Spans

If $\text{span}(v_1, \dots, v_m)$ equals V , we say that v_1, \dots, v_m **spans** V .

Definition 2.6: Finite-Dimensional Vector Space

A vector space is called **finite-dimensional** if some list of vectors in it spans the space.

Definition 2.7: Polynomial, $\mathcal{P}(\mathbb{F})$

- A function $p : \mathbb{F} \rightarrow \mathbb{F}$ is called a **polynomial** with coefficients in \mathbb{F} if there exist $a_0, \dots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \quad (2.3)$$

for all $z \in \mathbb{F}$.

- $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in \mathbb{F} .

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbb{F})$ is a vector space over \mathbb{F} .

Definition 2.8: Degree of a Polynomial, $\deg p$

- A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have **degree** m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \dots + a_m z^m \quad (2.4)$$

for all $z \in \mathbb{F}$. If p has degree m , we write $\deg p = m$.

- The polynomial that is identically 0 is said to have degree $-\infty$.

Definition 2.9: $\mathcal{P}_m(\mathbb{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in \mathbb{F} and degree at most m . We note that $\mathcal{P}_m(\mathbb{F})$ is a finite-dimensional vector space for each non-negative integer m .

Definition 2.10: Infinite-Dimensional Vector Space

A vector space is called **infinite-dimensional** if it is not finite-dimensional.

Example 2.1

$\mathcal{P}(\mathbb{F})$ is infinite dimensional.

Proof. Suppose by contradiction, that there exists a list of elements (p_1, \dots, p_n) that spans $\mathcal{P}(\mathbb{F})$. Then, let

$$m := \max_{j \in \mathbb{Z}_n} \deg(p_j) \quad (2.5)$$

Consider the polynomial $q(z) = z^{m+1}$. Clearly, $q \in \mathcal{P}(\mathbb{F})$ but $q \notin \text{span}(p_1, \dots, p_m)$. Hence, we have a contradiction. Therefore, $\mathcal{P}(\mathbb{F})$ must be infinite dimensional. \square

2.1.2 Linear Independence

Definition 2.11: Linear Independence

- A list v_1, \dots, v_m of vectors in V is said to be **linearly independent** if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that satisfies

$$a_1 v_1 + \dots + a_m v_m = 0 \quad (2.6)$$

is if $a_1 = \dots = a_m = 0$.

- The empty list $()$ is defined to be linearly independent.

This reasoning shows that v_1, \dots, v_m is linearly independent if and only if each vector in $\text{span}(v_1, \dots, v_m)$ has only one representation as a linear combination of v_1, \dots, v_m .

Proposition 2.2

Let (v_1, \dots, v_m) denote a linearly independent list. If some vectors are removed from this list, the remaining list is also linearly independent.

Proof. WLOG, consider the list (v_1, \dots, v_j) where $j < m$. Then, suppose by contradiction that there exists a set of values $b_1, \dots, b_j \in \mathbb{F}$ that are not all zero so that

$$\sum_{i=1}^j b_i v_i = 0. \quad (2.7)$$

Since (v_1, \dots, v_m) are linearly independent, we must have that the only solution to

$$\sum_{i=1}^m a_i v_i = 0 \quad (2.8)$$

is if $a_i = 0 \forall i$. However, we can clearly see that

$$\sum_{i=1}^m a_i v_i = \sum_{i=1}^j a_i v_i + \sum_{k=j+1}^m a_k v_k = 0 \quad (2.9)$$

is also satisfied if $a_i = b_i$ for $1 \leq i \leq j$ and $a_i = 0$ for $j+1 \leq i \leq m$, which by construction is not all zeroes. This is a contradiction. Hence, any new list obtained by the removal of some elements from a linearly independent list must itself be linearly independent. \square

Definition 2.12: Linearly Dependent

- A list of vectors in V is called **linearly dependent** if it is not linearly independent.
- In essence, a list v_1, \dots, v_m of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1 v_1 + \dots + a_m v_m = 0 \quad (2.10)$$

Lemma 2.1: Linear Dependence Lemma

Suppose that v_1, \dots, v_m is a linearly dependent list in V . Then, there exists $j \in \{1, 2, \dots, m\}$ such that the following hold:

- $v_j \in \text{span}(v_1, \dots, v_{j-1})$
- If the j^{th} term is removed from v_1, \dots, v_m , the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.

Theorem 2.1: Length of Linearly Independent List \leq Length of Spanning List

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proposition 2.3: Finite-Dimensional Subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional.

2.2 Bases

Definition 2.13: Basis

A **basis** of V is a list of vectors in V that is linearly independent and spans V .

Corollary 2.1: Basis Span

Let V be a finite-dimensional vector space. Suppose that v_1, \dots, v_n forms a basis for V , then $V = \text{span}(v_1, \dots, v_n)$.

Definition 2.14: Standard Basis

The list $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a basis of \mathbb{F}^n , called the **standard basis** of \mathbb{F}^n .

Proposition 2.4: Criterion for Basis

A list v_1, \dots, v_n of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \dots + a_n v_n \quad (2.11)$$

where $a_1, \dots, a_n \in \mathbb{F}$.

Proposition 2.5: Spanning List Contains a Basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proposition 2.6: Basis of Finite-Dimensional Vector Space

Every finite-dimensional vector space has a basis.

Proposition 2.7: Linearly Independent List Extends to a Basis

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Theorem 2.2: Every Subspace of V is part of a Direct Sum equal to V

Suppose that V is finite-dimensional and U is a subspace of V . Then there is a subspace W of V such that $V = U \oplus W$.

2.3 Dimension

Proposition 2.8: Basis Length does not Depend on Basis

Any two bases of a finite-dimensional vector space have the same length.

Definition 2.15: Dimension, $\dim V$

- The **dimension** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

Proposition 2.9: Dimension of a Subspace

If V is finite-dimensional and U is a subspace of V , then $\dim U \leq \dim V$.

Proposition 2.10: Linearly Independent List of the Right Length is a Basis

Suppose that V is finite-dimensional. Then, every linearly independent list of vectors in V with length $\dim V$ is a basis of V .

Proposition 2.11: Spanning List of the Right Length is a Basis

Suppose that V is finite-dimensional. Then, every spanning list of vectors in V with length $\dim V$ is a basis of V .

Theorem 2.3: Dimension of a Sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) \quad (2.12)$$

2.4 Exercises**Proposition 2.12**

Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Then, there exists 1-dimensional subspaces U_1, \dots, U_n of V such that

$$V = U_1 \oplus \dots \oplus U_n \quad (2.13)$$

Proof. Let v_1, \dots, v_n denote a basis for V . Then, we define $U_i = \text{span}(v_i)$. It's clear that U_i is a subset of V , $0 \in U_i$ and all U_i 's are closed under addition and scalar multiplication. Hence, we observe that U_i are subspaces of V .

The length of the list of vectors spanning U_i 's is 1, hence $\dim U_i = 1 \ \forall i$. We want to first show that $V = U_1 + \dots + U_n$. We note that

$$\sum_{i=1}^n \text{span}(v_i) = \text{span}(v_1, v_2, \dots, v_n), \quad (2.14)$$

Since v_1, \dots, v_n forms a basis for V , then $V = \text{span}(v_1, \dots, v_n)$. Hence, we have that $U_1 + \dots + U_n = V$.

Lastly, consider an element $u_1 + \dots + u_n \in U_1 + \dots + U_n$, where $u_j \in U_j$. Then, this can take the form $u_1 + \dots + u_n = \sum_{i=1}^n a_i v_i$ for some $a_i \in \mathbb{F}$. Since v_1, \dots, v_n form a basis for $U_1 + \dots + U_n$, then the only way to satisfy $\sum_{i=1}^n a_i v_i = 0$ is if $a_i = 0 \ \forall i$. In other words, $\sum_{i=1}^n u_i = \sum_{i=1}^n a_i v_i = 0$ is only satisfied if $u_i = 0 \ \forall i$. Then, by Proposition 1.8, $U_1 + \dots + U_n$ is a direct sum. Consequently,

$$V = U_1 \oplus \dots \oplus U_n \quad (2.15)$$

□

Proposition 2.13

Suppose U_1, \dots, U_m are finite-dimensional subspaces of V such that $U_1 + \dots + U_m$ is a direct sum. Then, $U_1 \oplus \dots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \dots \oplus U_m) = \dim(U_1) + \dots + \dim(U_m) \quad (2.16)$$

Chapter 3

Linear Maps

Definition 3.1: Notation: \mathbb{F}, V, W

- \mathbb{F} denotes \mathbb{R} or \mathbb{C} .
- V and W denote vector spaces over \mathbb{F} .

3.1 The Vector Space of Linear Maps

Definition 3.2: Linear Map

A **linear map** from V to W is a function $T : V \rightarrow W$ with the following properties:

1. **Additivity**

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V \quad (3.1)$$

2. **Homogeneity**

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbb{F} \text{ and all } v \in V. \quad (3.2)$$

Some mathematicians use the term **linear transformation** which means the same as linear map. For linear maps, we will often use the notation Tv as well as the more standard functional notation $T(v)$.

Definition 3.3: Notation: $\mathcal{L}(V, W)$ or $\text{Hom}_{\mathbb{F}}(V, W)$

Let V, W be vector spaces over \mathbb{F} . The set of all linear maps from V to W will be denoted by $\mathcal{L}(V, W)$. We note that a lot of literature also denotes this by $\text{Hom}_{\mathbb{F}}(V, W)$.

Common Linear Maps

- **Zero**

We'll let 0 denote the function that takes each element of some vector space to the additive identity of another vector space. Specifically, $0 \in \mathcal{L}(V, W)$ is defined by

$$0v = 0 \quad \forall v \in V \quad (3.3)$$

The zero on the left is a function from V to W , whereas the 0 on the right side is the additive identity in W .

- **Identity**

The **identity map**, denoted I , is the function on some vector space that takes each element to itself. Specifically, $I \in \mathcal{L}(V, V)$ is defined by

$$Iv = v \quad \forall v \in V \quad (3.4)$$

- **Differentiation**

Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$Dp = p' \quad (3.5)$$

Proposition 3.1: Linear Maps and Basis of Domain

Suppose that v_1, \dots, v_n is a basis of V and $w_1, \dots, w_n \in W$. Then, there exists a unique linear map $T : V \rightarrow W$ such that

$$Tv_j = w_j \quad (3.6)$$

for each $j = 1, \dots, n$.

3.1.1 Algebraic Operations on $\mathcal{L}(V, W)$

Definition 3.4: Addition on Scalar Multiplication on $\mathcal{L}(V, W)$

Suppose that $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$ and the **product** λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv) \quad (3.7)$$

for all $v \in V$.

Theorem 3.1: $\mathcal{L}(V, W)$ is a Vector Space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Definition 3.5: Product of Linear Maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the **product** $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu) \quad (3.8)$$

for $u \in U$. Hence, ST is the usual composition $S \circ T$ of two functions, but when both functions are linear, most mathematicians write ST instead of $S \circ T$.

Proposition 3.2: Algebraic Properties of Products of Linear Maps

- **Associativity**

$$(T_1 T_2) T_3 = T_1 (T_2 T_3) \quad (3.9)$$

whenever T_1, T_2 , and T_3 are linear maps such that the products make sense (meaning that T_3 maps into domain of T_2 and T_2 maps into the domain of T_1).

- **Identity**

$$T I_V = I_W T = T \quad (3.10)$$

whenever $T \in \mathcal{L}(V, W)$. We have denoted I_V as the identity map on V and I_W as the identity map on W .

- **Distributive Properties**

$$(S_1 + S_2)T = S_1T + S_2T \text{ and } S(T_1 + T_2) = ST_1 + ST_2 \quad (3.11)$$

whenever $T, T_1, T_2 \in \mathcal{L}(V, W)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Proposition 3.3: Linear Maps take 0 to 0

Let T be a linear map from V to W . Then $T(0) = 0$.

3.2 Null Spaces and Ranges

3.2.1 Null Space and Injectivity

Definition 3.6: Null Space, null T / Kernel

Let $T \in \mathcal{L}(V, W)$. The **null space** of T , denoted $\text{null } T$, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\} \quad (3.12)$$

Some mathematicians use the term **kernel** instead of null space.

Proposition 3.4: The Null Space is a Subspace

Let $T \in \mathcal{L}(V, W)$. Then $\text{null } T$ is a subspace of V .

Proposition 3.5: Injectivity is Equivalent to Null Space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $\text{null } T = \{0\}$.

Definition 3.7: Injective

A function $T : V \rightarrow W$ is called **injective** if $Tu = Tv$ implies that $u = v$. Many mathematicians also use the term *one-to-one* to mean injectivity.

3.2.2 Range and Surjectivity

Definition 3.8: Range

Let $T \in \mathcal{L}(V, W)$. The **range** of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\text{range } T = \{Tv : v \in V\} \quad (3.13)$$

Some mathematicians also use the term **image** to denote the range^a, though this is used in the wider context for any map $f : A \rightarrow B$, where A, B are sets and f is an arbitrary function.

^aIn the broader sense, range is equivalent to image but our consideration is restricted to linear maps between vector spaces.

Proposition 3.6: The Range is a Subspace

If $T \in \mathcal{L}(V, W)$, then $\text{range } T$ is a subspace of W .

Definition 3.9: Surjective

A function $T : V \rightarrow W$ is called **surjective** if its range equals W . Many mathematicians also use the term *onto* to mean surjectivity.

3.2.3 Fundamental Theorem of Linear Maps

Theorem 3.2: Fundamental Theorem of Linear Maps

Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\text{range } T$ is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T. \quad (3.14)$$

Proposition 3.7: A map to a smaller dimensional space is not injective

Suppose that V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Proposition 3.8: A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Definition 3.10: Homogeneous / Inhomogeneous System of Linear Equations

Fix positive integers m and n . Let $A_{j,k} \in \mathbb{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$. A system of linear equations is defined as the collection of equalities:

$$\sum_{k=1}^n A_{1,k} x_k = c_1 \quad (3.15)$$

$$: \quad (3.16)$$

$$: \quad (3.17)$$

$$\sum_{k=1}^n A_{m,k} x_k = c_m \quad (3.18)$$

where $x_i, c_i \in \mathbb{F}$. We say that this system of linear equations is **homogeneous** iff $c_1 = \dots = c_m = 0 \forall i$ and **inhomogeneous** otherwise.

Proposition 3.9: Homogeneous System of Linear Equations

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proposition 3.10: Inhomogeneous System of Linear Equations

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

3.3 Matrices

3.3.1 Representing a Linear Map by a Matrix

Definition 3.11: Matrix, $A_{j,k}$

Let m and n denote positive integers. An m -by- n **matrix** A is a rectangular array of elements of \mathbb{F} with m rows and n columns:

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \quad (3.19)$$

The notation $A_{j,k}$ denotes the entry in row j , column k of A .

Definition 3.12: Matrix of a Linear Map, $\mathcal{M}(T)$

Suppose that $T \in \mathcal{L}(V, W)$ and v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W . The **matrix of T** with respect to these bases is the m -by- n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m. \quad (3.20)$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ is used. We note that if T maps from an n -dimensional vector space to an m -dimensional vector space, then $\mathcal{M}(T)$ is an m -by- n matrix. If T is a linear map from \mathbb{F}^n to \mathbb{F}^m , then unless stated otherwise, assume the bases in question are the standard ones.

3.3.2 Addition and Scalar Multiplication of Matrices

In this section, we assume that V and W are finite-dimensional and that a basis has been chosen for each of these vector spaces. Hence, for each linear map from V to W , we can talk about its matrix (w.r.t chosen bases).

Definition 3.13: Matrix Addition

The **sum of two matrices of the same size** is the matrix obtained by adding the corresponding entries in the matrices:

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} + \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \vdots & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{bmatrix} = \begin{bmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & \vdots & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{bmatrix} \quad (3.21)$$

Proposition 3.11: The matrix of the sum of Linear Maps

Suppose that we fix the same basis for all three linear maps $S, T, S + T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Definition 3.14: Scalar Multiplication of a Matrix

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{bmatrix}. \quad (3.22)$$

In essence, $(\lambda A)_{j,k} = \lambda A_{j,k}$ for any $\lambda \in \mathbb{F}$.

Proposition 3.12: The Matrix of a Scalar times a Linear Map

Suppose that $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then, $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Definition 3.15: Notation: $\mathbb{F}^{m,n}$

For m and n positive integers, the set of all m -by- n matrices with entries in \mathbb{F} is denoted by $\mathbb{F}^{m,n}$.

Proposition 3.13: $\dim \mathbb{F}^{m,n} = mn$

Suppose that m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbb{F}^{m,n}$ is a vector space with dimension mn .

3.3.3 Matrix Multiplication**Definition 3.16: Matrix Multiplication**

Suppose that $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{n,p}$. Then AC is defined to be the m -by- p matrix whose entry in row j , column k , is given by the following equation

$$(AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k} \quad (3.23)$$

In other words, the entry in row j , column k of AC is computed by taking row j of A and column k of C , multiplying together the corresponding entries, and then summing.

Proposition 3.14: The Matrix of the Product of Linear Maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Definition 3.17: Notation $A_{j,\cdot}, A_{\cdot,k}$

Suppose A is an m -by- n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by- n matrix consisting of row j of A .
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m -by-1 matrix consisting of column k of A .

Proposition 3.15: Entry of Matrix Product Equals Row times Column

Suppose that $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{n,p}$. Then

$$(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k} \quad (3.24)$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$.

Proposition 3.16: Column of Matrix Product equals Matrix times Column

Suppose that $A \in \mathbb{F}^{m,n}$ and $C \in \mathbb{F}^{n,p}$. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k} \quad (3.25)$$

for $1 \leq k \leq p$.

Proposition 3.17: Linear Combination of Columns

Suppose that $A \in \mathbb{F}^{m,n}$ and $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an n -by-1 matrix. Then,

$$Ac = c_1 A_{\cdot,1} + \dots + c_n A_{\cdot,n} \quad (3.26)$$

In essence, Ac is a linear combination of the columns of A , with the scalars that multiply the columns coming from c .

3.4 Invertibility and Isomorphic Vector Spaces

3.4.1 Invertible Linear Maps

Definition 3.18: Invertible Linear Map / Inverse

- A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity map on V and TS equals the identity map on W .
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I_V$ and $TS = I_W$ is called an **inverse** of T . Here, I_V denotes the identity map on V and I_W denotes the identity map on W .

Proposition 3.18: Inverse is Unique

An invertible linear map has a unique inverse.

Definition 3.19: Notation: T^{-1}

If T is invertible, then its inverse is denoted by T^{-1} . In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(V, W)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Proposition 3.19: Invertibility is Equivalent to Injectivity and Surjectivity

A linear map is invertible if and only if it is injective and surjective.

3.4.2 Isomorphic Vector Spaces**Definition 3.20: Isomorphism, Isomorphic**

- An **isomorphism** is an invertible linear map.
- Two vector spaces are called **isomorphic** if there is an isomorphism from one vector space onto the other one.

Proposition 3.20: Dimension shows whether vector spaces are isomorphic

Two finite-dimensional vector spaces over \mathbb{F} are isomorphic if and only if they have the same dimension.

Proposition 3.21: $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are isomorphic

Suppose that v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W . Then \mathcal{M} is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

Proposition 3.22: $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Suppose that V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W) \quad (3.27)$$

3.4.3 Linear Maps Thought of as Matrix Multiplication**Definition 3.21: Matrix of a Vector, $\mathcal{M}(v)$**

Suppose that $v \in V$ and v_1, \dots, v_n is a basis of V . The **matrix of v** with respect to this basis is the n -by-1 matrix

$$\mathcal{M}(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad (3.28)$$

where c_1, \dots, c_n are the scalars such that

$$v = c_1 v_1 + \dots + c_n v_n \quad (3.29)$$

Hence, the matrix $\mathcal{M}(v)$ of a vector $v \in V$ depends on the basis $\{v_1, \dots, v_n\}$ of V as well as on v .

Proposition 3.23: $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$

Suppose that $T \in \mathcal{L}(V, W)$ and $\{v_1, \dots, v_n\}$ is a basis of V and w_1, \dots, w_m is a basis of W . Let $1 \leq k \leq n$. Then, the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$ equals $\mathcal{M}(v_k)$.

Proposition 3.24: Linear Maps act Like Matrix Multiplication

Suppose that $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W . Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v) \quad (3.30)$$

3.4.4 Operators

Definition 3.22: Operator, $\mathcal{L}(V)$

- A linear map from a vector space to itself is called an **operator**.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V . In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Definition 3.23: Injectivity is Equivalent to Surjectivity in Finite Dimensions

Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Then, the following are equivalent:

- T is invertible;
- T is injective;
- T is surjective.

3.5 Products and Quotients of Vector Spaces

3.5.1 Products of Vector Spaces

Definition 3.24: Product of Vector Spaces

Suppose that V_1, \dots, V_m are vector spaces over \mathbb{F} .

- The **product** $V_1 \times \dots \times V_m$ is defined by

$$V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\} \quad (3.31)$$

- Addition on $V_1 \times \dots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m) \quad (3.32)$$

- Scalar multiplication on $V_1 \times \dots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m) \quad (3.33)$$

Proposition 3.25: Product of Vector Spaces is a Vector Space

Suppose that V_1, \dots, V_m are vector spaces over \mathbb{F} . Then $V_1 \times \dots \times V_m$ is a vector space over \mathbb{F} .

Proposition 3.26: Dimension of a Product is the Sum of Dimensions

Suppose that V_1, \dots, V_m are finite-dimensional vector spaces. Then, $V_1 \times \dots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m \quad (3.34)$$

3.5.2 Products and Direct Sums**Proposition 3.27: Products and Direct Sums**

Suppose that U_1, \dots, U_m are subspaces of V . Define a linear map $\Gamma : U_1 \times \dots \times U_m \rightarrow U_1 + \dots + U_m$ by

$$\Gamma(u_1, \dots, u_m) = u_1 + \dots + u_m. \quad (3.35)$$

Then $U_1 + \dots + U_m$ is a direct sum if and only if Γ is injective.

Proposition 3.28: A Sum is a Direct Sum if and only if Dimensions add up

Suppose V is finite-dimensional and U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \dots + U_m) = \dim U_1 + \dots + \dim U_m \quad (3.36)$$

3.5.3 Quotients of Vector Spaces**Definition 3.25: Notation: $v + U$**

Suppose that $v \in V$ and U is a subspace of V . Then, $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\} \quad (3.37)$$

In the mathematical literature, this object is given the name of a left coset. Since, addition is abelian for the cases of U that we are considering, left or right coset need not be specified.

Example 3.1: Parallel Lines with Cosets

Suppose that

$$U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \quad (3.38)$$

Then U is the line in \mathbb{R}^2 through the origin with slope 2. Thus,

$$(17, 20) + U \quad (3.39)$$

is the line in \mathbb{R}^2 that contains the point $(17, 20)$ and has slope 2.

Definition 3.26: Affine Subset, Parallel

- An **affine subset** of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V .
- For $v \in V$ and U a subspace of V , the affine subset $v + U$ is said to be **parallel** to U .

Definition 3.27: Quotient Space, V/U

Suppose U is a subspace of V . Then the **quotient space** V/U is the set of all affine subsets of V parallel to U . In other words,

$$V/U = \{v + U : v \in V\} \quad (3.40)$$

Example 3.2: Quotient Space: Set of all Lines

If $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then \mathbb{R}^2/U is the set of all lines in \mathbb{R}^2 that have slope 2.

Proposition 3.29: Two Affine Subsets Parallel to U are Equal or Disjoint

Suppose that U is a subspace of V and $v, w \in V$. Then, the following are equivalent:

- (a) $v - w \in U$;
- (b) $v + U = w + U$;
- (c) $(v + U) \cap (w + U) \neq \emptyset$

Definition 3.28: Addition and Scalar Multiplication on V/U

Suppose that U is a subspace of V . Then, addition and scalar multiplication are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U \quad (3.41)$$

$$\lambda(v + U) = (\lambda v) + U \quad (3.42)$$

for $v, w \in V$ and $\lambda \in \mathbb{F}$.

Theorem 3.3: Quotient Space is a Vector Space

Suppose that U is a subspace of V . Then, V/U , with the operations of addition and scalar multiplication as defined above, is a vector space.

Definition 3.29: Quotient map, π

Suppose U is a subspace of V . The **quotient map** π is the linear map $\pi : V \rightarrow V/U$ defined by

$$\pi(v) = v + U \quad (3.43)$$

for $v \in V$.

Proposition 3.30: Dimension of a Quotient Space

Suppose that V is finite-dimensional and U is a subspace of V . Then

$$\dim V/U = \dim V - \dim U \quad (3.44)$$

Definition 3.30: \tilde{T}

Suppose that $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\text{null } T) \rightarrow W$ by

$$\tilde{T}(v + \text{null } T) = Tv \quad (3.45)$$

Since coset representations are not unique, we need to ensure that this map is well defined. To that end, suppose that $v, w \in V$ such that

$$v + \text{null } T = w + \text{null } T. \quad (3.46)$$

Then, we want $Tv = Tw$ or this map doesn't make sense. Since $v + \text{null } T = w + \text{null } T$, then by Proposition 3.29 we have $v - w \in \text{null } T$. Hence, we have $T(v - w) = 0$, from which we conclude that $T(v) = T(w)$. Thus, the definition of \tilde{T} makes sense.

Proposition 3.31: Null Space and Range of \tilde{T}

Suppose that $T \in \mathcal{L}(V, W)$. Then

- (a) \tilde{T} is a linear map from $V/(\text{null } T)$ to W ;
- (b) \tilde{T} is injective;
- (c) $\text{range } \tilde{T} = \text{range } T$;
- (d) $V/(\text{null } T)$ is isomorphic to $\text{range } T$.

3.6 Duality

3.6.1 The Dual Space and the Dual Map

Definition 3.31: Linear Functional

A **linear functional** on V is a linear map from V to \mathbb{F} . In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Definition 3.32: Dual Space, V'

The **dual space** of V , denoted V' , is the vector space of all linear functionals on V . In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Proposition 3.32: $\dim V' = \dim V$

Suppose that V is finite-dimensional. Then V' is also finite-dimensional and $\dim V' = \dim V$.

Definition 3.33: Dual Basis

If v_1, \dots, v_n is a basis for V , then the **dual basis** of v_1, \dots, v_n is the list ψ_1, \dots, ψ_n of elements of V' , where each ψ_j is the linear functional on V such that

$$\psi_j(v_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \quad (3.47)$$

Proposition 3.33: Dual Basis is a Basis of the Dual Space

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V' .

Definition 3.34: Dual Map, T'

If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by $T'(\psi) = \psi \circ T$ for $\psi \in W'$.

Proposition 3.34: Algebraic Properties of Dual Maps

Let U, V, W be vector spaces.

- $(S + T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$.
- $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and all $T \in \mathcal{L}(V, W)$.
- $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and all $S \in \mathcal{L}(V, W)$

Some books use the notation V^* and T^* for duality instead of V' and T' . However, this notation will be reserved for the adjoint.

3.6.2 The Null Space and Range of the Dual of a Linear Map**Definition 3.35: Annihilator, U^0**

For $U \subset V$, the **annihilator** of U , denoted U^0 , is defined by

$$U^0 = \{\psi \in V' : \psi(u) = 0 \text{ for all } u \in U\} \quad (3.48)$$

Proposition 3.35: The Annihilator is a Subspace

Suppose that $U \subset V$. Then U^0 is a subspace of V' .

Proposition 3.36: Dimension of the Annihilator

Suppose V is finite-dimensional and U is a subspace of V . Then

$$\dim U + \dim U^0 = \dim V \quad (3.49)$$

Proposition 3.37: The Null Space of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- (a) $\text{null } T' = (\text{range } T)^0$;
- (b) $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$.

Proposition 3.38: T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proposition 3.39: The Range of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- (a) $\dim \text{range } T' = \dim \text{range } T$;
- (b) $\text{range } T' = (\text{null } T)^0$.

Proposition 3.40: T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T' is surjective.

3.6.3 The Matrix of the Dual of a Linear Map**Definition 3.36: Transpose, A^t**

The **transpose** of a matrix A , denoted A^t , is the matrix obtained from A by interchanging the rows and columns. More specifically, if A is an m -by- n matrix, then A^t is the n -by- m matrix whose entries are given by the equation

$$(A^t)_{k,j} = A_{j,k} \quad (3.50)$$

Proposition 3.41: The Transpose of the Product of Matrices

If A is an m -by- n matrix and C is an n -by- p matrix, then

$$(AC)^t = C^t A^t \quad (3.51)$$

Theorem 3.4: The Matrix of T' is the Transpose of the Matrix of T

Suppose that $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

3.6.4 The Rank of a Matrix

Definition 3.37: Row Rank, Column Rank

Suppose that A is an m -by- n matrix with entries in \mathbb{F} .

- The **row rank** of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.
- The **column rank** of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.

Proposition 3.42: Dimension of range T equals column rank of $\mathcal{M}(T)$

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \text{range } T$ equals the column rank of $\mathcal{M}(T)$.

Proposition 3.43: Row Rank equals Column Rank

Suppose $A \in \mathbb{F}^{m,n}$. Then the row rank of A equals the column rank of A .

Definition 3.38: Rank

The **rank** of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A .

Chapter 4

Polynomials

4.1 Polynomials

4.1.1 Complex Conjugate and Absolute Value

Definition 4.1: Re z , Im z

Suppose that $z = a + bi$, where $a, b \in \mathbb{R}$.

- The **real part** of z , denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
- The **imaginary part** of z , denoted $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.

Hence, for every complex number z , we have

$$z = \operatorname{Re} z + (\operatorname{Im} z)i \quad (4.1)$$

Definition 4.2: Complex Conjugate, \bar{z}

Let $z \in \mathbb{C}$. The **complex conjugate** of $z \in \mathbb{C}$, denoted \bar{z} , is defined by

$$\bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i \quad (4.2)$$

Definition 4.3: Absolute Value, $|z|$

Let $z \in \mathbb{C}$. The **absolute value** of $z \in \mathbb{C}$, denoted $|z|$, is defined by

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \quad (4.3)$$

Proposition 4.1: Properties of Complex Numbers

Suppose that $w, z \in \mathbb{C}$. Then

1. **Sum of z and \bar{z}**

$$z + \bar{z} = 2\operatorname{Re} z; \quad (4.4)$$

2. **Difference of z and \bar{z}**

$$z - \bar{z} = 2(\operatorname{Im} z)i; \quad (4.5)$$

3. Product of z and \bar{z}

$$z\bar{z} = |z|^2 \quad (4.6)$$

4. Additivity and Multiplicativity of Complex Conjugate

$$\overline{w + z} = \bar{w} + \bar{z} \text{ and } \overline{wz} = \bar{w}\bar{z}; \quad (4.7)$$

5. Conjugate of Conjugate

$$\bar{\bar{z}} = z; \quad (4.8)$$

6. Real and Imaginary Parts are Bounded by $|z|$

$$|\operatorname{Re} z| \leq |z| \text{ and } |\operatorname{Im} z| \leq |z| \quad (4.9)$$

7. Absolute Value of the Complex Conjugate

$$|\bar{z}| = |z| \quad (4.10)$$

8. Multiplicativity of Absolute Value

$$|wz| = |w||z|; \quad (4.11)$$

9. Triangle Inequality

$$|w + z| \leq |w| + |z| \quad (4.12)$$

4.1.2 Uniqueness of Coefficients for Polynomials

Proposition 4.2: If a Polynomial is the Zero Function, then all Coefficients are 0

Suppose $a_0, \dots, a_m \in \mathbb{F}$. If

$$a_0 + a_1z + \dots + a_mz^m = 0 \quad (4.13)$$

for every $z \in \mathbb{F}$, then $a_0 = \dots = a_m = 0$.

4.1.3 The Division Algorithm for Polynomials

Proposition 4.3: Division Algorithm for Polynomials

Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that

$$p = sq + r \quad (4.14)$$

and $\deg r < \deg s$.

4.1.4 Zeros of Polynomials

Definition 4.4: Zero of a Polynomial

A number $\lambda \in \mathbb{F}$ is called a **zero** (or **root**) of a polynomial $p \in \mathcal{P}(\mathbb{F})$ if

$$p(\lambda) = 0 \quad (4.15)$$

Definition 4.5: Factor

A polynomial $s \in \mathcal{P}(\mathbb{F})$ is called a **factor** of $p \in \mathcal{P}(\mathbb{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p = sq$.

Proposition 4.4: Each Zero of a Polynomial Corresponds to a Degree-1 Factor

Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there is a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that

$$p(z) = (z - \lambda)q(z) \quad (4.16)$$

for every $z \in \mathbb{F}$.

Proposition 4.5: A Polynomial has At Most as Many Zeros as its Degree

Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zero in \mathbb{F} .

4.1.5 Factorization of Polynomials over \mathbb{C}

Theorem 4.1: Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has a zero.

Proposition 4.6: Factorization of a Polynomial over \mathbb{C}

If $p \in \mathcal{P}(\mathbb{C})$ is nonconstant polynomial, then p has a unique factorization (except for the order of the factors) of the form

$$p(z) = c(z - \lambda_1) \dots (z - \lambda_m) \quad (4.17)$$

where $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$.

4.1.6 Factorization of Polynomials over \mathbb{R}

Proposition 4.7: Polynomials with Real Coefficients have Zeros in Pairs

Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p , then so is $\bar{\lambda}$.

Proposition 4.8: Factorization of a Quadratic Polynomial

Suppose $b, c \in \mathbb{R}$. Then there is a polynomial factorization of the form

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2) \quad (4.18)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $b^2 \geq 4c$.

Proposition 4.9: Factorization of Polynomial over \mathbb{R}

Suppose $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial. Then p has a unique factorization (except for the order of the factors) of the form

$$p(x) = c(x - \lambda_1) \dots (x - \lambda_m)(x^2 + b_1x + c_1) \dots (x^2 + b_Mx + c_M) \quad (4.19)$$

where $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbb{R}$, with $b_j^2 < 4c_j$ for each j .

Chapter 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

5.1 Invariant Subspaces

Definition 5.1: Invariant Subspace

Suppose that $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T if $u \in U$ implies $Tu \in U$.

In essence, U is invariant under T if $T|_U$ is an operator on U .

5.1.1 Eigenvalues and Eigenvectors

Definition 5.2: Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Proposition 5.1: Equivalent Conditions to be an Eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

- (a) λ is an eigenvalue of T ;
- (b) $T - \lambda I$ is not injective;
- (c) $T - \lambda I$ is not surjective;
- (d) $T - \lambda I$ is not invertible.

Definition 5.3: Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T . A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Corollary 5.1

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ be an eigenvalue of T . A vector $v \in V$ such that $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \text{null}(T - \lambda I)$.

Proposition 5.2: Linearly Independent Eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and v_1, \dots, v_m are corresponding eigenvectors. Then v_1, \dots, v_m is linearly independent.

Proposition 5.3: Number of Eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

5.1.2 Restriction and Quotient Operators**Definition 5.4:** $T|_U$ and T/U

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T .

- The **restriction operator** $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U(u) = Tu \quad (5.1)$$

for $u \in U$.

- The **quotient operator** $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U \quad (5.2)$$

for $v \in V$.

5.2 Eigenvectors and Upper-Triangular Matrices**5.2.1 Polynomials Applied to Operators****Definition 5.5:** T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- T^m is defined by

$$T^m = \underbrace{T \cdots T}_{m \text{ times}}. \quad (5.3)$$

- T^0 is defined to be the identity operator I on V .
- If T is invertible with inverse T^{-1} , then T^{-m} is defined by

$$T^{-m} = (T^{-1})^m \quad (5.4)$$

Definition 5.6: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1z + \dots + a_mz^m \quad (5.5)$$

for $z \in \mathbb{F}$. Then $p(T)$ is the operator defined by

$$p(T) = a_0I + a_1T + a_2T^2 + \dots + a_mT^m \quad (5.6)$$

Corollary 5.2

Let $T \in \mathcal{L}(V)$, then the function $P_T : \mathcal{P}(\mathbb{F}) \rightarrow \mathcal{L}(V)$ given by

$$P_T(p) = p(T) \quad (5.7)$$

is linear.

Definition 5.7: Product of Polynomials

If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z) \quad (5.8)$$

for $z \in \mathbb{F}$.

Proposition 5.4: Multiplicative Properties

Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

- (a) $(pq)(T) = p(T)q(T)$;
- (b) $p(T)q(T) = q(T)p(T)$.

5.2.2 Existence of Eigenvalues**Proposition 5.5: Operators on Complex Vector Spaces have an Eigenvalue**

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

5.2.3 Upper-Triangular Matrices**Definition 5.8: Matrix of an Operator, $\mathcal{M}(T)$**

Suppose $T \in \mathcal{L}(V)$ and v_1, \dots, v_n is a basis for V . The **matrix of T** with respect to this basis is the n -by- n matrix

$$\mathcal{M}(T) = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \quad (5.9)$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \dots + A_{n,k}v_n \quad (5.10)$$

If the basis is not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n))$.

Definition 5.9: Diagonal of a Matrix

The **diagonal** of a square matrix consists of the entries along the line from the upper left corner to the bottom right corner.

Definition 5.10: Upper-Triangular Matrix

A matrix is called **upper-triangular** if all the entries below the diagonal equal 0.

Proposition 5.6: Conditions for Upper-Triangular Matrix

Suppose $T \in \mathcal{L}(V)$ and v_1, \dots, v_n is a basis of V . Then the following are equivalent:

- (a) the matrix of T with respect to v_1, \dots, v_n is upper triangular;
- (b) $Tv_j \in \text{span}(v_1, \dots, v_j)$ for each $j = 1, \dots, n$.
- (c) $\text{span}(v_1, \dots, v_j)$ is invariant under T for each $j = 1, \dots, n$.

Proposition 5.7: Over \mathbb{C} , every operator has an upper-triangular matrix

Suppose V is finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V .

Proposition 5.8: Determination of Invertibility from Upper-Triangular Matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V . Then T is invertible if and only if all the entries on the diagonal of that upper-triangular matrix are nonzero.

Proposition 5.9: Determination of Eigenvalues from Upper-Triangular Matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V . Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

5.3 Eigenspaces and Diagonal Matrices

Definition 5.11: Diagonal Matrix

A **diagonal matrix** is a square matrix that is 0 everywhere except possibly along the diagonal.

Definition 5.12: Eigenspace, $E(\lambda, T)$

Suppose that $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **eigenspace** of T corresponding to λ , denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I). \quad (5.11)$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ , along with the 0 vector.

Proposition 5.10: Sum of Eigenspaces is a Direct Sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T . Then

$$E(\lambda_1, T) + \dots + E(\lambda_m, T) \quad (5.12)$$

is a direct sum. Furthermore,

$$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T) \leq \dim V. \quad (5.13)$$

Definition 5.13: Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of V .

Theorem 5.1: Conditions Equivalent to Diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T . Then the following are equivalent:

- (a) T is diagonalizable;
- (b) V has a basis consisting of eigenvectors of T ;
- (c) There exists 1-dimensional subspaces U_1, \dots, U_n of V , each invariant under T , such that

$$V = U_1 \oplus \dots \oplus U_n \quad (5.14)$$

- (d) $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$;
- (e) $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$.

Proposition 5.11: Enough Eigenvalues Implies Diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Chapter 6

Inner Product Spaces

6.1 Inner Products and Norms

6.1.1 Inner Products

Definition 6.1: Dot Product

Let $x, y \in \mathbb{R}^n$. The **dot product** of x and y , denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n \quad (6.1)$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

Definition 6.2: Inner Product

An **inner product** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties:

- **Positivity^a**

$$\langle v, v \rangle \geq 0 \text{ for all } v \in V \quad (6.2)$$

- **Definiteness**

$$\langle v, v \rangle = 0 \text{ if and only if } v = 0 \quad (6.3)$$

- **Additivity in First Slot**

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ for all } u, v, w \in V \quad (6.4)$$

- **Homogeneity in First Slot**

$$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \text{ for all } \lambda \in \mathbb{F} \text{ and all } u, v \in V \quad (6.5)$$

- **Conjugate Symmetry**

$$\langle u, v \rangle = \overline{\langle v, u \rangle} \text{ for all } u, v \in V \quad (6.6)$$

It should be noted that although most mathematicians define an inner product as above, many physicists use a definition requiring homogeneity in the second slot instead of the first slot.

^aIf $\lambda \in \mathbb{C}$, then the notation $\lambda \geq 0$ means that λ is real and nonnegative.

Example 6.1: Euclidean Inner Product

The **Euclidean Inner product** on \mathbb{F}^n is defined by

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1 \overline{z_1} + \dots + w_n \overline{z_n} \quad (6.7)$$

Definition 6.3: Inner Product Space

An **inner product space** is a vector space V along with an inner product on V .

For the rest of the chapter, V denotes an inner product space over \mathbb{F} .

Proposition 6.1: Basic Properties of an Inner Product

- (a) For each fixed $u \in V$, the function that takes v to $\langle v, u \rangle$ is a linear map from V to \mathbb{F} .
- (b) $\langle 0, u \rangle = 0$ for every $u \in V$.
- (c) $\langle u, 0 \rangle = 0$ for every $u \in V$.
- (d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
- (e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and $u, v \in V$.

6.1.2 Norms**Definition 6.4: Norm, $\|v\|$**

For $v \in V$, the **norm** of v , denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle} \quad (6.8)$$

Proposition 6.2: Basic Properties of the Norm

Suppose that $v \in V$.

- (a) $\|v\| = 0$ if and only if $v = 0$.
- (b) $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{F}$.^a

^aSince we typically consider $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , $|\lambda|$ denotes the magnitude of the complex number.

Definition 6.5: Orthogonal

Two vectors $u, v \in V$ are said to be **orthogonal** if $\langle u, v \rangle = 0$.

Proposition 6.3: Orthogonality and 0

- (a) 0 is orthogonal to every vector in V .
- (b) 0 is the only vector in V that is orthogonal to itself.

Proposition 6.4: Pythagorean Theorem

Suppose that u, v are orthogonal vectors in V . Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \quad (6.9)$$

Proposition 6.5: Orthogonal Decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2}v$. Then

$$\langle w, v \rangle = 0 \text{ and } u = cv + w \quad (6.10)$$

Theorem 6.1: Cauchy-Schwartz Inequality

Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \|v\| \quad (6.11)$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Proof. Let $u, v \in V$ such that $u, v \neq 0$. By the definition of an inner product, we have that

$$\left\langle u - \frac{\langle u, v \rangle}{\|v\|^2}v, u - \frac{\langle u, v \rangle}{\|v\|^2}v \right\rangle \geq 0. \quad (6.12)$$

It follows from the additive properties of inner products that

$$0 \leq \langle u, u \rangle - \frac{\langle u, v \rangle \langle v, u \rangle}{\|v\|^2}. \quad (6.13)$$

Hence,

$$\langle u, v \rangle \langle v, u \rangle \leq \|u\|^2 \|v\|^2, \quad (6.14)$$

from which one finally takes a square root to achieve the desired result. \square

Proposition 6.6: Triangle Inequality

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|. \quad (6.15)$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

Proof. Let $u, v \in V$. Then we observe that

$$\begin{aligned}
 \|u + v\|^2 &= \langle u + v, u + v \rangle \\
 &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\
 &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}[\langle u, v \rangle] \\
 &\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v \rangle| \\
 &\leq \|u\|^2 + \|v\|^2 + 2\|u\| \|v\| \\
 &= (\|u\| + \|v\|)^2
 \end{aligned} \tag{6.16}$$

Hence, it follows that $\|u + v\| \leq \|u\| + \|v\|$. In the first inequality, we used the fact that $\operatorname{Re}(z) \leq |z|$ for all $z \in \mathbb{C}$. The second inequality follows from the Cauchy-Schwartz Inequality. \square

Proposition 6.7: Parallelogram Equality

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \tag{6.17}$$

6.2 Orthonormal Bases

Definition 6.6: Orthonormal

- A list of vectors is called **orthonormal** if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In essence, a list e_1, \dots, e_m of vectors in V is orthonormal if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \tag{6.18}$$

Proposition 6.8: The Norm of an Orthonormal Linear Combination

If e_1, \dots, e_m is an orthonormal list of vectors in V , then

$$\|a_1 e_1 + \dots + a_m e_m\|^2 = |a_1|^2 + \dots + |a_m|^2 \tag{6.19}$$

for all $a_1, \dots, a_m \in \mathbb{F}$.

Corollary 6.1: An Orthonormal List is Linearly Independent

Every orthonormal list of vectors is linearly independent.

Definition 6.7: Orthonormal Basis

An **orthonormal basis** of V is an orthonormal list of vectors in V that is also a basis of V .

Proposition 6.9: An Orthonormal List of the Right Length is an Orthonormal Basis

Every orthonormal list of vectors in V with length $\dim V$ is an orthonormal basis of V .

Proposition 6.10: Writing a Vector as Linear Combination of Orthonormal Basis

Suppose e_1, \dots, e_n is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n \quad (6.20)$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2 \quad (6.21)$$

Proposition 6.11: Gram-Schmidt Procedure

Suppose that v_1, \dots, v_m is a linearly independent list of vectors in V . Let $e_1 = v_1/\|v_1\|$. For $j = 2, \dots, m$ define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}. \quad (6.22)$$

Then e_1, \dots, e_m is an orthonormal list of vectors in V such that

$$\text{span}(v_1, \dots, v_j) = \text{span}(e_1, \dots, e_j) \quad (6.23)$$

for $j = 1, \dots, m$.

Proposition 6.12: Existence of Orthonormal Basis

Every finite-dimensional inner product space has an orthonormal basis.

Proposition 6.13: Orthonormal List Extends to Orthonormal Basis

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V .

Proposition 6.14: Upper-Triangular Matrix with respect to Orthonormal Basis

Suppose $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V , then T has an upper-triangular matrix with respect to some orthonormal basis of V .

Theorem 6.2: Schur's Theorem

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V .

6.2.1 Linear Functionals on Inner Product Spaces

Theorem 6.3: Riesz Representation Theorem

Suppose V is finite-dimensional and ψ is a linear functional on V . Then there is a unique vector $u \in V$ such that

$$\psi(v) = \langle v, u \rangle \quad (6.24)$$

for every $v \in V$.

6.3 Orthogonal Complements and Minimization Problems

6.3.1 Orthogonal Complements

Definition 6.8: Orthogonal Complement, U^\perp

If U is a subset of V , then the **orthogonal complement** of U , denoted U^\perp , is the set of all vectors in V that are orthogonal to every vector in U :

$$U^\perp = \{v \in V : \langle v, u \rangle = 0 \text{ for every } u \in U\} \quad (6.25)$$

Proposition 6.15: Basic Properties of Orthogonal Complement

Consider the inner product space V .

- (a) If U is a subset of V , then U^\perp is a subspace of V .
- (b) $\{0\}^\perp = V$.
- (c) $V^\perp = \{0\}$.
- (d) If U is a subset of V , then $U \cap U^\perp \subset \{0\}$.
- (e) If U and W are subsets of V and $U \subset W$, then $W^\perp \subset U^\perp$.

Proposition 6.16: Direct Sum of a Subspace and its Orthogonal Complement

Suppose U is a finite-dimensional subspace of V . Then

$$V = U \oplus U^\perp \quad (6.26)$$

Proposition 6.17: The Orthogonal Complement of the Orthogonal Complement

Suppose U is a finite-dimensional subspace of V . Then

$$U = (U^\perp)^\perp \quad (6.27)$$

Definition 6.9: Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V . The **orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For every $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$.

Proposition 6.18: Properties of the Orthogonal Projection P_U

Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

- (a) $P_U \in \mathcal{L}(V)$;
- (b) $P_U u = u$ for every $u \in U$;
- (c) $P_U w = 0$ for every $w \in U^\perp$;
- (d) $\text{range } P_U = U$;
- (e) $\text{null } P_U = U^\perp$;
- (f) $v - P_U v \in U^\perp$;
- (g) $(P_U)^2 = P_U$;
- (h) $\|P_U v\| \leq \|v\|$;
- (i) For every orthonormal basis e_1, \dots, e_m of U ,

$$P_U v = \langle v, e_1 \rangle e_1 + \cdots \langle v, e_m \rangle e_m. \quad (6.28)$$

6.3.2 Minimization Problems**Proposition 6.19: Minimizing the Distance to a Subspace**

Suppose that U is a finite-dimensional subspace of V , $v \in V$, and $u \in U$. Then

$$\|v - P_U v\| \leq \|v - u\|. \quad (6.29)$$

Furthermore, the inequality above is an equality of and only if $u = P_U v$.

Chapter 7

Operators on Inner Product Spaces

7.1 Self-Adjoint and Normal Operators

7.1.1 Adjoints

Definition 7.1: Adjoint, T^*

Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \rightarrow V$ such that

$$\langle Tv, w \rangle = \langle v, T^*w \rangle \quad (7.1)$$

for every $v \in V$ and every $w \in W$.

Lemma 7.1: Inner Product Property

Let V be an inner product space and let $x, y \in V$. If $\langle v, x \rangle = \langle v, y \rangle$ for all $v \in V$, then $x = y$.

Proof. If $\langle v, x \rangle = \langle v, y \rangle$ for all $v \in V$, then

$$\langle v, x - y \rangle = 0 \quad \forall v \in V \quad (7.2)$$

We note that if $\langle v, z \rangle = 0$ for all $v \in V$, then $z = 0$. One can see this from the fact that $V^\perp = \{0\}$. Hence, from our relation, we have that $x - y = 0$, concluding that $x = y$. \square

Proposition 7.1: The Adjoint is a Linear Map

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proposition 7.2: Properties of The Adjoint

Let U, V, W be inner product spaces over \mathbb{F} .

- (a) $(S + T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(V, W)$;
- (b) $(\lambda T)^* = \bar{\lambda}T^*$ for all $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$
- (c) $(T^*)^* = T$ for all $T \in \mathcal{L}(V, W)$;

- (d) $I^* = I$, where I is the identity operator on V ;
 (e) $(ST)^* = T^*S^*$ for all $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$

Proposition 7.3: Null Space and Range of T^*

Suppose that $T \in \mathcal{L}(V, W)$. Then

- (a) $\text{null } T^* = (\text{range } T)^\perp$;
 (b) $\text{range } T^* = (\text{null } T)^\perp$;
 (c) $\text{null } T = (\text{range } T^*)^\perp$;
 (d) $\text{range } T = (\text{null } T^*)^\perp$.

Definition 7.2: Conjugate Transpose

The **conjugate transpose** of an m -by- n matrix is the n -by- m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

Proposition 7.4: The Matrix of T^*

Let $T \in \mathcal{L}(V, W)$. Suppose e_1, \dots, e_n is an orthonormal basis of V and f_1, \dots, f_m is an orthonormal basis of W . Then

$$\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n)) \quad (7.3)$$

is the conjugate transpose of

$$\mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m)) \quad (7.4)$$

In essence, once a basis is fixed, then $T_{ij} = \overline{(T^*)_{ji}}$ for all i, j .

7.1.2 Self-Adjoint Operators

Definition 7.3: Self-Adjoint

An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle \quad (7.5)$$

for all $v, w \in V$. Some mathematicians use the term **Hermitian** instead of self-adjoint.

Proposition 7.5: Eigenvalues of Self-Adjoint Operators are Real

Every eigenvalue of a self-adjoint operator is real.

Proposition 7.6: Over \mathbb{C} , Tv is orthogonal to v for all v only for the 0 operator

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Suppose

$$\langle Tv, v \rangle = 0 \quad (7.6)$$

for all $v \in V$. Then $T = 0$.

Proposition 7.7: Over \mathbb{C} , $\langle Tv, v \rangle$ is real for all v only for self-adjoint operators

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if

$$\langle Tv, v \rangle \in \mathbb{R} \quad (7.7)$$

for every $v \in V$.

Proposition 7.8: If $T = T^*$ and $\langle Tv, v \rangle = 0$ for all v , then $T = 0$

Suppose T is a self-adjoint operator on V such that

$$\langle Tv, v \rangle = 0 \quad (7.8)$$

for all $v \in V$. Then $T = 0$.

7.1.3 Normal Operators

Definition 7.4: Normal Operator

- An operator on an inner product space is called **normal** if it commutes with its adjoint.
- In essence, $T \in \mathcal{L}(V)$ is normal if

$$TT^* = T^*T \quad (7.9)$$

Proposition 7.9: T is normal if and only if $\|Tv\| = \|T^*v\|$ for all v

An operator $T \in \mathcal{L}(V)$ is normal if and only if

$$\|Tv\| = \|T^*v\| \quad (7.10)$$

for all $v \in V$.

Proposition 7.10: For T normal, T and T^* have the same eigenvectors

Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of T with eigenvalue λ . Then v is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

Proposition 7.11: Orthogonal Eigenvectors for Normal Operators

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

7.2 The Spectral Theorem

7.2.1 The Complex Spectral Theorem

Theorem 7.1: Complex Spectral Theorem

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) T is normal.
- (b) V has an orthonormal basis consisting of eigenvectors of T .
- (c) T has a diagonal matrix with respect to some orthonormal basis of V .

7.2.2 The Real Spectral Theorem

Proposition 7.12: Invertible Quadratic Expressions

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI \tag{7.11}$$

is invertible.

Proposition 7.13: Self-Adjoint Operators have Eigenvalues

Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

Proposition 7.14: Self-Adjoint Operators and Invariant Subspaces

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T . Then

- (a) U^\perp is invariant under T .
- (b) $T|_U \in \mathcal{L}(U)$ is self-adjoint.
- (c) $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

Theorem 7.2: Real Spectral Theorem

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) T is self-adjoint.
- (b) V has an orthonormal basis consisting of eigenvectors of T .
- (c) T has a diagonal matrix with respect to some orthonormal basis of V .

7.3 Positive Operators and Isometries

7.3.1 Positive Operators

Definition 7.5: Positive Operator

An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0 \quad (7.12)$$

for all $v \in V$. Some mathematicians also use the term positive semidefinite operator.

Definition 7.6: Square Root

An operator R is called a **square root** of an operator T if $R^2 = T$.

Proposition 7.15: Characterization of Positive Operators

Let $T \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) T is positive;
- (b) T is self-adjoint and all the eigenvalues of T are nonnegative;
- (c) T has a positive square root;
- (d) T has a self-adjoint square root;
- (e) There exists an operator $R \in \mathcal{L}(V)$ such that $T = R^*R$.

Proposition 7.16: Each Positive Operator has Only One Positive Square Root

Every positive operator on V has a unique positive square root.

7.3.2 Isometries

Definition 7.7: Isometry

- An operator $S \in \mathcal{L}(V)$ is called an **isometry** if

$$\|Sv\| = \|v\| \quad (7.13)$$

for all $v \in V$.

- In other words, an operator is an isometry if it preserves norms.

Theorem 7.3: Characterization of Isometries

Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) S is an isometry;

- (b) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$;
- (c) Se_1, \dots, Se_n is orthonormal for every orthonormal list of vectors e_1, \dots, e_n in V ;
- (d) There exists an orthonormal basis e_1, \dots, e_n of V such that Se_1, \dots, Se_n is orthonormal;
- (e) $S^*S = I$;
- (f) $SS^* = I$;
- (g) S^* is an isometry;
- (h) S is invertible and $S^{-1} = S^*$

Proposition 7.17: Description of Isometries when $\mathbb{F} = \mathbb{C}$

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) S is an isometry.
- (b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.