**Path**

$P ::= X_i \mid P.X$

$\mathcal{P} ::= X_i \mid \mathcal{P}.X \mid \mathcal{P}_1(\mathcal{P}_2) \mid (\mathcal{P} :> \mathcal{M})$

**Module Expressions**

$M ::= P$ (Variables)

$\mid (M : \mathcal{M})$ (Opaque Ascription)

$\mid (M :> \mathcal{M})$ (Transparent Ascription)

$\mid M_1(M_2)$ (Functor application)

$\mid (X_i : \mathcal{M}) \to M$ (Functor)

$\mid \texttt{struct}\ S\ \texttt{end}$ (Structure)

**Structures**

$S ::= \varepsilon \mid D; S$

$D ::= \texttt{let}\ x_i = e$ (Values)

$\mid \texttt{type}\ t = \tau$ (Types)

$\mid \texttt{module}\ X_i = M$ (Modules)

$\mid \texttt{module type}\ \mathcal{X}_i = \mathcal{M}$ (Module types)

**Core language**

$e ::= P.x$ (Qualified variable)

$\mid \ldots$ (Other expressions)

$\tau ::= \mathcal{P}.t$ (Qualified type)

$\mid \ldots$ (Other types)

**Module types**

$\mathcal{M}^\circ ::= \mathcal{X}_i \mid \mathcal{P}.\mathcal{X}$ (Variables)

$\mid (= \mathcal{P})$ (Alias)

$\mid \mathcal{M}\ \texttt{with}\ \mathcal{C}$ (Substitution)

$\mid (X_i : \mathcal{M}_1) \to \mathcal{M}_2$ (Functor)

$\mid \texttt{sig}\ \mathcal{S}\ \texttt{end}$ (Signature)

$\mathcal{M} ::= \mathcal{M}/_l\mathcal{P}$ (Strengthening)

$\mid \mathcal{M}_{[\mathcal{C}]}$ (Substution)

$\mid (\mathcal{M} :> \mathcal{M}')$ (Ascription)

$\mid \mathcal{M}^\circ$

**Substitutions**

$\mathcal{C} ::= P.t = \tau$

$\mid P = P'$

**Signatures**

$\mathcal{S} ::= \varepsilon \mid \mathcal{D}; \mathcal{S}$

$\mathcal{D} ::= \texttt{val}\ x_i : \tau$ (Values)

$\mid \texttt{type}\ t = \tau$ (Types)

$\mid \texttt{type}\ t$ (Abstract types)

$\mid \texttt{module}\ X_i : \mathcal{M}$ (Modules)

$\mid \texttt{module type}\ \mathcal{X}_i = \mathcal{M}$ (Module types)

**Environments**

$\Gamma ::= \mathcal{S}$

Figure 1: Module language

Judgements:

| | |
|---|---|
| $\Gamma \blacktriangleright M : \mathcal{M}$ | The module $M$ is of type $\mathcal{M}$ in $\Gamma$. See fig. 5. |
| $\Gamma \blacktriangleright \mathcal{M} <: \mathcal{M}'$ | The module type $\mathcal{M}$ is a subtype of $\mathcal{M}'$ in $\Gamma$. See fig. 6. |
| $\Gamma \vDash \mathcal{M}$ | The module type $\mathcal{M}$ is well-formed in $\Gamma$. See **??**. |

Operations:

| | |
|---|---|
| $\mathcal{M}/_l\mathcal{P}$ | Strenghten $\mathcal{M}$ by $\mathcal{P}$ while ignoring the list of module paths $l$. |
| $\mathcal{M}_{[\mathcal{C}]}$ | Applies the substitution $\mathcal{C}$ in $\mathcal{M}$. |
| $(\mathcal{M} :> \mathcal{M}')$ | Computes the transparent ascription of $\mathcal{M}$ by $\mathcal{M}'$. |
| $\mathrm{force}_\Gamma(\mathcal{M})$ | Forces all the operations in $\mathcal{M}$ until obtaining a simple module type $\mathcal{M}^\circ$. |

Environment accesses:

| | |
|---|---|
| $\Gamma(P)$ | Lookup the module or module type $P$ in the environment (i.e., the signature) $\Gamma$. |
| $\mathrm{resolve}_\Gamma(\mathcal{M}^\circ)$ | Resolve $\mathcal{M}^\circ$ until it's not a path (i.e., either an arrow or a signature). |
| $\mathrm{normalize}_\Gamma(\mathcal{P})$ | Normalizes the path $\mathcal{P}$ in $\Gamma$ |

$$
\begin{aligned}
\mathcal{P}'/_l\mathcal{P} &\rightarrow \mathrm{resolve}_\Gamma(\mathcal{P}')/_l\mathcal{P} \\
\texttt{sig } \mathcal{S} \texttt{ end}/_l\mathcal{P} &\rightarrow \texttt{sig } \mathcal{S}/_l\mathcal{P} \texttt{ end} \\
(\mathcal{M} \texttt{ with } \mathcal{C})/_l\mathcal{P} &\rightarrow (\mathcal{M}/_{l,\mathrm{id}(\mathcal{C})}\mathcal{P}) \texttt{ with } \mathcal{C} \\
((X:\mathcal{M}) \rightarrow \mathcal{M}')/_l\mathcal{P} &\rightarrow (X:\mathcal{M}) \rightarrow \mathcal{M}'/_l\mathcal{P}(X) \\
(=\mathcal{P}')/_l\mathcal{P} &\rightarrow (=\mathcal{P}) \\
(\mathcal{M} :> \mathcal{M}')/_l\mathcal{P} &\rightarrow (\mathcal{M}/_l\mathcal{P} :> \mathcal{M}')
\end{aligned}
$$

$$
\begin{aligned}
\texttt{type } t = \tau; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{type } t = \tau; (\mathcal{S}/_l\mathcal{P}) && \text{when } t \notin l \\
\texttt{type } t; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{type } t; (\mathcal{S}/_l\mathcal{P}) && \text{when } t \in l \\
\texttt{type } t = \tau; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{type } t = \mathcal{P}.t; (\mathcal{S}/_l\mathcal{P}) \\
\texttt{type } t; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{type } t = \mathcal{P}.t; (\mathcal{S}/_l\mathcal{P}) \\
\texttt{module } X : \mathcal{M}; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{module } X : \mathcal{M}/_{\mathrm{chop}(l,X)}\mathcal{P}.X; (\mathcal{S}/_l\mathcal{P}) && \text{when } X \notin l \\
\texttt{module } X : \mathcal{M}; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{module } X : \mathcal{M}; (\mathcal{S}/_l\mathcal{P}) && \text{when } X \in l \\
\texttt{module type } \mathcal{X} = \mathcal{M}; \mathcal{S}/_l\mathcal{P} &\rightarrow \texttt{module type } \mathcal{X} = \mathcal{M}; (\mathcal{S}/_l\mathcal{P})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}/_l\mathcal{P}/_{l'}\mathcal{P}' &\rightarrow \mathcal{M}/_l\mathcal{P} \\
\mathcal{M}_{[\mathcal{C}]}/_l\mathcal{P} &\rightarrow \mathcal{M}_{[\mathcal{C}]}/_l\mathcal{P}
\end{aligned}
$$

Figure 2: Module strengthening operation – $\mathcal{M}/_l\mathcal{P}$

$$
\begin{aligned}
\mathcal{P}_{[\mathcal{C}]} &\rightarrow \mathrm{resolve}_\Gamma(\mathcal{P})_{[\mathcal{C}]} \\
\texttt{sig } \mathcal{S} \texttt{ end}_{[\mathcal{C}]} &\rightarrow \texttt{sig } \mathcal{S}_{[\mathcal{C}]} \texttt{ end} \\
(\mathcal{M} \texttt{ with } \mathcal{C}')_{[\mathcal{C}]} &\rightarrow \mathcal{M}_{[\mathcal{C}']}{}_{[\mathcal{C}]} \\
((X:\mathcal{M}) \rightarrow \mathcal{M}')_{[\mathcal{C}]} &\rightarrow \texttt{fail} \\
(=\mathcal{P})_{[\mathcal{C}]} &\rightarrow (=\mathcal{P}) && \text{when } \mathrm{resolve}_\Gamma(\mathcal{P})_{[\mathcal{C}]} \rightarrow \_ \\
(\mathcal{M} :> \mathcal{M}')_{[\mathcal{C}]} &\rightarrow (\mathcal{M}_{[\mathcal{C}]} :> \mathcal{M}')
\end{aligned}
$$

$$
\begin{aligned}
\texttt{type } t = \tau; \mathcal{S}_{[t=\tau']} &\rightarrow \texttt{type } t = \tau'; \mathcal{S} && \text{when } \Gamma \triangleright \tau' <: \tau \\
\texttt{module } X : \mathcal{M}; \mathcal{S}_{[X=\mathcal{P}]} &\rightarrow \texttt{module } X : (=\mathcal{P}); \mathcal{S} && \text{when } \Gamma \blacktriangleright (=\mathcal{P}) <: \mathcal{M} \\
\texttt{module } X : \mathcal{M}; \mathcal{S}_{[X.P.t=\tau]} &\rightarrow \texttt{module } X : \mathcal{M}_{[P.t=\tau]}; \mathcal{S} \\
\texttt{module } X : \mathcal{M}; \mathcal{S}_{[X.P=\mathcal{P}']} &\rightarrow \texttt{module } X : \mathcal{M}_{[P=\mathcal{P}']}; \mathcal{S} \\
(\mathcal{D}; \mathcal{S})_{[\mathcal{C}]} &\rightarrow \mathcal{D}; \mathcal{S}_{[\mathcal{C}]} && \text{when } id(\mathcal{D}) \text{ is not a prefix of } id(\mathcal{C})
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{M}/_lP)_{[\mathcal{C}]} &\rightarrow (\mathcal{M}/_lP)_{[\mathcal{C}]} \\
\mathcal{M}_{[\mathcal{C}']}{}_{[\mathcal{C}]} &\rightarrow \mathcal{M}_{[\mathcal{C}']}{}_{[\mathcal{C}]}
\end{aligned}
$$

Figure 3: Substitution operation – $\mathcal{M}_{[\mathcal{C}]}$

$$
\begin{aligned}
(\mathcal{M} \mathrel{:>} \mathcal{M}') &\rightarrow (\mathrm{resolve}_\Gamma(\mathcal{M}) \mathrel{:>} \mathrm{resolve}_\Gamma(\mathcal{M}')) & \\
(\texttt{sig}\ \mathcal{D}_1; \ldots; \mathcal{D}_n\ \texttt{end} \mathrel{:>} \texttt{sig}\ \mathcal{D}'_1; \ldots; \mathcal{D}'_m\ \texttt{end}) &\rightarrow \texttt{sig}\ (\mathcal{D}_{\pi(i)} \mathrel{:>} \mathcal{D}_i)\ \texttt{end} & \text{with } \pi : [1; m] \rightarrow [1; n] \\
((\texttt{val}\ x_i : \tau) \mathrel{:>} (\texttt{val}\ x_i : \tau')) &\rightarrow (\texttt{val}\ x_i : \tau) & \text{when } \Gamma \triangleright \tau <: \tau' \\
((\texttt{val}\ x_i : \tau) \mathrel{:>} \varepsilon) &\rightarrow \varepsilon & \\
((\texttt{module}\ X_i : \mathcal{M}) \mathrel{:>} (\texttt{module}\ X_i : \mathcal{M}')) &\rightarrow (\texttt{module}\ X_i : (\mathcal{M} \mathrel{:>} \mathcal{M}')) & \\
((\texttt{module}\ X_i : \mathcal{M}) \mathrel{:>} \varepsilon) &\rightarrow (\texttt{module}\ X_i : (\mathcal{M} \mathrel{:>} \varepsilon)) & \\
((\texttt{type}\ t_i = \tau) \mathrel{:>} (\texttt{type}\ t_i)) &\rightarrow (\texttt{type}\ t_i = \tau) & \\
((\texttt{type}\ t_i = \tau) \mathrel{:>} (\texttt{type}\ t_i = \tau')) &\rightarrow (\texttt{type}\ t_i = \tau) & \text{when } \Gamma \triangleright \tau <: \tau' \\
((\texttt{type}\ t_i = \tau) \mathrel{:>} \varepsilon) &\rightarrow (\texttt{type}\ t_i = \tau) & \\
((\texttt{type}\ t_i) \mathrel{:>} (\texttt{type}\ t_i)) &\rightarrow (\texttt{type}\ t_i) & \\
((\texttt{type}\ t_i) \mathrel{:>} (\texttt{type}\ t_i = \tau')) &\rightarrow (\texttt{type}\ t_i = \tau') & \text{when } \Gamma \triangleright t <: \tau' \\
((\texttt{type}\ t_i) \mathrel{:>} \varepsilon) &\rightarrow (\texttt{type}\ t_i) &
\end{aligned}
$$

Figure 4: Module transparent ascription – $(\mathcal{M} \mathrel{:>} \mathcal{M}')$

$$\textsc{ModVar}\quad \frac{\Gamma(P) = \_}{\Gamma \blacktriangleright P :(= P)} \qquad \frac{\Gamma \blacktriangleright M : \mathcal{M}' \quad \Gamma \blacktriangleright \mathcal{M}' <: \mathcal{M}}{\Gamma \blacktriangleright M : \mathcal{M}} \qquad \frac{\Gamma \vDash \mathcal{M} \quad \Gamma \blacktriangleright M : \mathcal{M}}{\Gamma \blacktriangleright (M : \mathcal{M}) : \mathcal{M}} \qquad \frac{\Gamma \vDash \mathcal{M} \quad \Gamma \blacktriangleright M : \mathcal{M}_i}{\Gamma \blacktriangleright (M :> \mathcal{M}) : (\mathcal{M}_i :> \mathcal{M})}$$

$$\frac{\Gamma \blacktriangleright M_f : (X_i : \mathcal{M}_a) \to \mathcal{M}_r \quad \Gamma \blacktriangleright M_a : \mathcal{M}_a}{\Gamma \blacktriangleright M_f(M_a) : \mathcal{M}_{r[X=(M_a :> \mathcal{M}_a)]}} \qquad \frac{\Gamma \vDash \mathcal{M} \quad X_i \notin \mathrm{BV}(\Gamma) \quad \Gamma; \texttt{module } X_i : \mathcal{M} \blacktriangleright M : \mathcal{M}'}{\Gamma \blacktriangleright (X_i : \mathcal{M}) \to M : (X_i : \mathcal{M}) \to \mathcal{M}'}$$

$$\frac{\Gamma \blacktriangleright S : \mathcal{S}}{\Gamma \blacktriangleright \texttt{struct } S \texttt{ end} : \texttt{sig } \mathcal{S} \texttt{ end}} \qquad \frac{}{\Gamma \blacktriangleright \varepsilon : \varepsilon}$$

$$\frac{\Gamma \triangleright e : \tau \quad x_i \notin \mathrm{BV}(\Gamma) \quad \Gamma; \texttt{val } x_i : \tau \blacktriangleright S : \mathcal{S}}{\Gamma \blacktriangleright (\texttt{let } x_i = e; S) : (\texttt{val } x_i : \tau; \mathcal{S})} \qquad \frac{\Gamma \blacktriangleright M : \mathcal{M} \quad X_i \notin \mathrm{BV}(\Gamma) \quad \Gamma; \texttt{module } X_i : \mathcal{M} \blacktriangleright S : \mathcal{S}}{\Gamma \blacktriangleright (\texttt{module } X_i = M; S) : (\texttt{module } X_i : \mathcal{M}; \mathcal{S})}$$

$$\frac{\Gamma \vDash \tau \quad t_i \notin \mathrm{BV}(\Gamma) \quad \Gamma; \texttt{type } t_i = \tau \blacktriangleright S : \mathcal{S}}{\Gamma \blacktriangleright (\texttt{type } t = \tau; S) : (\texttt{type } t = \tau; \mathcal{S})} \qquad \frac{\Gamma \vDash \mathcal{M} \quad \mathcal{X}_i \notin \mathrm{BV}(\Gamma) \quad \Gamma; \texttt{module type } \mathcal{X}_i = \mathcal{M} \blacktriangleright S : \mathcal{S}}{\Gamma \blacktriangleright (\texttt{module type } \mathcal{X}_i = \mathcal{M}; S) : (\texttt{module type } \mathcal{X}_i = \mathcal{M}; \mathcal{S})}$$

Figure 5: Module typing rules – $\Gamma \blacktriangleright M : \mathcal{M}$

$$\frac{\Gamma \blacktriangleright \mathrm{force}_\Gamma(\mathcal{M}) <: \mathrm{force}_\Gamma(\mathcal{M}')}{\Gamma \blacktriangleright \mathcal{M} <: \mathcal{M}'} \qquad \frac{\mathrm{normalize}_\Gamma(\mathcal{P}) = \mathrm{normalize}_\Gamma(\mathcal{P}')}{\Gamma \blacktriangleright (= \mathcal{P}) <: (= \mathcal{P}')} \qquad \frac{\Gamma \blacktriangleright \mathrm{resolve}_\Gamma(\mathcal{M}) <: \mathrm{resolve}_\Gamma(\mathcal{M}')}{\Gamma \blacktriangleright \mathcal{M} <: \mathcal{M}'}$$

$$\frac{\Gamma \blacktriangleright \mathcal{M}'_a <: \mathcal{M}_a \quad \Gamma, \texttt{module } X : \mathcal{M}'_a \blacktriangleright \mathcal{M}_r <: \mathcal{M}'_r}{\Gamma \blacktriangleright (X : \mathcal{M}_a) \to \mathcal{M}_r <: (X : \mathcal{M}'_a) \to \mathcal{M}'_r}$$

$$\frac{\pi : [1;m] \to [1;n] \quad \forall i \in [1;m],\ \Gamma; \mathcal{D}_1; \ldots; \mathcal{D}_n \blacktriangleright \mathcal{D}_{\pi(i)} <: \mathcal{D}'_i}{\Gamma \blacktriangleright \texttt{sig } \mathcal{D}_1; \ldots; \mathcal{D}_n \texttt{ end} <: \texttt{sig } \mathcal{D}'_1; \ldots; \mathcal{D}'_m \texttt{ end}} \qquad \frac{\Gamma \triangleright \tau_1 <: \tau_2}{\Gamma \blacktriangleright (\texttt{val } x_i : \tau_1) <: (\texttt{val } x_i : \tau_2)}$$

$$\frac{\Gamma \blacktriangleright \mathcal{M}_1 <: \mathcal{M}_2}{\Gamma \blacktriangleright (\texttt{module } X_i : \mathcal{M}_1) <: (\texttt{module } X_i : \mathcal{M}_2)} \qquad \frac{\Gamma \triangleright \tau_1 <: \tau_2}{\Gamma \blacktriangleright (\texttt{type } t_i = \tau_1) <: (\texttt{type } t_i = \tau_2)}$$

$$\frac{}{\Gamma \blacktriangleright (\texttt{type } t_i) <: (\texttt{type } t_i)} \qquad \frac{\Gamma \triangleright t <: \tau}{\Gamma \blacktriangleright (\texttt{type } t_i) <: (\texttt{type } t_i = \tau)} \qquad \frac{}{\Gamma \blacktriangleright (\texttt{type } t_i = \tau_1) <: (\texttt{type } t_i)}$$

Figure 6: Module subtyping rules – $\Gamma \blacktriangleright \mathcal{M} <: \mathcal{M}'$

$$\frac{\texttt{module}\ X_i : \mathcal{M} \in \Gamma}{\Gamma(X_i) = \mathcal{M}} \qquad \frac{\texttt{module type}\ \mathcal{X}_i = \mathcal{M} \in \Gamma}{\Gamma(\mathcal{X}_i) = \mathcal{M}} \qquad \frac{\Gamma(P) = \texttt{sig}\ \mathcal{S}_1; \texttt{module}\ X : \mathcal{M}; \mathcal{S}_2\ \texttt{end}}{\Gamma(P.X) = \mathcal{M}[n_i \mapsto P.n \mid n_i \in \mathrm{BV}(\mathcal{S}_1)]}$$

$$\frac{\Gamma(\mathcal{P}) = \texttt{sig}\ \mathcal{S}_1; \texttt{module type}\ \mathcal{X} = \mathcal{M}; \mathcal{S}_2\ \texttt{end}}{\Gamma(\mathcal{P}.\mathcal{X}) = \mathcal{M}[n_i \mapsto \mathcal{P}.n \mid n_i \in \mathrm{BV}(\mathcal{S}_1)]} \qquad \frac{\Gamma(\mathcal{P}_f) = (X_i : \mathcal{M}_a) \to \mathcal{M}_r \qquad \Gamma(\mathcal{P}_a) = \mathcal{M}_a}{\Gamma(\mathcal{P}_f(\mathcal{P}_a)) = \mathcal{M}_{r[X=(\mathcal{P}_a :> \mathcal{M}_a)]}}$$

$$\frac{\Gamma(\mathcal{P}) = \mathcal{M}}{\Gamma((\mathcal{P} :> \mathcal{M}')) = (\mathcal{M} :> \mathcal{M}')}$$

Figure 7: Lookup rules – $\Gamma(P) = \mathcal{M}$

$$\begin{aligned}
\mathrm{force}_\Gamma(\mathcal{M}/_l\mathcal{P}) &= \mathrm{force}_\Gamma(\mathcal{M}{\color{red}/_l\mathcal{P}}) \\
\mathrm{force}_\Gamma(\mathcal{M}_{[\mathcal{C}]}) &= \mathrm{force}_\Gamma(\mathcal{M}_{\color{red}[\mathcal{C}]}) \\
\mathrm{force}_\Gamma((\mathcal{M} :> \mathcal{M}')) &= \mathrm{force}_\Gamma((\mathcal{M} {\color{red}:>} \mathcal{M}')) \\
\mathrm{force}_\Gamma(\mathcal{M}^\circ) &= \mathcal{M}^\circ
\end{aligned}$$

$$\begin{aligned}
\mathrm{resolve}_\Gamma(\mathcal{P}) &= \mathrm{resolve}_\Gamma(\Gamma(\mathcal{P}))/\mathcal{P} \\
\mathrm{resolve}_\Gamma((= \mathcal{P})) &= \mathrm{resolve}_\Gamma(\Gamma(\mathcal{P}))/\mathcal{P} \\
\mathrm{resolve}_\Gamma(\mathcal{M}^\circ) &= \mathcal{M}^\circ && \text{otherwise}
\end{aligned}$$

$$\mathrm{normalize}_\Gamma(\mathcal{P}) = \begin{cases} \mathrm{normalize}_\Gamma(\mathcal{P}') & \text{when } \Gamma(\mathcal{P}) = (= \mathcal{P}') \\ \mathcal{P} & \text{otherwise} \end{cases}$$