Compilation and Program Analysis (#2a): Semantics

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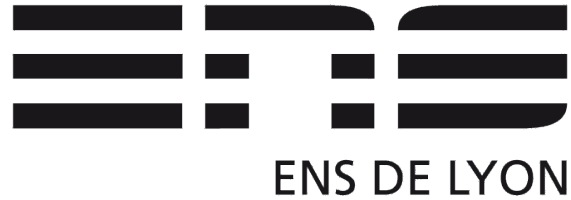

Intro

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Note on organisation:

1: Course

2: **exercises and proofs during the course** ;

3: **exercises and proofs done at the end the course if we have the time**

Semantics

We will first define an abstract syntax for our language.

Example : arithmetic expressions, $x \in V$ a set of variables

$$e ::= x \mid n \mid e + e \mid e * e \mid \ \ldots$$

This is just another view of the AST obtained after parsing.

On the abstract syntax we will define one or several semantics.

Different kinds of semantics:

- axiomatic
- denotational
- by translation
- **operational semantics (natural, structural)**

Axiomatic Semantics (Hoare logic)

(*An axiomatic basis for computer programming*, 1969)

Characterisation by properties on variables, using triples of the form:

$$\{P\}\ i\ \{Q\}$$

"if P is true before the instruction i, then Q is true afterwards"

Example :

$$\{x \geq 0\}\ x := x + 1\ \{x > 0\}$$

Example of generating rule:

$$\{P[x \leftarrow E]\}\ x := E\ \{P(x)\}$$

- proving properties of programs.

Denotational Semantics

Associates to an expression e its mathematical meaning $[\![e]\!]$ that represents its computation in a mathematical domain $\mathcal{D}$.

Example : arithmetic expressions, $x \in V$ a set of variables

$$e ::= x \mid n \mid e + e \mid e * e \mid \ \ldots$$

You must choose a domain for the mathematical meaning with adequate operations.

Trivial example for expressions with $\mathcal{D} = \texttt{env} \to \mathbb{N}$.

$$[\![x]\!]\ \rho = \rho(x)$$
$$[\![n]\!]\ \rho = \mathcal{N}(n)$$
$$[\![e_1 + e_2]\!]\ \rho = [\![e_1]\!]\ \rho + [\![e_2]\!]\ \rho$$
$$[\![e_1 * e_2]\!]\ \rho = [\![e_1]\!]\ \rho \times [\![e_2]\!]\ \rho$$

Semantics by translation

(*Definitional interpreters for higher-order programming languages*, Reynolds, 1972)

We can define the semantics of a language by translation into a language whose semantics is already known.

$$\begin{aligned}[\![x = v + v']\!] = \quad & y = \text{get } v; \\ & z = \text{get } v'; \\ & x = y + z\end{aligned}$$

- Inherit for free the meta-theory from the host language.
- Not always very illuminating in terms of behaviour
- ... but sometimes a good specification in terms of implementation or compilation.

Operational Semantics

Describes the computation as an evaluation from the program to its computed value. Operates directly on the abstract syntax. 2 kinds:

- "natural" or "*big-steps semantics*", evaluates the program in one step

$$e \Downarrow v$$

- "by reduction" or "*small-steps semantics*", repeat the evaluation until a result is obtained:

$$e \to e_1 \to e_2 \to \cdots \to v$$

  A relation describes an atomic reduction, and the semantics consider the transitive (reflexive) closure of this relation.

Results do not need to be a value.

Note: different notations (arrows) exist: $\Downarrow$ / $\Rightarrow$ / ... $\vdash$ ... $\to$ ...

- language specification and proving properties of languages.

mini-while

$$e \in \mathcal{A} ::= x \mid n \mid e + e \mid e * e \mid \ \ldots$$

(abstract) grammar:

$S(Smt)$	$::=$	$x := e$	assign
	$\mid$	$skip$	do nothing
	$\mid$	$S_1; S_2$	sequence
	$\mid$	`if` b `then` S_1 `else` S_2	test
	$\mid$	`while` b `do` S `done`	loop

Semantics of expressions

We denote $State = Var \to \mathbf{Z}$.

This kind of state is sometimes called “store”. We denote them by σ.

Access is denoted $\sigma(x)$. Update is denoted by $\sigma[y \mapsto n]$.

Semantics of arithmetic expressions – Val: $\mathcal{A} \to State \to \mathbf{Z}$ (in each state an integer value): **On board**

$$Val(n, \sigma) = \mathcal{N}(n)$$
$$Val(x, \sigma) =$$
$$Val(e + e', \sigma) =$$
$$Val(e \times e', \sigma) =$$

Note: what kind of semantics is this? big step? denotational?

Semantics of boolean expressions

$Val : \mathcal{B} \to State \to \mathbf{Z}$ **Not now**

$(b ::= tt \mid ff \mid x \mid b \wedge b \mid ... \mid e < e \mid ...)$

First properties and exercise

Semantics of arithmetic expressions

Substitution is denoted $e[e'/x]$.

Show the two following properties (first one at the end of the course):

1. Let $e \in \mathcal{A}$ a given arithmetic expression. Let σ, σ' be two states. Show that if $(\forall x \in Vars(e), \sigma(x) = \sigma'(x))$, then $Val(e, \sigma) = Val(e, \sigma')$. **At the end of course?**
2. Let $e, e' \in \mathcal{A}$, show that:

$$Val(e[e'/x], \sigma) = Val(e, \sigma[x \mapsto Val(e', \sigma)])$$

now

Natural semantics (big step) for mini-while 1/2

In one step from the source program to the final result.

$\Downarrow : Stm \times State \to State$

$$(x := e, \sigma) \Downarrow \sigma[x \mapsto Val(e, \sigma)]$$

$$(\texttt{skip}, \sigma) \Downarrow \sigma$$

$$\frac{(S_1, \sigma) \Downarrow \sigma' \qquad (S_2, \sigma') \Downarrow \sigma''}{\big((S_1; S_2), \sigma\big) \Downarrow \sigma''}$$

Natural semantics (big step) for mini-while 2/2

$$\frac{Val(b,\sigma) = tt \qquad (S_1,\sigma) \Downarrow \sigma'}{(\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2,\sigma) \Downarrow \sigma'}$$

$$\frac{Val(b,\sigma) = ff \qquad (S_2,\sigma) \Downarrow \sigma'}{(\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2,\sigma) \Downarrow \sigma'}$$

$$\frac{Val(b,\sigma) = tt \qquad (S,\sigma) \Downarrow \sigma' \qquad (\texttt{while } b \texttt{ do } S \texttt{ done},\sigma') \Downarrow \sigma''}{(\texttt{while } b \texttt{ do } S \texttt{ done},\sigma) \Downarrow \sigma''}$$

$$\frac{Val(b,\sigma) = ff}{(\texttt{while } b \texttt{ do } S \texttt{ done},\sigma) \Downarrow \sigma}$$

Example

Compute the semantics (leaves are axioms, nodes are rules) of:

- $x := 2; \texttt{while}\ x > 0\ \texttt{do}\ x := x - 1\ \texttt{done}$
- $x := 2; \texttt{while}\ x > 0\ \texttt{do}\ x := x + 1\ \texttt{done}$

Using the semantics to prove properties

Example: determinism

In mini-while there is a single way to evaluate a program.

Theorem: Determinism

For all S, for all $\sigma, \sigma', \sigma''$:

- If $(S, \sigma) \Downarrow \sigma'$ and $(S, \sigma) \Downarrow \sigma''$ then $\sigma' = \sigma''$.
- If $(S, \sigma) \Downarrow \sigma'$, there is no infinite derivation.

The Proof is by induction on the structure of the derivation tree.

WE do a proof sketch

Structural Op. Semantics (SOS = small step) for mini-while 1/2

Evaluating one statement at a time.

$\Rightarrow$: $Stm \times State \to Stm \times State$ OR $Stm \times State \to State$ (we could have a **done** statement to avoid the two cases).

$$(x := e, \sigma) \to \sigma[x \mapsto Val(e, \sigma)] \qquad (\texttt{skip}, \sigma) \to \sigma$$

$$\frac{(S_1, \sigma) \to \sigma'}{\big((S_1; S_2), \sigma\big) \to (S_2, \sigma')} \qquad \frac{(S_1, \sigma) \to (S_1', \sigma')}{\big((S_1; S_2), \sigma\big) \to (S_1'; S_2, \sigma')}$$

Structural Op. Semantics (SOS = small step) for mini-while 2/2

$$\frac{Val(b,\sigma) = tt}{(\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2, \sigma) \to (S_1, \sigma)}$$

$$\frac{Val(b,\sigma) = ff}{(\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2, \sigma) \to (S_2, \sigma)}$$

$$(\texttt{while } b \texttt{ do } S \texttt{ done}, \sigma) \to$$
$$(\texttt{if } b \texttt{ then } (S; \texttt{while } b \texttt{ do } S \texttt{ done}) \texttt{ else skip}, \sigma)$$

Exercises

Compute the semantics of:

- $x := 2; \texttt{while}\ x > 0\ \texttt{do}\ x := x - 1\ \texttt{done}$
- $x := 2; \texttt{while}\ x > 0\ \texttt{do}\ x := x + 1\ \texttt{done}$

How to prove determinism for the SOS semantics? What is the structure of the proof? do the proof

Comparison: divergence

In general a program diverges if it runs forever.

In mini-while, a program diverges in state σ iff:

- NAT: no successor to (S, σ).
- SOS: infinite sequence begining with (S, σ).

In other languages/semantics there might be other reasons to have no successor (see later in course), and you could have no successor in the SOS without reaching a final state.

Comparison: equivalence of programs

Semantics is also useful for defining program equivalence, in mini-while it is quite simple:

Two mini-while programs S_1 and S_2 are semantically equivalent if:

- NAT: $\forall \sigma, \sigma', (S_1, \sigma) \Downarrow \sigma'$ iff $(S_2, \sigma) \Downarrow \sigma'$
- SOS: $\forall \sigma$:
  - for all config (blocking or not): $(S_1, \sigma) \rightarrow^* \sigma'$ iff $(S_2, \sigma) \rightarrow^* \sigma'$
  - (S_1, σ) diverges iff (S_2, σ) diverges

Are the two semantics equivalent?

$$\mathcal{S}_{NS}[S]\sigma = \begin{cases} \sigma' & \text{If } (S,\sigma) \Downarrow \sigma' \\ undef & \text{else} \end{cases}$$

$$\mathcal{S}_{SOS}[S]\sigma = \begin{cases} \sigma' & \text{If } (S,\sigma) \Rightarrow^* \sigma' \\ undef & \text{else} \end{cases}$$

Theorem

$$\mathcal{S}_{NS} = \mathcal{S}_{SOS}$$

Proof: see next slides ...

Equivalence of semantics 1/2

Proposition

If $(S, \sigma) \Downarrow \sigma'$ then $(S, \sigma) \rightarrow^* \sigma'$.

Proof relies on:

Lemma

If $(S_1, \sigma) \rightarrow^k \sigma'$ then $((S_1; S_2), \sigma) \rightarrow^k (S_2, \sigma')$

Proof: structural induction on the derivation tree for $(S, \sigma) \Downarrow$.

Equivalence of semantics 2/2

Proposition

If $(S, \sigma) \rightarrow^k \sigma'$ then $(S, \sigma) \Downarrow \sigma'$.

Proof relies on:

Lemma

If $(S_1; S_2, \sigma) \rightarrow^k \sigma'')$ then there exists σ', k_1 such that $(S_1, \sigma) \rightarrow^{k_1} \sigma'$ and $(S_2, \sigma') \rightarrow^{k-k_1} \sigma''$

Proof: induction on k.

Expressing parallelism

SOS can express interleaving, NAT cannot:

$$\frac{(S_1, \sigma) \to (S_1', \sigma')}{\big((S_1 || S_2), \sigma\big) \to (S_1' || S_2, \sigma')} \qquad \frac{(S_2, \sigma) \to (S_2', \sigma')}{\big((S_1 || S_2), \sigma\big) \to (S_1 || S_2', \sigma')}$$

... more later in the course.

Mini-while is not exactly mini-C

variable initialisation!

- **variable declarations**
  - Main problem is scope of variables (x may not refer to the same variable depending on the point in the program)
  - see course on typing
- Expression **evaluation**

  restricted to expressions without side-effect, the val function has to be encoded as a set of instructions (a more precise semantics would define several reduction steps)
- **print-int and print-string** (operational semantics not much interesting)
- Mini-C will have **functions** ... defined later in the course

Conclusion

We have seen different kinds of semantics and compared them briefly.

We have shown how to define operational semantics.

- For expression evaluation
- On mini-while

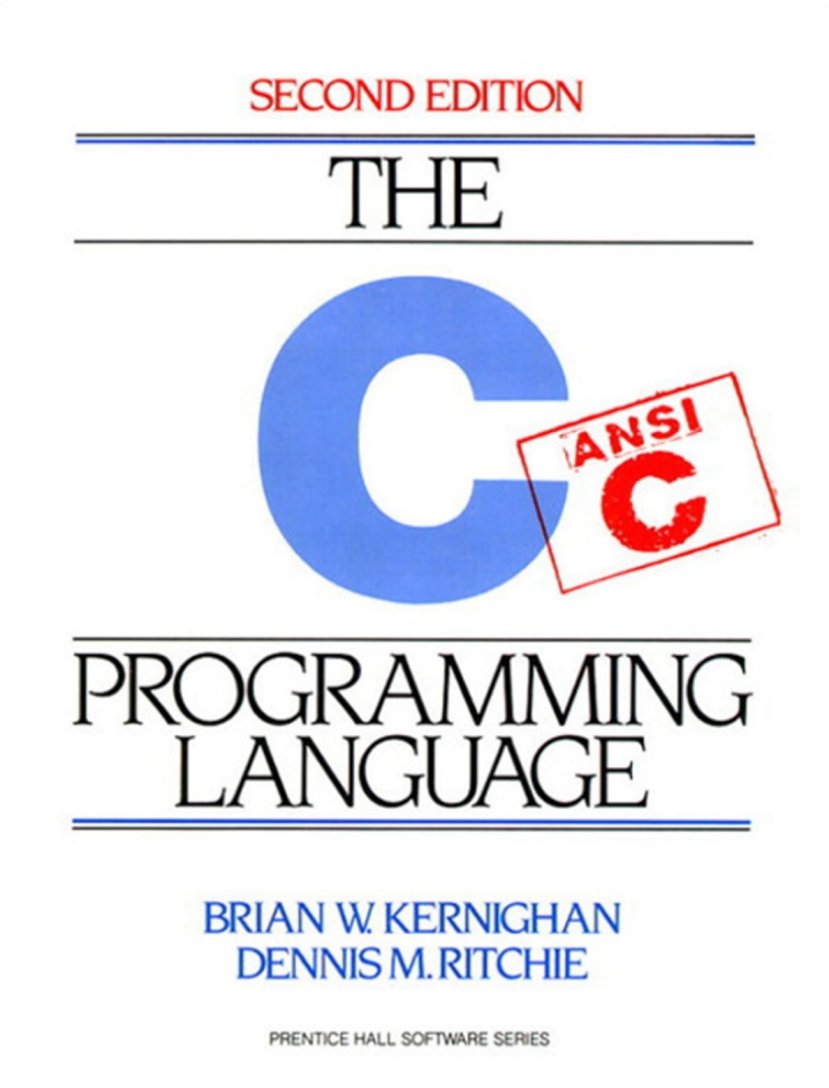
And how to reason on them to derive language properties (or at least properties of the semantics).

Next course on typing will illustrate more properties.

Additional exercise: **repeat**.

Final words: Different degrees of precision

Semi-formal specification in natural language



Final words: Different degrees of precision

Formal semantics

$$
\begin{array}{lllll}
\text{(store)} & s & ::= & \{\text{inst } inst^*,\ \text{tab } tabinst^*,\ \text{mem } meminst^*\} & \\
\text{(instances)} & inst & ::= & \{\text{func } cl^*,\ \text{glob } v^*,\ \text{tab } i^?,\ \text{mem } i^?\} & \\
 & tabinst & ::= & cl^* & \\
 & meminst & ::= & b^* & \\
\text{(closures)} & cl & ::= & \{\text{inst } i,\ \text{code } f\} & \text{(where } f \text{ is not an import and has all exports } ex^* \text{ erased)} \\
\text{(values)} & v & ::= & t.\text{const } c & \\
\text{(administrative operators)} & e & ::= & \ldots \mid \textbf{trap} \mid \textbf{call}\ cl \mid \textbf{label}_n\{e^*\}\ e^*\ \textbf{end} \mid \textbf{local}_n\{i; v^*\}\ e^*\ \textbf{end} & \\
\text{(local contexts)} & L^0 & ::= & v^*\ [.]\ e^* & \\
 & L^{k+1} & ::= & v^*\ \textbf{label}_n\{e^*\}\ L^k\ \textbf{end}\ e^* &
\end{array}
$$

Reduction

$$\frac{s; v^*; e^* \hookrightarrow_i s'; v'^*; e'^*}{s; v^*; L^k[e^*] \hookrightarrow_i s'; v'^*; L^k[e'^*]} \qquad \frac{s; v^*; e^* \hookrightarrow_i s'; v'^*; e'^*}{s; v_0^*; \textbf{local}_n\{i; v^*\}\ e^*\ \textbf{end} \hookrightarrow_j s'; v_0^*; \textbf{local}_n\{i; v'^*\}\ e'^*\ \textbf{end}} \qquad \boxed{s; v^*; e^* \hookrightarrow_i s; v^*; e^*}$$

$$
\begin{array}{rcll}
L^0[\textbf{trap}] & \hookrightarrow & \textbf{trap} & \text{if } L^0 \neq [.] \\
(t.\text{const } c)\ t.unop & \hookrightarrow & t.\text{const } unop_t(c) & \\
(t.\text{const } c_1)\ (t.\text{const } c_2)\ t.binop & \hookrightarrow & t.\text{const } c & \text{if } c = binop_t(c_1, c_2) \\
(t.\text{const } c_1)\ (t.\text{const } c_2)\ t.binop & \hookrightarrow & \textbf{trap} & \text{otherwise} \\
(t.\text{const } c)\ t.testop & \hookrightarrow & \textbf{i32.const } testop_t(c) & \\
(t.\text{const } c_1)\ (t.\text{const } c_2)\ t.relop & \hookrightarrow & \textbf{i32.const } relop_t(c_1, c_2) & \\
(t_1.\text{const } c)\ t_2.\textbf{convert } t_1\_sx^? & \hookrightarrow & t_2.\text{const } c' & \text{if } c' = \text{cvt}^{sx^?}_{t_1,t_2}(c) \\
(t_1.\text{const } c)\ t_2.\textbf{convert } t_1\_sx^? & \hookrightarrow & \textbf{trap} & \text{otherwise} \\
(t_1.\text{const } c)\ t_2.\textbf{reinterpret } t_1 & \hookrightarrow & t_2.\text{const } \text{const}_{t_2}(\text{bits}_{t_1}(c)) & \\
\textbf{unreachable} & \hookrightarrow & \textbf{trap} & \\
\textbf{nop} & \hookrightarrow & \epsilon & \\
v\ \textbf{drop} & \hookrightarrow & \epsilon & \\
v_1\ v_2\ (\textbf{i32.const } 0)\ \textbf{select} & \hookrightarrow & v_2 & \\
v_1\ v_2\ (\textbf{i32.const } k+1)\ \textbf{select} & \hookrightarrow & v_1 & \\
v^n\ \textbf{block}\ (t_1^n \to t_2^m)\ e^*\ \textbf{end} & \hookrightarrow & \textbf{label}_m\{\epsilon\}\ v^n\ e^*\ \textbf{end} & \\
v^n\ \textbf{loop}\ (t_1^n \to t_2^m)\ e^*\ \textbf{end} & \hookrightarrow & \textbf{label}_n\{\textbf{loop}\ (t_1^n \to t_2^m)\ e^*\ \textbf{end}\}\ v^n\ e^*\ \textbf{end} & \\
(\textbf{i32.const } 0)\ \textbf{if } tf\ e_1^*\ \textbf{else } e_2^*\ \textbf{end} & \hookrightarrow & \textbf{block } tf\ e_2^*\ \textbf{end} & \\
(\textbf{i32.const } k+1)\ \textbf{if } tf\ e_1^*\ \textbf{else } e_2^*\ \textbf{end} & \hookrightarrow & \textbf{block } tf\ e_1^*\ \textbf{end} & \\
\textbf{label}_n\{e^*\}\ v^*\ \textbf{end} & \hookrightarrow & v^* & \\
\textbf{label}_n\{e^*\}\ \textbf{trap end} & \hookrightarrow & \textbf{trap} & \\
\textbf{label}_n\{e^*\}\ L^j[v^n\ (\textbf{br } j)]\ \textbf{end} & \hookrightarrow & v^n\ e^* & \\
(\textbf{i32.const } 0)\ (\textbf{br\_if } j) & \hookrightarrow & \epsilon & \\
(\textbf{i32.const } k+1)\ (\textbf{br\_if } j) & \hookrightarrow & \textbf{br } j & \\
(\textbf{i32.const } k)\ (\textbf{br\_table } j_1^k\ j\ j_2^*) & \hookrightarrow & \textbf{br } j & \\
(\textbf{i32.const } k+n)\ (\textbf{br\_table } j_1^k\ j) & \hookrightarrow & \textbf{br } j & \\
s; \textbf{call } j & \hookrightarrow_i & \textbf{call } s_{\text{func}}(i, j) & \\
s; (\textbf{i32.const } j)\ \textbf{call\_indirect } tf & \hookrightarrow_i & \textbf{call } s_{\text{tab}}(i, j) & \text{if } s_{\text{tab}}(i, j)_{\text{code}} = (\textbf{func } tf\ \textbf{local } t^*\ e^*) \\
s; (\textbf{i32.const } j)\ \textbf{call\_indirect } tf & \hookrightarrow_i & \textbf{trap} & \text{otherwise} \\
v^n\ (\textbf{call } cl) & \hookrightarrow & \textbf{local}_m\{cl_{\text{inst}}; v^n\ (t.\text{const } 0)^k\}\ \textbf{block}\ (\epsilon \to t_2^m)\ e^*\ \textbf{end end} & \ldots \\
\textbf{local}_n\{i; v_l^*\}\ v^n\ \textbf{end} & \hookrightarrow & v^n & \mid \ \ldots \text{if } cl_{\text{code}} = (\textbf{func}\ (t_1^n \to t_2^m)\ \textbf{local}\ t^*\ e^*) \\
\textbf{local}_n\{i; v_l^*\}\ \textbf{trap end} & \hookrightarrow & \textbf{trap} & \\
\textbf{local}_n\{i; v_l^*\}\ L^k[v^n\ \textbf{return}]\ \textbf{end} & \hookrightarrow & v^n & \\
v_1^j\ v\ v_2^k; \textbf{get\_local } j & \hookrightarrow & v & \\
v_1^j\ v\ v_2^k; v'\ (\textbf{set\_local } j) & \hookrightarrow & v_1^j\ v'\ v_2^k; \epsilon & \\
v\ (\textbf{tee\_local } j) & \hookrightarrow & v\ v\ (\textbf{set\_local } j) & \\
s; \textbf{get\_global } j & \hookrightarrow_i & s_{\text{glob}}(i, j) & \\
s; v\ (\textbf{set\_global } j) & \hookrightarrow_i & s'; \epsilon & \text{if } s' = s \text{ with } \text{glob}(i, j) = v \\
s; (\textbf{i32.const } k)\ (t.\textbf{load } a\ o) & \hookrightarrow_i & t.\text{const } \text{const}_t(b^*) & \text{if } s_{\text{mem}}(i, k+o, |t|) = b^* \\
s; (\textbf{i32.const } k)\ (t.\textbf{load } tp\_sx\ a\ o) & \hookrightarrow_i & t.\text{const } \text{const}_t^{sx}(b^*) & \text{if } s_{\text{mem}}(i, k+o, |tp|) = b^* \\
s; (\textbf{i32.const } k)\ (t.\textbf{load } tp\_sx^?\ a\ o) & \hookrightarrow_i & \textbf{trap} & \text{otherwise} \\
s; (\textbf{i32.const } k)\ (t.\text{const } c)\ (t.\textbf{store } a\ o) & \hookrightarrow_i & s'; \epsilon & \text{if } s' = s \text{ with } \text{mem}(i, k+o, |t|) = \text{bits}_t^{|t|}(c) \\
s; (\textbf{i32.const } k)\ (t.\text{const } c)\ (t.\textbf{store } tp\ a\ o) & \hookrightarrow_i & s'; \epsilon & \text{if } s' = s \text{ with } \text{mem}(i, k+o, |tp|) = \text{bits}_t^{|tp|}(c) \\
s; (\textbf{i32.const } k)\ (t.\text{const } c)\ (t.\textbf{store } tp^?\ a\ o) & \hookrightarrow_i & \textbf{trap} & \text{otherwise} \\
s; \textbf{current\_memory} & \hookrightarrow_i & \textbf{i32.const } |s_{\text{mem}}(i, *)|/64\,\text{Ki} & \\
s; (\textbf{i32.const } k)\ \textbf{grow\_memory} & \hookrightarrow_i & s'; \textbf{i32.const } |s_{\text{mem}}(i, *)|/64\,\text{Ki} & \text{if } s' = s \text{ with } \text{mem}(i, *) = s_{\text{mem}}(i, *)\ (0)^{k \cdot 64\,\text{Ki}} \\
s; (\textbf{i32.const } k)\ \textbf{grow\_memory} & \hookrightarrow_i & \textbf{i32.const } (-1) &
\end{array}
$$

Figure 2. Small-step reduction rules

Final words: Different degrees of precision

Mechanized formal semantics in a proof assistant

```
(** One step of execution *)

Inductive step: state -> trace -> state -> Prop :=

  | step_skip_seq: forall f s k sp e m,
      step (State f Sskip (Kseq s k) sp e m)
        E0 (State f s k sp e m)
  | step_skip_block: forall f k sp e m,
      step (State f Sskip (Kblock k) sp e m)
        E0 (State f Sskip k sp e m)
  | step_skip_call: forall f k sp e m m',
      is_call_cont k ->
      Mem.free m sp 0 f.(fn_stackspace) = Some m' ->
      step (State f Sskip k (Vptr sp Ptrofs.zero) e m)
        E0 (Returnstate Vundef k m')

  | step_assign: forall f id a k sp e m v,
      eval_expr sp e m a v ->
      step (State f (Sassign id a) k sp e m)
        E0 (State f Sskip k sp (PTree.set id v e) m)

  | step_store: forall f chunk addr a k sp e m vaddr v m',
      eval_expr sp e m addr vaddr ->
      eval_expr sp e m a v ->
      Mem.storev chunk m vaddr v = Some m' ->
      step (State f (Sstore chunk addr a) k sp e m)
        E0 (State f Sskip k sp e m')

  | step_call: forall f optid sig a bl k sp e m vf vargs fd,
      eval_expr sp e m a vf ->
      eval_exprlist sp e m bl vargs ->
      Genv.find_funct ge vf = Some fd ->
      funsig fd = sig ->
      step (State f (Scall optid sig a bl) k sp e m)
```