Calculus: Early Transcendentals Notes

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Contents

Chapter 2: Limits and Derivatives

2.1: Limit of a Function

Definition 2.1: Limit [ϵ-δ Formalism]

Let $f : A \to B$ be a function. We write

$$\lim_{x \to a} f(x) = L \tag{2.1}$$

if we can make the values of f(x) arbitrarily close to L. Formally, $\forall \epsilon > 0, \exists \delta$ such that $\forall x \in A$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

2.2: Calculating Limits using the Limit Laws

Proposition 2.1: Limit Laws

Let $c \in \mathbb{R}$ be a constant and n a positive integer. If the limits

$$\lim_{x \to a} f(x), \qquad and \qquad \lim_{x \to a} g(x) \tag{2.2}$$

exist. Then,

$$1. \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \tag{2.3}$$

$$2. \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \tag{2.4}$$

$$3. \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \tag{2.5}$$

$$4. \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \tag{2.6}$$

$$5. \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \qquad if \lim_{x \to a} g(x) \neq 0 \tag{2.7}$$

$$6. \lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \tag{2.8}$$

$$7. \lim_{x \to a} [f(x)]^{1/n} = [\lim_{x \to a} f(x)]^{1/n} \tag{2.9}$$

Theorem 2.1: Squeeze Theorem

Let $f, g, h : \mathbb{R} \to \mathbb{R}$. If $f(x) \leq g(x) \leq h(x)$ when x is near[a] *a, and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \tag{2.10}$$

then

$$\lim_{x \to a} g(x) = L \tag{2.11}$$

[a] i.e on some neighbourhood of a

2.3: Continuity

Definition 2.2: Continuity

Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is continuous at a if

$$\lim_{x\to a} f(x) = f(a) \tag{2.12}$$

Theorem 2.2

Let $f, g : \mathbb{R} \to \mathbb{R}$. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$1.\ f+g, \qquad 2.\ f-g, \qquad 3.\ cf, \qquad 4.\ fg, \qquad 5.\ \frac{f}{g} \text{ if } g(a) \neq 0 \tag{2.13}$$

Theorem 2.3: Continuity of Common Functions

(a) Polynomials are continuous everywhere, i.e. they're continuous on $\mathbb{R}$.
(b) Rational functions are continuous wherever they're defined.

In addition to the above, the following types of functions are continuous at every number in their domain:

- *Root Functions*
- *Trig Functions*
- *Inverse Trig Functions*
- *Exponential Functions*
- *Logarithmic Functions*

Theorem 2.4: Limit Composition

Let $f, g : \mathbb{R} \to \mathbb{R}$. If f is continuous at $b \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f\big(g(x)\big) = f(b)$. In other words,

$$\lim_{x\to a} f\big(g(x)\big) = f\big(\lim_{x\to a} g(x)\big) \tag{2.14}$$

Theorem 2.5: Continuity of Composite Functions

Let $f, g : \mathbb{R} \to \mathbb{R}$. If g is continuous at $a \in \mathbb{R}$ and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 2.6: Intermediate Value Theorem

Suppose that f is continuous on the closed interval [a,b] and let N be any number between f(a) and f(b), where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

2.4: Derivatives and Rates of Change

Definition 2.3: Derivative

The derivative of a function f at a number a, denote by $f'(a)$ *is*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.15}$$

If this limit exists.

Definition 2.4: Differentiability [Higher-Dimensional Generalization]

A function $f : R^n \to R^m$ *is differentiable at point* $\mathbf{a}$ *if the following holds:*

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})||}{||\mathbf{x} - \mathbf{a}||} = 0 \tag{2.16}$$

Where $D\mathbf{f}(\mathbf{a})$ *is the matrix of first partials evaluated at* $\mathbf{a}$.

Chapter 3: Differentiation Rules

Proposition 3.1: Tangent and Normal line at a Point

Given some function $y = f(x)$, we can establish the tangent, y_t and normal lines, y_n at some point (p_x, p_y) as follows:

$$y_n = \frac{-1}{f'(p_x)}(x - p_x) + p_y \tag{3.1}$$

$$y_t = f'(p_x)(x - p_x) + p_y, \tag{3.2}$$

provided that $f'(p_x) \neq 0$.

Chapter 5: Integrals

5.1: The Definite Integral

Proposition 5.1: Comparison Properties of the Integral

Let $f, g : [a, b] \to \mathbb{R}$.
1. If $f(x) \geq 0$ *for* $a \leq x \leq b$, *then* $\int_a^b f(x)dx \geq 0$
2. If $f(x) \geq g(x)$ *for* $a \leq x \leq b$, *then* $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
3. If $m \leq f(x) \leq M$ *for* $a \leq x \leq b$, *then*

$$m(b-a) \leq \int_a^b f(x)dx \ \leq M(b-a) \tag{5.1}$$

5.2: The Fundamental Theorem of Calculus

Theorem 5.1: Leibniz Rule

If f is continuous and h and g are differentiable functions, then the Leibniz Rule is given by:

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(x,t)\ dt\right) = f(h(x))h'(x) - f(g(x))g'(x) + \int_{g(x)}^{h(x)} \frac{df(x,t)}{dx}dt \tag{5.2}$$

5.3: The Substitution Rule

Proposition 5.2: Symmetric Functions

Suppose that f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, *then* $\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)dx$

(b) If f is odd $[f(-x) = -f(x)]$, *then* $\int_{-a}^a f(x)\ dx = 0$

Chapter 6: Applications of Integration

6.1: Areas between Curves

Proposition 6.1: Area Between two Curves

The area between the curves $y = f(x)$ and $y = g(x)$ on the interval [a,b] is given by:

$$A = \int_a^b |f(x) - g(x)|dx \tag{6.1}$$

6.2: Volumes

Definition 6.1: Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the volume of S is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\ \Delta x = \int_a^b A(x)dx \tag{6.2}$$

Definition 6.2: Solids of Revolution

If we consider solids that are obtained by rotating around an axis, then the resulting volume would be given by:

$$V = \int_a^b A(x)dx, \qquad or \qquad V = \int_c^d A(y)dy \tag{6.3}$$

• *If the cross section is a disk, we find the radius, R of the disk (in terms of x or y) and use:*

$$A = \pi R^2 \tag{6.4}$$

• *If cross-section is a washer, then the outer and inner radius would have to be used, so cross-sectional area would be given by:*

$$A = \pi(R_{out})^2 - \pi(R_{in})^2 \tag{6.5}$$

6.3: Volumes by Cylindrical Shells

Sometimes, the solids obtained from revolving around an axis have volumes that are difficult to calculate via the disk method of Def 6.2.

Proposition 6.2: Method of Cylindrical Shells

The volume of a solid, obtained by rotating about the y-axis for the region under the curve $y = f(x)$ on $[a, b]$, is

$$V = \int_a^b 2\pi x f(x)dx, \qquad where\ 0 \le a < b. \tag{6.6}$$

One can consider $2\pi x$, $f(x)$, dx as the circumference, height and thickness respectively.

Chapter 7: Techniques of Integration

7.1: Trigonometric Integrals

Strategy for Evaluating $\int sin^m x\ cos^n x\ dx$

(a) If power of cosine is odd (n = 2k+1), you save one cosine factor and use $cos^2(x) = 1 - sin^2(x)$ to express remaining factors in terms of sine:

$$\int sin^m(x)\ cos^{2k+1}(x)\ dx = \int sin^m(x)(1 - sin^2(x))^k cos(x)\ dx \tag{7.1}$$

You can then perform a u-substitution of u = sin(x).

(b) If power of sine odd (m = 2l+1), save one sine factor and use $sin^2(x) = 1 - cos^2(x)$ to express remaining factors in terms of cosine.

$$\int sin^{2l+1}(x) cos^n(x) dx = \int (1 - cos^2(x))^l cos^n(x) sin(x)\ dx \tag{7.2}$$

You can then substitute u = cos(x). [If both powers are odd, then you can make the choice of using sin(x) or cos(x)]

(c) If powers of both sine and cosine are even, use the half-angle identities

$$sin^2(x) = \frac{1}{2}(1 - cos(2x)), \qquad cos^2(x) = \frac{1}{2}(1 + cos(2x)) \tag{7.3}$$

It's also sometimes helpful to employ the following identity:

$$sin(x) cos(x) = \frac{1}{2} sin(2x) \tag{7.4}$$

Strategy for Evaluating $\int tan^m x\ sec^n x\ dx$

(a) If the power of secant is even (n = 2k, $k \geq 2$), save factor of $sec^2(x)$ and use $sec^2(x) = 1 + tan^2(x)$ to express remaining factors in terms of tan(x):

$$\int tan^m(x) sec^{2k}(x) dx = \int tan^m(x)(1 + tan^2(x))^{k-1} sec^2(x) dx \tag{7.5}$$

Then substitute u = tan(x). Reminder that $\frac{d}{dx} tan(x) = sec^2(x)$.

(b) If power of tangent is odd (m = 2k+1), save factor of sec(x)tan(x) and use $tan^2(x) = sec^2(x) - 1$:

$$\int tan^{2k+1}(x) sec^n(x) = \int (sec^2(x) - 1)^k sec^{n-1}(x) sec(x) tan(x)\ dx \tag{7.6}$$

Then substitute u = sec(x). Reminder that $\frac{d}{dx} sec(x) = sec(x) tan(x)$.

Core Trigonometric Integrals

$$\int tan(x)dx = ln|sec(x)|, \qquad \int sec(x)dx = ln|sec(x)+tan(x)|, \qquad \int csc(x)dx = ln|csc(x)-cot(x)| \quad (7.7)$$

Strategy for Evaluating Integrals of sine-cosine products with different arguments

To evaluate integrals such as (a) $\int sin(mx)cos(nx)dx$, (b) $\int sin(mx)sin(nx)dx$, or (c) $\int cos(mx)cos(nx)dx$, we can use the following identities:

$$(a)\ sin(A)cos(B) = \frac{1}{2}\big[sin(A-B) + sin(A+B)\big] \quad (7.8)$$

$$(b)\ sin(A)sin(B) = \frac{1}{2}\big[cos(A-B) - cos(A+B)\big] \quad (7.9)$$

$$(c)\ cos(A)cos(B) = \frac{1}{2}\big[cos(A-B) + cos(A+B)\big] \quad (7.10)$$

7.2: Integration of Rational Functions by Partial Fractions

If we have some rational function of the form $R(x)/Q(x)$, we'd like to reexpress it as a sum of *partial fractions* of the form:

$$\frac{A}{(ax+b)^i}, \qquad or \qquad \frac{Ax+B}{(ax^2+bx+c)^j} \quad (7.11)$$

Rational Function Decomposition

Suppose that we have a rational function of the form $R(x)/Q(x)$.

Case 1: Denominator Q(x) is a product of distinct linear factors
This means that we can write:

$$Q(x) = (a_1x+b_1)(a_2x+b_2)...(a_kx+b_k) \quad (7.12)$$

Thus, we can express our integrand as:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_k}{a_kx+b_k} \quad (7.13)$$

Then solve for the A_j's.

Case 2: Q(x) is product of linear factors, some of which are repeated
Suppose that the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of Q(x). Then we would use:

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + ... + \frac{A_r}{(a_1x+b_1)^r} = \frac{C_1x^{r-1} + C_2x^{r-2} + ... + C_r}{(a_1x+b_1)^r} \quad (7.14)$$

Case 3: Q(x) contains irreducible quadratic factors, none of which is repeated
If Q(x) contains a factor ax^2+bx+c, where $b^2-4ac<0$, then in addition to partial fractions, $R(x)/Q(x)$ will have term of the form:

$$\frac{Ax+B}{ax^2+bx+c} \quad (7.15)$$

Note that we can integrate the above by completing the square, and then using:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\ tan^{-1}\left(\frac{x}{a}\right) \tag{7.16}$$

Case 4: Q(x) contains a repeated irreducible quadratic factor
If Q(x) has factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of a single partial fraction, we'd use the sum:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_rx+B_r}{(ax^2+bx+c)^r} = \frac{C_1x^{2r-1}+C_2x^{2r-2}+..+C_{2r}}{(ax^2+bx+c)^r} \tag{7.17}$$

7.3: Strategy for Integration

Integration Strategy

1. Simplify the Integrand if Possible.
Algebraic manipulations and trig identities can reduce an integrals seemingly complexity.

2. Look for an obvious substitution.
Try to find some function $u = g(x)$ in integrand whose differential $du = g'(x)dx$ also occurs, apart from some constant factor.

3. Classify Integrand according to its form.
If the integrand is f(x), then we attempt classification.
(a) Trig Functions. If f(x) is product of sin(x), cos(x) or tan(x) and sec(x) or cot(x) and csc(x), you can use standard techniques.

(b) Rational Functions. If f is a rational function, then use techniques from 7.4.

(c)Integration by parts. If f(x) is a product of a polynomial and a transcendental function (such as trig, exponential or logarithmic), then try tabular integration.

(d) Radicals.
(i) If $\sqrt{\pm x^2 \pm a^2}$ occurs, then use trig substitution.
(ii) If $(ax+b)^{1/n}$ occurs, we can use rationalizing substitution $u = (ax+b)^{1/n}$. This also sometimes works for the more general $(g(x))^{1/n}$

4. Try Again.

7.4: Approximate Integration

Midpoint Rule

Let $f : [a,b] \to \mathbb{R}$ be continuous over $[a,b]$. Then, we define it's nth midpoint integral approximation by

$$\int_a^b f(x)dx \approx M_n = \Delta x[f(\bar{x_1}) + f(\bar{x_2}) + ... + f(\bar{x_n})], \tag{7.18}$$

where

$$\Delta x = \frac{b-a}{n}, \qquad \bar{x_i} = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of [x_{i-1}, x_i] \tag{7.19}$$

Trapezoidal Rule

Let $f : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$. Then, we define it's n^{th} trapezoidal integral approximation by

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)], \tag{7.20}$$

where

$$\Delta x = \frac{b-a}{n}, \qquad and\ x_i = a + i\Delta x \tag{7.21}$$

Simpsons Rule

Let $f : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$. Then, we define it's integral approximation through Simpson's rule

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)], \tag{7.22}$$

where n is even and $\Delta x = \frac{b-a}{n}$.

Proposition 7.1: Error Bounds

Suppose that $|f''(x)| \leq K$ for all $x \in [a, b]$. If E_T and E_M are the errors in trapezoidal and midpoint rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \qquad and \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2} \tag{7.23}$$

Suppose that $|f^{(4)}(x)| \leq K$ for all $x \in [a, b]$. Then, the error in simpsons rule, E_S is:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} \tag{7.24}$$

7.5: Improper Integrals

Definition 7.1: Improper Integrals

The first type of improper integrals are those with an unbounded domain, such as:

$$\int_a^\infty f(x)\ dx, \qquad \text{or} \qquad \int_{-\infty}^b f(x)\ dx \tag{7.25}$$

The second type of improper integral are those that are discontinuous on the boundary of integration, i.e. continuous on [a,b) or (a,b] but discontinuous at b or a respectively. In which case, the following integral would be improper:

$$\int_a^b f(x)\ dx \tag{7.26}$$

Theorem 7.1: Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\ dx$ is convergent, then $\int_a^\infty g(x)$ is convergent.

(b) If $\int_a^\infty g(x)\ dx$ is divergent, then $\int_a^\infty f(x)\ dx$ is divergent.

Chapter 8: Further Applications of Integration

8.1: Arc Length

Definition 8.1: Arc Length Formula

f f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$ on $a \leq x \leq b$, is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{8.1}$$

8.2: Area of Surface of Revolution

Definition 8.2: Surface Area

If f is positive and has continuous derivative, we define the surface area of the surface obtained by rotating the curve y =f(x), $a \leq x \leq b$, about the x-axis as:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \tag{8.2}$$

We can express this more compactly as:

$$S = \int 2\pi y\ ds, \qquad S = \int 2\pi x\ ds \tag{8.3}$$

With the second integral being for a rotation about the y-axis.

Chapter 9: Differential Equations

9.1: Modeling with differential equations

Definition 9.1: Logistic Differential Equation

Logistic Differential Equation given by:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \tag{9.1}$$

Equilibrium solutions are values of P that result in $dP/dt = 0$. For the above form of the logistic equation, this occurs for $P = 0$ and $P = M$.

9.2: Separable Equations

Definition 9.2: Separable Equation

*A **separable** equation is a first-order ODE for dy/dx that can be factored into a product of function of x and a function of y:*

$$\frac{dy}{dx} = g(x)f(y) \tag{9.2}$$

*An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally.*

9.3: Linear Equations

Definition 9.3: Linear ODE

first order linear ODE is one that can be placed into the form:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{9.3}$$

*To solve such an equation, we multiply both sides by an **integrating factor** $I(x) = e^{\int P(x)dx}$ and integrate both sides.*

Definition 9.4: Bernoulli ODE

Bernoulli differential equation is one of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{9.4}$$

We can perform the substitution $u = y^{1-n}$ to transform it into a linear equation.

Chapter 10: Parametric Equations and Polar Coordinates

10.1: Curves Defined by Parametric Equations

Definition 10.1: Parameters & Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a ***parameter****) by the equations*

$$x = f(t), \qquad y = g(t) \tag{10.1}$$

These would be called parametric equations with the curve C that is traced out by $\{(x, y) : x = f(t), y = g(t)\}$ *referred to as a parametric curve.*

10.2: Calculus with Parametric Curves

Definition 10.2: Chain Rule for Tangents

Suppose that f and g are differentiable functions and we want to find tangent line at point on curve where y is also differentiable function of x. If $dx/dt \neq 0$*, then we can establish*[a]

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{10.2}$$

[a]This is quite trivial, one can observe that it follows from $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

Theorem 10.1: Arc Length

If a curve C is described by the parametric equations x = f(t), y = g(t), $\alpha \leq t \leq \beta$*, where f' and g' are continuous on* $[\alpha, \beta]$ *and C is traversed exactly once as t increases from* α *to* β*, then the length of C is:*

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{10.3}$$

Theorem 10.2: Surface Area

Similarly, if the curve is given by the parametric equations x = f(t), y = g(t), $\alpha \leq t \leq \beta$*, rotated about x-axis or y-axis, with f', g' being continuous and* $g(t) \geq 0$*, then area of resulting surface is given by:*

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, \qquad or \qquad S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{10.4}$$

10.3: Polar Coordinates

Definition 10.3: Polar Coordinates

$$x = rcos(\theta), \qquad y = rsin(\theta) \tag{10.5}$$

With the following satisfied:

$$r^2 = x^2 + y^2, \qquad tan(\theta) = \frac{y}{x} \tag{10.6}$$

Definition 10.4: Polar Curve

To find the tangent line to a polar curve $r = f(\theta)$, we instead write:

$$x = f(\theta)cos(\theta), \qquad y = f(\theta)sin(\theta) \tag{10.7}$$

and solve for the tangent line just as in 10.2, taking θ to be our parameter.

10.4: Areas and Lengths in Polar Coordinates

Definition 10.5: Polar Area

If $r = f(\theta)$, then the formula for the area A of the polar region R is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta \tag{10.8}$$

Definition 10.6: Polar Arc Length

If we go through the same story in the parameterization of x and y with $r = f(\theta)$, then the length of a curve over $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \tag{10.9}$$

10.5: Conic Sections

Definition 10.7: Parabolas

A parabola would be defined as a set of points in a plane that are equidistant from a fixed point F, called the focus and a fixed line, called the directrix.

An equation of the parabola with focus (0, p) and directrix y = -p is

$$x^2 = 4py \tag{10.10}$$

Definition 10.8: Ellipses

An ellipse is a set of points in a plane, the sum of whose distance from two fixed points F_1 and F_2 is a constant. These two fixed points are called the foci.

(1) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a \geq b > 0 \tag{10.11}$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

(2) The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \qquad a \geq b > 0 \tag{10.12}$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

Definition 10.9: Hyperbolas

A hyperbola is a set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant.

(1) The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{10.13}$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

(2) The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{10.14}$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

Chapter 11: Infinite Sequences and Series

11.1: Sequences

Definition 11.1: Sequence

Formally, a sequence can be defined as a function whose domain is either the set of the natural numbers (for infinite sequences) or the set of the first n natural numbers (for a sequence of finite length n).

Definition 11.2: Limit of a Sequence

A sequence $\{a_n\}$ has the limit L and we write:

$$\lim_{n\to\infty} a_n = L \qquad \textit{or } a_n \to L \ \ \textit{as} \ \ n \to \infty \tag{11.1}$$

If the limit exists, we then say that the sequence converges, otherwise we say that it diverges.

If the sequence has the limit L, then the above holds if for every $\epsilon > 0$ there is a corresponding integer N such that

$$if\ n > N \qquad then \qquad |a_n - L| < \epsilon \tag{11.2}$$

Theorem 11.1

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Def 11.2: $a_n \to \infty$
$\lim_{n\to\infty} a_n = \infty$ *means that for every positive number M there is an integer N such that*

$$if \qquad n > N \qquad then \qquad a_n > M \tag{11.3}$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then the limit laws in 2.3 from $\#1-6$ hold, this time allowing $n > 0$ with $a_n > 0$.

Theorem 11.2

$$If \lim_{n\to\infty} |a_n| = 0 \qquad then \qquad \lim_{n\to\infty} a_n = 0 \tag{11.4}$$

Theorem 11.3

If $\lim_{n\to\infty} |a_n| = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L) \tag{11.5}$$

Definition 11.3: Boundedness

A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \quad \forall n \geq 1 \tag{11.6}$$

It is bounded below if there is a number m such that

$$m \leq a_n \quad \forall n \geq 1 \tag{11.7}$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Theorem 11.4: Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

11.2: Series

Definition 11.4: Convergent/Divergent Series

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i \tag{11.8}$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = s \tag{11.9}$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

Theorem 11.5

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Corollary 11.1: Test for Divergence

If $\lim_{n\to\infty} a_n$ DNE or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 11.6

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are:

$$(i) \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n, \qquad (ii) \sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \qquad (iii) \sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \tag{11.10}$$

11.3: The Integral Test and Estimates of Sums

Theorem 11.7: The Integral Test

Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ *and let* $a_n = f(n)$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent iff the improper integral* $\int_1^{\infty} f(x)dx$ *is convergent. In essence:*

$$(i)\text{If } \int_1^{\infty} f(x)dx \text{ is convergent then} \sum_{n=1}^{\infty} a_n \text{ is convergent.} \tag{11.11}$$

$$(ii)\text{If } \int_1^{\infty} f(x)dx \text{ is divergent, then} \sum_{n=1}^{\infty} a_n \text{ is divergent.} \tag{11.12}$$

Proposition 11.1: p-Series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$.

Proposition 11.2: Remainder Estimate for the Integral Test

Let a be a sequence. Suppose that $f(k) = a_k$, *where f is continuous over* $\mathbb{R}$ *and positive, decreasing function for* $x \geq n$. *Suppose that* $\sum a_n$ *is convergent. If* $R_n = s - s_n$, *then*

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx \tag{11.13}$$

11.4: The Comparison Tests

Proposition 11.3: The Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms.*

(i) If $\sum_{n=1}^{\infty} b_n$ *is convergent and* $a_n \leq b_n$ *for all n, then* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

(ii) If $\sum_{n=1}^{\infty} b_n$ *is divergent and* $a_n \geq b_n$ *for all n, then* $\sum_{n=1}^{\infty} a_n$ *is also divergent.*

Proposition 11.4: The Limit Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. If*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c \tag{11.14}$$

where c is a finite number and $c > 0$, *then either both series converge or both diverge.*

11.5: Alternating Series

Proposition 11.5: Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + ..., \qquad b_n > 0 \tag{11.15}$$

satisfies[a]

$$(i)\ b_{n+1} \leq b_n \ \forall\ n \tag{11.16}$$

$$(ii) \lim_{n\to\infty} b_n = 0 \tag{11.17}$$

Then the series is convergent.

[a]Where we assume that every b_n is finite. If this isn't specified, a counter example such as $b_n = \frac{1}{n-1}$ also satisfies $(i), (ii)$ but diverges at b_1. Property (i) is the statement that the sequence b is monotonically decreasing.

Theorem 11.8: Alternating Series Estimation Theorem

If $s = \sum(-1)^{n-1}b_n$ *is sum of alternating series that satisfies the above requirements for convergence, then*

$$|R_n| = |s - s_n| \leq b_{n+1} \tag{11.18}$$

11.6: Absolute Convergence and Ratio / Root Tests

Definition 11.5: Absolute Convergence

A series $\sum a_n$ *is called absolutely convergent if the series of absolute values* $\sum |a_n|$ *is convergent.*

Definition 11.6: Conditional Convergence

A series $\sum a_n$ *is called conditionally convergent if it is convergent but not absolutely convergent.*

Theorem 11.9

If a series $\sum a_n$ *is absolutely convergent, then it is convergent.*

Proposition 11.6: The Ratio Test

(i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

(ii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ *[or* ∞*], then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, *then the ratio test is inconclusive.*

Proposition 11.7: The Root Test

(i) If $\lim_{n\to\infty}(|a_n|)^{1/n} = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

(ii) If $\lim_{n\to\infty}(|a_n|)^{1/n} = L > 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(iii) If $\lim_{n\to\infty}(|a_n|)^{1/n} = 1$, *then the root test is inconclusive.*

11.7: Strategy for Testing Series

Strategy for Testing Series

1. If series is of the form $\sum 1/n^p$, *then it's a p-series.*

2. If series has form $\sum ar^{n-1}$, *it's a geometric series. Some algebraic manipulation might be first required to bring it into this form.*

3. If series has form similar to p-series or geometric series, then one of the comparison tests should be employed. Particularly, if a_n *is a rational or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Comparison tests apply to series with positive terms but if* $\sum a_n$ *has some negative terms then we can apply comparison test to* $\sum |a_n|$ *for absolute convergence test.*

4. If you can immediately see that $\lim_{n\to\infty} a_n \neq 0$, *then test for divergence should be used.*

5. If series has form $\sum (-1)^n b_n$, *then alternating series test is a good possibility.*

6. Series that involve factorials or other products (such as constant raised to nth power) are often conveniently tested using the ratio test.

7. If a_n *is of the form* $(b_n)^n$, *then root test could be useful.*

8. If $a_n = f(n)$, *where* $\int_1^\infty f(x)dx$ *is easily evaluated, then Integral test is effective.*

11.8: Power Series

Definition 11.7: Power Series

A power series (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n \tag{11.19}$$

where a_n *represents the coefficient of the* n^{th} *term and* c *is a constant.*

Theorem 11.10: Radius of Convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilities:*

(i) The series converges only when x = a.
(ii) The series converges for all x.
(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is referred to as the radius of convergence of the power series.

11.9: Representations of Functions as Power Series

Theorem 11.11: Differentiation and Integration of Power Series

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by:

$$f(x) = c_0 + c_1(x-a) + ... = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{11.20}$$

is differentiable (and therefore continuous) on the interval (a-R, a+R) and

$$(i) f'(x) = c_1 + 2c_2(x-a) + ... = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{11.21}$$

$$(ii) \int f(x)dx = C + c_0(x-a) = c_1(x-a)^2/2 + ... = C + \sum_{n=0} c_n \frac{(x-a)^{n+1}}{n+1} \tag{11.22}$$

The radii of convergence for the power series in (i) and (ii) is also R.

11.10: Taylor and Maclaurin Series

Theorem 11.12

f f(x) = $T_n(x) + R_n(x)$, where T_n is the nth degree Taylor Polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0 \tag{11.23}$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Proposition 11.8: Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M for$ $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad for \qquad |x-a| \leq d \tag{11.24}$$

Chapter 12: Vectors and the Geometry of Space

12.1: The Dot Product

Theorem 12.1

If θ is the angle between the vectors **a** and **b**, *then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|cos(\theta) \tag{12.1}$$

Definition 12.1: Scalar, Vector & Orthogonal Projections

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

The scalar projection of **b** *onto* **a** *is:*

$$comp_a\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \tag{12.2}$$

The vector projection of **b** *onto* **a** *is:*

$$proj_a\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} \tag{12.3}$$

The orthogonal projection of **b** *onto* **a** would be given by:

$$orth_a\mathbf{b} = \mathbf{b} - proj_a\mathbf{b} \tag{12.4}$$

12.2: The Cross Product

Theorem 12.2

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. *If θ is the angle between* **a** and **b** (where $0 \leq \theta \leq \pi$), *then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|sin(\theta). \tag{12.5}$$

The magnitude of the cross product $\mathbf{a} \times \mathbf{b}$ *is equal to area of the parallelogram determined by* **a** and **b**.

Proposition 12.1: Parallelepiped Volume

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. *The volume of the parallelepiped determined by the vectors* **a**, **b**, and **c** *is the magnitude of their scalar triple product:*

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \tag{12.6}$$

12.3: Equations of Lines and Planes

Definition 12.2: Parametric / Symmetric Equations for Line

Suppose that a line, L passes through some point given by the vector $\mathbf{r_0}$, *then the vector equation for the line L is given by*

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v} \tag{12.7}$$

With t being the parameter that varies to trace out the entire line. Here, $\mathbf{v}$ is a parallel vector.

If $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$, then we can establish the following parametric equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct \tag{12.8}$$

The symmetric equations of L would be the following:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{12.9}$$

Definition 12.3: Vector Equation for Line Segment

The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation:

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1}, \qquad 0 \leq t \leq 1 \tag{12.10}$$

Definition 12.4: Skew Lines

Skew Lines are lines that do not intersect one another and are not parallel.

Definition 12.5: Vector / Scalar Equation of a Plane

If given some plane that has some point $\mathbf{r_0}$ and a vector $\mathbf{r}$ that would trace out all points on the plane, then the normal vector $\mathbf{n}$ to the plane gives us the vector equation of the plane:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \tag{12.11}$$

It follows that the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{12.12}$$

If given two lines, one can find the plane traced out by both of them by taking cross product and recognizing it as the normal vector.

Proposition 12.2: Distance from Point to a Plane

One can find the distance, D from a point $P_1(x_1, y_1, z_1)$ to the plane ax + by + cz + d = 0 as:

$$D = \frac{|ax_1 + by + 1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{12.13}$$

Chapter 13: Vector Functions

13.1: Derivatives and Integrals of Vector Functions

Definition 13.1: Derivative of Vector Function / Tangent Vector

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$. The derivative $\mathbf{r}'$ of the vector function $\mathbf{r}$ is defined in much the same way as for real valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{13.1}$$

The vector $\mathbf{r}'$ is called the tangent vector to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$.

Definition 13.2: Unit Tangent Vector

Suppose that $\mathbf{r}'(t) \neq \mathbf{0}$. The unit tangent vector at t is defined by

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{13.2}$$

Theorem 13.1

If $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle$, where f,g and h are differentiable functions, then:

$$\mathbf{r}'(t) = \big\langle f'(t), g'(t), h'(t) \big\rangle \tag{13.3}$$

Definition 13.3: The Definite Integral of Vector Functions

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$. We write $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,[a] where $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous. We can define the definite integral of a continuous vector function $\mathbf{r}(t)$ in much the same way as for real-valued functions. This would be seen as:

$$\int_a^b \mathbf{r}(t)dt = \bigg(\int_a^b f(t)dt\bigg)\hat{\mathbf{i}} + \bigg(\int_a^b g(t)dt\bigg)\hat{\mathbf{j}} + \bigg(\int_a^b h(t)dt\bigg)\hat{\mathbf{k}} \tag{13.4}$$

[a] Where we have defined $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ as the standard Euclidean basis in $\mathbb{R}^3$.

13.2: Arc Length and Curvature

Proposition 13.1: Arc Length of Vector Curve

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$. The length of a space curve defined by the vector equation $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle$ is given by

$$L = \int_a^b \sqrt{\big(f'(t)\big)^2 + \big(g'(t)\big)^2 + \big(h'(t)\big)^2} = \int_a^b |\mathbf{r}'(t)|dt \tag{13.5}$$

Definition 13.4: Smooth Curves

A parameterization $\mathbf{r}(t)$ *is called smooth on an interval* I *if* $\mathbf{r}'$ *is continuous and* $\mathbf{r}'(t) \neq 0$. *A curve is called smooth if it has a smooth parameterization.*

Definition 13.5: Curvature

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be a differentiable function and* $\mathbf{T}$ *denote its unit tangent vector. The curvature of the curve* $\mathbf{r}$ *is defined as*

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{13.6}$$

Definition 13.6: Principle Unit Normal Vector

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be a differentiable function and* $\mathbf{T}$ *denote its unit tangent vector. The principle unit normal vector,* $\boldsymbol{N}(t)$ *(or simply unit normal) is defined as:*

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \tag{13.7}$$

Definition 13.7: Binormal Vector

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be a differentiable function,* $\mathbf{T}$ *denote its unit tangent vector and* $\mathbf{N}$ *its principle unit normal vector. The binormal vector is perpendicular to both* $\boldsymbol{T}$ and $\boldsymbol{N}$, *it is defined as:*

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \tag{13.8}$$

Definition 13.8: Normal Plane

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be a differentiable function,* $\mathbf{T}$ *denote its unit tangent vector,* $\mathbf{N}$ *its principle unit normal vector and* $\mathbf{B}$ *its binormal vector. The plane determined by the normal and binormal vectors* $\boldsymbol{N}$ and $\boldsymbol{B}$ *at a point* P *on a curve* C *is called the normal plane of* C at P. *It would consist of all lines orthogonal to the tangent vector* $\boldsymbol{T}$.

Definition 13.9: Osculating Plane

The plane determined by the vectors $\boldsymbol{T}$ and $\boldsymbol{N}$ *is called the osculating plane of* C at P.

Chapter 14: Partial Derivatives

14.1: Functions of Several Variables

Definition 14.1: Level Curves

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Definition 14.2: Level Surfaces

The level surfaces of a function f of three variables are the curves with equations $f(x, y, z) = k$, where k is a constant (in the range of f).

14.2: Limits and Continuity

Definition 14.3: Limit of Two-Variable Function

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{14.1}$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$If\ (x, y) \in D \qquad and \qquad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \qquad then \qquad |f(x, y) - L| < \epsilon \tag{14.2}$$

Definition 14.4: Limit DNE

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Definition 14.5: Limit of Multivariate Function

Consider the metric space $(\mathbb{R}^n, d)$. Let f be defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that:

$$\text{If } \mathbf{x} \in D \qquad and \qquad 0 < d(\mathbf{x}, \mathbf{a}) < \delta \qquad then \qquad |f(\mathbf{x}) - L| < \epsilon, \tag{14.3}$$

where d is the metric on $\mathbb{R}^n$.

Definition 14.6: Continuity (Multivariate)

*Continuity of f at **a** would be defined if:*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}) \tag{14.4}$$

14.3: Partial Derivatives

Definition 14.7: Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivatives of f are the functions f_x and f_y, defined by:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{14.5}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} \tag{14.6}$$

Theorem 14.1: Clairaut's Theorem

Suppose that f is defined on a disk D that with $(a, b) \in D$. If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{14.7}$$

14.4: Tangent Planes and Linear Approximations

Definition 14.8: Tangent Plane

Suppose that a surface S has equation z = f(x,y), where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. If T_1 and T_2 are two tangent lines to the curves C_1 and C_2 lying on the surface at point P, then the tangent plane to the surface S at point P is defined to be the plane that contains both tangent lines T_1 and T_2.

It follows that if f has continuous partial derivatives, then an equation of the tangent plane to the surface z = f(x,y) at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{14.8}$$

Definition 14.9: Linearization / Linear Approximation

The linear function whose graph is the tangent plane, given by:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{14.9}$$

is called the linearization of f at (a, b), if you replace $L(x, y) \to f(x, y)$, then the RHS of the equation serves as the linear approximation or tangent plane approximation of f at (a,b).

Theorem 14.2: Theorem 14.2

Let $f : \mathbb{R}^2 \to \mathbb{R}$. If the partial derivatives f_x and f_y exist near $(a, b) \in \mathbb{R}^2$ and are continuous at (a, b), then f is differentiable at (a, b).

Definition 14.10: Differential

Given some function $w(\mathbf{x})$ of n variables, with $\mathbf{x} \in R^n$. The differential dw is defined as:

$$dw = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} dx_i \tag{14.10}$$

14.5: The Chain Rule

14.6: Directional Derivatives and the Gradient Vector

Definition 14.11: Gradient

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then, its gradient[a] *is defined as*

$$\nabla f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \hat{\mathbf{e}}_i, \tag{14.11}$$

where $\{\hat{\mathbf{e}}_i\}$ is the standard Euclidean basis in $\mathbb{R}^n$.

[a] If one is interested in Differential Geometry, the gradient is similarly defined for vector valued functions on Riemannian manifolds but is modified by terms coming from the Riemannian metric. One can look at the Wikipedia page for Gradient to find further information.

Definition 14.12: Directional Derivative

Let $f : \mathbb{R}^2 \to \mathbb{R}$. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is defined to be

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{14.12}$$

If we introduce the gradient of f, then we can compactly express the directional derivative as:

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} \tag{14.13}$$

Theorem 14.3: Maximizing Directional Derivative

Suppose that f is differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has same direction as the gradient vector $\nabla f(\mathbf{x})$. The minimum value of the directional derivative occurs when $\mathbf{u}$ has the opposite direction to the gradient vector $\nabla f(\mathbf{x})$.[a]

[a] The popular parameter learning algorithm *Gradient Descent* is based upon the method of steepest descent. It is tasked with finding the global minima of some vector valued function as quickly as possible. The locally optimal choice at each step is to go in the direction of steepest descent (i.e where the directional derivative is minimized). However, better algorithms exist for faster convergence as vanilla Gradient Descent is a greedy algorithm; it makes locally optimal choices that generally are not globally optimal.

Definition 14.13: Tangent Plane to Level Surface

The natural definition of the plane that passes through $P(x_0, y_0, z_0)$ with normal vector $\nabla F(x_0.y_0, z_0)$ is given by:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{14.14}$$

14.7: Maximum and Minimum Values

Definition 14.14: Local / Absolute Max / Min

A function of two variables has a local maximum at (a,b) if $f(x, y) \leq f(a, b)$ when (x,y) is near (a,b). The number f(a,b) would be called the local maximum value. If $f(x, y) \geq f(a, b)$ when (x,y) is near (a,b), then f has a local minimum at (a,b) and f(a,b) is a local minimum value.

*If the above inequalities are satisfied for all points (x,y) in the domain of f, then f has an absolute maximum (or **absolute minimum**) at (a,b).*

Theorem 14.4

Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f has a local maximum or minimum at $(a, b) \in \mathbb{R}^2$ and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proposition 14.1: Second Derivatives Test

Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that the second partial derivatives of f are continuous on a disk with center $(a, b) \in \mathbb{R}^2$, and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [(a, b) is a critical point of f.] Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \tag{14.15}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is neither a local max or local min. The point (a, b) would be referred to as a saddle point.

Theorem 14.5: Extreme Value Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Algorithm for Locating max / mins

Let $f : D \to \mathbb{R}$ where D is a compact domain. The algorithm for finding absolute max and mins for a continuous function on a compact domain is to:

(1) Find values of f at the critical points of f in D.
(2) Find extreme values of f on boundary of D.
(3) The largest and smallest values from steps 1) and 2) are the absolute max and mins respectively.

14.8: Lagrange Multipliers

Definition 14.15: Method of Lagrange Multipliers

Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions. Suppose that extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$. To find the max and min values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, we need to

(a) Find all values of $(x, y, z) \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \tag{14.16}$$

and

$$g(x, y, z) = k \tag{14.17}$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f, the smallest is the minimum value of f.

Definition 14.16: Lagrange Multipliers for Two Constraints

Let $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions. If you have two constraints, given by $g(x, y, z) = k$ and $h(x, y, z) = c$, you can then use the following equation instead:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \tag{14.18}$$

where λ and μ would be the Lagrange multipliers.

Chapter 15: Chapter 15: Multiple Integrals

15.1: Surface Area

Definition 15.1: Surface Area

The area of the surface with equation z=f(x,y), $(x, y) \in D$, where f_x and f_y are continuous is

$$A(S) = \int\int_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA \tag{15.1}$$

15.2: Change of Variables in Multiple Integrals

Definition 15.2: Jacobian

Suppose that the integrand variables are denoted by $\{x, y\}$. If we perform a transformation into a new set of integrand variables $\{u, v\}$ related to $\{x, y\}$ by the transformation $x = g(u, v)$ and $y = h(u, v)$, then the Jacobian determinant is given by

$$det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) = det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{15.2}$$

Proposition 15.1: Change of Variables in a Double Integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type 1 or type 2 plane regions. Suppose also that T is one-to-one, except perhaps on boundary of S. Then

$$\int\int_D f(x,y)dA = \int\int_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \tag{15.3}$$

Chapter 16: Chapter 16: Vector Functions

16.1: Vector Fields

Definition 16.1: Conservative Vector Field

A conservative vector field is a vector field $\boldsymbol{F}$ *that is the gradient of some scalar function. Thus, if there exists a function f, such that* $\mathbf{F} = \nabla f$*, then F is conservative.*

16.2: Line Integrals

Definition 16.2: Line Integral

If we have some plane curve C given by the parametric equations

$$x = x(t), \qquad y = y(t), \qquad a \leq t \leq b \tag{16.1}$$

Then the line integral of f along C is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \tag{16.2}$$

The above integral would be called the line integral with respect to arc length. We can also construct integrals with respect to x and y, depicted as follows:

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt \tag{16.3}$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt \tag{16.4}$$

Note that these integrals follows analogously for plane curves parameterized in three dimensions.

Definition 16.3: Line integral of Vector Field

Let $\boldsymbol{F}$ *be a continuous vector field defined on a smooth curve C given by a vector function* $\mathbf{r}'(t)$*, with* $a \leq t \leq b$*. Then the line integral of F along C is*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt = \int_C \mathbf{F} \cdot \mathbf{T}\ ds. \tag{16.5}$$

If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$*, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz \tag{16.6}$$

16.3: 16.3: The Fundamental Theorem for Line Integrals

Theorem 16.1

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{16.7}$$

Definition 16.4: Path Independence

If we define C_1 and C_2 as two piecewise-smooth curves (called paths) that have same initial point A and terminal point B, then in general, we have:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \tag{16.8}$$

However, if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then for any two paths C_1 and C_2 [same initial and terminal points], we would obtain the same value of the integral.

One consequence of Theorem 16.1 is that line integrals of conservative vector fields are independent of path.

Definition 16.5: Closed Curve

A curve is called closed if its terminal point coincides with its initial point, that is $\mathbf{r}(a) = \mathbf{r}(b)$.

Theorem 16.2

If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D iff $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Definition 16.6: Open Domain

If D is open, it means that for every point P in D there is a disk with center P that lies entirely in D. [D doesn't contain any of its boundary points]

Definition 16.7: Connected

If D is connected, it means that any two points in D can be joined by a path that lies in D.

Theorem 16.3

Suppose that $\boldsymbol{F}$ is vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\boldsymbol{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem 16.4

If $\boldsymbol{F}$(x,y) $= P$(x,y)$\boldsymbol{i}$ + Q(x,y)$\boldsymbol{j}$ is a conservative vector field where P and Q have continuous first-order partial *derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{16.9}$$

Definition 16.8: Simple Curve

A simple curve is a curve that doesn't intersect itself anywhere between its endpoints.

Definition 16.9: Simply Connected

A simply connected region in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.

Theorem 16.5

Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \textit{throughout } D, \tag{16.10}$$

then $\mathbf{F}$ is conservative.

16.4: 16.4: Greens' Theorem

Definition 16.10: Positive Orientation

We define the positive orientation of a simple closed curve C as referring to a single counterclockwise traversal of C.

Theorem 16.6: Green's Theorem

Let $P, Q : \mathbb{R} \to \mathbb{R}$. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{16.11}$$

16.5: Curl and Divergence

Theorem 16.7

If $\boldsymbol{F}$ is a vector field defined on all of $\mathbb{R}^3$, whose component functions have continuous partial derivatives and curl $\boldsymbol{F} = 0$, then $\boldsymbol{F}$ is a conservative vector field.

Definition 16.11: Divergence

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$*. We write* $\mathbf{F} = \sum_{i=1}^{n} F_i \hat{\mathbf{e}}_i$*, where* F_i *denotes the component functions of* $\mathbf{F}$ *with* $\{\hat{\mathbf{e}}_i\}$ *representing the standard Euclidean basis on* $\mathbb{R}^n$*. Then the divergence of* $\mathbf{F}$ *is defined as*

$$div\ \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} \tag{16.12}$$

Theorem 16.8

If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ *is a vector field on* R^3 *and P, Q and R have continuous second-order partial derivatives, then*

$$div\ curl \mathbf{F} = 0 \tag{16.13}$$

Proposition 16.1: Vector form of Green's Theorem

If $\mathbf{n}$ *refers to the normal vector of* $\mathbf{F}$ *then,*

$$\int_C \mathbf{F} \cdot \mathbf{n}\ ds = \int \int_D \nabla \cdot \mathbf{F}(x, y)\ dA \tag{16.14}$$

16.6: Parametric Surfaces and their Areas

Definition 16.12: Parametric Surface / Parametric Equations

The set of all points (x,y,z) in $\mathbb{R}^3$ *such that*

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v) \tag{16.15}$$

with (u,v) varying throughout D, is called the parametric surface *S with these equations referred to as the* parametric equations*.*

Definition 16.13: Grid Curves

Consider the parametric equations $x = x(u, v), y = y(u, v), z = z(u, v)$*. If we set one of these parameters to be a constant, such as* $v = v_0$*, then the resulting curve is referred to as a* grid curve*.*

Definition 16.14: Surfaces of Revolution

Surfaces of revolution can be represented parametrically by considering the surface S obtained by rotating the curve $y = f(x), a \leq x \leq b$ *about the x-axis, where* $f(x) \geq 0$*. If (x,y,z) is a point on S, then*

$$x = x, \qquad y = f(x)cos(\theta), \qquad z = f(x)sin(\theta) \tag{16.16}$$

Definition 16.15: Tangent Planes

We can find the tangent plane to the parametric surface S traced out by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \tag{16.17}$$

at a point P_0 by finding partial derivatives with respect to each of these parameters:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k} \tag{16.18}$$

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k} \tag{16.19}$$

Hence, we are able to construct the equation of the tangent plane by recognizing that $r_u \times r_v$ would be normal to the plane.

Definition 16.16: Surface Area

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \qquad (u,v) \in D \tag{16.20}$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| dA \tag{16.21}$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \tag{16.22}$$

Definition 16.17: Surface Area of the Graph of a Function

Natural parameterization for a surface S defined by z = f(x,y), where $(x,y) \in D$ are:

$$x = x, \qquad y = y, \qquad z = f(x,y) \tag{16.23}$$

16.7: Surface Integrals

Definition 16.18: Surface Integral

Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \qquad (u,v) \in D \tag{16.24}$$

We then define the surface integral of f over the surface S as:

$$\int\int_S f(x,y,z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}*)\Delta S_{ij} = \int\int_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA \tag{16.25}$$

Definition 16.19: Surface Integral of Vector Fields

If $\mathbf{F}$ *is continuous vector field defined on an oriented surface* S *with unit normal vector* $\mathbf{n}$*, then the surface integral of* $\mathbf{F}$ *over* S *is*

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S \mathbf{F} \cdot \mathbf{n}\; dS = \int\int_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\; dA \tag{16.26}$$

This integral is also called the flux of $\mathbf{F}$ *across* S*. The second equality holds of* S *is given by a vector function* $\mathbf{r}(u, v)$*.*

16.8: Stokes Theorem

Theorem 16.9: Stokes Theorem

Let S *be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve* C *with positive orientation. Let* $\mathbf{F}$ *be a vector field whose components have continuous partial derivatives on an open region in* R^3 *that contains* S*. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S curl\; \mathbf{F} \cdot d\mathbf{S} \tag{16.27}$$

16.9: The Divergence Theorem

Theorem 16.10: Divergence Theorem

Let E *be a simple solid region and let* S *be the boundary surface of* E*, given with positive (outward) orientation. Let* $\mathbf{F}$ *be a vector field whose component functions have continuous partial derivatives on an open region that contains* E *Then*

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E div\; \mathbf{F}\; dV \tag{16.28}$$