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The necessary condition and sufficient conditions for wavelet frame on local fields ☆

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ABSTRACT

In the paper entitled "Multiresolution analysis on local fields" [H.K. Jiang, D.F. Li, N. Jin, Multiresolution analysis on local fields, J. Math. Anal. Appl. 294 (2) (2004) 523–532], we establish the orthonormal wavelet construction from multiresolution analysis on local fields. The objective of this paper is to construct wavelet frame on local fields. A necessary condition and four sufficient conditions for wavelet frame on local fields are given. An example is presented at the end.

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0. Introduction

One of the principal goals of wavelet theory has been the construction of useful orthonormal bases for $L^2(\mathbb{R}^d)$. The group $\mathbb{R}^d$ has been an appropriate setting both because of its applications and its special property of containing lattices such as $\mathbb{Z}^d$ which induce discrete groups of translation operators on $L^2(\mathbb{R}^d)$. Generally, if $\mathbb{R}^d$ is replaced by a group G with certain properties, many authors have proposed a theory of wavelets on $L^2(G)$, for example, S. Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [1], M. Holschneider gave wavelet analysis over Abelian groups [2], M.C. Lang investigated wavelet analysis and related topics on the Cantor dyadic group [3–5], and so on [6,7].

Recently, R.L. Benedetto et al. considered the constructions of wavelets on some specific groups G which include the p-adic rational group $\mathbb{Q}_p$ and the Cantor dyadic group $\mathbb{F}_2((t))$, and in particular, they showed that Shannon wavelets on G are the same as Haar wavelets on G [8,9], whereas in $L^2(\mathbb{R})$, the Haar and Shannon wavelets are at opposite extremes, in the sense that the Haar wavelet is localized in time but not in frequency, while the Shannon wavelet is localized in frequency but not in time. Our interest is on local fields. It is well known that a local field has the same algebraic structure as the real number field and complex number field. But their topologies are different. Any local field is totally disconnected, and is of the geometrical structure of a tree. The Fourier analysis on local fields has been widely studied [10].

Motivated by the above described work, we established the orthonormal wavelet construction from multiresolution analysis on local field in [11]. In this paper, we turn to investigate wavelet frame on local fields. Frames differ from orthogonal systems in that their elements may be linearly dependent. The concept of orthogonal wavelets on $\mathbb{R}^d$ is restrictive because of the strong geometrical relations. The frame concept allows for redundancy and gives more flexibility for the construction of wavelets, which can be used to incorporate additional features such as symmetries, a high number of vanishing moments,

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a small support, and a high order of smoothness. In the univariate setting, this flexibility was successfully applied in [12,13], and there is successful multivariate construction in [14].

A necessary condition and four sufficient conditions for wavelet frame on local fields are given in this paper. An example given here shows that there are lots of tight wavelet frames which can be easily constructed on local fields on basis of the sufficient conditions we have established.

This paper is organized as follows. Section 1 briefly introduces some notations of local fields needed throughout the paper. A necessary condition for wavelet frame on local fields is established in Section 2. Section 3 is devoted to the discussion of sufficient conditions for wavelet frame on local fields, and four such conditions are given. In the last section, an example is presented.

1. Some notations of local fields

In this section, we list some notations of local fields needed throughout the paper. More details are referred to [10] and [11].

A local field, denoted by K, is an algebraic field and a topological space with the topological properties of locally compact, non-discrete, complete and totally disconnected. The additive and multiplicative groups of K are denoted by K^+ and K^*, respectively. dx is *the normalized Haar measure* on K^+. $|\alpha|$ is the *absolute value* or *valuation* of α in K, which is also a *non-archimedian norm* on K. But $|E|$ is the Haar measure of $E \subset K$. $\mathfrak{p}$ is a fixed prime element of K. We have the fact $|\mathfrak{p}| = q^{-1}$ with that $q = p^c$, p is a prime number and c is a positive integer.

Every $\mathfrak{P}^k = \{x \in K\colon\ |x| \leqslant q^{-k}\}$ is a compact subgroup of K^+. $\mathfrak{O} = \mathfrak{P}^0$ is the *ring of integers* in K. So, $|\mathfrak{O}| = 1$ and $|\mathfrak{P}^k| = q^{-k}$.

χ is a fixed character on K^+ that is trivial on $\mathfrak{O}$ but is non-trivial on $\mathfrak{P}^{-1}$. It follows that χ is constant on cosets of $\mathfrak{O}$ and that if $y \in \mathfrak{P}^k$, then χ_y $(\chi_y(x) = \chi(yx))$ is constant on cosets of $\mathfrak{P}^{-k}$.

Definition 1. If $f \in L^1(K)$, then the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(y) = \int_K f(x)\overline{\chi_y(x)}\,dx = \int_K f(x)\chi(-yx)\,dx. \tag{1}$$

The Fourier transform in $L^p(K)$, $1 < p \leqslant 2$, can be defined similarly as in $L^p(\mathbb{R})$.

The inner product is denoted by

$$\langle f, g\rangle = \int_K f(x)\overline{g(x)}\,dx \quad \text{for } f, g \in L^2(K). \tag{2}$$

The "natural" order on the sequence $\{u(n) \in K\}_{n=0}^{\infty}$ is endowed as follows.

We recall that $\mathfrak{P}$ is the prime ideal in $\mathfrak{O}$, $\mathfrak{O}/\mathfrak{P} \cong \mathrm{GF}(q) = \Gamma$, $q = p^c$, p a prime, c a positive integer and $\rho : \mathfrak{O} \to \Gamma$ the canonical homomorphism of $\mathfrak{O}$ on to Γ. Note that $\Gamma = \mathrm{GF}(q)$ is a c-dimensional vector space over $\mathrm{GF}(p) \subset \Gamma$. We choose a set $\{1 = \varepsilon_0, \ldots, \varepsilon_{c-1}\} \subset \mathfrak{O}^* = \mathfrak{O}\backslash\mathfrak{P}$ such that $\{\rho(\varepsilon_k)\}_{k=0}^{c-1}$ is a basis of $\mathrm{GF}(q)$ over $\mathrm{GF}(p)$.

Definition 2. For n, $0 \leqslant n < q$, $n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}$, $0 \leqslant a_k < p$ and $k = 0, \ldots, c-1$, we define

$$u(n) = (a_0 + a_1\varepsilon_1 + \cdots + a_{c-1}\varepsilon_{c-1})\mathfrak{p}^{-1} \quad (0 \leqslant n < q). \tag{3}$$

For $n = b_0 + b_1 q + \cdots + b_s q^s$, $0 \leqslant b_k < q$, $n \geqslant 0$, we set

$$u(n) = u(b_0) + \mathfrak{p}^{-1}u(b_1) + \cdots + \mathfrak{p}^{-s}u(b_s).$$

Note that for $n, m \geqslant 0$, it is not true that $u(n+m) = u(n) + u(m)$. However, it is true that for all $r, k \geqslant 0$, $u(rq^k) = \mathfrak{p}^{-k}u(r)$, and for $r, k \geqslant 0$, $0 \leqslant t < q^k$, $u(rq^k + t) = u(rq^k) + u(t) = \mathfrak{p}^{-k}u(r) + u(t)$.

Hereafter we will denote $\chi_{u(n)}$ by χ_n $(n \geqslant 0)$. We also denote the test function space on K by $\mathcal{S}$, i.e., each function φ in $\mathcal{S}$ is a finite linear combination of functions of the form $\Phi_k(\cdot - h)$, $h \in K$, $k \in \mathbb{Z}$, where Φ_k is the characteristic function of $\mathfrak{P}^k$. It is clear that $\mathcal{S}$ is dense in $L^p(K)$, $1 \leqslant p < \infty$, and each function in $\mathcal{S}$ is of compact support and so is its Fourier transform. We also often use the following number sets throughout this paper: $\mathbb{Z}$ = the set of all integers; $\mathbb{P} = \{0, 1, 2, \ldots\}$; $\mathbb{N} = \{1, 2, \ldots\}$.

2. A necessary condition of wavelet frame for $L^2(K)$

Let

$$\psi \in L^2(K), \qquad \psi_{j,k}(x) = q^{\frac{j}{2}}\psi\big(\mathfrak{p}^{-j}x - u(k)\big), \quad j \in \mathbb{Z},\ k \in \mathbb{P},$$

we have

$$\widehat{\psi_{j,k}}(\xi) = q^{-\frac{j}{2}}\,\bar{\chi}_k\big(\mathfrak{p}^j\xi\big)\hat{\psi}\big(\mathfrak{p}^j\xi\big). \tag{4}$$

We call the function system $\{\psi_{j,k}(x)\}_{(j,k)\in\mathbb{Z}\times\mathbb{P}}$ a wavelet frame for $L^2(K)$ if there are two constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2 \leqslant B\|f\|^2 \tag{5}$$

hold for all $f \in L^2(K)$.

Remark 1. Since $\mathcal{S}$ is dense in $L^2(K)$ and closed under Fourier transform, the set $\mathcal{S}^0 = \{f \in \mathcal{S}\colon \operatorname{supp}\hat{f} \subset K\backslash\{0\}\}$ is also dense in $L^2(K)$. So it is sufficient to say that $\{\psi_{j,k}(x)\}_{(j,k)\in\mathbb{Z}\times\mathbb{P}}$ is a wavelet frame for $L^2(K)$ if the inequalities in (5) hold for all $f \in \mathcal{S}^0$.

We first prove two lemmas which will be used in the proofs of main results.

Lemma 1. *If $f \in \mathcal{S}$ and $\psi \in L^2(K)$, then*

$$\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2 = \int_K \overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\bigg\{\sum_{l\in\mathbb{P}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\bigg\}\,d\xi. \tag{6}$$

Proof. Applying Parseval equality and (4), we have

$$\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2 = \sum_{k\in\mathbb{P}}q^j\int_K\bigg\{\sum_{l\in\mathbb{P}}\int_{\mathfrak{D}}\hat{f}\big(\mathfrak{p}^{-j}\big(t+u(l)\big)\big)\chi_k\big(t+u(l)\big)\overline{\hat{\psi}\big(t+u(l)\big)}\,dt\bigg\}\overline{\hat{f}\big(\mathfrak{p}^{-j}t\big)}\,\bar{\chi}_k(t)\hat{\psi}(t)\,dt.$$

Since $\sum_{l\in\mathbb{P}}$ contains only finite non-zero terms for $f \in \mathcal{S}$ (notice that $\hat{f}$ has compact support), and $\chi_k(u(l)) \equiv 1$ for all $k, l \in \mathbb{P}$ [11, Proposition 2], the last quantity is equal to

$$\sum_{k\in\mathbb{P}}q^j\int_K\bigg(\int_{\mathfrak{D}}\bigg\{\sum_{l\in\mathbb{P}}\hat{f}\big(\mathfrak{p}^{-j}\big(t+u(l)\big)\big)\overline{\hat{\psi}\big(t+u(l)\big)}\bigg\}\chi_k(t)\,dt\bigg)\bar{\chi}_k(x)\hat{f}\big(\mathfrak{p}^{-j}x\big)\cdot\overline{\hat{\psi}(x)}\,dx.$$

By the convergence theorem of Fourier series on $\mathfrak{D}$, we obtain

$$\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2 = \int_K \overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\bigg\{\sum_{l\in\mathbb{P}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\bigg\}\,d\xi.$$

This is just (6). The proof is complete. □

Lemma 2. *Let f be in $\mathcal{S}^0$ and ψ be in $L^2(K)$. If* $\operatorname{ess\,sup}\{\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\xi)|^2\colon \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\} < +\infty$, *then*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2 = \int_K\big|\hat{f}(\xi)\big|^2\sum_{j\in\mathbb{Z}}\big|\hat{\psi}\big(\mathfrak{p}^j\xi\big)\big|^2\,d\xi + R_\psi(f), \tag{7}$$

where

$$\begin{aligned} R_\psi(f) &= \sum_{j\in\mathbb{Z}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\bigg[\sum_{l\in\mathbb{N}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\bigg]d\xi \\ &= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\,d\xi. \end{aligned} \tag{8}$$

Furthermore, the iterated series in (8) *is absolutely convergent.*

Proof. Firstly, we remark again that, $\sum_{l\in\mathbb{N}}$ in (8) contains only finite non-zero terms for $f \in \mathcal{S}$ mentioned above. Hence

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\bigg[\sum_{l\in\mathbb{N}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\bigg]d\xi \\ &\quad= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\,d\xi. \end{aligned}$$

We claim that

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2=\sum_{j\in\mathbb{Z}}\int_K\left|\hat f(\xi)\right|^2\left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2d\xi+R_\psi(f) \tag{9}$$

hold for all $f\in\mathcal{S}^0$. In fact, by (6), we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2&=\sum_{j\in\mathbb{Z}}\int_K\left\{\left|\hat f(\xi)\right|^2\left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2+\overline{\hat f(\xi)}\cdot\hat\psi\left(\mathfrak{p}^j\xi\right)\cdot\sum_{l\in\mathbb{N}}\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right\}d\xi\\&=\sum_{j\in\mathbb{Z}}\int_K\left|\hat f(\xi)\right|^2\left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2d\xi+R_\psi(f).\end{aligned}$$

This is just (9). Finally, by the condition that $\operatorname{ess\,sup}\{\sum_{j\in\mathbb{Z}}|\hat\psi(\mathfrak{p}^j\xi)|^2\colon\ \xi\in\mathfrak{P}^{-1}\backslash\mathfrak{D}\}<+\infty$ and Levi Lemma, we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2=\int_K\left|\hat f(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2d\xi+R_\psi(f).$$

Thus (7) and (8) are valid.

Now, we prove that the iterated series in (8) is absolutely convergent. For this, we put

$$\begin{aligned}Q&=\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_K\left|\hat f(\xi)\cdot\hat\psi\left(\mathfrak{p}^j\xi\right)\cdot\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\cdot\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)\right|d\xi\\&=\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}q^{-j}\int_K\left|\hat f\left(\mathfrak{p}^{-j}t\right)\cdot\hat f\left(\mathfrak{p}^{-j}\left(t+u(l)\right)\right)\cdot\hat\psi(t)\cdot\hat\psi\left(t+u(l)\right)\right|dt.\end{aligned}$$

Note that

$$\left|\hat\psi(t)\cdot\hat\psi\left(t+u(l)\right)\right|\leqslant\frac{1}{2}\left(\left|\hat\psi(t)\right|^2+\left|\hat\psi\left(t+u(l)\right)\right|^2\right).$$

Hence it suffices to verify that the series

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}q^{-j}\int_K\left|\hat f\left(\mathfrak{p}^{-j}t\right)\cdot\hat f\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right)\right|\cdot\left|\hat\psi(t)\right|^2dt \tag{10}$$

is convergent. Since $u(l)\neq0$ ($l\in\mathbb{N}$) and $f\in\mathcal{S}^0$, there exists a constant $J>0$ such that for all $|j|>J$,

$$\hat f\left(\mathfrak{p}^{-j}t\right)\cdot\hat f\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right)=0.$$

On the other hand, for each fixed $|j|\leqslant J$, there exists a constant L such that for all $l>L$,

$$\hat f\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right)=0.$$

These mean that only finite terms of the iterated series in (10) are non-zero. Consequently, there exists a constant C such that

$$Q\leqslant C\|\hat f\|_\infty^2\|\hat\psi\|_2^2.$$

So the iterated series in (8) is absolutely convergent. The proof of Lemma 2 is complete. □

Now we establish a necessary condition for wavelet frame for $L^2(K)$. Similar to the $L^2(\mathbb{R})$ case in [15], we have

Theorem 1. *If $\{\psi_{j,k}(x)\colon\ j\in\mathbb{Z},\ k\in\mathbb{P}\}$ is a wavelet frame for $L^2(K)$ with bounds A and B, then*

$$A\leqslant S_\psi(\xi)\leqslant B,\quad \text{a.e. }\xi\in K, \tag{11}$$

where $S_\psi(\xi)=\sum_{j\in\mathbb{Z}}|\hat\psi(\mathfrak{p}^j\xi)|^2$.

Proof. For $f\in\mathcal{S}$ and $\psi\in L^2(K)$, by Lemma 1, (6) holds, i.e.,

$$\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2=\int_K\overline{\hat f(\xi)}\cdot\hat\psi\left(\mathfrak{p}^j\xi\right)\left\{\sum_{l\in\mathbb{P}}\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\cdot\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right\}d\xi.$$

Assume E_j is the set of regular points of $|\psi(\mathfrak{p}^j\xi)|^2$, which means that for each $x \in E_j$,

$$q^n \int_{\xi - x \in \mathfrak{P}^n} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 d\xi \to \left|\hat{\psi}\left(\mathfrak{p}^j x\right)\right|^2, \quad \text{as } n \to +\infty.$$

Then $|E_j^c| = 0$ [10, Theorem 1.14]. Thus $|\bigcup_{j\in\mathbb{Z}} E_j^c| = 0$.

Let $\xi_0 \in K \setminus \bigcup_{j\in\mathbb{Z}} E_j^c$. For each fixed positive integer M, set

$$\hat{f}(\xi) = q^{\frac{m}{2}} \Phi_m(\xi - \xi_0), \quad m \geqslant M,$$

where $\Phi_m(\xi - \xi_0)$ is in fact the characteristic function of $\xi_0 + \mathfrak{P}^m$. Then it follows that for $l \in \mathbb{N}$ and $j \geqslant -M$, $\overline{\hat{f}(\xi)}\hat{f}(\xi + \mathfrak{p}^{-j}u(l)) \equiv 0$, since ξ and $\xi + \mathfrak{p}^{-j}u(l)$ cannot be in $\xi_0 + \mathfrak{P}^m$ simultaneously. Obviously $\|f\|_2^2 = 1$. Now we have

$$\sum_{j\geqslant -M} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 = \sum_{j\geqslant -M} \int_{\xi_0 + \mathfrak{P}^m} q^m \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 d\xi \leqslant B.$$

Let $m \to +\infty$ and $M \to +\infty$ consecutively, we have

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi_0\right)\right|^2 \leqslant B.$$

This is the right inequality of (11).

To prove the left inequality of (11), let

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 = I + II,$$

where

$$I = \sum_{j>-M} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \quad \text{and} \quad II = \sum_{j\leqslant -M} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2.$$

Let ξ_0, m, M and $\hat{f}$ be as above. By the definition of the frame, $I \geqslant A - II$. Since we have already obtained that $I = \sum_{j>-M} |\hat{\psi}(\mathfrak{p}^j\xi_0)|^2$, it is enough to prove that $\lim_{M\to+\infty} II = 0$.

By (6) and Schwarz's inequality, we have

$$0 \leqslant II \leqslant \sum_{j\leqslant -M} \sum_{l\in\mathbb{P}} \left\{ \int_K \left|\overline{\hat{f}(\xi)} \cdot \hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 d\xi \right\}^{\frac{1}{2}} \left\{ \int_K \left|\hat{f}\left(\xi + \mathfrak{p}^{-j}u(l)\right) \cdot \overline{\hat{\psi}\left(\mathfrak{p}^j\xi + u(l)\right)}\right|^2 d\xi \right\}^{\frac{1}{2}}.$$

Observe that for fixed $j \leqslant -M$, if $\xi + \mathfrak{p}^{-j}u(l) \in \xi_0 + \mathfrak{P}^m$, then $|\mathfrak{p}^{-j}u(l)| \leqslant q^{-m}$, hence $|u(l)| \leqslant q^{-m-j}$. Consequently, by the hypothesis of $\hat{f}$, the number of the summation index l is bounded by q^{-m-j}. Therefore

$$II \leqslant \sum_{j\leqslant -M} q^{-m-j} \int_K \left|\overline{\hat{f}(\xi)} \cdot \hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 d\xi \leqslant \sum_{j\leqslant -M} \int_{\mathfrak{p}^{-j}\xi_0 + \mathfrak{P}^{-j+m}} \left|\hat{\psi}(\xi)\right|^2 d\xi.$$

Suppose that $\xi_0 \neq 0$. For any given $\varepsilon > 0$, choose M to satisfy the following two inequalities:

$$q^{-M} < |\xi_0| = q^s \quad \text{and} \quad \int_{\mathfrak{P}^{M-s}} \left|\hat{\psi}(\xi)\right|^2 d\xi < \varepsilon.$$

We have then

$$\mathfrak{p}^{-j}\xi_0 + \mathfrak{P}^{-j+m} \subset \mathfrak{P}^{M-s} \quad \text{for all } j \leqslant -M, \tag{12}$$

since $|\mathfrak{p}^{-j}\xi_0| = q^j q^s \leqslant q^{-M} q^s$ and $\mathfrak{P}^{-j+m} \subset \mathfrak{P}^{M-s}$.

On the other hand, for any $j_1 < j_2 \leqslant -M$, we claim that

$$\left\{\mathfrak{p}^{-j_1}\xi_0 + \mathfrak{P}^{-j_1+m}\right\} \cap \left\{\mathfrak{p}^{-j_2}\xi_0 + \mathfrak{P}^{-j_2+m}\right\} = \phi. \tag{13}$$

In fact, for any $x \in \mathfrak{p}^{-j_1}\xi_0 + \mathfrak{P}^{-j_1+m}$ and $y \in \mathfrak{p}^{-j_2}\xi_0 + \mathfrak{P}^{-j_2+m}$, write $x = \mathfrak{p}^{-j_1}\xi_0 + x_1$ and $y = \mathfrak{p}^{-j_2}\xi_0 + y_1$, then $|x - y| = \max\{|\mathfrak{p}^{-j_1}\xi_0 - \mathfrak{p}^{-j_2}\xi_0|, \ |x_1 - y_1|\} = q^{j_2+s} \neq 0$. This shows that (13) holds. Applying (12) and (13) to the last inequality for II, we obtain

$$II \leqslant \int_{\mathfrak{P}^{M-s}} \left|\hat{\psi}(t)\right|^2 dt < \varepsilon.$$

This is what we desired. The proof of Theorem 1 is complete. $\square$

3. Sufficient conditions of wavelet frame for $L^2(K)$

To establish the first sufficient condition of wavelet frame for $L^2(K)$, we put

$$\underline{S}_\psi = \operatorname{ess\,inf}\{S_\psi(\xi):\ \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}, \qquad \overline{S}_\psi = \operatorname{ess\,sup}\{S_\psi(\xi):\ \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\},$$

where $S_\psi(\xi)$ is the same as the one in (11), and set

$$\beta_\psi\big(u(m)\big) = \operatorname{ess\,sup}\left\{\sum_{j\in\mathbb{Z}}\big|t_\psi\big(u(m), \mathfrak{p}^j\xi\big)\big|:\ \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\right\},$$

where $t_\psi(u(m),\xi) = \sum_{k\in\mathbb{P}} \hat{\psi}(\mathfrak{p}^{-k}\xi)\cdot\overline{\hat{\psi}(\mathfrak{p}^{-k}(\xi+u(m))}$. We also use the following two sets:

$$q\mathbb{P} = \{qk:\ k = 0, 1, 2, \ldots\}, \qquad \widetilde{Q} = \{1, 2, \ldots, q-1\}.$$

As on $\mathbb{R}^d$ (see [15] or Section 3.3.2 in [16]), we give the first sufficient condition as follows.

Theorem 2. *Suppose $\psi \in L^2(K)$ such that*

$$A_1(\psi) = \underline{S}_\psi - \sum_{m\in q\mathbb{P}+\widetilde{Q}} \big[\beta_\psi\big(u(m)\big)\cdot\beta_\psi\big(-u(m)\big)\big]^{\frac{1}{2}} > 0,$$

$$B_1(\psi) = \overline{S}_\psi + \sum_{m\in q\mathbb{P}+\widetilde{Q}} \big[\beta_\psi\big(u(m)\big)\cdot\beta_\psi\big(-u(m)\big)\big]^{\frac{1}{2}} < +\infty.$$

Then $\{\psi_{j,k}(x):\ j\in\mathbb{Z},\ k\in\mathbb{P}\}$ is a wavelet frame for $L^2(K)$ with bounds $A_1(\psi)$ and $B_1(\psi)$.

Proof. Let $f \in \mathcal{S}^0$. We estimate $R_\psi(f)$ by means of a decomposition of the set $\{u(l)\}_{l\in\mathbb{N}}$. For given $l\in\mathbb{N}$, there is a unique pair (k,m) with $k\in\mathbb{P}$ and $m\in q\mathbb{P}+\widetilde{Q}$, such that $l = q^k m$. The inverse is obvious, that is, for any pair (k,m) with $k\in\mathbb{P}$ and $m\in q\mathbb{P}+\widetilde{Q}$, $l=q^k m$ is in $\mathbb{N}$. Thus we have that $\{u(l)\}_{l\in\mathbb{N}} = \{\mathfrak{p}^{-k}u(m)\}_{(k,m)\in\mathbb{P}\times\{q\mathbb{P}+\widetilde{Q}\}}$ by Definition 2. Since the last series in (8) is absolutely convergent, we can estimate $R_\psi(f)$ by rearranging the series, changing the orders of summation and integration by Levi Lemma as follows.

$$\begin{aligned}
R_\psi(f) &= \sum_{j\in\mathbb{Z}}\int_K \overline{\hat{f}(\xi)}\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\Big[\sum_{l\in\mathbb{N}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\Big]d\xi\\
&= \sum_{j\in\mathbb{Z}}\int_K \overline{\hat{f}(\xi)}\Big[\sum_{k\in\mathbb{P}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\hat{\psi}\big(\mathfrak{p}^j\xi\big)\hat{f}\big(\xi+\mathfrak{p}^{-j-k}u(m)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+\mathfrak{p}^{-k}u(m)\big)}\Big]d\xi\\
&= \int_K \overline{\hat{f}(\xi)}\Big[\sum_{k\in\mathbb{P}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\sum_{j\in\mathbb{Z}}\hat{\psi}\big(\mathfrak{p}^{j-k}\xi\big)\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\overline{\hat{\psi}\big(\mathfrak{p}^{j-k}\xi+\mathfrak{p}^{-k}u(m)\big)}\Big]d\xi\\
&= \int_K \overline{\hat{f}(\xi)}\Big[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\cdot\sum_{k\in\mathbb{P}}\hat{\psi}\big(\mathfrak{p}^{j-k}\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^{-k}\big(\mathfrak{p}^j\xi+u(m)\big)\big)}\Big]d\xi\\
&= \int_K \overline{\hat{f}(\xi)}\Big[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\cdot t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\Big]d\xi\\
&= \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\int_K \overline{\hat{f}(\xi)}\cdot\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\cdot t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\,d\xi.
\end{aligned}$$

We deduce further that

$$\begin{aligned}
\big|R_\psi(f)\big| &\leqslant \int_K\big|\hat{f}(\xi)\big|\Big[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\big|\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\big|\big|t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\big|\Big]d\xi\\
&= \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\int_K\big(\big|\hat{f}(\xi)\big|\big|t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\big|^{\frac{1}{2}}\big)\big(\big|\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\big|\big|t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\big|^{\frac{1}{2}}\big)\,d\xi\\
&\leqslant \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{P}+\widetilde{Q}}\Big(\int_K\big|\hat{f}(\xi)\big|^2\big|t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\big|\,d\xi\Big)^{\frac{1}{2}}\Big(\int_K\big|\hat{f}\big(\xi+\mathfrak{p}^{-j}u(m)\big)\big|^2\big|t_\psi\big(u(m),\mathfrak{p}^j\xi\big)\big|\,d\xi\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\leqslant \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left\{ \sum_{j\in\mathbb{Z}} \int_K \left|\hat f(\xi)\right|^2 \left|t_\psi\left(u(m), \mathfrak{p}^j\xi\right)\right| d\xi \right\}^{\frac12} \left\{ \sum_{j\in\mathbb{Z}} \int_K \left|\hat f\left(\xi+\mathfrak{p}^{-j}u(m)\right)\right|^2 \left|t_\psi\left(u(m), \mathfrak{p}^j\xi\right)\right| d\xi \right\}^{\frac12}$$
$$= \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left\{ \sum_{j\in\mathbb{Z}} \int_K \left|\hat f(\xi)\right|^2 \left|t_\psi\left(u(m), \mathfrak{p}^j\xi\right)\right| d\xi \right\}^{\frac12} \left\{ \sum_{j\in\mathbb{Z}} \int_K \left|\hat f(\eta)\right|^2 \left|t_\psi\left(-u(m), \mathfrak{p}^j\eta\right)\right| d\eta \right\}^{\frac12}$$
$$\leqslant \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left\{ \int_K \left|\hat f(\xi)\right|^2 \beta_\psi\left(u(m)\right) d\xi \right\}^{\frac12} \left\{ \int_K \left|\hat f(\xi)\right|^2 \beta_\psi\left(-u(m)\right) d\xi \right\}^{\frac12}$$
$$= \int_K \left|\hat f(\xi)\right|^2 d\xi \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left\{\beta_\psi\left(u(m)\right)\cdot\beta_\psi\left(-u(m)\right)\right\}^{\frac12}.$$

Consequently, it follows from the expression (7) in Lemma 2 that

$$\int_K \left|\hat f(\xi)\right|^2 \left\{ \sum_{j\in\mathbb{Z}} \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2 - \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left[\beta_\psi\left(u(m)\right)\cdot\beta_\psi\left(-u(m)\right)\right]^{\frac12} \right\} d\xi \leqslant \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2, \tag{14}$$

and

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \leqslant \int_K \left|\hat f(\xi)\right|^2 \left\{ \sum_{j\in\mathbb{Z}} \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2 + \sum_{m\in q\mathbb{P}+\widetilde{\mathbb{Q}}} \left[\beta_\psi\left(u(m)\right)\cdot\beta_\psi\left(-u(m)\right)\right]^{\frac12} \right\} d\xi. \tag{15}$$

Taking infimum in (14) and supremum in (15), respectively, we obtain that

$$A_1(\psi)\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \leqslant B_1(\psi)\|f\|_2^2$$

hold for all $f \in \mathcal{S}^0$. The proof of Theorem 2 is complete. □

Using the idea in [17] (see also [18–20]), we are able to give the second sufficient condition of wavelet frame for $L^2(K)$ as follows.

Theorem 3. *Suppose $\psi \in L^2(K)$ such that*

$$A_2(\psi) = \operatorname*{ess\,inf}_{\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\right|^2 - \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| \right\} > 0,$$
$$B_2(\psi) = \operatorname*{ess\,sup}_{\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(k)\right)}\right| < +\infty.$$

Then $\{\psi_{j,k}(x)\colon j \in \mathbb{Z}, k \in \mathbb{P}\}$ is a wavelet frame for $L^2(K)$ with bounds $A_2(\psi)$ and $B_2(\psi)$.

Proof. We apply Lemma 2 to estimate $R_\psi(f)$ in (8) for $f \in \mathcal{S}^0$ with another technique. We first deduce that

$$\left|R_\psi(f)\right| = \left| \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \int_K \overline{\hat f(\xi)}\cdot\hat\psi\left(\mathfrak{p}^j\xi\right)\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)} d\xi \right|$$
$$\leqslant \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \left\{ \int_K \left|\hat f(\xi)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi \right\}^{\frac12} \left\{ \int_K \left|\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi \right\}^{\frac12}$$
$$\leqslant \sum_{j\in\mathbb{Z}} \left\{ \sum_{l\in\mathbb{N}} \int_K \left|\hat f(\xi)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi \right\}^{\frac12} \left\{ \sum_{l\in\mathbb{N}} \int_K \left|\hat f\left(\xi+\mathfrak{p}^{-j}u(l)\right)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi \right\}^{\frac12}$$
$$= \sum_{j\in\mathbb{Z}} \left\{ \sum_{l\in\mathbb{N}} \int_K \left|\hat f(\xi)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi \right\}^{\frac12} \left\{ \sum_{l\in\mathbb{N}} \int_K \left|\hat f(\eta)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\eta-u(l)\right)\overline{\hat\psi\left(\mathfrak{p}^j\eta\right)}\right| d\eta \right\}^{\frac12}.$$

Since $\{u(k)\colon k \in \mathbb{N}\} = \{-u(k)\colon k \in \mathbb{N}\}$, we have

$$\sum_{l\in\mathbb{N}} \int_K \left|\hat f(\xi)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\xi\right)\overline{\hat\psi\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi = \sum_{l\in\mathbb{N}} \int_K \left|\hat f(\eta)\right|^2 \left|\hat\psi\left(\mathfrak{p}^j\eta-u(l)\right)\overline{\hat\psi\left(\mathfrak{p}^j\eta\right)}\right| d\eta.$$

Therefore

$$\left|R_\psi(f)\right| \leqslant \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \int_K \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right| d\xi.$$

By Levi Lemma again,

$$\left|R_\psi(f)\right| \leqslant \int_K \left|\hat{f}(\xi)\right|^2 \left\{\sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right|\right\} d\xi. \tag{16}$$

Applying (7), we have

$$\int_K \left|\hat{f}(\xi)\right|^2 \left\{\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2 - \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right|\right\} d\xi \leqslant \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2, \tag{17}$$

and

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \leqslant \int_K \left|\hat{f}(\xi)\right|^2 \left\{\sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right|\right\} d\xi. \tag{18}$$

Taking infimum in (17) and supremum in (18), respectively, we obtain again that

$$A_2(\psi)\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{P}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \leqslant B_2(\psi)\|f\|_2^2$$

hold for all $f \in \mathcal{S}^0$. The proof of Theorem 3 is complete. □

In what follows, we give better bounds for wavelet frame for $L^2(K)$ in comparing with Theorem 3. We first present the concept of q-adic elements in local field K, which is inspired by the concept of a-adic numbers in $\mathbb{R}$ introduced by Chui and Shi in [21] (see also [22]).

An element $u(\alpha) \in K$ is called q-adic element if it is of the form $u(\alpha) = \mathfrak{p}^{-j}u(m)$ for some $j \in \mathbb{Z}$ and $m \in \mathbb{P}$, where $\alpha = \frac{m}{q^j}$ is called a nonnegative q-adic number. We always make q-adic elements in K corresponded to nonnegative q-adic numbers. We also need the following notations.

$$\Lambda^+(q) = \left\{\alpha \in \mathbb{R}\text{: there exist } (j,m) \in \mathbb{Z}\times P \text{ such that } \alpha = \frac{m}{q^j}\right\},$$

and for all $\alpha \in \Lambda^+(q)$,

$$I(\alpha) = \left\{(j,m) \in \mathbb{Z}\times P\text{: } \alpha = \frac{m}{q^j}\right\},$$

$$\Delta_\alpha^+(\xi) = \sum_{(j,m)\in I(\alpha)} \hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(m)\right)},$$

$$\Delta_\alpha^-(\xi) = \sum_{(j,m)\in I(\alpha)} \hat{\psi}\left(\mathfrak{p}^j\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi-u(m)\right)}.$$

It is easy to verify the following facts:

1° $I(0) = \mathbb{Z}\times\{0\}$;
2° For any $\alpha_1, \alpha_2 \in \Lambda^+(q)$, if $u(\alpha_1) \neq u(\alpha_2)$, then $\alpha_1 \neq \alpha_2$;
3° For any $\alpha_1, \alpha_2 \in \Lambda^+(q)$, if $\alpha_1 \neq \alpha_2$, then $I(\alpha_1) \neq I(\alpha_2)$;
4° $\mathbb{Z}\times\mathbb{P} = \bigcup_{\alpha\in\Lambda^+(q)} I(\alpha)$ and $\mathbb{Z}\times\mathbb{N} = \bigcup_{\alpha\in\Lambda^+(q)\setminus\{0\}} I(\alpha)$.

With the notations above we state the following result.

Theorem 4. *Suppose $\psi \in L^2(K)$ such that*

$$A_3(\psi) = \operatorname*{ess\,inf}_{\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}} \left\{\Delta_0^+(\xi) - \sum_{\alpha\in\Lambda^+(q)\setminus\{0\}} \left|\Delta_\alpha^+(\xi)\right|\right\} > 0,$$

$$B_3(\psi) = \operatorname*{ess\,sup}_{\xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}} \sum_{\alpha\in\Lambda^+(q)} \left|\Delta_\alpha^+(\xi)\right| < +\infty.$$

Then $\{\psi_{j,k}(x)\text{: } j \in \mathbb{Z},\ k \in \mathbb{P}\}$ is a wavelet frame for $L^2(K)$ with bounds $A_3(\psi)$ and $B_3(\psi)$.

Proof. We first note that $\Delta_0^+(\xi)=\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\xi)|^2$ by the definition of $\Delta_\alpha^+(\xi)$ and the fact 1° above. Let $f\in\mathcal{S}^0$, and we estimate (8). Note that the series on the right-hand side of (8) is absolutely convergent, hence we can rearrange the series if necessary,

$$\begin{aligned}R_\psi(f)&=\sum_{j\in\mathbb{Z}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{\psi}(\mathfrak{p}^j\xi)\Big[\sum_{l\in\mathbb{N}}\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\Big]d\xi\\&=\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{f}\big(\xi+\mathfrak{p}^{-j}u(l)\big)\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(l)\big)}\,d\xi\\&=\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\ \sum_{(j,k)\in I(\alpha)}\int_K\overline{\hat{f}(\xi)}\cdot\hat{f}\big(\xi+\mathfrak{p}^{-j}u(k)\big)\cdot\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(k)\big)}\,d\xi.\end{aligned}$$

For all $(j,k)\in I(\alpha)$, $\alpha=\frac{k}{q^j}$ represents the same constant number. Therefore $u(\alpha)=\mathfrak{p}^{-j}u(k)$ are the same element in K by the fact 2° above. We have

$$\begin{aligned}R_\psi(f)&=\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{f}\big(\xi+u(\alpha)\big)\cdot\Big[\sum_{(j,k)\in I(\alpha)}\hat{\psi}\big(\mathfrak{p}^j\xi\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(k)\big)}\Big]d\xi\\&=\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\overline{\hat{f}(\xi)}\cdot\hat{f}\big(\xi+u(\alpha)\big)\cdot\Delta_\alpha^+(\xi)\,d\xi.\end{aligned}\tag{19}$$

Consequently, by (19),

$$\begin{aligned}\big|R_\psi(f)\big|&\leqslant\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\Big\{\int_K\big|\hat{f}(\xi)\big|^2\big|\Delta_\alpha^+(\xi)\big|\,d\xi\Big\}^{\frac12}\Big\{\int_K\big|\hat{f}\big(\xi+u(\alpha)\big)\big|^2\big|\Delta_\alpha^+(\xi)\big|\,d\xi\Big\}^{\frac12}\\&\leqslant\Big\{\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\big|\hat{f}(\xi)\big|^2\big|\Delta_\alpha^+(\xi)\big|\,d\xi\Big\}^{\frac12}\Big\{\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\big|\hat{f}\big(\xi+u(\alpha)\big)\big|^2\big|\Delta_\alpha^+(\xi)\big|\,d\xi\Big\}^{\frac12}.\end{aligned}\tag{20}$$

Let $\eta=\xi+u(\alpha)$. We deduce from $u(\alpha)=\mathfrak{p}^{-j}u(k)$ for $(j,k)\in I(\alpha)$ that

$$\Delta_\alpha^+(\xi)=\sum_{(j,m)\in I(\alpha)}\hat{\psi}\big(\mathfrak{p}^j\big(\eta-u(\alpha)\big)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\big(\eta-u(\alpha)\big)+u(k)\big)}=\sum_{(j,m)\in I(\alpha)}\hat{\psi}\big(\mathfrak{p}^j\eta-u(k)\big)\cdot\overline{\hat{\psi}\big(\mathfrak{p}^j\eta\big)}=\overline{\Delta_\alpha^-(\eta)}.$$

Furthermore, since

$$\big\{u(k)\colon\ \alpha\in\Lambda^+(q)\backslash\{0\},\ (j,k)\in I(\alpha)\big\}=\big\{-u(k)\colon\ \alpha\in\Lambda^+(q)\backslash\{0\},\ (j,k)\in I(\alpha)\big\},$$

we have

$$\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\big|\Delta_\alpha^+(\xi)\big|=\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\big|\overline{\Delta_\alpha^-(\xi)}\big|.\tag{21}$$

Replacing $\xi+u(\alpha)$ by η in the last integration of (20), we derive from (20) and (21) that

$$\begin{aligned}\big|R_\psi(f)\big|&\leqslant\Big\{\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\big|\hat{f}(\xi)\big|^2\big|\Delta_\alpha^+(\xi)\big|\,d\xi\Big\}^{\frac12}\Big\{\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\int_K\big|\hat{f}(\eta)\big|^2\big|\overline{\Delta_\alpha^-(\eta)}\big|\,d\eta\Big\}^{\frac12}\\&=\int_K\big|\hat{f}(\xi)\big|^2\Big\{\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\big|\Delta_\alpha^+(\xi)\big|\Big\}d\xi.\end{aligned}\tag{22}$$

So, by (22) and (7), we have

$$\int_K\big|\hat{f}(\xi)\big|^2\Big\{\sum_{j\in\mathbb{Z}}\big|\hat{\psi}\big(\mathfrak{p}^j\xi\big)\big|^2-\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\big|\Delta_\alpha^+(\xi)\big|\Big\}d\xi\leqslant\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2,$$

or, equivalently

$$\int_K\big|\hat{f}(\xi)\big|^2\Big\{\Delta_0^+(\xi)-\sum_{\alpha\in\Lambda^+(q)\backslash\{0\}}\big|\Delta_\alpha^+(\xi)\big|\Big\}d\xi\leqslant\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\big|\langle f,\psi_{j,k}\rangle\big|^2,\tag{23}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2\leqslant\int_K\left|\hat{f}(\xi)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2+\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left|\Delta_\alpha^+(\xi)\right|\right\}d\xi,$$

or, equivalently

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2\leqslant\int_K\left|\hat{f}(\xi)\right|^2\left\{\sum_{\alpha\in\Lambda^+(q)}\left|\Delta_\alpha^+(\xi)\right|\right\}d\xi. \tag{24}$$

Taking infimum in (23) and supremum in (24), respectively, we obtain again that

$$A_3(\psi)\|f\|_2^2\leqslant\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2\leqslant B_3(\psi)\|f\|_2^2$$

hold for all $f\in\mathcal{S}^0$. The proof of Theorem 4 is complete. □

Set

$$\gamma_\alpha^+=\operatorname{ess\,sup}\left\{\left|\Delta_\alpha^+(\xi)\right|:\ \xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}\right\},\qquad\gamma_\alpha^-=\operatorname{ess\,sup}\left\{\left|\Delta_\alpha^-(\xi)\right|:\ \xi\in\mathfrak{P}^{-1}\setminus\mathfrak{D}\right\}.$$

Modifying the proof of Theorem 4, we have

Theorem 5. *Suppose $\psi\in L^2(K)$ such that*

$$A_4(\psi)=\underline{S}_\psi-\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left(\gamma_\alpha^+\gamma_\alpha^-\right)^{\frac12}>0,\qquad B_4(\psi)=\overline{S}_\psi+\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left(\gamma_\alpha^+\gamma_\alpha^-\right)^{\frac12}<+\infty.$$

Then $\{\psi_{j,k}(x):\ j\in\mathbb{Z},\ k\in\mathbb{P}\}$ is a wavelet frame for $L^2(K)$ with bounds $A_4(\psi)$ and $B_4(\psi)$.

Proof. By (19), we have

$$\begin{aligned}\left|R_\psi(f)\right|&\leqslant\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left\{\int_K\left|\hat{f}(\xi)\right|^2\left|\Delta_\alpha^+(\xi)\right|d\xi\right\}^{\frac12}\left\{\int_K\left|\hat{f}\left(\xi+u(\alpha)\right)\right|^2\left|\Delta_\alpha^+(\xi)\right|d\xi\right\}^{\frac12}\\&=\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left\{\int_K\left|\hat{f}(\xi)\right|^2\left|\Delta_\alpha^+(\xi)\right|d\xi\right\}^{\frac12}\left\{\int_K\left|\hat{f}(\eta)\right|^2\left|\Delta_\alpha^-(\eta)\right|d\eta\right\}^{\frac12}\\&\leqslant\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left(\gamma_\alpha^+\gamma_\alpha^-\right)^{\frac12}\int_K\left|\hat{f}(\xi)\right|^2d\xi.\end{aligned}$$

Using similar for (14) and (15), we have

$$\int_K\left|\hat{f}(\xi)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2-\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left(\gamma_\alpha^+\gamma_\alpha^-\right)^{\frac12}\right\}d\xi\leqslant\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2,$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2\leqslant\int_K\left|\hat{f}(\xi)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2+\sum_{\alpha\in\Lambda^+(q)\setminus\{0\}}\left(\gamma_\alpha^+\gamma_\alpha^-\right)^{\frac12}\right\}d\xi.$$

These imply that

$$A_4(\psi)\|f\|_2^2\leqslant\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{P}}\left|\langle f,\psi_{j,k}\rangle\right|^2\leqslant B_4(\psi)\|f\|_2^2$$

hold for all $f\in\mathcal{S}^0$. The proof of Theorem 5 is complete. □

Remark 2. For orthogonal wavelets, an expansion is provided by the Parseval formula. In case of frames, the situation is more sophisticated. Given a wavelet frame, each element of the underlying Hilbert space has a frame expansion by applying the canonical dual frame. However, the canonical dual frame might not have the wavelet structure and it is often very complicated to determine since it requires the inversion of the frame operator. The inversion is void if the frame is tight. The sufficient conditions obtained here yield tight wavelet frame if the lower and upper frame bounds are equal. We have established characterizations of tight wavelet frames on local field which will be elaborated in a forthcoming paper.

Remark 3. Considerable work has already been done on dual wavelet frames on $\mathbb{R}^d$ in [23,24], thus it seems worthwhile to think about sufficient conditions for pairs of dual wavelet frames on local field in a further research.

4. An example of wavelet frame

Let s be a positive integer,

$$\psi_1(x) = \begin{cases} 1, & x \in \mathfrak{D}, \\ 0, & x \bar{\in} \mathfrak{D}, \end{cases} \quad \text{and} \quad \psi_2(x) = \begin{cases} q^{-s}, & x \in \mathfrak{P}^{-s}, \\ 0, & x \bar{\in} \mathfrak{P}^{-s}. \end{cases}$$

Set $\psi(x) = \psi_1(x) - \psi_2(x)$. Since $\hat{\psi}_1(\xi) = \psi_1(\xi)$ and

$$\hat{\psi}_2(\xi) = \begin{cases} 1, & \xi \in \mathfrak{P}^s, \\ 0, & \xi \bar{\in} \mathfrak{P}^s, \end{cases}$$

we have

$$\hat{\psi}(\xi) = \begin{cases} 1, & \xi \in \mathfrak{D} \backslash \mathfrak{P}^s, \\ 0, & \text{otherwise}. \end{cases}$$

Then for $\xi \neq 0$ we can calculate that

$$\sum_{j \in \mathbb{Z}} \left| \hat{\psi}\left(\mathfrak{p}^j \xi\right) \right|^2 = s, \tag{25}$$

and since

$$\sum_{m \in q\mathbb{P} + \widetilde{Q}} \left[\beta_\psi\left(u(m)\right) \cdot \beta_\psi\left(-u(m)\right) \right]^{\frac{1}{2}} = 0, \tag{26}$$

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left| \hat{\psi}\left(\mathfrak{p}^j \xi\right) \overline{\hat{\psi}\left(\mathfrak{p}^j \xi + u(l)\right)} \right| = 0, \tag{27}$$

$$\left| \Delta_\alpha^+(\xi) \right| = 0, \quad \alpha \in \Lambda^+(q) \backslash \{0\}, \tag{28}$$

this ψ satisfies all the theorems mentioned in this paper, and generates a tight wavelet frame with the bounds $A_\psi = B_\psi = s$. This fact shows that we can easily find wavelet frames on local fields.

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References

[1] S. Dahlke, Multiresolution analysis and wavelets on locally compact Abelian groups, in: P.J. Laurent, A. Lemehaute, L.L. Schumaker (Eds.), Wavelets, Images and Surface Fitting, AK Peters, Wellesley, 1994, pp. 141–156.
[2] M. Holschneider, Wavelet analysis over Abelian groups, Appl. Comput. Harmon. Anal. 2 (1995) 52–60.
[3] M.C. Lang, Orthogonal wavelets on the Cantor dyadic group, SIAM J. Math. Anal. 27 (1996) 305–312.
[4] M.C. Lang, Fractal multiwavelets related to the Cantor dyadic group, Int. J. Math. Sci. 21 (1998) 307–314.
[5] M.C. Lang, Wavelet analysis on the Cantor dyadic group, Houston J. Math. 24 (1998) 533–544 and 757–758.
[6] K. Frorns, A. Grossman, M. Holschneider, B. Torreseni, Wavelet on discrete fields, Appl. Comput. Harmon. Anal. 1 (1994) 137–146.
[7] T.P. Lukashenko, Wavelets on topological groups, Izv. Ross. Akad. Nauk Ser. Mat. 58 (3) (1994) 88–102.
[8] R.L. Benedetto, Examples of wavelets for some groups, in: C. Heil, P.E.T. Jorgensen, D.R. Larson (Eds.), Wavelets, Frames and Operator Theory, in: Contemp. Math., vol. 345, 2004, pp. 27–46.
[9] J.J. Benedetto, R.L. Benedetto, Wavelet theory for related groups, J. Geom. Anal. 14 (2004) 424–456.
[10] M.H. Taibleson, Fourier Analysis on Local Fields, Princeton University Press, Princeton, 1975.
[11] H.K. Jiang, D.F. Li, N. Jin, Multiresolution analysis on local fields, J. Math. Anal. Appl. 294 (2) (2004) 523–532.
[12] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.
[13] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based construction of wavelet frames, Appl. Comput. Harmon. Anal. 14 (1) (2003) 1–46.
[14] M. Ehler, On multivariate compactly supported bi-frames, J. Fourier Anal. Appl. 13 (5) (2007) 511–532.
[15] C.K. Chui, X.L. Shi, Inequalities of Littlewood–Paley type for frames and wavelets, SIAM J. Math. Anal. 24 (1993) 263–277.
[16] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.
[17] P.G. Casazza, O. Christensen, Weyl–Heisenberg frames for subspaces of $L^2(R)$, Proc. Amer. Math. Soc. 129 (2001) 145–154.
[18] O. Christensen, Frames, Riesz bases, and discrete Gabor/wavelet expansions, Bull. Amer. Math. Soc. (N.S.) 38 (2001) 273–291.
[19] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[20] P.G. Casazza, The art of frame theory, Taiwanese J. Math. 4 (2000) 129–201.
[21] C.K. Chui, X.L. Shi, Orthonormal wavelets and tight frames with arbitrary real dilations, Appl. Comput. Harmon. Anal. 9 (2000) 243–264.
[22] X.L. Shi, Q.L. Shi, A new criterion of affine frame, Chinese Ann. Math. Ser. A 26 (2005) 257–262.
[23] B. Han, On dual wavelet tight frames, Appl. Comput. Harmon. Anal. 4 (4) (1997) 380–413.
[24] A. Ron, Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$ II: Dual systems, J. Fourier Anal. Appl. 3 (1997) 617–637.