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Olivier Lablée

# Spectral Theory in Riemannian Geometry

European Mathematical Society

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Saint Martin d'Hères, September 2014 Olivier Lablée

# Contents

# Chapter 1
# Introduction to Spectral Geometry

## From P.-S. Laplace to E. Beltrami

The *Laplace operator* was first introduced by P.-S. Laplace (1749–1827) for describing celestial mechanics (the notation $\Delta$ is due to G. Lamé). For example, in our three-dimensional (Euclidean) space the Laplace operator (or just *Laplacian*) is the linear differential operator:

$$\Delta \colon \begin{cases} \mathcal{C}^2(U) \longrightarrow \mathcal{C}^0(U) \\ f \longmapsto \Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \end{cases}$$

where $U$ is an open set of $\mathbb{R}^3$. This operator can be generalized to a Riemannian manifold $(M, g)$: this generalization is called the *Laplace–Beltrami operator* and is denoted by the symbol $\Delta_g$. The study of this operator and in particular the study of its spectrum is called *spectral geometry*. The Laplace–Beltrami operator is very useful in many fields of physics, in particular in all diffusion processes:

- Fluid mechanics: the *Navier–Stokes equations*

$$\begin{cases} \frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u = -\nabla p \\ \mathrm{div}(u) = 0; \end{cases}$$

- potential theory and gravity theory (with Newton potential): the *Laplace equation* and the *Poisson equation*

$$\Delta u = f;$$

- heat diffusion process: the *heat equation*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t);$$

- wave physics: the *wave equation*

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0;$$

- quantum physics: the *Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t}(x,t) = -\Delta \psi(x,t) + V(x)\psi(x,t);$$

- etc. ...

Let us mention that this operator appears also in:

- computer science: in particular, in computer vision (blob detection)
- economy: financial models, *Black–Scholes equations*
- etc. ...

Moreover, spectral geometry is an inter-disciplinary field of mathematics; it involves

- analysis of ODE and PDE
- dynamical systems: classical and quantum completely integrable systems, quantum chaos, geodesic flows and Anosov flows on (negatively curved) manifolds (for example, Ruelle resonances are related to the spectrum of the Laplacian, see the recent articles [Fa-Ts1], [Fa-Ts2]).
- geometry and topology (the main purpose of these notes is to explain this)
- geometric flow: the parabolic behaviour of scalar curvature

$$\frac{\partial R_{g(t)}}{\partial t} = \Delta R_{g(t)} + 2\left|\mathrm{Ric}_{g(t)}\right|^2;$$

  the Ricci flow (see also Sections 7.8.3 and 7.8.4)

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}$$

- probability: the Brownian motion on a Riemannian manifold $(M, g)$ is defined to be a diffusion on $M$ (generator operator is given by $\frac{1}{2}\Delta_g$).
- etc. ...

## Meaning of the Laplacian operator

To get a better feeling about the Laplacian operator, consider for example a one-dimensional $\mathcal{C}^3$ function $u\colon \mathbb{R} \to \mathbb{R}$. The mean value of $u$ on the compact set $[-h, h]$ is given by

$$\overline{u} := \frac{1}{2h}\int_{-h}^{h} u(x)\,dx.$$

Now, using the Taylor expansion of $u$ around the origin we get: for all $x \in [-h, h]$,

$$u(x) = u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6} + o(x^4).$$

Therefore,

$$\overline{u} = \frac{1}{2h}\left(\int_{-h}^{h} u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6}\,dx\right) + o(h^4)$$

i.e., $\overline{u} = u(0) + \frac{u''(0)}{12}h^2 + o(h^4)$, hence $\overline{u} - u(0) = \frac{u''(0)}{12}h^2 + o(h^4)$. In other words,

$$\Delta u(0) = u''(0) = \frac{12}{h^2}\left(\overline{u} - u(0)\right) + o(h^2),$$

thus the Laplacian of $u$ measures the difference between the function $u$ at 0 and the mean value of $u$ on the neighbourhood $[-h, h]$.

Another way to understand this interpretation is to use the *finite difference method*: for example, on the domain $I = [0, 1]$ and for an integer $N > 0$ consider the discretization grid of $I$ given by $\{t_i := ih,\ i \in \{0 \dots N\}\}$ with $h := \frac{1}{N}$. Thus using the Taylor expansion of $u$ around the point $t_i$ we have

$$u(t_{i+1}) = u(t_i + h) = u(t_i) + u'(t_i)h + u''(t_i)\frac{h^2}{2} + u'''(t_i)\frac{h^3}{6} + o(h^4)$$

and

$$u(t_{i-1}) = u(t_i - h) = u(t_i) - u'(t_i)h + u''(t_i)\frac{h^2}{2} - u'''(t_i)\frac{h^3}{6} + o(h^4),$$

whence

$$\Delta u(t_i) = u''(t_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + o(h^2)$$

where $u_k := u(t_k)$. The principal term $u_{i-1} - 2u_i + u_{i+1}$ represents the difference between the value of the function $u$ at the point $t_i$ and the values of $u$ on the grid-neighborhood of $t_i$. In particular, the discretization of the problem $-u'' = f$ on $I$ with the boundary conditions $u(0) = u(1) = 0$ is the linear equation $AX = B$ with

$$A = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

and

$$X = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad B = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

More generally: *the Laplace–Beltrami operator at a point $x$ measures the difference of the mean value of $f$ on a neighbourhood of $x$ and the value of the function at the point $x$.*

## Main Topics in Spectral Geometry

Given a (compact) Riemannian manifold $(M, g)$, we can associate to it the (linear) unbounded operator $-\Delta_g$. This operator is self-adjoint and its spectrum is discrete (see Chapter 4): namely, the spectrum consists of an increasing sequence $(\lambda_k(M))_k$ of real eigenvalues of finite multiplicity such that $\lambda_k(M)$ as $k \to +\infty$. We denote this spectrum by

$$\mathrm{Spec}(M, g) := (\lambda_k(M))_k \,.$$

In others words, for any integer $k \geq 0$ there exists a non trivial function $u_k$ on $M$ (eigenfunction) such that

$$-\Delta_g u_k = \lambda_k(M) u_k.$$

The spectral theory of the Laplacian on a compact Riemannian manifold $(M, g)$ is in particular interested in the connection between the spectrum $\mathrm{Spec}(M, g)$ and the geometry of the manifold $(M, g)$. Indeed there are many deep connections between spectrum and geometry. Therefore spectral geometry studies such connections. The main topics in spectral geometry can be split into two types.

### Direct problems

The main questions in direct problems are (see Chapter 5):

**Question.** *Can we compute* (*exactly or not*) *the spectrum* $\mathrm{Spec}(M, g)$*? And* (*or*)*: can we find properties on the spectrum* $\mathrm{Spec}(M, g)$*?*

Then the principle of direct problems is to compute or find some properties on the spectrum of a compact Riemannian manifold $(M, g)$. Obviously the question to compute the spectrum of Laplacian (or for other operators) arises in a lot of problems: analysis of PDE, dynamical systems, mathematical physics, differential geometry, probability, etc. See, for example, Section 4.1.3 for a concrete application in quantum dynamics.

For example, the first non-null eigenvalue $\lambda_\star(M)$ plays a very important role in Riemannian geometry, and one of the main questions is to find a lower bound for $\lambda_\star(M)$ depending on the geometric properties of the manifold, e.g. the dimension $n$, the volume $\mathrm{Vol}(M,g)$, the curvature $R$, etc. (see Section 5.2.1):

$$\lambda_\star(M) \geq a(n, \mathrm{Vol}(M,g), R, \cdots).$$

Another example of (asymptotic) computation (based on the Weyl formula, see Section 7.6) is

$$\lambda_k(M) \underset{k\to+\infty}{\sim} \left(\frac{(2\pi)^n}{B_n \mathrm{Vol}(M,g)}\right)^{\frac{2}{n}} k^{\frac{2}{n}};$$

here $n$ is the dimension of the compact manifold $(M,g)$ and $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

## Inverse problems

The data of a metric on a Riemannian manifold determine completely the Laplacian $\Delta_g$ and therefore its spectrum $\mathrm{Spec}(M,g)$. Hence the spectrum is a *Riemannian invariant*: if two Riemannian manifolds $(M,g)$ and $(M',g')$ are isometric, then they are isospectral, i.e., $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$. Conversely the main question is

**Question.** *Does the data of the spectrum* $\mathrm{Spec}(M,g)$ *determine the "shape" of the manifold* $(M,g)$*?*

In other words: which geometric information can we deduce from the spectrum? For example, we have the classical heat invariants; indeed the spectrum of the manifold $(M,g)$ determines

- the dimension of $(M,g)$
- the volume of $(M,g)$
- the integral of the scalar curvature $\mathrm{Scal}_g$ over $(M,g)$.

Another main topic in inverse problems is

**Question.** *What sequences of real numbers can be spectra of a compact manifold?*

A simpler version of this question is: Let $M$ be a fixed manifold, given a finite increasing sequence of real numbers $0 < a_1 \leq a_2 \leq \cdots \leq a_N$ does, there exists a metric $g$ such that the $k^{th}$ first eigenvalues of $(M,g)$ are equal to $0 < a_1 \leq a_2 \leq \cdots \leq a_N$? Y. Colin de Verdière proved in 1987 that the answer is positive (see Section 7.4).

The next important example of inverse problem concerns the length spectrum (see Section 7.2). The length spectrum of a compact Riemannian manifold $(M,g)$

is the set of lengths of closed geodesics on $(M, g)$ counted with multiplicities. As it turns out, the spectrum of the manifold determines the length spectrum. Spectral theory is also an important tool for understanding the relationships between the formalism of classical mechanics and that of quantum mechanics:

- The formalism of classical mechanics on a Riemannian manifold is expressed in terms of geodesics.
- The formalism of quantum mechanics on a Riemannian manifold is expressed in terms of the Laplace–Beltrami operator.

The question of isospectrality in Riemannian geometry may be traced back to H. Weyl in 1911–1912 and became popularized thanks to M. Kac's article of 1966 [Kac1]. The famous sentence of Kac "*Can one hear the shape of a drum?*" refers to this type of isospectral problem. The exact formulation of the isospectrality question is

**Question.** *If two Riemannian manifolds $(M, g)$ and $(M', g')$ are isospectral, are they isometric?*

The answer is negative and was given first by J. Milnor in 1964 (see Section 7.3). In 1984 and 1985, respectively C. Gordon, E.N. Wilson [Go-Wi] and T. Sunada [Sun] gave a systematic construction of counter-examples. In 1992, C. Gordon, D. Webb, and S. Wolpert [GWW1] gave the first planar counter-example.

The main classical references on spectral geometry are the book of M. Berger, P. Gauduchon and E. Mazet [BGM], the book of P. Bérard [Bér8][Bér], of I. Chavel [Cha1] and the book of S. Rosenberg [ROS].

## Organization of the book

The present book is a basic introduction to spectral geometry. The reader is assumed to have a good grounding in functional analysis and differential calculus. Chapter 2 discusses the fundamental notions of spectral theory for compact and unbounded operators. Chapter 3 is a review of differentiable manifolds and Riemannian geometry, including the definition of Laplace–Beltrami operator. In Chapter 4 we define the spectrum of the Laplace–Beltrami operator on a Riemannian manifold and then we present the minimax principle and some geometrical consequences. Chapter 5 discusses principles underlying the treatment of direct problems of spectral geometry, including some exact computations of spectrum. In Chapter 6 we present a topological perturbation result for eigenvalues of a manifold. Chapter 7 is devoted to inverse problems in spectral geometry; in particular, at the end of the chapter we introduce briefly some results of conformal geometry in dimension 2 and 3.

# Chapter 2
# Detailed introduction to abstract spectral theory

In order to study the Laplace–Beltrami operator on a compact Riemannian manifold, we need to introduce some basic material of abstract spectral theory. In this chapter we give some classical results concerning spectral theory in Hilbert spaces for bounded and unbounded operators. We focus on the case of compact self-adjoint operators.

For a complete course on Hilbert space and compact operators see for example the book of H. Brézis [Bré]. For the general case of unbounded operators see for example the bible of T. Kato [Kat] and the books of M. Reed and B. Simon [Re-Si], see also [Dav], [Yos], [Le-Br]. For a pure algebraic approach see N. Bourbaki [Bou] (in French) or W. Arverson [Arv1], [Arv2].

*Throughout this section $H$ is a separable Hilbert space over the field $\mathbb{C}$ and for the inner product we use the conventions*

$$\begin{aligned}\left\langle \lambda x + x', y \right\rangle_H &= \lambda \left\langle x, y \right\rangle_H + \left\langle x', y \right\rangle_H \\ \left\langle x, \mu y + y' \right\rangle_H &= \overline{\mu} \left\langle x, y \right\rangle_H + \left\langle x, y' \right\rangle_H\end{aligned}$$

## 2.1 Linear operators

### 2.1.1 Bounded operators

A *bounded operator* on $H$ is a linear map $T \colon H \to H$ with the property that there exists $M \geq 0$, such that for any vector $x \in H$ we have

$$\|Tx\|_H \leq M \, \|x\|_H \, .$$

In other words, a bounded operator is a continuous linear map. In this book we denote by $B\left(H, \|\cdot\|_H\right))$, or simply $B(H)$, the set of bounded operators of $H$. It is easy to see that $B(H)$ is an algebra. The space $B(H)$ can be equipped with the operator norm

$$|||\cdot||| : \begin{cases} B(H) \longrightarrow \mathbb{R}_+ \\ T \longmapsto |||T||| := \sup\limits_{x \in H, x \neq 0} \dfrac{\|Tx\|_H}{\|x\|_H}. \end{cases}$$

This norm is also an *algebra norm*, i.e., for all $T, S \in B(H)$ we have

$$|||TS||| \leq |||T|||\,|||S|||\,.$$

Since $H$ is complete, the space $(B(H), |||\cdot|||)$ is a Banach space. Observe that we also have

$$|||T||| = \sup_{\|x\|_H \leq 1} \|Tx\|_H = \sup_{\|x\|_H, \|y\|_H \leq 1} |\langle Tx, y\rangle_H|. \tag{2.1}$$

Similarly, a linear form $\ell \colon H \to \mathbb{C}$ is bounded if that there exists $M \geq 0$, such that for any vector $x \in H$

$$|\ell x| \leq M \|x\|_H\,.$$

In this case we put

$$|||\ell||| := \sup_{x \in H, x \neq 0} \frac{|\ell x|}{\|x\|_H}.$$

*Remark* 2.1.1. Note that every operators on a finite-dimensional space is bounded.

### 2.1.2 Lax–Milgram theorem

We start this section by a very useful theorem in functional analysis and PDE analysis.

**Theorem 2.1.2** (Lax–Milgram theorem). *Let $V$ be a Hilbert space and consider a bilinear symmetric form $a(\cdot,\cdot)$ on $V^2$ which is continuous and elliptic (namely, there exists a constant $C \geq 0$ such that for all $(u,v) \in V^2$ we have $|a(u,v)| \leq C\|u\|_V \|v\|_V$, and there exists a constant $\alpha > 0$ such that for all $u \in V$ we have $|a(u,u)| \geq \alpha \|u\|_V^2$). Also, let $\ell(\cdot)$ be a bounded linear form on $V$. Then there exists a unique vector $u \in V$ such that for all $v \in V$ we have $a(u,v) = \ell(v)$. Moreover, $\|u\|_V \leq \frac{|||\ell|||}{\alpha}$.*

*Proof.* Since $\ell$ is a bounded linear form on the Hilbert space $V$, the Riesz–Fréchet representation theorem provides a unique $\tau_\ell \in V$ such that for all $v \in V$ we have $\ell(v) = \langle \tau_\ell, v\rangle_V$. Similarly, for a fixed vector $u \in V$ the map $v \in V \mapsto a(u,v) \in \mathbb{R}$ is a bounded linear form, so there exists a unique $f_u \in V$ such that for all $v \in V$ we have $a(u,v) = \langle f_u, v\rangle_V$. So we can consider the map

$$f \colon \begin{cases} V \longrightarrow V \\ u \longmapsto f_u. \end{cases}$$

Using the linearity of $a$ it is clear that this map is linear. Moreover,

$$\|f_u\|_V^2 = \langle f_u, f_u\rangle_V = a(u, f_u) \leq C\,\|u\|_V\,\|f_u\|_V\,,$$

therefore

$$\|f(u)\|_V = \|f_u\|_V \leq C\,\|u\|_V\,.$$

Thus the map $f\colon (V, \|\cdot\|_V) \to (V, \|\cdot\|_V)$ is bounded and $|||f||| \leq C$ (here $C = |||a|||$).

Now let us show that $f$ is bijective. First for $u \in \ker(f)$ we have

$$\langle f_u, u\rangle_V = a(u, u) = 0,$$

and since $a(u, f_u) \geq \alpha\,\|u\|_V^2$ we deduce that $u = 0$. Hence $f$ is injective.

Now to show that $f$ is surjective observe first that $\operatorname{im}(f)^\perp = \{0\}$ and $\operatorname{im}(f)$ is a closed subspace of $V$. The first claim is clear: for $v \in \operatorname{im}(f)^\perp$ we have in particular $\langle v, f_v\rangle_V = a(v, v) = 0$, and using also that $a(v, v) \geq \alpha\,\|v\|_V^2$ we deduce that $v = 0$. Consequently $\operatorname{im}(f)$ is dense in $V$. Next to see that $\operatorname{im}(f) = V$ it suffices to prove that $\operatorname{im}(f)$ is a closed subspace of $V$. Consider a Cauchy sequence $(x_n)_n := (f(v_n))_n$ in the space $\operatorname{im}(f)$. Since $V$ is closed, the sequence $(x_n)_n$ converge to some vector $x \in V$. Using the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\alpha\left\|v_p - v_q\right\|_V^2 &\leq a\left(v_p - v_q, v_p - v_q\right) = \left\langle f\left(v_p - v_q\right), v_p - v_q\right\rangle_V \\
&= \left\langle f\left(v_p\right) - f\left(v_q\right), v_p - v_q\right\rangle_V \\
&= \left\langle x_p - x_q, v_p - v_q\right\rangle_V \leq C\left\|x_p - x_q\right\|_V\left\|v_p - v_q\right\|_V.
\end{aligned}$$

Since $(x_n)_n$ is a Cauchy sequence in $V$, $(v_n)_n$ is also a Cauchy sequence in the Hilbert space $V$, therefore it converges to some vector $v \in V$. Further, since the map $f$ is continuous,

$$\|x_n - f(v)\|_V = \|f(v_n) - f(v)\|_V \leq |||f|||\,\|v_n - v\|_V\,,$$

so the sequence $(x_n)_n$ converges to the vector $f(v) \in \operatorname{im}(f)$, which shows that the subspace $\operatorname{im}(f)$ is closed. Hence, $\operatorname{im}(f) = V$.

Since the map $f$ is bijective, there exists a unique $u \in V$ such that

$$f(u) = \tau_\ell.$$

Therefore, for all $v \in V$

$$\langle f(u), v\rangle_V = \langle \tau_\ell, v\rangle_V\,,$$

i.e.,

$$a(u, v) = \ell(v).$$

To finish the proof remark that

$$\alpha \|u\|_V^2 \le a(u,u) = \ell(u) \le |||\ell||| \, \|u\|_V \,,$$

and so

$$\|u\|_V \le \frac{|||\ell|||}{\alpha} \|u\|_V \,. \qquad \square$$

### 2.1.3 Unbounded operators

A generalization of bounded operators is provided by the notion of unbounded operators (or operators with domains). In fact, it is a necessity to deal with such operators if one wishes to study Quantum Mechanics since they appear as soon as one wishes to consider, say, a free quantum particle in $\mathbb{R}$.

We said $A$ is an *unbounded linear operator* with domain $D(A)$ if $D(A)$ is a subspace of $H$ and the map

$$A\colon \begin{cases} D(A) \longrightarrow H \\ \\ x \longmapsto Ax, \end{cases}$$

is linear.

*Remark* 2.1.3. *Be careful*: unbounded operators are not continuous.

**Example 2.1.4.** Consider the operator of multiplication by $x$ on the space $L^2(\mathbb{R})$,

$$M_x\colon \begin{cases} D\left(M_x\right) \longrightarrow L^2(\mathbb{R}) \\ \\ f \longmapsto xf(x), \end{cases}$$

with the domain $D\left(M_x\right) := \left\{f \in L^2(\mathbb{R});\ xf(x) \in L^2(\mathbb{R})\right\}$. Obviously, $D\left(M_x\right)$ is a subspace of $L^2(\mathbb{R})$ and $M_x$ is linear. Let us remark that here $D\left(M_x\right) \neq L^2(\mathbb{R})$. Note that $M_x$ is the *position operator* of Quantum Mechanics.

**Definition 2.1.5.** We said that an unbounded linear operator $(A, D(A))$ is *included* in another operator $(B, D(B))$ if $D(A) \subset D(B)$ and for all $x \in D(A)$, $Bx = Ax$. If $A$ is included in $B$ we write $A \subset B$.

In others words, $A \subset B$ if and only if the graph of $A$ is included into the graph of $B$. Now, recall the following usual definition:

**Definition 2.1.6.** Let $(A, D(A))$ be an unbounded linear operator. We define:

(i) $\ker(A)$, the *kernel* of $A$, by $\ker(A) := \{x \in D(A);\ Ax = 0\}$.

(ii) $\operatorname{im}(A)$, the *image* of $A$, by $\operatorname{im}(A) := \{Ax,\ x \in D(A)\}$.

A bounded operator $A$ is said to be *invertible* if there exists $B \in B(H)$ such that $AB = I$ and $BA = I$. In this case we denote by $A^{-1} = B$ the inverse of $A$. Now, let us recall a classical result on bounded operators.

**Lemma 2.1.7** (von Neumann lemma). *Let $T$ be a bounded operator on $H$ such that $|||T||| < 1$. Then the operator $I - T$ is invertible and*

$$(I-T)^{-1} = \sum_{k=0}^{+\infty} T^k \in B(H).$$

*Proof.* Consider the sequence $S_n := \sum_{k=0}^{n} T^k$, $n \geq 0$. It is clear that for all $n \geq 0$ the operator $S_n$ is bounded. Next, for all integers $(n, m)$ with $m > n$ we have

$$|||S_m - S_n||| \leq \sum_{k=n+1}^{+\infty} |||T|||^k = |||T|||^{n+1} \frac{1}{1 - |||T|||},$$

so $|||S_m - S_n|||$ converges to 0 as $n \to \infty$, hence $(S_n)_n$ is a Cauchy sequence in the Banach algebra of bounded operators. Consequently, there exists a bounded operator $S$ such that $\lim_{n\to\infty} S_n = S$ in the norm $|||\cdot|||$. It immediately follows that $\lim_{n\to\infty}(I-T)S_n = (I-T)S$ and $\lim_{n\to\infty} S_n(I-T) = S(I-T)$ for the norm $|||\cdot|||$. And, observe that for all $n \geq 0$

$$\begin{aligned}(I-T)S_n = S_n - \sum_{k=0}^{n} T^{k+1} &= \sum_{k=0}^{n} T^k - \sum_{k=1}^{n+1} T^k \\ &= I - T^{n+1}.\end{aligned}$$

Since $|||T||| < 1$ we have $\lim_{n\to\infty}(I-T)S_n = I_d$ in the norm $|||\cdot|||$. Similarly, $\lim_{n\to\infty}(I-T)S_n = I$. By the unicity of the limits, $(I-T)S = I$ and $S(I-T) = I$. In other words, $I - T$ is invertible and

$$(I-T)^{-1} = S := \sum_{k=0}^{+\infty} T^k \in B(H). \qquad \square$$

A very useful consequence of Lemma 2.1.7 is

**Corollary 2.1.8.** *Let $T$ be a bounded operator on $H$. For all $z \in \mathbb{C}$ such that $|z| > |||T|||$ the operator $zI - T$ is invertible with bounded inverse, and*

$$(zI - T)^{-1} = \sum_{k=0}^{+\infty} \frac{1}{z^{k+1}} T^k \in B(H).$$

*Proof.* It suffices to remark that $(zI - T) = z\left(I - \frac{1}{z}T\right)$ and apply the von Neumann lemma. $\square$

## 2.2 Closed operators

We have seen that an unbounded linear operator is not continuous. The analogue of boundedness for unbounded linear operators is the notion of closed operators:

**Definition 2.2.1.** Let $(A, D(A))$ be an unbounded linear operator. We say that $(A, D(A))$ is *closed* if the graph $\mathrm{Gr}(A) := \{(x, Ax)\,;\, x \in D(A)\}$ is closed in $H^2$ with respect to the *graph norm* $\|(x, Ax)\|^2 := \|x\|_H^2 + \|Ax\|_H^2$.

For the practice we use:

**Proposition 2.2.2.** *An unbounded linear operator $(A, D(A))$ is closed if and only if for any sequence $(x_n)_n \in D(A)^{\mathbb{N}}$ such that*

$$\begin{cases} \lim\limits_{n\to\infty} x_n = x & \text{in the norm } \|\cdot\|_H \\ \quad and & \\ \lim\limits_{n\to\infty} Ax_n = y & \text{in the norm } \|\cdot\|_H \end{cases}$$

*we have $x \in D(A)$ and $y = Ax$.*

*Proof.* Suppose $(A, D(A))$ is closed and let $(x_n)_n \in H^{\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$. Then the sequence $(x_n, Ax_n)_n \in \mathrm{Gr}(A)^{\mathbb{N}}$ converges in the graph norm as $n \to \infty$ to the couple $(x, y)$. Since the graph is closed we deduce that $x \in D(A)$ and $y = Ax$.

Conversely, suppose that for any sequence $(x_n)_n \in D(A)^{\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ we have $x \in D(A)$ and $y = Ax$. Let $(x_n, Ax_n) \in \mathrm{Gr}(A)^{\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty}(x_n, Ax_n) = (x, y)$ in $H^2$ in the graph norm. In particular, $(x_n)_n$ converge as $n \to \infty$ to $x$ and $(Ax_n)_n$ converge as $n \to \infty$ to $y$, hence $x \in D(A)$ and $y = Ax$, that is

$$\lim_{n\to\infty}(x_n, Ax_n) = (x, Ax) \in \mathrm{Gr}(A).$$

Therefore, the graph $\mathrm{Gr}(A) := \{(x, Ax)\,,\, x \in D(A)\}$ is closed (in the graph norm) in $H^2$. $\square$

We have also the following criterion.

**Proposition 2.2.3.** *An unbounded linear operator $(A, D(A))$ is closed if and only if the set $D(A)$ is complete for the norm $\|\cdot\|_A$, where $\|x\|_A^2 := \|x\|_H^2 + \|Ax\|_H^2$.*

*Proof.* Suppose first that $(A, D(A))$ is closed. Let $(x_n)_n$ be a Cauchy sequence in $(D(A), \|\cdot\|_A)$. Then $(x_n)_n$ and $(Ax_n)_n$ are Cauchy sequences in the Hilbert space $H$, so there exists $(x, y) \in H^2$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$

in the norm $\|\cdot\|_H$. Since $(A, D(A))$ is closed we deduce that $x \in D(A)$ and $y = Ax$. Consequently,

$$\|x_n - x\|_A^2 = \|x_n - x\|_H^2 + \|Ax_n - Ax\|_H^2$$

converge as $n \to \infty$ to 0, i.e., $\lim_{n\to\infty} x_n = x$ in the norm $\|\cdot\|_A$. So $D(A)$ is complete in the norm $\|\cdot\|_A$.

Conversely, suppose that $D(A)$ is complete in the norm $\|\cdot\|_A$. Let $(x_n)_n$ be a sequence in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ in $\|\cdot\|_H$. In particular $(x_n)_n$ and $(Ax_n)_n$ are Cauchy sequences, so $(x_n)_n$ is a Cauchy sequence in $(D(A), \|\cdot\|_A)$. Since $D(A)$ is supposed to be complete, there exists a $x_0 \in D(A)$ such that $\lim_{n\to\infty} x_n = x_0$ in the norm $\|\cdot\|_A$ and we deduce that $x_0 = x \in D(A)$ and $y = Ax$. □

**Example 2.2.4.** The operator

$$A\colon \begin{cases} D(A) := \mathcal{C}^1([0,1]) \longrightarrow \mathcal{C}^0([0,1]), \\ \qquad\qquad \varphi \longmapsto \varphi' \end{cases}$$

is closed. Indeed, $A$ is well defined and linear. $\left(\mathcal{C}^0([0,1]), \|\cdot\|_\infty\right)$ is a Banach space. Moreover $\mathbb{R}[X]|[0,1] \subset \mathcal{C}^1([0,1]) \subset \mathcal{C}^0([0,1])$ and since the space of real polynomials $\mathbb{R}[X]$ restricted to $[0,1]$ is dense in $\mathcal{C}^0([0,1])$ in the norm $\|\cdot\|_\infty$, we have

$$\overline{\mathcal{C}^1([0,1])} = \mathcal{C}^0([0,1]),$$

where the closure is taken in the norm $\|\cdot\|_\infty$. Next, let $(\varphi_n)_n$ be a sequence in $\left(\mathcal{C}^1([0,1])\right)^{\mathbb{N}}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ and $\lim_{n\to\infty} \varphi_n' = \psi$ in the norm $\|\cdot\|_\infty$. In particular, the sequence $(\varphi_n)_n$ converges pointwise to $\varphi$ and the sequence $\left(\varphi_n'\right)_n$ converges uniformly to $\psi$, so since every $\varphi_n$ is $\mathcal{C}^1$, it follows that $\varphi$ is $\mathcal{C}^1$ and $\psi = \varphi'$. Therefore, the operator $(A, D(A))$ is closed.

Now consider an operator $(A, D(A))$ with $D(A) = H$. Using the closed graph theorem we have immediately that $(A, D(A))$ is closed if and only if $A$ is bounded. Hence, every bounded operator is closed.

**Example 2.2.5.** Consider the differentiation operators

$$A_k : \begin{cases} D(A_k) \longrightarrow L^2\left(]0,1[\right) \\ \qquad u \longmapsto iu', \end{cases}$$

where

$$\begin{aligned}
D(A_0) &:= H^1(]0,1[)\,,\\
D(A_1) &:= \left\{u \in H^1(]0,1[)\,;\, u(0)=0\right\},\\
D(A_2) &:= \left\{u \in H^1(]0,1[)\,;\, u(1)=0\right\},\\
D(A_3) &:= \left\{u \in H^1(]0,1[)\,;\, u(0)=u(1)\right\} \text{ and}\\
D(A_4) &:= H_0^1(]0,1[) = \left\{u \in H^1(]0,1[)\,;\, u(0)=u(1)=0\right\}.
\end{aligned}$$

We claim that all these operators are closed with dense domains.

For the first operator: since $H^1(]0,1[)$ is dense in $L^2(]0,1[)$ for the $L^2$-norm and $H^1(]0,1[)$ is a Banach space for the $H^1$-norm, therefore the space $D(A_0)$ equipped with the norm $\|\cdot\|_{A_0}$ is complete, consequently the operator $A_0, D(A_0)$ is closed. For the second operator: recall that the injection of $H^1(]0,1[)$ into $\mathcal{C}^0([0,1])$ is continuous (see [Bré]) hence $D(A_1)$ is a closed subset of $D(A_0)$, consequently the space $D(A_1)$ equipped with the norm $\|\cdot\|_{A_1}$ is complete, it follows that $A_1, D(A_1)$ is closed. For the cases $D(A_2)$, $D(A_3)$ and $D(A_4)$ proceed in the same manner.

## 2.3 Adjoint operators

### 2.3.1 Introduction

Let $(A, D(A))$ be a closed unbounded linear operator with domain dense in $H$. The adjoint of $(A, D(A))$ is the closed unbounded linear operator $A^\star$ with domain $D(A^\star)$ defined by:

$$D(A^\star) := \{y \in H;\, \exists \eta \in H, \text{ such that } \forall x \in D(A),\ \langle Ax, y\rangle_H = \langle x, \eta\rangle_H\}.$$

In fact, here the vector $\eta$ is unique (see below). So the *adjoint operator* $A^\star$ is defined by

$$A^\star\colon \begin{cases} D(A^\star) \longrightarrow H \\ y \longmapsto A^\star y := \eta. \end{cases}$$

Let us explain the uniqueness of $\eta$. Suppose that there exists $\eta_1, \eta_2 \in H^2$ such that for all $x \in D(A)$,

$$\langle y, Ax\rangle_H = \langle \eta_1, x\rangle_H = \langle \eta_2, x\rangle_H\,.$$

Then $\langle \eta_1 - \eta_2, x\rangle_H = 0$ for all $x \in D(A)$, therefore $\eta_1 - \eta_2 \in D(A)^\perp$ and since $D(A)$ is dense in $E$ we have $D(A)^\perp = \{0\}$, consequently $\eta_1 = \eta_2$.

Note that the operator $(A^\star, D(A^\star))$ is always closed (even $A$ is not closed) and $A^{\star\star} = A$. If we have two unbounded linear operators $A, B$ such that $A \subset B$, then $B^\star \subset A^\star$. Indeed, let $y \in D(B^\star)$. Then $\langle y, Bx\rangle = \langle B^\star y, x\rangle$ for all $x \in D(B)$. Since the domain $D(A)$ is included in $D(B)$ and $B$ coincides with $A$ on $D(A)$,

$$\langle y, Ax\rangle = \left\langle B^\star y, x\right\rangle$$

for all $x \in D(A)$, therefore $y \in D(A^\star)$ and $A^\star y = B^\star y$, i.e.: $B^\star \subset A^\star$.

**Example 2.3.1.** Consider the differentiation operators

$$A_k : \begin{cases} D(A_k) \longrightarrow L^2\left(]0,1[\right) \\ \\ \qquad u \longmapsto iu', \end{cases}$$

where

$$\begin{aligned}
&D(A_0) := H^1\left(]0,1[\right), \\
&D(A_1) := \left\{u \in H^1\left(]0,1[\right);\, u(0) = 0\right\}, \\
&D(A_2) := \left\{u \in H^1\left(]0,1[\right);\, u(1) = 0\right\}, \\
&D(A_3) := \left\{u \in H^1\left(]0,1[\right);\, u(0) = u(1)\right\} \text{ and} \\
&D(A_4) := H_0^1\left(]0,1[\right) = \left\{u \in H^1\left(]0,1[\right);\, u(0) = u(1) = 0\right\}.
\end{aligned}$$

We have seen in a previous example that all these operators are closed with dense domains. Here we determine their adjoints: we claim that $A_0^\star = A_4$, $A_1^\star = A_2$, $A_2^\star = A_1$, $A_3^\star = A_3$, and $A_4^\star = A$.

Let us show only the first equality. For all $v \in D(A_4^\star)$ and all $u \in \mathcal{C}_c^\infty\left(]0,1[\right) \subset D(A_4)$, we have

$$\left\langle A_4^\star v, u\right\rangle = \langle v, A_4 u\rangle\,,$$

i.e.,

$$\left\langle A_4^\star v, u\right\rangle = -i\left\langle v, u'\right\rangle.$$

Therefore, for all $u \in \mathcal{C}_c^\infty\left(]0,1[\right) \subset D(A_4)$,

$$\left\langle v, u'\right\rangle = i\left\langle A_4^\star v, u\right\rangle$$

which means that $v \in H^1\left(]0,1[\right)$ and $v' = -iA_4^\star v$. In others words $A_4^\star v = iv'$, and so $A_4^\star \subset A_0$.

Now let $g \in D(A_4) = H_0^1\left(]0,1[\right)$. Then for all $u \in D(A_0)$ we have

$$\langle A_0 u, g\rangle = i\left\langle u', g\right\rangle = -i\left\langle u, g'\right\rangle,$$

which shows that $g \in D(A_0^\star)$ and $A_0^\star g = ig'$. So $A_4 \subset A_0^\star$. It follows that $A_0 = A_0^{\star\star} \subset A_4^\star$ and since $A_4^\star \subset A_0$ we finally have $A_0 = A_4^\star$.

**Proposition 2.3.2** (von Neumann). *For every unbounded linear operator $A$ with domain $D(A)$ dense in $H$ we have*

$$\ker\left(A^\star\right) = (\mathrm{im}(A))^\perp . \tag{2.2}$$

*Proof.* Let $x \in \ker(A^\star)$, i.e., $A^\star x = 0$. Then for all $y \in D(A)$ we have $\langle x, Ay\rangle = \langle A^\star x, y\rangle = 0$ and so $x \in (\mathrm{im}(A))^\perp$.

Conversely, let $x \in (\mathrm{im}(A))^\perp$, i.e., $\langle x, Ay\rangle = 0$ for all $y \in D(A)$. Then $x \in D(A^\star)$ and $A^\star x = 0$. □

What happens if $A$ is bounded? Using the Riesz–Fréchet representation theorem we have $D(A^\star) = H$ and the operator $A^\star$ is also bounded, with $|||A^\star||| = |||A|||$.

### 2.3.2 Symmetry and self-adjointness

**Definition 2.3.3.** An operator $(A, D(A))$ is *symmetric* if $A \subset A^\star$. The operator $(A, D(A))$ is said to be *self-adjoint* if $A = A^\star$.

Thus an operator $(A, D(A))$ is symmetric if and only if for any couple $(x, y) \in D(A)^2$ we have

$$\langle Ax, y\rangle_H = \langle x, Ay\rangle_H .$$

In particular, for all $x \in D(A)$

$$\overline{\langle x, Ax\rangle_H} = \langle Ax, x\rangle_H = \langle x, Ay\rangle_H$$

and so $\langle Ax, x\rangle_H \in \mathbb{R}$.

Every self-adjoint operator is symmetric. If an operator $(A, D(A))$ is symmetric and $D(A^\star) \subset D(A)$ then $(A, D(A))$ is self-adjoint.

**Definition 2.3.4.** A unbounded symmetric operator $(A, D(A))$ is said to be *positive* if $\langle Ax, x\rangle_H \geq 0$ for all $x \in D(A)$.

*Remark* 2.3.5. In the case of bounded operators symmetry and self-adjointness coincide.

**Proposition 2.3.6.** *For every bounded symmetric* (*self-adjoint*) *operator $T$ on $H$ we have*

$$|||T||| = \sup_{\|x\|_H \leq 1} |\langle Tx, x\rangle_H| .$$

*Proof.* By (2.1), $|||T||| = \sup_{\|x\|_H, \|y\|_H \leq 1} |\langle Tx, y\rangle_H|$, whence

$$\sup_{\|x\|_H \leq 1} |\langle Tx, x\rangle_H| \leq |||T||| .$$

Using the polarization identity, we see that for all vectors $x, y \in H$ such that $\|x\|_H, \|y\|_H \leq 1$

$$\begin{aligned}
|\langle Tx, y\rangle_H| &\leq \frac{1}{4}\|x+y\|_H^2 \left\langle T\left(\frac{x+y}{\|x+y\|_H}\right), \frac{x+y}{\|x+y\|_H}\right\rangle_H \\
&\quad + \frac{1}{4}\|x-y\|_H^2 \left\langle T\left(\frac{x-y}{\|x-y\|_H}\right), \frac{x-y}{\|x-y\|_H}\right\rangle_H \\
&\leq \frac{1}{4}\|x+y\|_H^2\, c + \frac{1}{4}\|x-y\|_H^2\, c \\
&= \frac{c}{4}\left(2\|x\|_H^2 + 2\|y\|_H^2\right) \leq c,
\end{aligned}$$

where $c := \sup\limits_{\|x\|_H \leq 1} |\langle Tx, x\rangle_H|$. □

We have also

**Proposition 2.3.7.** *For every bounded positive symmetric (self-adjoint) operator $T$ on $H$,*

$$\|Tx\|_H^2 \leq |||T|||\, |\langle Tx, x\rangle_H|$$

*for all $x \in H$.*

*Proof.* Consider the bilinear form $a(x, y) := \langle Tx, y\rangle_H$ on $H^2$. Since $T$ is positive and symmetric we have, by the Cauchy–Schwarz inequality,

$$|a(x, y)|^2 \leq a(x, x) \times a(y, y)$$

for all $x, y \in H$, whence

$$\begin{aligned}
|\langle Tx, y\rangle_H|^2 &\leq \langle Tx, x\rangle_H \times \langle Ty, y\rangle_H \\
&\leq \langle Tx, x\rangle_H \times |||T|||\, \|y\|_H^2
\end{aligned}$$

for all $x, y \in H$. Hence, for the normed vector $y := \frac{Tx}{\|Tx\|_H}$ we get

$$\|Tx\|_H^2 \leq \langle Tx, x\rangle_H \times |||T|||\,.$$

□

### 2.3.3 An important result concerning self-adjoint operators

Observe first that for every symmetric operator $(A, D(A))$ of $H$, for all $x \in D(A)$ and for all $\lambda = \mu + i\gamma \in \mathbb{C}$ such that $\gamma \neq 0$ we have

$$\begin{aligned}
\|(\lambda I - A)x\|_H^2 &= \langle(\mu I - A)x + i\gamma x, (\mu I - A)x + i\gamma x\rangle_H \\
&= \|(\mu I - A)x\|_H^2 - i\gamma\langle(\mu I - A)x, x\rangle_H \\
&\quad + i\gamma\langle x, (\mu I - A)x\rangle_H + \gamma^2\|x\|_H^2 \\
&= \|(\mu I - A)x\|_H^2 + \gamma^2\|x\|_H^2 \geq \gamma^2\|x\|_H^2,
\end{aligned}$$

because $A$ is symmetric and $\mu$ is real. Therefore,

$$\|(\lambda I - A)\, x\|_H^2 \geq \gamma^2 \, \|x\|_H^2 \, . \tag{2.3}$$

**Lemma 2.3.8.** *Let $(A, D(A))$ be a symmetric operator of $H$. For all $\lambda = \mu + i\gamma \in \mathbb{C}$ such that $\gamma \neq 0$, $\ker(\lambda I - A) = \{0\}$ and $\operatorname{im}(\lambda I - A)$ is closed.*

*Proof.* Using (2.3) we have

$$\|(\lambda I - A)\, u\|_H^2 \geq \gamma^2 \, \|u\|_H^2$$

for all $u \in D(A)$. Thus it is clear that $\ker(\lambda I - A) = \{0\}$. Next, consider a Cauchy sequence $y_n = (\lambda I - A)\, x_n$ in $\operatorname{im}(\lambda I - A)$ (obviously the sequence $x_n$ belongs to $D(A)$). Using (2.3) we see that $(x_n)_n$ is a Cauchy sequence of $D(A)$. Since $H$ is complete, there exists a couple $(x, y) \in H^2$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in the norm $\|\cdot\|_H$. Therefore,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \lambda x_n - y_n = \lambda x - y$$

and since $A$ is closed we have $x \in D(A)$ and $Ax = \lambda x - y$. Consequently, $y = (\lambda I - A)x \in \operatorname{im}(\lambda I - A)$. This means that the sequence $y_n = (\lambda I - A)x_n$ converges in $\operatorname{im}(\lambda I - A)$, thus $\operatorname{im}(\lambda I - A)$ is closed. □

So we deduce

**Theorem 2.3.9.** *Let $(A, D(A))$ be a symmetric operator of $H$. The following properties are equivalent:*

(i) *$A$ is self-adjoint.*

(ii) *$A$ is closed and $\ker(A^\star \pm iI) = \{0\}$.*

(iii) $\operatorname{im}(A \pm iI) = H$.

*Proof.*

- First we show (i) $\Rightarrow$ (ii). Suppose $A$ self-adjoint, then $A$ is closed, because $A = A^\star$. Next if $w \in \ker(A^\star - iI)$ then $A^\star w = iw$, and we get $\langle A^\star w, w\rangle_H = i\,\|w\|_H^2$. On the other hand, since $A = A^\star$, we also have

$$\langle A^\star w, w\rangle_H = \langle Aw, w\rangle_H = \langle w, A^\star w\rangle_H = -i\,\|w\|_H^2 \, ,$$

whence $w = 0$. The case $\ker(A^\star + iI)$ is similar.

- Now let us prove (ii) $\Rightarrow$ (iii). Assume (ii) holds. Then $(\operatorname{im}(\pm iI - A))^\perp = \{0\}$ because of Proposition 2.3.2, so we have $\overline{\operatorname{im}(\pm iI - A)} = H$. By Lemma 2.3.8 (with $\lambda = i$) the subspace $\operatorname{im}(\pm iI - A)$ is closed.

- Finally, we show (iii) $\Rightarrow$ (i). Since $\operatorname{im}(A \pm iI) = H$ we have (by Proposition 2.3.2)
$$\ker\left(A^\star \pm iI\right) = \{0\}.$$
Let $v \in D(A^\star)$. Since $(A^\star - iI)\,v \in H$, there exists $u \in D(A)$ such that $(A^\star - iI)\,v = (A - iI)\,u$. Since $A$ is symmetric, $(A^\star - iI)\,(v - u) = 0$ and it follows that $v = u$ (because $\ker(A^\star \pm iI) = \{0\}$). This shows that $v \in D(A)$, hence $A$ is self-adjoint. $\square$

## 2.4 Spectrums and resolvent set

### 2.4.1 Spectrum versus point spectrum

**Definition 2.4.1.** Let $A\colon D(A) \subset H \to H$ be a closed unbounded linear operator.

(i) A scalar $\lambda$ is an *eigenvalue* of $A$ if $\ker(\lambda I - A) \neq \{0\}$. The *point spectrum* of $A$ is the set of eigenvalues of $A$, denoted by $\operatorname{Spec}(A)$ or $\sigma_{\mathrm{p}}(A)$.

(ii) A scalar $\lambda$ is said to belong to *the resolvent set* of $A$ if the operator $A - \lambda I$ is an isomorphism from $D(A)$ onto $H$ and the inverse $(\lambda I - A)^{-1}$ is a bounded operator in $H$. The resolvent set of $A$ is denoted by $\rho(A)$.

(iii) The *spectrum* of $A$ is the complement of the set $\rho(A)$ in $\mathbb{C}$. The spectrum of $A$ is denoted by $\sigma(A)$.

Let $\lambda$ be a fixed eigenvalue of $A$. An *eigenvector associated* to $\lambda$ is a vector $\varphi \in D(A) \subset H$ such that $A\varphi = \lambda\varphi$. The set of eigenvectors associated to $\lambda$ is a vector space called the *eigenspace associated* to $\lambda$.

In the case of bounded operators the definitions of the spectra are similar (for (ii) just replace $D(A)$ by $H$). We have the inclusion
$$\operatorname{Spec}(A) \subset \sigma(A).$$
Indeed, if $\lambda \notin \sigma(A)$ i.e., $\lambda \in \rho(A)$, then in particular the operator $(\lambda I - A)$ is injective; in other words, $\ker(\lambda I - A) = \{0\}$, i.e., $\lambda \notin \operatorname{Spec}(A)$.

The spectrum of an operator is a closed set of $\mathbb{C}$ (see the following theorem). Moreover, if the operator $A$ is bounded, we have also the inclusion
$$\sigma(A) \subset \overline{B_{\mathbb{C}}(0, |||A|||)}$$
where $\overline{B_{\mathbb{C}}(0,r)} := \{z \in \mathbb{C};\ |z| \le r\}$. Indeed, for each $z \in \mathbb{C}$ such that $|\lambda| > |||A|||$ the operator $zI - A$ is invertible with bounded inverse (Corollary 2.1.8 of von Neumann's lemma), i.e., $z \in \rho(A)$. Therefore if $A$ is bounded the spectrum $\sigma(A)$ is a compact subset of $\mathbb{C}$.

*Remark* 2.4.2. In the case of finite-dimensional spaces the spectrum and the point spectrum coincide.

For any $\lambda \in \rho(A)$ we denote by $R_A(\lambda)$ the bounded operator $(\lambda I - A)^{-1}$. This operator is called *the resolvent operator* of $A$ at the point $\lambda$.

**Example 2.4.3.** Consider the family of derivation operators

$$A_k : \begin{cases} D(A_k) \longrightarrow L^2\left(]0,1[\right) \\ \\ \qquad u \longmapsto iu', \end{cases}$$

where

$$\begin{aligned} &D(A_0) := H^1\left(]0,1[\right), \\ &D(A_1) := \left\{u \in H^1\left(]0,1[\right); u(0) = 0\right\}, \\ &D(A_2) := \left\{u \in H^1\left(]0,1[\right); u(1) = 0\right\}, \\ &D(A_3) := \left\{u \in H^1\left(]0,1[\right); u(0) = u(1)\right\} \text{ and} \\ &D(A_4) := H_0^1\left(]0,1[\right) = \left\{u \in H^1\left(]0,1[\right); u(0) = u(1) = 0\right\}. \end{aligned}$$

We have: $\sigma(A_0) = \mathbb{C}$, $\sigma(A_1) = \sigma(A_2) = \emptyset$, $\sigma(A_3) = \{2i\pi n, n \in \mathbb{Z}\}$, $\sigma(A_4) = \mathbb{C}$.

- Start with $A_0$ and note that for any constant $\lambda \in \mathbb{C}$ the function $x \mapsto u(x) = e^{-\lambda i x}$ with $x \in [0,1]$ is a solution of the ODE

  $$iu' - \lambda u = 0.$$

  This means that for all $\lambda \in \mathbb{C}$ we have

  $$\ker(A_0 - \lambda I) \neq \{0\},$$

  hence the point spectrum of $A_0$ is "full", i.e., $\mathrm{Spec}(A_0) = \mathbb{C}$, therefore $\sigma(A_0) = \mathbb{C}$.
- For $A_1$ (and similarly for $A_2$), for any function $f$ and any $\lambda \in \mathbb{C}$, using classical ODE techniques one can verify that the equation

  $$iu' - \lambda u = f$$

  with $u(0) = 0$ has a unique solution, namely

  $$\begin{aligned} u(x) &= u(0)e^{-\lambda i x} - e^{-\lambda i x}\int_0^x ie^{\lambda i t} f(t)\,dt \\ &= -e^{-\lambda i x}\int_0^x ie^{\lambda i t} f(t)\,dt. \end{aligned}$$

  Hence, for any $\lambda \in \mathbb{C}$ we have $\lambda \in \rho(A_1)$, and so $\sigma(A_1) = \emptyset$ and for all $\lambda \in \mathbb{C}$

  $$(R_\lambda A_1)\,u(x) = -e^{-\lambda i x}\int_0^x ie^{\lambda i t} f(t)\,dt.$$

- For $A_3$: for any function $f$ and for any $\lambda \in \mathbb{C}$, using classical ODE techniques one can verify that the solutions of the equation

$$iu' - \lambda u = 0$$

are given by $u(x) = Ce^{-\lambda ix}$ with $C \in \mathbb{C}$. The condition

$$u(0) = u(1) \Longleftrightarrow \lambda = 2\pi n,\ n \in \mathbb{Z},$$

hence $\mathrm{Spec}(A_3) = \{2i\pi n,\ n \in \mathbb{Z}\}$, and $\{2i\pi n,\ n \in \mathbb{Z}\} \subset \sigma(A_3)$. Next for $\lambda \in \mathbb{C}$ such that $\lambda \neq 2i\pi n$ for any integer $n$, the equation

$$iu' - \lambda u = f$$

has a unique solution, namely

$$u(x) = u(0)e^{-\lambda ix} - e^{-\lambda ix}\int_0^x ie^{\lambda it} f(t)\,dt.$$

Imposing the boundary conditions ($u(0) = u(1)$) we get

$$u(0) = \frac{1}{e^{-\lambda ix} - 1}\int_0^x ie^{\lambda i(t-1)} f(t)\,dt,$$

so we deduce (arguing as for the operator $A_1$) the expression of the solution $x \mapsto u(x)$ and the expression of $R_\lambda A_3$. We see that $\mathrm{im}(A_3 - \lambda I) = L^2(]0,1[)$. Finally, if $\lambda \in \mathbb{C}$, such that $\lambda \neq 2i\pi n$ for any integer $n$, then $\lambda \in \rho(A_3)$, therefore $\sigma(A_3) \subset \{2i\pi n,\ n \in \mathbb{Z}\}$.
- To finish we look at the operator $A_4$. Since for all $\lambda \in \mathbb{C}$

$$\mathrm{im}\,(A_0 - \lambda I) = \left\{iu' - \lambda u;\ u \in H_0^1\,(]0,1[)\right\}$$

and since for all $u \in H_0^1\,(]0,1[)$ we have, using integrating by parts, that

$$\begin{aligned}\int_0^1 e^{\lambda ix}\left(iu'(x) - \lambda u(x)\right)\,dx &= i\int_0^1 e^{\lambda ix}u'(x)\,dx - \lambda\int_0^1 e^{\lambda ix}u(x)\,dx\\ &= \lambda\int_0^1 e^{\lambda ix}u(x)\,dx - \lambda\int_0^1 e^{\lambda ix}u(x)\,dx\\ &= 0,\end{aligned}$$

it follows that for any $\lambda \in \mathbb{C}$ we have $(\mathrm{im}\,(A_0 - \lambda I))^\perp \neq \{0\}$, so $\lambda \in \sigma(A_4)$. We conclude that $\sigma(A_4) = \mathbb{C}$.

**Theorem 2.4.4.** *Let $A\colon D(A) \subset H \to H$ be a closed unbounded linear operator.*

(i) *For all $(\lambda, \mu) \in \rho(A)^2$ we have the resolvent equation*

$$R_A(\lambda) - R_A(\mu) = (\mu - \lambda) R_A(\lambda) R_A(\mu). \tag{2.4}$$

(ii) *For all $(\lambda, \mu) \in \rho(A)^2$ the resolvents of $A$ in $\lambda$ and in $\mu$ commutes*

$$R_A(\lambda) R_A(\mu) = R_A(\mu) R_A(\lambda).$$

(iii) *The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$.*

(iv) *The mapping*

$$R_A\colon \begin{cases} \rho(A) \longrightarrow B(H) \\ \lambda \longmapsto R_A(\lambda), \end{cases}$$

*is holomorphic on $\rho(A)$ and for all $\lambda \in \rho(A)$ we have $R'_A(\lambda) = -(R_A(\lambda))^2$.*

(v) *For all $\lambda \in \rho(A)$ we have*

$$A R_A(\lambda) = \lambda R_A(\lambda) - I \in B(H). \tag{2.5}$$

*Proof.*

- Start with (i). For all $(\lambda, \mu) \in \rho(A)^2$,

$$\begin{aligned} R_A(\lambda) - R_A(\mu) &= (\lambda I - A)^{-1} - (\mu I - A)^{-1} \\ &= (\lambda I - A)^{-1} \left( I - (\lambda I - A)(\mu I - A)^{-1} \right) \\ &= (\lambda I - A)^{-1} ((\mu I - A) - (\lambda I - A)) (\mu I - A)^{-1} \\ &= (\mu - \lambda) (\lambda I - A)^{-1} (\mu I - A)^{-1} . \end{aligned}$$

- For item (ii): the case $\lambda = \mu$ is clear. So, suppose $\lambda \neq \mu$. Using (i) we have

$$\begin{aligned} R_A(\lambda) R_A(\mu) &= \frac{R_A(\lambda) - R_A(\mu)}{\mu - \lambda} \\ &= \frac{R_A(\mu) - R_A(\lambda)}{\lambda - \mu} = R_A(\mu) R_A(\lambda). \end{aligned}$$

- The argument for (iii) is the following: for each fixed $\lambda_0 \in \rho(A)$ and for all $\mu \in \mathbb{C}$,

$$(\lambda_0 + \mu) I - A = \lambda_0 I - A + \mu I \quad = (\lambda_0 I - A)(I + R_A(\lambda_0) \mu I).$$

Now for all $\mu \in \mathbb{C}$ such that $|\mu| < \frac{1}{|||R_A(\lambda_0)|||}$ (i.e., $|||\mu R_A(\lambda_0)||| < 1$), the von Neumann's lemma shows that $(\lambda_0 + \mu) I - A$ is invertible with a bounded inverse (so $\lambda_0 + \mu \in \rho(A)$) and

$$R_A(\lambda_0 + \mu) = \sum_{k=0}^{+\infty} (-\mu)^k \, (R_A(\lambda_0))^k.$$

In particular, for each $\lambda_0 \in \rho(A)$ there exists a radius $r := \frac{1}{|||R_A(\lambda_0)|||}$ such that the open disc centered at $\lambda_0$ and of radius $r$ is a subset of $\rho(A)$; in other words, $\rho(A)$ is an open set.

- Next, item (iv): for any $\lambda_0 \in \rho(A)$ and for any complex number $z$ such that $|z - \lambda_0| < \frac{1}{|||R_A(\lambda_0)|||}$ we have seen that $(\lambda_0 + \mu) I - A$ is invertible with a bounded inverse and

$$R_A(z) = (zI - A)^{-1} = \sum_{k=0}^{+\infty} (\lambda_0 - z)^k \, (R_A(z))^{k+1}.$$

This formula show that $\lambda \in \rho(A) \mapsto R_A(\lambda)$ is analytic, hence holomorphic. Moreover, using the proof of (ii) we get

$$R_A'(\lambda) = \lim_{z \to \lambda} \frac{R_A(z) - R_A(\lambda)}{z - \lambda} = -\left(R_A(\lambda)\right)^2.$$

- Finally, let us prove (v). For any $\lambda \in \rho(A)$ we have $(\lambda I - A)\, R_A(\lambda) = I$, whence

$$\lambda R_A(\lambda) - I = A R_A(\lambda). \qquad \square$$

Next we give an important theorem concerning the spectrum of resolvent operators:

**Theorem 2.4.5.** *Let $A\colon D(A) \subset H \to H$ be a closed unbounded linear operator such that $\rho(A) \neq \emptyset$. For any $\lambda_0 \in \rho(A)$ we have*

(i) $\operatorname{Spec}(R_A(\lambda_0)) - \{0\} = \left\{\frac{1}{\lambda_0 - \mu};\ \mu \in \operatorname{Spec}(A)\right\}$

(ii) $\sigma\left(R_A(\lambda_0)\right) - \{0\} = \left\{\frac{1}{\lambda_0 - \mu};\ \mu \in \sigma(A)\right\}$.

*Proof.*

- Start with item (i): if $\lambda \in \operatorname{Spec}(R_A(\lambda_0))$ with $\lambda \neq 0$, then there exists $\varphi \in H$, $\varphi \neq 0$, such that $R_A(\lambda_0)\varphi = \lambda\varphi$, so $\varphi \in D(A)$ and

$$\begin{aligned} A\varphi &= A\left(\frac{1}{\lambda}\left(R_A(\lambda_0)\varphi\right)\right) = \frac{1}{\lambda}\left(\lambda_0 R_A(\lambda_0) - I\right)\varphi \\ &= \frac{\lambda_0}{\lambda}\lambda\varphi - \frac{1}{\lambda}\varphi = \left(\lambda_0 - \frac{1}{\lambda}\right)\varphi, \end{aligned}$$

so with $\mu := \lambda_0 - \frac{1}{\lambda}$ we have $\mu \in \mathrm{Spec}(A)$ and $\lambda = \frac{1}{\lambda_0 - \mu}$.
Conversely, arguing in the same way we have also

$$\left\{\frac{1}{\lambda_0 - \mu};\ \mu \in \mathrm{Spec}(A)\right\} \subset \mathrm{Spec}\left(R_A(\lambda_0)\right) - \{0\}.$$

- Let us prove item (ii): for all $\mu \in \mathbb{C}^\star$ and for all $\lambda_0 \in \rho(A)$ we have

$$(\mu - R_A(\lambda_0))\,\varphi = \mu\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) R_A(\lambda_0)\varphi$$

for any vector $\varphi \in H$ (because $((\mu\lambda_0 - 1)\, I - \mu A)\, R_A(\lambda_0) = \mu\lambda_0 R_A(\lambda_0) - R_A(\lambda_0) - \mu\lambda_0 R_A(\lambda_0) + \mu I$. Moreover, for any vector $\psi \in H$,

$$(\mu - R_A(\lambda_0))\,\psi = \mu\left(R_A(\lambda_0)\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right)\varphi$$

hence if $x \in \ker(\mu I - R_A(\lambda_0))$ then $x \in D(A)$ and $\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) x \in \ker\left(R_A(\lambda_0)\right) = \{0\}$. Consequently,

$$x \in \ker\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right).$$

We also have $\ker\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) \subset \ker(\mu I - R_A(\lambda_0))$.
Next, let $y \in \mathrm{im}\,(\mu I - R_A(\lambda_0))$. Then there exists $x \in H$ such that $y = (\mu I - R_A(\lambda_0))\,x$ hence $y \in \mathrm{im}\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right)$. Now, if $y \in \mathrm{im}\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right)$, there exists $x \in D(A)$ such that $y = \left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) x$ and there exists a unique $w \in H$ such that $x = R_A(\lambda_0)w$, therefore $y \in \mathrm{im}\,(\mu I - R_A(\lambda_0))$. So we have proved that

$$\ker\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) = \ker(\mu I - R_A(\lambda_0))$$

and

$$\mathrm{im}\left(\left(\lambda_0 - \frac{1}{\lambda}\right) I - A\right) = \mathrm{im}\,(\mu I - R_A(\lambda_0)).$$

It follows that

$$\mu \in \rho\left(R_A(\lambda_0)\right) \Longleftrightarrow \left(\lambda_0 - \frac{1}{\mu}\right) \in \rho(A) \text{ with } \mu \neq 0,$$

i.e.,

$$\mu \in \sigma\left(R_A(\lambda_0)\right) \Longleftrightarrow \left(\lambda_0 - \frac{1}{\mu}\right) \in \sigma(A) \text{ with } \mu \neq 0,$$

so we get $\sigma\left(R_A(\lambda_0)\right) - \{0\} = \left\{\frac{1}{\lambda_0 - \mu},\ \mu \in \sigma(A)\right\}$. $\square$

### 2.4.2 The self-adjoint case

**Theorem 2.4.6.** *If an unbounded operator $A$ is self-adjoint, then* $\operatorname{Spec}(A) \subset \sigma(A) \subset \mathbb{R}$. *Moreover, the eigenspaces of $A$ are orthogonal to one another.*

*Proof.* Let $\lambda \in \operatorname{Spec}(A)$, i.e., there exists $x \neq 0$ in $D(A)$ such that $Ax = \lambda x$. We have

$$\langle Ax, x\rangle_H = \lambda \|x\|_H^2 .$$

On the other hand,

$$\langle Ax, x\rangle_H = \langle x, Ax\rangle_H = \overline{\lambda} \|x\|_H^2$$

so we deduce that $\lambda \in \mathbb{R}$, hence $\operatorname{Spec}(A) \subset \mathbb{R}$. Now for $\lambda \in \mathbb{C}$ such that $\operatorname{im}(\lambda) \neq 0$, Lemma 2.3.8 shows that

$$\ker(\lambda I - A) = \{0\},$$

hence

$$(\operatorname{im}(\lambda I - A))^{\perp} = \{0\}$$

because $\lambda \in \mathbb{R}$ and $A = A^{\star}$. So we have $\overline{\operatorname{im}(\lambda I - A)} = H$ and by Lemma 2.3.8 the subspace $\operatorname{im}(\lambda I - A)$ is closed. So we have $\operatorname{im}(\lambda I - A) = H$ and the operator $\lambda I - A$ is surjective. Consequently, if $\lambda \in \sigma(A)$, then $\lambda I - A$ is not surjective, it follows that $\operatorname{Im}(\lambda) = 0$. Hence $\sigma(A) \subset \mathbb{R}$.

To finish, let $\lambda_1$ and $\lambda_2$ be two distinct (real) eigenvalues of $A$ and consider two associated eigenvectors $x_1$ and $x_2$. We have

$$\langle Ax_1, x_2\rangle_H = \lambda_1 \langle x_1, x_2\rangle_H .$$

On the other hand,

$$\langle Ax_1, x_2\rangle_H = \langle x_1, Ax_2\rangle_H = \lambda_2 \langle x_1, x_2\rangle_H .$$

Therefore, $\langle x_1, x_2\rangle_H = 0$. □

We have also

**Proposition 2.4.7.** *For every bounded self-adjoint operator $T$ on $H$ let us denote $m := \inf_{\|x\|_H \le 1} |\langle Tx, x\rangle_H|$ and $M := \sup_{\|x\|_H \le 1} |\langle Tx, x\rangle_H|$. Then $\sigma(T) \subset [m, M]$ and the numbers $m, M$ belong to $\sigma(T)$.*

*Proof.* Let us prove the statement for $M$ (for $m$ the proof is similar). Consider a fixed $f \in H$ and $\lambda \in \mathbb{R}$ with $\lambda > M$. Then for all $x \in H$ we have

$$\langle Tx, x\rangle_H \le M \|x\|_H^2 ,$$

hence

$$\langle (\lambda I - T)\, x, x \rangle_H \geq \lambda \, \|x\|_H^2 - M \, \|x\|_H^2 = (\lambda - M) \, \|x\|_H^2 .$$

Now consider the symmetric (because $T$ is self-adjoint) bilinear form $a(x, y) := \langle (\lambda I - T)\, x, y \rangle_H$. It is clear that this form is bounded (i.e., continuous) and for all $x \in H$ we have $a(x, x) \geq (\lambda - M) \, \|x\|_H^2$, so this form *is* elliptic (coercive) on $H$. Next, consider the bounded linear form $\ell(y) := \langle f, y \rangle_H$. Using the Lax–Milgram theorem we see that for all $f \in H$ there exists a unique vector $x^\star \in H$ (which depends on $f$) such that for all $y \in H$

$$a(x^\star, y) = \ell(y)$$

and $\|x^\star\|_H \leq K \, \|f\|_H$, with a constant $K \geq 0$. In other words the problem: for a fixed $f \in H$, find $x \in H$ such that

$$\langle (\lambda I - T)\, x, y \rangle_H = \langle f, y \rangle_H$$

for all $y \in H$, admits a unique solution $x^\star \in H$. This problem is equivalent to the problem: for a fixed $f \in H$, find $x \in H$ such that

$$(\lambda I - T)\, x = f.$$

In fact, we have shown that for every $f \in H$ there exists an unique solution $x^\star \in H$, consequently the operator $\lambda I - T$ is bijective and $x^\star = (\lambda I - T)^{-1} f$. Moreover, $\|x^\star\|_H = \|(\lambda I - T)^{-1} f\|_H \leq K \, \|f\|_H$, so we deduce that $(\lambda I - T)^{-1}$ is bounded, thus the operator $(\lambda I - T)$ is invertible. Summing up, if $\lambda > M$, then $\lambda \in \rho(T)$, i.e., if $\lambda \in \sigma(T)$ then $\lambda \leq M$.

Next let us verify that $M \in \sigma(T)$: if $M \in \rho(T)$, then the operator $(MI - T)$ is invertible and consequently $(MI - T)^{-1}$ is bounded. Then by the definition of $M$ there exists a sequence $(x_n)_n$ in $H$ with $\|x_n\|_H = 1$ such that

$$\lim_{n \to \infty} \langle (MI - T)\, x_n, x_n \rangle_H = 0.$$

On the other hand since for all $\|x\|_H \leq 1$ we have

$$\langle (MI - T)\, x, x \rangle_H = M \, \langle x, x \rangle_H - \langle Tx, x \rangle_H \geq 0,$$

the operator $MI - T$ is self-adjoint and positive, consequently for all $\|x\|_H \leq 1$ (Proposition 2.3.7)

$$\|(MI - T)\, x\|_H^2 \leq |||MI - T||| \, |\langle (MI - T)\, x, x \rangle_H| .$$

Applying this to the sequence $(x_n)_n$ we have

$$\lim_{n \to \infty} \|(MI - T)\, x_n\|_H = 0,$$

hence since $x_n = (MI - T)^{-1}(MI - T)x_n$, we get

$$\|x_n\|_H \leq \left|\left|\left|(MI - T)^{-1}\right|\right|\right| \|(MI - T)x_n\|_H .$$

Finally we deduce that $\lim_{n\to\infty} \|x_n\|_H = 0$, which is absurd (because $\|x_n\|_H = 1$). It follows that $M \in \sigma(T)$. □

**Corollary 2.4.8.** *Let $T$ be a bounded self-adjoint operator on $H$. If $\sigma(T) = \{0\}$, then $T = 0$.*

*Proof.* In particular, $M = 0$, so using Proposition 2.3.6 we get

$$0 = \sup_{\|x\|_H \leq 1} |\langle Tx, x\rangle_H| = |||T||| . \qquad \square$$

## 2.5 Spectral theory of compact operators

### 2.5.1 The notion of compact operators

**Definition 2.5.1.** Let $A$ be a bounded operator on $H$ and denote by $B_H(0,1) := \{x \in H;\ \|x\|_H < 1\}$ the open ball of radius 1 in $H$. We say that $A$ is a *compact operator* if the closed set $\overline{A(B_H(0,1))}$ is compact.

**Proposition 2.5.2.** *A bounded operator $A$ is compact if and only if for any bounded subset $X$ of $H$, the closed set $\overline{A(X)}$ is a compact subset of $H$.*

*Proof.* Suppose $A$ is compact and let $X$ be a bounded subset of $H$. There exists $r > 0$ such that $X \subset B_H(0,r) := \{x \in H;\ \|x\|_H < r\}$. Let $(y_n)_n$ be a sequence in $\overline{A(X)}$, so there exists a sequence $(x_n)_n$ in $X$ such that for all $n$, $y_n = A(x_n)$. For any $n$ we have $A\left(\frac{x_n}{r}\right) = \frac{1}{r}A(x_n)$ and $\frac{x_n}{r} \in B_H(0,1)$, thus the sequence $\left(A\left(\frac{x_n}{r}\right)\right)_n$ belong to $A(B_H(0,1))$. Since $\overline{A(B_H(0,1))}$ is compact one can suppose, by extracting a subsequence if necessary, that the sequence $\left(A\left(\frac{x_n}{r}\right)\right)_n$ converge in $\overline{A(B_H(0,1))}$. It follows that the sequence $y_n = (A(x_n))_n$ converge in $\overline{A(X)}$.

To prove the converse, it suffices to take $X = B_H(0,1)$. □

The *Riesz lemma* asserts that the closed unit ball $\overline{B_H(0,1)} := \{x \in H;\ \|x\|_H \leq 1\}$ is compact if and only if $\dim(H) < +\infty$. Thus every *finite rank operator* (i.e., a bounded operator $A$ such that $\dim(\mathrm{im}(A)) < +\infty$) is compact.

## 2.5.2 Algebraic properties of compact operators

The algebraic structure of the set of compact operators is described as follows.

**Theorem 2.5.3.** *The set of compact operators of $H$ is a closed subspace of the space of bounded operators. Moreover, this set is also a (two-sided) ideal of the algebra of bounded operators of $H$.*

*Proof.* Let $T, S$ be two compact operators and $\alpha \in \mathbb{C}$. Obviously,

$$(\alpha T + S)\left(B_H(0,1)\right) \subset \alpha T\left(B_H(0,1)\right) + S\left(B_H(0,1)\right),$$

so

$$\overline{(\alpha T + S)\left(B_H(0,1)\right)} \subset \alpha \overline{T\left(B_H(0,1)\right)} + \overline{S\left(B_H(0,1)\right)}.$$

Hence, since $T, S$ are compact operators the set $\alpha \overline{T\left(B_H(0,1)\right)} + \overline{S\left(B_H(0,1)\right)}$ is compact; indeed, if $K_1, K_2$ are two compact sets, the set $\alpha K_1 + K_2$ is also compact, because $K_1 \times K_2$ is compact and the map

$$\begin{cases} K_1 \times K_2 \longrightarrow \alpha K_1 + K_2 \\ \\ (x, y) \longmapsto \alpha x + y, \end{cases}$$

is continuous. Therefore, the set $\overline{(\alpha T + S)\left(B_H(0,1)\right)}$ is compact, i.e., the operator $\alpha T + S$ is compact.

Now let us consider a sequence of compact operators $(T_n)_n$ such that $\lim_{n\to\infty} T_n = T$ in the norm $|||\cdot|||$. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $|||T_N - T||| < \frac{\varepsilon}{2}$. Since $\bigcup_{y\in H} B_H\left(y, \frac{\varepsilon}{2}\right) \supset H$ we have, in particular,

$$\overline{T_N\left(B_H(0,1)\right)} \subset \bigcup_{y\in H} B_H\left(y, \frac{\varepsilon}{2}\right).$$

By the compactness of $\overline{T_N\left(B_H(0,1)\right)}$, there is a finite set $(y_1, y_2, \ldots, y_k)$ in $H$ such that

$$\overline{T_N\left(B_H(0,1)\right)} \subset \bigcup_{i=1}^{k} B_H\left(y_i, \frac{\varepsilon}{2}\right).$$

Let $x \in B_H(0,1)$, then $\|(T_N - T)\,x\|_F < \frac{\varepsilon}{2}$ and

$$T_N x \in T_N\left(B_H(0,1)\right) \subset \overline{T_N\left(B_H(0,1)\right)},$$

so there exists $\ell \in \{1, \ldots, k\}$ such that $T_N x \in B_H\left(y_\ell, \frac{\varepsilon}{2}\right)$. Therefore, $Tx \in B_H\left(y_\ell, \varepsilon\right)$, hence

$$\overline{T\left(B_H(0,1)\right)} \subset \bigcup_{i=1}^{k} B_H\left(y_i, \varepsilon\right).$$

Consequently, the set $\overline{T(B_H(0,1))}$ is compact, i.e., the operator $T$ is compact. It follows that set of compact operators is closed.

Let $A$ be a bounded operator on $H$ and $T$ a compact operator on $H$. First,

$$A(T(B_H(0,1))) \subset A\left(\overline{T(B_H(0,1))}\right),$$

thus

$$\overline{A(T(B_H(0,1)))} \subset \overline{A\left(\overline{T(B_H(0,1))}\right)}$$

and since $\overline{T(B_H(0,1))}$ is compact and $A$ is continuous (because is bounded), the set $A\left(\overline{T(B_H(0,1))}\right)$ is compact. Hence the closed set $\overline{A(T(B_H(0,1)))}$ is compact, i.e., $AT$ is compact. To finish, (since $A$ is bounded) we have the inclusion $A(B_H(0,1)) \subset A(B_H(0,|||A|||))$, therefore

$$T(A(B_H(0,1))) \subset T(A(B_H(0,|||A|||)))$$

thus

$$\overline{T(A(B_H(0,1)))} \subset \overline{T(A(B_H(0,|||A|||)))}.$$

Here the set $\overline{T(A(B_H(0,|||A|||)))}$ is compact (see the previous proposition), so the closed set $\overline{T(A(B_H(0,1)))}$ is compact, consequently $TA$ is compact. □

Another useful result is the following proposition.

**Proposition 2.5.4.** *Let $T$ be a compact operator in an infinite-dimensional Hilbert space $H$. Then for every orthonormal sequence $(e_n)_n$ in $H$ we have $T(e_n) \to 0$ in the norm $\|\cdot\|_H$ as $n \to \infty$.*

*Proof.* Suppose $T(e_n)$ does not converge to 0, i.e., there exists $m > 0$ such that for any integer $n$ we have $\|Te_n\|_H^2 \geq m$. Since the sequence $(e_n)_n$ is bounded (it is orthonormal) and since $T$ is compact, there exists a compact subspace $X$ of $H$ such that for any $n$ we have $Te_n \in X$. Thus one can suppose, by extracting a subsequence if necessary, that the sequence $(Te_n)_n$ converges to a vector $y \in X \subset H$. In particular, $\|y\|_H^2 \geq m$. Moreover,

$$0 < m \leq \langle y, y\rangle_H = \lim_{n\to+\infty} \langle Te_n, y\rangle_H = \lim_{n\to+\infty} \left\langle e_n, T^\star y\right\rangle_H$$

and $\langle e_n, T^\star y\rangle_H \to 0$ as $n \to +\infty$ because $\langle T^\star y, e_n\rangle_H = \sum_{n=0}^{+\infty} |\langle T^\star y, e_n\rangle_H|^2 < +\infty$, and this is absurd. □

For finish recall (without proof) the:

**Theorem 2.5.5** (Schauder theorem). *Let $T$ be a compact operator on $H$. Then the adjoint $T^\star$ is a compact operator on $H$.*

### 2.5.3 Main spectral theorem for compact self-adjoint operators

The main result concerning compact self-adjoint operators is the following

**Theorem 2.5.6.** *Let $T$ be a compact self-adjoint operator on a infinite-dimensional separable Hilbert space $H$ such that $0 \notin \operatorname{Spec}(T)$. Then the spectrum of $T$ is a real discrete set: it consists of an infinite sequence $(\mu_k)_k$ of real eigenvalues with finite multiplicity such that $\mu_k \to 0$ as $k \to +\infty$. Moreover, the associated eigenfunctions $(e_k)_{k\geq 0}$ form a Hilbert basis of the space $H$.*

In particular, if the operator $T$ is positive the eigenvalues $(\mu_k)_k$ are positive. In this case we can suppose that

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{k-1} \geq \mu_k \to 0.$$

### 2.5.4 A proof using the Riesz–Schauder theory

The key ingredient of our proof for Theorem 2.5.6 is the following result:

**Lemma 2.5.7.** *Let $T$ be a compact operator on a separable Hilbert space $H$. Then*

(i) $\ker(I - T)$ *is a finite-dimensional space.*

(ii) *The bounded operator $I - T$ restricted to the subspace $(\ker(I - T))^{\perp}$ is invertible (with continuous inverse) from $(\ker(I - T))^{\perp}$ onto $\operatorname{im}(I - T)$.*

(iii) *The subspace $\operatorname{im}(I - T)$ is closed.*

(iv) $\ker(I - T) = \{0\} \Leftrightarrow \operatorname{im}(I - T) = H$.

*Proof.*

- Start with (i). Denote $B := \overline{B_H(0,1)} \cap \ker(I - T)$. Then $T(B) = B$. Since $T$ is compact and $B$ is bounded, $\overline{T(B)}$ is a compact subset of $H$, so $\overline{B}$ is also compact. Thus the Riesz lemma[1] applied to space $\ker(I - T)$ implies that the dimension of $\ker(I - T)$ is finite.
- Item (ii): It is clear that the operator $I - T$ restricted to the subspace $(\ker(I - T))^{\perp}$ is injective and surjective onto image $\operatorname{im}(I_d - T)$. Thus the operator:

$$S := I - T \colon (\ker(I - T))^{\perp} \longrightarrow \operatorname{im}(I - T)$$

is bijective. Now let us verify that its inverse

$$S^{-1} = (I - T)^{-1} \colon \operatorname{im}(I - T) \longrightarrow (\ker(I - T))^{\perp}$$

[1] Riesz lemma: in a Hilbert space $H$ the closed unit ball is compact if and only if the dimension of $H$ is finite.

is continuous. Suppose $S^{-1}$ is not bounded: there exists a sequence $(y_n)_n \in (\mathrm{im}(I-T))^{\mathbb{N}}$ with $\|y_n\|_H = 1$ for all $n$ and such that $\left\|S^{-1}y_n\right\|_H \to +\infty$ as $n \to +\infty$. So for any $n$ we have

$$y_n = (I-T)\,S^{-1}\,(y_n) = S^{-1}y_n - TS^{-1}y_n,$$

and $\frac{y_n}{\|S^{-1}y_n\|_H} \to 0$ as $n \to +\infty$. Let

$$x_n := S^{-1}\left(\frac{y_n}{\|S^{-1}y_n\|_H}\right) = \frac{S^{-1}y_n}{\|S^{-1}y_n\|_H}.$$

Then for any $n$, $\|x_n\| = 1$ and $x_n \in \mathrm{im}(S^{-1}) = (\ker(I-T))^{\perp}$. Moreover, $x_n - Tx_n \to 0$ as $n \to +\infty$. Since the sequence $(Tx_n)_n$ belongs to $T(B)$ and since $\overline{T(B)}$ is compact one can assume, by extracting a subsequence if necessary, that the sequence $(Tx_n)_n$ converge to a vector $x \in \overline{T(B)}$. Thus the sequence $(x_n)_n$ converge also to $x$ and since $(x_n)_n$ belongs to the closed subspace $(\ker(I-T))^{\perp}$ we have $x \in (\ker(I-T))^{\perp}$. But we also have $x - Tx = 0$; in other words, $x \in \ker(I-T)$, therefore $x = 0$ and this is absurd (because for any $n$, $\|x_n\| = 1$, so $\|x\| = 1$).

- Item (iii): The subspace

$$\mathrm{im}(I-T) = S\left((\ker(I-T))^{\perp}\right)$$

is closed because by (ii) the operator $S$ is invertible (in particular bijective and bi-continuous) and $(\ker(I-T))^{\perp}$ is closed.

- Item (iv): First, suppose $\ker(I-T) = \{0\}$ and $\mathrm{im}(I-T) \neq H$. Let consider

$$H_1 := S(H) = \mathrm{im}(S) \subset H.$$

By hypothesis, $H_1 \neq H$, in fact since $H_1 = \mathrm{im}(I-T)$ is closed by (iii), this space admits an orthogonal complement in $H$. Therefore, there exists a vector $e_1 \in H$ with $\|e_1\| = 1$ such that $e_1 \in H_1^{\perp}$. Now consider the space

$$H_2 := S(H_1) = S(S(H)) \subset H_1.$$

We have $H_2 \neq H_1$: otherwise, $S(S(H)) = S(H)$ whence $S(H) = H$ because $S$ is bijective by (ii), and then $H_1 = H$, which is absurd. Moreover, the space $H_2$ is closed: indeed, $H_2 = S(H_1)$ with $H_1$ closed and $S^{-1}$ is continuous. Consequently since $H_2$ is closed this space admits an orthogonal complement in $H_1$. Therefore, there exists a vector $e_2 \in H$ with $\|e_2\| = 1$ such that $e_2 \in H_1 \cap H_2^{\perp}$. By induction we construct a sequence of closed subspaces

$$H_n := S(H_{n-1})$$

such that $H_n \subset H_{n-1}$ with $H_n \neq H_{n-1}$, as well as a sequence of vectors $e_n \in H$ with $\|e_n\| = 1$ such that $e_n \in H_n \cap H_{n+1}^{\perp}$. By construction, $Se_n \in H_{n+1}$. It follows that $Se_n$ is orthogonal to $e_n$ and by the Pythagorean theorem

$$\|Te_n\|_H^2 = \|(I - S)\,e_n\|_H^2 = \|e_n\|_H^2 + \|Se_n\|_H^2 \geq \|e_n\|_H^2 = 1.$$

But this is absurd, because the operator $T$ is compact (see Proposition 2.5.4). Finally, $\operatorname{im}(I - T) = H$.

Conversely, suppose $\operatorname{im}(I - T) = H$. Since $\overline{\operatorname{im}(I - T)} = \ker(I - T^\star)^\perp$ (see Proposition 2.3.2) we get $\ker(I - T^\star) = \{0\}$ and therefore (with the implication $\Rightarrow$ of (iv) applied to the compact operator $T^\star$) we deduce that $\operatorname{im}(I - T^\star) = H$, so $\ker(I - T) = \{0\}$. □

**Corollary 2.5.8** (Riesz–Schauder). *Let $T$ be a compact operator on a separable Hilbert space $H$. Then $\sigma(T) - \{0\} = \operatorname{Spec}(T) - \{0\}$ and every non-null eigenvalue has a finite multiplicity.*

*Proof.* Obviously, $\operatorname{Spec}(T) - \{0\} \subset \sigma(T) - \{0\}$. Conversely, let $\lambda \in \mathbb{C}^\star$. Without loss generality one can suppose, upon replacing $T$ by the compact operator $\frac{1}{\lambda}T$ if necessary, that $\lambda = 1$. If $1 \notin \operatorname{Spec}(T)$, then using item (iv) of the previous theorem we have $\operatorname{im}(I - T) \neq H$ and thus $1 \notin \sigma(T)$. The assertion concerning the finite multiplicity is a also a direct consequence of the previous theorem. □

**Proposition 2.5.9.** *Let $T$ be a compact operator on a separable infinite-dimensional Hilbert space $H$. Then $0 \in \sigma(T)$.*

*Proof.* If $0 \notin \sigma(T)$, then 0 is in the resolvent set of $T$, the operator $T$ is invertible, and $I = TT^{-1}$. It follows that $I$ is compact (because $T$ is compact and $T^{-1}$ bounded). Consequently, $\overline{I\,(B_H(0,1))} = \overline{(B_H(0,1))}$ is closed, but this is absurd because $\dim(H) = +\infty$ (using the Riesz lemma). □

The last part in our proof of Theorem 2.5.6 is

**Lemma 2.5.10.** *Let $T$ be a compact operator on a separable infinite-dimensional Hilbert space $H$. Then every element of $\sigma(T) - \{0\}$ is isolated.*

*Proof.* Suppose $\lambda \in \sigma(T) - \{0\}$ is not isolated in $\sigma(T) - \{0\}$. Then there exists a sequence $(\lambda_n)_n \in (\sigma(T) - \{0\})^{\mathbb{N}}$ such that $\lambda_n \to \lambda$ as $n \to +\infty$. By Corollary 2.5.8, for all $n$ the scalar $\lambda_n$ is a non-null eigenvalue of $T$, so let us denote by $e_n$ an associated eigenvector (with $\|e_n\|_H = 1$) and consider the sets $E_n$, $E$ defined by $E_n := \operatorname{span}\{e_k, 1 \leq k \leq n\}$ and

$$E := \bigcup_{n=0}^{+\infty} E_n = \operatorname{span}\{e_n, n \in \mathbb{N}\}.$$

For any $n \geq 1$ we have $E_n \subset E_{n+1}$ and $E_n \neq E_{n+1}$. Indeed, assuming that $E_n = E_{n+1}$, there exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that

$$e_{n+1} = \sum_{i=1}^{n} \alpha_i e_i,$$

whence

$$\lambda_{n+1} e_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} e_i,$$

and also

$$Te_{n+1} = \sum_{i=1}^{n} \alpha_i Te_i,$$

i.e.,

$$\lambda_{n+1} e_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i e_i.$$

It follows that

$$\sum_{i=1}^{n} \alpha_i \left(\lambda_{n+1} - \lambda_i\right) e_i = 0$$

and consequently $\alpha_i = 0$ for all $i \in \{1, \ldots, n\}$. Therefore, $e_{n+1} = 0$, which is absurd. Now since for any integer $n \geq 1$ we have $E_n \subset E_{n+1}$ and $E_n \neq E_{n+1}$, we deduce that $\dim(E) = +\infty$.

Next, using the *Gram–Schmidt orthogonalization process* we construct a Hilbert basis $(f_n)_n$ of $E$: $f_1 := e_1$ and for all $n \geq 2$ we have $f_n \in E_n \cap E_{n+1}^{\perp}$. Note that $Tf_n - \lambda_n f_n \in E_{n-1}$: indeed if $f_n = \sum_{i=1}^{n} \alpha_i e_i$, then

$$Tf_n = \sum_{i=1}^{n} \alpha_i \lambda_i e_i$$

so

$$Tf_n - \lambda_n f_n = \sum_{i=1}^{n-1} \alpha_i \left(\lambda_i - \lambda_n\right) e_i \in E_{n-1}.$$

It follows that the vector $Tf_n - \lambda_n f_n$ is orthogonal to the vector $f_n$.

To finish we consider the sequence $\left(\lambda_n^{-1} f_n\right)_n$. This sequence belong the closed ball $\overline{\left(B_H(0, \frac{1}{\min \lambda_i})\right)}$. Since $Tf_n - \lambda_n f_n \in E_{n-1}$ and $Tf_n - \lambda_n f_n$ is orthogonal to $f_n$, for all $m < n$:

$$\frac{Tf_n}{\lambda_n} - \frac{Tf_m}{\lambda_m} = \left(\frac{T - \lambda_n I_d}{\lambda_n} f_n - \frac{T - \lambda_m I_d}{\lambda_m} f_m - f_m\right) + f_n \in E_{n-1} \overset{\perp}{\oplus} \operatorname{span}\{f_n\}.$$

Therefore, by the Pythagorean theorem,

$$\left\| \frac{Tf_n}{\lambda_n} - \frac{Tf_m}{\lambda_m} \right\|_H \geq \|f_n\|_H = 1.$$

Consequently, the sequence $\left(T\left(\lambda_n^{-1} f_n\right)\right)_n$ does not have Cauchy subsequences, so the operator $T$ is not compact, which is absurd. Finally every $\lambda \in \sigma(T) - \{0\}$ is isolated. $\square$

**Corollary 2.5.11.** *Let $T$ be a compact operator on a separable infinite-dimensional Hilbert space $H$. Then*

- $\sigma(T) = \{0\}$*, or*
- $\sigma(T) - \{0\}$ *is a sequence converging to* $0$.

*Proof.* If $\varepsilon > 0$ is given, using the previous lemma and the fact that the spectrum $\sigma(T)$ is compact, we conclude that the intersection of $\sigma(T)$ with the set $\{\lambda \in \mathbb{C};\ |\lambda| \geq \varepsilon\}$ is empty or finite. $\square$

All this leads to the following result.

**Theorem 2.5.12.** *Let $T$ be a compact self-adjoint operator on a infinite-dimensional separable Hilbert space $H$ such that $0 \notin \mathrm{Spec}(T)$. Then the spectrum of $T$ is a real discrete set, namely, it consists of an infinite sequence $(\mu_k)_k$ of real eigenvalues with finite multiplicity such that $\mu_k \to 0$ as $k \to +\infty$. Moreover, the associated eigenfunctions $(e_k)_{k\geq 0}$ form a Hilbert basis of $H$.*

*Proof.* If $\sigma(T) = \{0\}$, then by Corollary 2.4.8, $T = 0$, and the result is trivial, so we suppose that $\sigma(T) \neq \{0\}$. First, since $\sigma(T) - \{0\} = \mathrm{Spec}(T) - \{0\}$ (Corollary 2.5.8), we have $\sigma(T) - \{0\} = \mathrm{Spec}(T)$. Moreover, by Lemma 2.5.10, the bounded set $\sigma(T) - \{0\}$ is discrete. In other words,

$$\sigma(T) - \{0\} = \mathrm{Spec}(T) = \{\mu_n, n \geq 1\}$$

and every eigenvalue $\mu_k$ has a finite multiplicity (Corollary 2.5.8). Next, the eigenspaces $E_k := \ker(T - \mu_k I)$ are all finite-dimensional and orthogonal to one another (Theorem 2.4.6). For every eigenspace $E_k := \ker(T - \mu_k I_d)$ we can choose a finite orthogonal basis (here the eigenvalues are repeated according to their multiplicities). We thus get a family of vectors $(e_n)_{n\geq 1}$ such that $i \neq j \Rightarrow e_i \perp e_j$. Denote $F := \mathrm{span}\{e_n,\ n \geq 1\}$. We claim that $F$ is dense in $H$, i.e., $F^\perp = \{0\}$. By construction, the family $(e_n)_{n\geq 1}$ is orthonormal. Moreover, $T(F) \subset F$: if $x = \sum_{i=1}^{n} \alpha_i e_i \in F$, then $Tx = \sum_{i=1}^{n} \alpha_i \lambda_i e_i \in F$. And since $T$ is self-adjoint, we also have $T(F^\perp) \subset F^\perp$: if $x \in F^\perp$, then for all $y \in F$

$$0 = \langle x, Ty\rangle_H = \langle Tx, y\rangle_H$$

hence $Tx \in F^{\perp}$. Therefore, we can consider the restriction $S$ of the operator $T$ to $F^{\perp}$

$$S : \begin{cases} F^{\perp} \longrightarrow F^{\perp} \\ x \longmapsto Sx := Tx. \end{cases}$$

The operator $S$ is compact and self-adjoint (because $T$ is compact and self-adjoint). Then by Corollary 2.5.11 we have two cases: if $\sigma(S) = \{0\}$, then $S = 0$ (by Corollary 2.4.8), hence $F = H$. Else, if $\sigma(S) \neq \{0\}$, then $\operatorname{Spec}(S) \neq \{0\}$, thus there exists an eigenvector $e \in F^{\perp}$, $e \neq 0$, of $S$, $Se = \mu e$. But then

$$Te = \mu e,$$

i.e., the vector $e$ is an eigenvector of $T$, thus $e \in F$. Therefore $e \in F \cap F^{\perp}$, which implies that $e = 0$. This is absurd, so $F^{\perp} = \{0\}$.

Finally, the sequence $(e_n)_{n\geq 1}$ is a Hilbert basis of $H$, and since the operator $T$ is compact, $T(e_n) \to 0$ as $n \to \infty$ (Proposition 2.5.4), i.e., $\lambda_n e_n \to 0$ as $n \to \infty$ and we deduce that $\lambda_n \to 0$ as $n \to \infty$. $\square$

## 2.6 The spectral theorem multiplication operator form for unbounded operators on a Hilbert space

### 2.6.1 Spectral theorem multiplication operator form

Start by the following lemma:

**Lemma 2.6.1.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and $F$ a real-valued function finite $\mu$-almost everywhere on $X$. Then the set $D(M_F) := \{\varphi \in L^2(X);\ F\varphi \in L^2(X)\}$ is dense in $L^2(X)$ and the multiplication operator defined by:*

$$M_F : D(M_F) := \{\varphi \in L^2(X);\ F\varphi \in L^2(X)\} \longrightarrow L^2(X)$$
$$\varphi \longmapsto F\varphi$$

*is self-adjoint. Moreover, if $F$ is bounded, then the operator $M_F$ is bounded on $L^2(X)$ with $|||M_F||| = \|F\|_\infty$.*

*Proof.* First, let us verify that the domain $D(M_F)$ is dense in $L^2(X)$. Let $\varphi \in L^2(X)$ and consider the sequence $(\varphi_n)_n$ of $L^2(X)$ defined by $\varphi_n := \varphi\chi_{(|F|\leq n)}$ where $\chi$ denotes the indicator function. Since for any $n \in \mathbb{N}$, $|F\varphi_n| \leq n|\varphi|$, the sequence $(\varphi_n)_n$ belongs to $D(M_F)$. Next, since $F$ is finite almost everywhere, the sequence $(\varphi_n)_n$ converge almost everywhere on $X$ to $\varphi$. Since for all $n \in \mathbb{N}$, $|\varphi_n| \leq |\varphi| \in L^2(X)$, using the Lebesgue dominated convergence theorem shows that the sequence $(\varphi_n)_n$ converges in the $L^2(X)$-norm on $X$ to $\varphi$. Hence the domain $D(M_F)$ is dense in $L^2(X)$.

Now, let us verify that the operator $M_F$ is closed. Consider a sequence $(\varphi_n)_n$ of $D(M_F)$ such that $\varphi_n$ converge in the $L^2(X)$-norm on $X$ to some function $\varphi$. And suppose also that the sequence $(M_F\varphi_n)_n$ converge in the $L^2(X)$-norm on $X$ to a function $\psi \in L^2(X)$. Using the Lebesgue dominated convergence theorem one can suppose, by extracting a subsequence if necessary, that $(\varphi_n)_n$ converge almost everywhere on $X$ to $\varphi$. Consequently, the sequence $(F\varphi_n)_n$ converge almost everywhere on $X$ to $F\varphi$, hence $\psi = F\varphi$ almost everywhere on $X$. Therefore, $\varphi \in D(M_F)$ and $M_F(\varphi) = \psi$.

To complete the proof, let us show that the operator $M_F$ is self-adjoint. Since $F$ is a real-valued function, the operator $M_F$ is symmetric on its domain $D(M_F)$. To see that $M_F$ is self-adjoint we use the criterion $\ker\big(M_F^* \pm iI\big) = \{0\}$ (Theorem 2.3.9): let $\varphi \in \ker\big(M_F^* - iI\big)$; then $M_F^*(\varphi) = i\varphi$. On the other hand, since for all $\psi \in D(M_F)$ we have

$$\big\langle M_F^*\varphi, \psi\big\rangle_{L^2} = \langle \varphi, M_F\psi\rangle_{L^2}$$

we get for all $\psi \in D(M_F)$

$$\int_X i\varphi\overline{\psi}\, d\mu = \int_X \varphi\overline{F\psi}\, d\mu$$

i.e., for all $\psi \in D(M_F)$, $\int_X \varphi(i-F)\overline{\psi}\, d\mu = 0$. Since $D(M_F)$ is dense in $L^2(X)$ and $F$ is a real-valued function, we obtain that $\varphi = 0$, hence $\ker\big(M_F^* - iI\big) = \{0\}$. By the same argument, $\ker\big(M_F^* + iI\big) = \{0\}$. The assertion for the case $F \in L^\infty$ is trivial. $\square$

In the spectral theory of $\mathbb{C}^*$ *algebras* [Arv1], [Arv2] the following result plays an important role: (Recall that a bounded operator $A$ is *normal* if $AA^\star = A^\star A$ and a *unitary map* is an isometric and surjective map).

**Theorem 2.6.2** (Spectral theorem for bounded normal operators). *Let $N$ be a bounded normal operator on a separable Hilbert space $H$. There exist a finite measure space $(X,\mu)$ and $F$ a bounded function on $X$ such that $N$ is unitarily equivalent to the operator of multiplication by $F$ in $L^2(X)$ in the sense of the previous lemma.*

*Notation* 2.6.3. Let $A$ be a self-adjoint operator with domain $D(A)$ on a separable Hilbert space $H$. Since $\mathrm{Spec}(A) \subset \mathbb{R}$, the complex numbers $\pm i$ are not in the spectrum of $A$. We denote $R_{\pm i} = (\pm iI + A)^{-1}$.

Theorem 2.6.2 has the following

**Corollary 2.6.4.** *With the previous notations, there exist a finite measure space $(X,\mu)$, a unitary operator $U\colon H \to L^2(X)$, and a bounded function $F$ with*

*$F \neq 0$ almost everywhere on $X$, such that*

$$UR_i U^{-1} = M_F.$$

*Proof.* First let us verify that the operators $R_{\pm i}$ are normal. Since Im $(\pm iI + A) = H$, for any couple of vectors $(u, v) \in H^2$ there exists an unique couple $(z, w) \in D(A)^2$ such that $v = (-iI + A)\, z$ and $u = (iI + A)\, w$. Hence we get

$$\begin{aligned}\langle v, R_i u\rangle_H &= \langle(-iI + A)\, z, w\rangle_H = \left\langle z, (-iI + A)^*\, w\right\rangle_H \\ &= \langle z, (iI + A)\, w\rangle_H = \langle R_{-i}(v), u\rangle_H\,.\end{aligned}$$

Consequently, $R^*_{\pm i} = R_{\mp i}$. On the other hand, by the resolvent equation (2.4) the operator $R_i$ and $R_{-i}$ commute, therefore $R_i$ and $R_{-i}$ are bounded normal operators.

Next we use the Theorem 2.6.2: we must verify that the function $F \neq 0$ almost everywhere on $X$. By its definition the operator $R_i$ is injective, hence by unitary conjugation $M_F$ is also injective, i.e., $F \neq 0$ almost everywhere on $X$. □

From this result we get the usual result:

**Corollary 2.6.5** (Spectral theorem multiplication operator form). *In the previous notations, there exist a finite measure space $(X, \mu)$, a unitary operator $U\colon H \to L^2(X)$, and $f$ a real-valued function finite $\mu$-almost everywhere on $X$ such that: $v \in D(A) \Leftrightarrow U(v) \in D(M_f)$ and on the set $U(D(A))$*

$$UAU^{-1} = M_f.$$

*Proof.* Let $F$ be the function provided by Corollary 2.6.4. Consider the function $f := \frac{1}{F} - i$. Then $f$ is finite $\mu$-almost everywhere on $X$ (because $F \neq 0$ almost everywhere on $X$). By construction, $F(f + i) = 1$ and $UR_i = M_F U$. First we show that $v \in D(A) \Leftrightarrow U(v) \in D(M_f)$. Let $v \in D(A)$; since the operator $R_i$ is invertible, there exists a unique $u \in H$ such that $v = R_i u$. Consequently, $Uv = M_F U u$. Multiplying this equality by $f$ we obtain $fUv = (1 - iF)Uu$. Since $Uu \in L^2(X)$ and $F$ is bounded, we get $fUv \in L^2(X)$, i.e., $Uv \in D(M_f)$.

Conversely, let $Uv \in D(M_f)$ be a vector, so we have $(f + i)Uv \in L^2(X)$. On the other hand, since the operator $U$ between the spaces $H$ and $L^2(X)$ is unitary, there exists a unique vector $u \in H$ such that $(f + i)Uv = Uu$. Multiplying this equality by $F$ we obtain $Uv = FUu$, whence

$$v = U^{-1} FUu = R_i u,$$

which in particular shows that $v \in D(A)$.

Now we must verify that $UAU^{-1} = M_f$ on $U(D(A))$. Note that for all $u \in \mathcal{H}$ we have $FUu = UR_i u$, therefore

$$Uu = \frac{1}{F} UR_i u.$$

Let $v \in D(A)$. Then there exists a unique $u \in H$ such that $v = R_i u$. By the resolvent equation, $Av = u - iv$, hence we deduce that $UAv = (\frac{1}{F} - i)Uv$. In other words, $UAv = fUv$ and this establishes the needed equality.

To finish we must verify that $f$ is a real-valued function. Suppose that the imaginary part of $f$ is strictly non-negative on a subset $\Omega$ of $X$ such that $\mu(\Omega) > 0$. Since the measure space $(X, \mu)$ is finite, $\chi_\Omega \in D(M_f)$. By Lemma 2.6.1, the operator $M_f$ is self-adjoint, so $\langle \chi_\Omega, f\chi_\Omega \rangle_{L^2} \in \mathbb{R}$. This is absurd by the definition of the set $\Omega$. Hence, $f$ is real-valued almost everywhere. □

### 2.6.2 Functional calculus

Using Corollary 2.6.5 we construct the standard functional bounded calculus

**Definition 2.6.6.** With the previous notations, for $h \in L^\infty(\mathbb{R})$ we define the bounded operator $h(A)$ on $H$ by

$$h(A) := U^{-1} M_{h \circ f} U.$$

An obvious consequence of this definition is the following result.

**Theorem 2.6.7** (Bounded functional calculus). *Let $(A, D(A))$ be a bounded self-adjoint operator on $H$. The map $\phi$ defined by:*

$$\phi : \begin{cases} L^\infty(\mathbb{R}) \longrightarrow B(H) \\ \\ h \longmapsto h(A) \end{cases}$$

*has the following properties.*

(i) *$\phi$ is a bounded $\star$-normed algebra homomorphism, and for any function $h \in L^\infty(\mathbb{R})$*
$$|||h(A)||| \leq \|h\|_\infty .$$

(ii) *If $(h_n)_n \in (L^\infty(\mathbb{R}))^{\mathbb{N}}$ is such that $(x \mapsto h_n(x))_n$ converges to $x \mapsto x$ and such that for any $(x, n) \in \mathbb{R} \times \mathbb{N}$ we have $|h_n(x)| \leq |x|$, then for any $u \in D(A)$*
$$h_n(A)u \longrightarrow Au \; in \, \mathcal{H}.$$

(iii) *If $(\lambda, u) \in \operatorname{Spec}(A) \times (H - \{0\})$ is such that $Au = \lambda u$ and if $h \in L^\infty(\mathbb{R})$, then $h(\lambda) \in \operatorname{Spec}(h(A))$ and*

$$h(A)u = h(\lambda)u.$$

### 2.6.3 Functional calculus in Quantum Mechanics

The functional calculus of operators has numerous applications in the analysis of PDE and physics. One of them concerns the mathematical formalism of Quantum Mechanics (see for example [Car], [Gu-Si]). Quantum mechanics is ubiquitous, in the study of elementary particles, semi-conductors, optical physics, etc. Quantum mechanics is one of the greatest intellectual revolution and one of the most fantastic theory in history. The mathematical formalism of Hamiltonian classical mechanics is grounded in symplectic geometry. In Hamilton's formalism physical particles are describes by positions and velocities: for example, in the Euclidean space $\mathbb{R}^3$ every point is characterized by a vector

$$(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^6.$$

For a Hamiltonian $f \in \mathcal{C}^\infty(\mathbb{R}^6, \mathbb{R})$, the classical dynamics is governed by Hamilton's equations

$$\begin{cases} \dot{\xi}_j = -\frac{\partial f}{\partial x_j} \\ \dot{x}_j = \frac{\partial f}{\partial \xi_j}. \end{cases}$$

In quantum physics the description of a particle $M \in \mathbb{R}^n$ at time $t \geq 0$ is given by a *wave function* i.e., a vector:

$$\psi(x, t) \in L^2(\mathbb{R}^n)$$

with the following interpretation: for any subset $\Omega$ of $\mathbb{R}^n$ the real number

$$\int_\Omega |\psi(x,t)|^2 dx_1 \dots dx_n$$

is the probability of finding the particle $M$ in $\Omega$ at time $t$. In quantum mechanics the space of all possible states of the particle at a given time is called the state space. This set is the Hilbert space $L^2(\mathbb{R}^n)$.

Another principle of quantum mechanics governs the dynamics (the analogue of Hamilton's equations): if the particle is placed in a force field that derives from a potential $V$, the time evolution of the wave function $\psi(x,t)$ is governed by the famous *Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t}(x,t) = -\frac{\hbar^2}{2} \Delta_g \psi(x,t) + V(x)\psi(x,t).$$

More generally, for a self-adjoint operator $(A, D(A))$ on a Hilbert space $H$, using Theorem 2.6.7 we can define the one-parameter family

$$U(t) = \{e^{itA}\}_{t \in \mathbb{R}}.$$

This one-parameter family is a continuous unitary group of bounded operators with generator $(iA, D(A))$. In quantum mechanics the behavior of a physical system $S$ is described by a unitary group with a generator $(iA, D(A))$. More precisely, if at time $t = 0$ the system is represented by the initial state $\psi_0 \in D(A)$, then at time $t \geq 0$ it is represented by the state $\psi(t) = U(t)\psi_0$. In other words, the quantum dynamics is given by the formula

$$\psi(t) = U(t)\psi_0 \in B(H).$$

The operator $P_\hbar := -\hbar A$ is called the *quantum Hamiltonian* of the system $S$ and it represents the total energy of $S$. Thus for all $\psi_0 \in D(A)$ we have

$$\psi(t) = e^{-i\frac{t}{\hbar}P_\hbar}\psi_0.$$

Taking the derivative of this equation we obtain

$$\frac{\partial\psi(t)}{\partial t} = -\frac{i}{\hbar}P_\hbar\psi(t),$$

so if $P_\hbar = -\frac{\hbar^2}{2}\Delta + V$ we get

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\Delta_g\psi(x,t) + V(x)\psi(x,t).$$

## 2.7 Some complements on operators theory

### 2.7.1 Closable operators

**Definition 2.7.1.** An operator $(A, D(A))$ is said to be *closable* if $(A, D(A))$ has a closed extension. In this case we denote by $\overline{A}$ the minimal (in the sense of inclusion of domains) extension. This operator is called the *closure* of $(A, D(A))$.

We have:

**Proposition 2.7.2.** *The unbounded linear operator $(A, D(A))$ is closable if and only if for any sequence $(x_n)_n \in D(A)^{\mathbb{N}}$ the conditions*

$$\begin{cases} \lim\limits_{n\to\infty} x_n = 0 & \text{in the norm } \|\cdot\|_E \\ \quad and & \\ \lim\limits_{n\to\infty} Ax_n = y & \text{in the norm } \|\cdot\|_F \end{cases}$$

*implies that $y = 0$.*

*Proof.* Suppose $(A, D(A))$ is closable, i.e., there exists a closed operator $(B, D(B))$ such that $A \subset B$. Let $(x_n)_n \in D(A)^{\mathbb{N}}$ satisfy the two conditions in the proposition. Since the sequence $(x_n)_n \in D(A)^{\mathbb{N}}$ we have

$$\begin{cases} \lim_{n\to\infty} x_n = 0 \quad \text{in the norm } \|\cdot\|_E \\ \quad\text{and} \\ \lim_{n\to\infty} Bx_n = y \quad \text{in the norm } \|\cdot\|_F , \end{cases}$$

thus $y = B0 = 0$ (because $B$ is closed).

Conversely: suppose there exists a sequence $(x_n)_n \in D(A)^{\mathbb{N}}$ satisfying the hypothesis. Consider the operator $\widetilde{A}$ with domain

$$D(\widetilde{A}) = \left\{ x \in H;\ \exists\, ((x_n)_n, y) \in D(A)^{\mathbb{N}} \times H;\ \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} Ax_n = y \right\}$$

and for all $x \in D(\widetilde{A})$, we define the operator $\widetilde{A}$ by $\widetilde{A}x := \lim_{n\to\infty} Ax_n$. Now if we have two sequences $(x_n)_n, (x'_n)_n \in D(A)^{\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x'_n = x$ and $\lim_{n\to\infty} Ax_n = y$, $\lim_{n\to\infty} Ax'_n = y$, then $\lim_{n\to\infty}(x_n - x'_n) = 0$ and $\lim_{n\to\infty}(Ax_n - Ax'_n) = y - y'$, so using the hypothesis we get $y - y' = 0$. Thus $\widetilde{A}x = \lim_{n\to\infty} Ax_n$ does not depend on the sequence $(x_n)_n$, therefore the operator $\widetilde{A}$ is well defined and linear. Moreover we have $A \subset \widetilde{A}$ and $\widetilde{A}$ is closed. $\square$

**Proposition 2.7.3.** *Every symmetric operator is closable.*

**Definition 2.7.4.** An operator $A$ is said to be *essentially self-adjoint* if its closure $\overline{A}$ is self-adjoint.

### 2.7.2 Unbounded operators with compact resolvents

**Definition 2.7.5.** Let $A, D(A)$ be a closed unbounded linear operator of $H$ such that $\rho(A) \neq \emptyset$. We said that $A$ has a *compact resolvent* if for all $\lambda \in \rho(A)$ the bounded operator $(\lambda I - A)^{-1}$ is compact.

In fact, we have

**Proposition 2.7.6.** *If there exists $\mu \in \rho(A)$ such that $(\mu I - A)^{-1}$ is compact, then $A$ has a compact resolvent.*

*Proof.* Using the resolvent equation (2.4) we have

$$R_A(\lambda) = (\mu - \lambda) R_A(\lambda) R_A(\mu) + R_A(\mu)$$

for all $\lambda \in \rho(A)$, and since the set of compact operators is a vector space and an ideal in the algebra bounded operator, it is clear that $R_A(\lambda) = (\lambda I - A)^{-1}$ is also compact. $\square$

Recall that the following result

**Proposition 2.7.7.** *If* $\dim(H) = +\infty$ *and if the operator $A$ has compact resolvent, then $A$ is not bounded.*

*Proof.* We have for all $\lambda \in \rho(A)$ the equality $(\lambda I - A)(\lambda I - A)^{-1} = I$ thus if the operator $A$ is bounded, then $I = (\lambda I - A)(\lambda I - A)^{-1}$ is compact, which is absurd because $\dim(H) = +\infty$. $\square$

A main result concerning operators with compact resolvents is

**Theorem 2.7.8.** *Let $(A, D(A))$ be a closed unbounded linear operator with compact resolvent in a Banach space $E$. Then the spectrum of $A$ is a discrete set and coincides with the point spectrum of $A$, i.e.,*

$$\sigma(A) = \mathrm{Spec}(A).$$

*Moreover, every eigenvalue has a finite multiplicity.*

*Proof.* For a fixed $z \in \rho(A)$, the operator $B := (zI - A)^{-1}$ is compact, and for all $\lambda \in \mathbb{C}$ we have seen in Theorem 2.4.5 that

$$\sigma(B) - \{0\} = \left\{\frac{1}{z - \mu},\ \mu \in \sigma(A)\right\}$$

and

$$\mathrm{Spec}(B) - \{0\} = \left\{\frac{1}{z - \mu},\ \mu \in \mathrm{Spec}(A)\right\}.$$

So we have

$$\sigma(A) = \left\{z - \frac{1}{\lambda},\ \lambda \in \sigma(B) - \{0\}\right\}$$

and

$$\mathrm{Spec}(A) = \left\{z - \frac{1}{\lambda},\ \lambda \in \mathrm{Spec}(B) - \{0\}\right\}.$$

Since the operator $B$ is compact, Corollary 2.5.8 shows that $\sigma(B) - \{0\} = \mathrm{Spec}(B) - \{0\}$. Consequently,

$$\sigma(A) = \left\{z - \frac{1}{\lambda},\ \lambda \in \mathrm{Spec}(B) - \{0\}\right\},$$

thus $\sigma(A) = \mathrm{Spec}(A)$ and this proves the theorem because every eigenvalue in $\mathrm{Spec}(B) - \{0\}$ is isolated and has a finite multiplicity (Corollary 2.5.8 and Lemma 2.5.10). $\square$

Finally, the analogue of Theorem 2.5.6 for unbounded self-adjoint operators with compact resolvent reads:

**Theorem 2.7.9.** *Let $H$ be a separable Hilbert space and $(A, D(A))$ be a self-adjoint operator with compact resolvent. Then the spectrum and the point spectrum of $A$ coincide. This spectrum is a real discrete set, namely it consists of a sequence $(\lambda_k)_k$ of infinite real eigenvalues with finite multiplicity such that $\lambda_k \to +\infty$ as $k \to +\infty$. Moreover, the associated eigenfunctions $(e_k)_{k\geq 0}$ form a Hilbert basis of the space $H$.*

*Proof.* Since $A$ is self-adjoint, $\mathrm{Spec}(A) \subset \sigma(A) \subset \mathbb{R}$. Let $\lambda \in \rho(A)$. Then the bounded operator $R := (\lambda I - A)^{-1}$ is compact, thus $\sigma(R) - \{0\} = \mathrm{Spec}(R) - \{0\}$ and this is a discrete subset of $\mathbb{R}$. Consequently, the sets $\mathrm{Spec}(A)$ and $\sigma(A)$ are equal and discrete (see Lemma 2.5.10). Hence there exists $\lambda \in \mathbb{R}$ such that $\lambda \in \rho(A)$, and so we can assume now that $\lambda \in \mathbb{R}$. It follows that the operator $R$ is compact, self-adjoint and $0 \notin \mathrm{Spec}(R)$ (else there exists $x \in H$ with $x \neq 0$ such that $Rx = 0$, whence, $(\lambda I - A)^{-1} x = 0$, i.e., $x = 0$ which is absurd). By the spectral theory of compact self-adjoint operators (Theorem 2.5.6), the spectrum of $R$ is a real discrete set, more precisely, the spectrum is an infinite sequence $(\mu_k)_k$ of real non-null eigenvalues with finite multiplicity such that $\mu_k \to 0$ as $k \to +\infty$. Moreover, the associated eigenfunctions $(e_k)_{k\geq 0}$ form a Hilbert basis of the space $H$. In other words, for any integer $n$

$$Re_n = \mu_n e_n,$$

i.e.,

$$(\lambda I - A)^{-1} e_n = \mu_n e_n,$$

whence

$$e_n = \lambda \mu_n e_n - \mu_n A e_n.$$

Therefore,

$$Ae_n = \left(\lambda - \frac{1}{\mu_n}\right) e_n$$

so, with $\lambda_n := \lambda - \frac{1}{\mu_n}$ we have $\lambda_n \to \pm\infty$ as $n \to +\infty$, $\sigma(A) = \mathrm{Spec}(A) = \{\lambda_n,\, n \geq 0\}$, and for any integer $n$

$$Ae_n = \lambda_n e_n. \qquad \square$$

In particular, if the operator $A$ is positive the eigenvalues $(\lambda_n)_n$ are positive. In this case we can arrange the eigenvalues so that

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \to +\infty.$$

### 2.7.3 Numerical range and applications

**Definition 2.7.10.** Let $(A, D(A))$ be an unbounded linear operator on $H$. The *numerical range* $\Theta(A)$ of $A$ is the subset of $\mathbb{C}$ defined by

$$\Theta(A) := \{\langle Au, u\rangle_H \,;\, u \in D(A) \text{ with } \|u\|_H = 1\}\,.$$

A theorem from Hausdorff asserts that this set is convex. If the operator $(A, D(A))$ is closed, $\Gamma := \overline{\Theta(A)}$ is a closed convex subset of $\mathbb{C}$. It is easy to see that the set $\Delta := \mathbb{C} - \Gamma$ is

- connected (here $\Gamma$ is compact), or
- non-connected if $\Gamma$ is a band bounded by two parallel lines (here $\Gamma$ is not compact).

**Lemma 2.7.11.** *For any $\lambda \in \Delta$:*

(i) *The inequality $\|(\lambda I - A)u\|_H \geq d(\lambda, \Gamma)\,\|u\|_H$ holds for any $u \in D(A)$ (here $d(\lambda, \Gamma)$ denotes the distance between $\lambda$ and $\Gamma$).*

(ii) $\ker(\lambda I - A) = \{0\}$ *and* $\operatorname{im}(\lambda I - A)$ *is closed.*

(iii) *For any $\lambda \in \rho(A)$,*

$$|||(\lambda I - A)^{-1}||| \leq \frac{1}{d(\lambda, \Gamma)}.$$

*Proof.* Here $d(\lambda, \Gamma) > 0$ because $\lambda \in \Delta$.

- Item (i): for all $u \in D(A)$ with $\|u\|_H = 1$, using the Cauchy–Schwarz inequality we get

  $$\begin{aligned} d(\lambda, \Gamma) &\leq |\lambda - \langle Au, u\rangle_H| = |\langle(\lambda I - A)\,u, u\rangle_H| \\ &\leq \|(\lambda I - A)u\|_H\,. \end{aligned}$$

  Therefore we obtain (i).
- Item (ii): using (i) it is clear that $\ker(\lambda I - A) = \{0\}$. Next consider a Cauchy sequence $y_n = (\lambda I - A)\,x_n$ in $\operatorname{im}(\lambda I - A)$ (obviously the sequence $(x_n)_n$ belongs to $D(A)$). Using (i) we see that $(x_n)_n$ is a Cauchy sequence in $D(A)$, and since $H$ is complete there exists a couple $(x, y) \in H^2$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in the norm $\|\cdot\|_H$. Therefore,

  $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \lambda x_n - y_n = \lambda x - y.$$

  So, since $A$ is closed, $x \in D(A)$ and $Ax = \lambda x - y$, whence $y = (\lambda I - A)\,x \in \operatorname{im}(\lambda I - A)$. This means that the sequence $y_n = (\lambda I - A)\,x_n$ converges in $\operatorname{im}(\lambda I - A)$, thus $\operatorname{im}(\lambda I - A)$ is a complete subspace of $H$. It follows that $\operatorname{im}(\lambda I - A)$ is closed.

- Finally, let us prove (iii). If $\lambda \in \rho(A)$, then $\operatorname{im}(\lambda I - A) = H$, so using (i) we have for all $u \in D(A)$ with $u \neq 0$

$$d(\lambda, \Gamma) \leq \frac{\|(\lambda I - A)\, u\|_H}{\|u\|_H}.$$

For all $v \in H$ with $v \neq 0$,

$$d(\lambda, \Gamma) \leq \frac{\|v\|_H}{\left\|(\lambda I - A)^{-1}\, v\right\|_H},$$

so for all $v \in H$ with $v \neq 0$,

$$\frac{\left\|(\lambda I - A)^{-1}\, v\right\|}{\|v\|_H} \leq \frac{1}{d(\lambda, \Gamma)}.$$

□

Now consider the map

$$d: \begin{cases} \mathbb{C} \longrightarrow \mathbb{N} \cup \{+\infty\} \\ \lambda \longmapsto \dim\left(\ker\left(\overline{\lambda} I - A^\star\right)\right). \end{cases}$$

Using (2.2), we have also

$$d(\lambda) = \dim\left((\operatorname{im}(\lambda I - A))^\perp\right).$$

If $\lambda \in \rho(A)$, then we immediately get $\operatorname{im}(\lambda I - A) = H$, hence $d(\lambda) = 0$. In fact we can be more precise

**Proposition 2.7.12.** *For any $\lambda \in \Delta = \mathbb{C} - \overline{\Theta(A)}$,*

$$d(\lambda) = 0 \Longleftrightarrow \lambda \in \rho(A).$$

*Proof.* Let $\lambda \in \Delta = \mathbb{C} - \overline{\Theta(A)}$. If $d(\lambda) = 0$, then $\ker\left(\overline{\lambda} I - A^\star\right) = \{0\}$, and using (2.2) we have

$$\operatorname{im}(\lambda I - A)^\perp = \{0\}.$$

Applying here $\perp$, we get

$$\overline{\operatorname{im}(\lambda I - A)} = H$$

and since $\ker(\lambda I - A) = \{0\}$ and $\operatorname{im}(\lambda I - A)$ is closed (Lemma 2.7.11), we obtain $\operatorname{im}(\lambda I - A) = H$, hence $\lambda \in \rho(A)$. □

**Lemma 2.7.13** (Perturbation lemma). *Let $(A, D(A))$ be a closed unbounded linear operator of $H$ with the property that there exists $\gamma > 0$ such that $\|Ax\|_H \geq \gamma \|x\|_H$ for all $x \in D(A)$. Then for every complex number $\lambda$ such that $|\lambda| < \frac{\gamma}{2}$ we have:*

$$\dim\left(\ker\left(A^\star - \overline{\lambda} I\right)\right) = \dim\left(\ker\left(A^\star\right)\right).$$

*Proof.* Let $\lambda$ be a complex such that $|\lambda| < \gamma$ and consider the sets $M := \ker(A^\star)$, $N := \ker\left(A^\star - \overline{\lambda} I\right)$. Our proof is carried out in three steps:

- **Step 1**. We claim that $M^\perp \cap N = \{0\}$. Indeed, if $u \in M^\perp \cap N$, since $M^\perp = \overline{\mathrm{im}(A)} = \mathrm{im}(A)$ (the subspace $\mathrm{im}(A)$ is closed because for all $x \in D(A)$ we have $\|Ax\|_H \geq \gamma \|x\|_H$, see the proof of Lemma 2.3.8), there exists $v \in D(A)$ such that $u = Av$, so by hypothesis, we have $\|u\|_H \geq \gamma \|v\|_H$. Now, since $u$ belongs to $N$,

$$\begin{aligned} 0 &= \left\langle \left(A^\star - \overline{\lambda} I\right) u, v \right\rangle_H = \langle u, (A - \lambda I)\, v\rangle_H \\ &= \langle u, u - \lambda v\rangle_H = \|u\|_H^2 - \overline{\lambda}\,\langle u, v\rangle_H \\ &\geq \|u\|_H^2 - |\lambda|\,\|u\|_H\,\|v\|_H \geq \|u\|_H^2\left(1 - \frac{|\lambda|}{\gamma}\right) \end{aligned}$$

and since we have supposed that $|\lambda| < \gamma$, we have $\left(1 - \frac{|\lambda|}{\gamma}\right) > 0$. It follows that $u = 0$.

- **Step 2**. We claim that $\dim(N) \leq \dim(M)$. Indeed, if $\dim(M) = +\infty$, this is clear. If now $\dim(M) = m < +\infty$, let $(e_1, \ldots, e_m, e_{m+1})$ be a family of $m+1$ vectors in $N$ and denote by $P$ the orthogonal projector onto $M$. Then $(Pe_1, \ldots, Pe_m, Pe_{m+1})$ is a family of $m+1$ vectors in $M$, so these vectors are not linearly independent: there exists $(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}) \in \mathbb{C}^{m+1}$ with $(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}) \neq (0, 0, \ldots, 0)$ such that

$$0 = \sum_{i=1}^{m+1} \lambda_i P e_i = P\left(\sum_{i=1}^{m+1} \lambda_i e_i\right).$$

Thus, $\sum_{i=1}^{m+1} \lambda_i e_i \in N \cap M^\perp$ (because $\ker(P) = M^\perp$), consequently using Step 1, we get

$$\sum_{i=1}^{m+1} \lambda_i e_i = 0$$

with $(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}) \neq (0, 0, \ldots, 0)$, i.e., that the vectors $(e_1, \ldots, e_m, e_{m+1})$ are not linearly independent, and since the family was arbitrary, we conclude that $\dim(N) \leq m = \dim(M)$. In other words,

$$\dim\left(\ker\left(A^\star - \overline{\lambda} I\right)\right) \leq \dim\left(\ker(A^\star)\right).$$

In the same way, if we replace $\lambda$ by $-\lambda$, we have also

$$\dim\left(\ker\left(A^\star + \overline{\lambda} I\right)\right) \leq \dim(\ker\left(A^\star\right)).$$

- **Step 3**. We claim that for all $\lambda$ such that $|\lambda| < \frac{\gamma}{2}$ and for all $x \in D(A)$ we have $\|(A - \lambda I)\, x\|_H \geq \frac{\gamma}{2}\, \|x\|_H$. Indeed, for all $u \in D(A)$

$$\begin{aligned}\|(A - \lambda I)\, x\|_H &\geq |\|Ax\|_H - |\lambda|\, \|x\|_H| \\ &\geq |\gamma\, \|x\|_H - |\lambda|\, \|x\|_H| \geq \frac{\gamma}{2}\, \|x\|_H\,,\end{aligned}$$

so if we consider the operator $B := A - \lambda I$ we have for all $x \in D(A)$, $\|Bx\|_H \geq \frac{\gamma}{2}\, \|x\|_H$. By step 2, applied to the operator $B$, for any complex number $\lambda$ such that $|\lambda| < \frac{\gamma}{2}$ it holds that

$$\dim\left(\ker\left(B^\star + \overline{\lambda} I\right)\right) \leq \dim\left(\ker\left(B^\star\right)\right),$$

i.e.,

$$\dim\left(\ker\left(A^\star\right)\right) \leq \dim\left(\ker\left(A^\star - \overline{\lambda} I\right)\right).$$

Finally, we get $\dim\left(\ker\left(A^\star - \overline{\lambda} I\right)\right) = \dim\left(\ker\left(A^\star\right)\right)$. □

**Corollary 2.7.14.** *The function $\lambda \mapsto d(\lambda)$ is constant on every connected components of $\Delta = \mathbb{C} - \overline{\Theta(A)}$.*

*Proof.* We have seen (Lemma 2.7.11) that for all $\lambda \in \Delta$ and for all $u \in D(A)$,

$$\|(\lambda I - A)\, u\|_H \geq \gamma\, \|u\|_H\,,$$

where $\gamma := d(\lambda, \Gamma) > 0$. Here for any complex number $\mu$ we have

$$\begin{aligned}d(\mu) &= \dim\left(\ker\left(A^\star - \overline{\mu} I\right)\right) = \dim\left(\ker\left(\overline{\mu} I - A^\star\right)\right) \\ &= \dim\left(\ker\left(\overline{\lambda} I - A^\star + (\overline{\mu} - \overline{\lambda}) I\right)\right).\end{aligned}$$

Consider the operator $B := \lambda I - A$. Then $B^\star = \overline{\lambda} I - A^\star$, and we have $\|Bu\|_H \geq \gamma\, \|u\|_H$. Therefore, by Lemma 2.7.13: if $|\mu - \lambda| < \frac{\gamma}{2}$,

$$\dim\left(\ker\left(B^\star - \overline{(\mu - \lambda)} I\right)\right) = \dim\left(\ker\left(B^\star\right)\right),$$

i.e., for any complex number $\mu$ such that $|\mu - \lambda| < \frac{\gamma}{2}$ we have

$$d(\mu) = d(\lambda).$$

Hence, the dimension of $\overline{\lambda} I - A^\star$ is locally constant, so the dimension of $\overline{\lambda} I - A^\star$ is constant on every connected components of $\Delta = \mathbb{C} - \overline{\Theta(A)}$. □

**Proposition 2.7.15.** *Let $A$ be a bounded operator of $H$. We have $\sigma(A) \subset \overline{\Theta(A)}$.*

*Proof.* Here

$$\Theta(A) = \{\langle Au, u\rangle_H \,;\, u \in H \text{ with } \|u\|_H = 1\}$$

thus for all $u \in H$ with $\|u\|_H \leq 1$ we have $|\langle Au, u\rangle_H| \leq |||A|||$. Therefore $\Theta(A) \subset B$, where $B$ is the closed disc of $\mathbb{C}$ with center 0 and radius $|||A|||$, so

$$\overline{\Theta(A)} \subset B.$$

Thus $\Delta := \mathbb{C} - \overline{\Theta(A)}$ is an open and connected subset of $\mathbb{C}$ (because $\overline{\Theta(A)}$ is compact) and $\{\lambda \in \mathbb{C};\, |\lambda| > |||A|||\} \subset \Delta$ (because $\overline{\Theta(A)} \subset B$). By the classical theory of bounded operators in Hilbert spaces, we have also $\sigma(A) \subset B$, hence if $|\lambda| > |||A|||$ then $\lambda \in \rho(A)$. It follows that $d(\lambda) = 0$ (because $\operatorname{im}(\lambda I - A) = H$, i.e., $\ker\left(A^\star - \overline{\lambda} I\right) = \{0\}$). Now, since the map $\lambda \mapsto d(\lambda)$ is constant on every connected component of $\Delta$, we get for all $\lambda \in \Delta$ that $d(\lambda) = 0$. Finally, using Proposition 2.7.12 we have $\Delta \subset \rho(A)$, i.e., $\sigma(A) \subset \overline{\Theta(A)}$. □

Consider now a symmetric unbounded operator $(A, D(A))$. First, it is clear that $\Theta(A) \subset \mathbb{R}$, Hence for a symmetric operator we always have $\pm i \in \Delta$. Denote

$$m_+ := \dim\left(\ker\left(\overline{\lambda} I - A^\star\right)\right) \quad \text{for } \operatorname{im}(\lambda) > 0.$$

In fact the number $m_+$ does not depend on the choice of $\lambda \in \mathbb{C}$ such that $\operatorname{im}(\lambda) > 0$. Indeed, since $\overline{\Theta(A)} \subset \mathbb{R}$, the set $P_+ := \{\lambda \in \mathbb{C};\, \operatorname{im}(\lambda) > 0\}$ is included in one of the connected components of $\Delta$, and we conclude that the map $\lambda \mapsto d(\lambda)$ is constant on $P_+$. Similarly, define

$$m_- := \dim\left(\ker\left(\overline{\lambda} I - A^\star\right)\right) \quad \text{for } \operatorname{im}(\lambda) < 0$$

$m_-$ does not depend on the choice of $\lambda \in \mathbb{C}$ such that $\operatorname{im}(\lambda) < 0$. Note that if the set $\Delta = \mathbb{C} - \overline{\Theta(A)}$ is connected then $m_+ = m_-$. Hence for a symmetric operator $\pm i \in \Delta$ and $m_+ = d(i)$, $m_- = d(-i)$.

**Theorem 2.7.16.** *Let $(A, D(A))$ be a closed symmetric operator of $H$. $A$ is self-adjoint if and only if $m_+ = m_- = 0$.*

*Proof.* Since $A$ is closed and symmetric, using Theorem 2.3.9 we get

$$m_+ = m_- = 0 \iff d(\pm i) = 0 \iff \ker\left(A^\star \pm iI\right) = \{0\} \iff A = A^\star. \quad \square$$

## 2.8 Exercises

**Exercise 2.8.1.** *Prove that the set $B(H)$ of bounded operators on a Hilbert space $H$ is a Banach space for the norm $|||\cdot|||$. Show that for all $T, S \in B(H)$ we have*

$$|||TS||| \leq |||T|||\,|||S|||\,.$$

**Exercise 2.8.2.** *Prove that for a bounded operator the spectrum and point spectrum coincide.*

**Exercise 2.8.3.** *Prove Theorem 2.5.5.*

**Exercise 2.8.4.** *Show that the operator*

$$T\colon \begin{cases} \ell^2 \longrightarrow \ell^2 \\ u_n \longmapsto \lambda_n u_n \end{cases}$$

*is bounded if and only if the sequence $(\lambda_n)_n$ is bounded. In this case show that $T$ is invertible with a not-bounded inverse.*

**Exercise 2.8.5.** *Show that the operator*

$$T\colon \begin{cases} \ell^2 \longrightarrow \ell^2 \\ u_n \longmapsto \frac{1}{n} u_n \end{cases}$$

*is invertible with a not-bounded inverse.*

**Exercise 2.8.6.** *Show that the operator*

$$S\colon \begin{cases} \ell^2 \longrightarrow \ell^2 \\ u_n \longmapsto u_{n+1} \end{cases}$$

*is a bounded.*

**Exercise 2.8.7.** *Let us denote by $B_{\mathrm{inv}}(H)$ the set of bounded invertible operators with bounded inverse. Show that the mapping*

$$\begin{cases} B_{\mathrm{inv}}(H) \longrightarrow B_{\mathrm{inv}}(H) \\ T \longmapsto T^{-1} \end{cases}$$

*is continuous (for the norm $|||\cdot|||$).*

**Exercise 2.8.8.** *Show that if an operator is symmetric then its eigenvalues are real.*

**Exercise 2.8.9.** *Prove that the point spectrum of the operator $T$ from the Exercise 2.8.4 is* $\mathrm{Spec}(T) = \{\lambda_n, n \geq 1\}$ *and that the spectrum of $T$ is* $\sigma(T) = \overline{\mathrm{Spec}(T)}$.

**Exercise 2.8.10.** *Consider the operator*

$$M_x \colon \begin{cases} L^2[0,1] \longrightarrow L^2[0,1] \\ \\ \quad f(x) \longmapsto x f(x). \end{cases}$$

*Show that $M_x$ is bounded and* $|||M_x||| \leq 1$. *Next prove that* $\mathrm{Spec}(M_x) = \emptyset$ *and* $\sigma(T) = [0,1]$.

**Exercise 2.8.11.** *Show that the operator of Exercise 2.8.10 is self-adjoint.*

**Exercise 2.8.12.** *Show that the operator of Exercise 2.8.4 is compact if and only if* $\lambda_n \to 0$.

**Exercise 2.8.13.** *Show that operator of Exercise 2.8.10 is not compact.*

**Exercise 2.8.14** (Cayley transform). *Let $(T, D(T))$ be a closed symmetric operator on a Hilbert space $H$.*

1. Show that $T + iI$ is injective with a closed image.
2. Show that the operator $U_T := (T - iI)\,(T + iI)^{-1}$ with domain $D(U_T) = \mathrm{Im}\,(T + iI)$ is closed with a closed domain.
3. Prove that $U_T$ is isometric with a closed image.
4. Prove that $\ker\,(I - U_T) = \{0\}$.
5. Show that $T = i\,(I + U_T)\,(I - U_T)^{-1}$ .

# Chapter 3
# The Laplacian on a compact Riemannian manifold

The goal of this chapter is to introduce some fundamental notions of Riemannian Geometry and analysis on manifold. At the end of the chapter we define the main object of this book: the Laplace–Beltrami operator on a compact Riemannian manifold.

**Assumption.** *All manifolds we work with are supposed to be connected, smooth, and countable at infinity.*

## 3.1 Basic Riemannian Geometry

One of the principal characteristics of modern physics is the geometry: general relativity, string theory, quantum field theory ... Riemannian geometry is a generalisation of Euclidean geometry, Riemannian geometry is in particular associated to general relativity and gauge theory.

For an introduction on differentiable manifolds see for example [War] or [DRh] and for a complete course on Riemannian Geometry see for example the books of I. Chavel [Cha2], Gallot–Hullin–Lafontaine [GHL], J. Jost [Jos], M. Do Carmo [DoC], T. Sakai [Sak]. See also the bible of M. Berger [Ber].

### 3.1.1 Differential Geometry: conventions and notations

First we need to introduce some notations: start with some very basics facts (without proofs) on differential geometry.

*From now on, all manifolds are supposed to be connected.*

**Definition 3.1.1.** A Hausdorff (i.e., possessing a countable basis of topology) separable topological space $M$ is a *$n$-dimensional topological manifold* if for each point $x \in M$ there exists an open neighbourhood $U$ of $x$ and a function $\varphi: U \to \mathbb{R}^n$ such that $\varphi$ is an homeomorphism from $U$ onto on open set of $\mathbb{R}^n$.

The pair $(U, \varphi)$ is called a *chart* of $M$ and for $y \in U$ the components of the vector

$$\varphi(y) = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

are called the *coordinates* of the point $y$ in the chart $(U, \varphi)$. An *atlas* of $M$ is a collection of charts $(U_i, \varphi_i)$ such that $M = \bigcup_i U_i$. We say that a topological manifold $M$ is a *smooth manifold* if for every pair of charts $(U, \varphi)$ and $(V, \psi)$ such that $U \cap V \neq \emptyset$, the map

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \subset \mathbb{R}^n \to \psi(U \cap V) \subset \mathbb{R}^n$$

is smooth in the classical sense of differential calculus in $\mathbb{R}^n$. This type of maps $\psi \circ \varphi^{-1}$ are called *transition functions*.

**Example 3.1.2.**

- The first simple example is the Euclidean space $\mathbb{R}^n$: here the atlas is reduced to the open set $U = \mathbb{R}^n$ and the unique chart is the identity map.
- The second usual example is the $n$-sphere $\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1};\ \|x\| = 1\right\}$: here one can consider an atlas with two charts: $U = \mathbb{S}^n - \{N\}$ and $V = \mathbb{S}^n - \{S\}$, where $N, S$ are the north and the south poles of the sphere. The homeomorphisms are the *stereographic projections*

$$\varphi_N \colon \begin{cases} U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n \\ (x_1, x_2, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), \end{cases}$$

and

$$\varphi_S \colon \begin{cases} V \subset \mathbb{R}^{n+1} \to \mathbb{R}^n \\ (x_1, x_2, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{1+x_{n+1}}, \frac{x_2}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}}\right). \end{cases}$$

**Definition 3.1.3.** A map $f \colon M \to N$ between two smooth manifolds is said to be *smooth* (or $\mathcal{C}^\infty$) if for any two charts $(U, \varphi)$ of $M$ and $(V, \psi)$ of $N$ the map

$$\psi \circ f \circ \varphi^{-1} \colon \varphi\left(U \cap f^{-1}(V)\right) \subset \mathbb{R}^n \to \psi\left(f(U) \cap V\right) \subset \mathbb{R}^n$$

is smooth in the classical sense.

In the case $N = \mathbb{R}$ we denote by $\mathcal{C}^\infty(M)$ the *set of smooth functions* between $M$ and $\mathbb{R}$.

**Definition 3.1.4.** A *tangent vector* to a smooth manifold $M$ at a point $x \in M$ is a *derivation* on $\mathcal{C}^\infty(M)$, that is, a linear form $X \colon \mathcal{C}^\infty(M) \to \mathbb{R}$ such that the *Leibniz rule*

$$X(fg) = X(f)g(x) + f(x)X(g)$$

holds for all $(f, g) \in \mathcal{C}^\infty(M)^2$.

For a fixed point $x$ on $M$ the set of derivations can be equipped with the following operations: if $X, Y$ are two derivations and $\lambda$ a scalar, for all $f \in \mathcal{C}^\infty(M)$ we put
$$(\lambda X + Y)(f) = \lambda X(f) + Y(f).$$
Thus the set of derivations is a $n$-dimensional vector space, called the *tangent space* of $M$ at the point $x$. This vector space is denoted by $T_x M$. The relation between this definition and the geometric intuitive notion of tangent vector is the following: for a curve $\gamma\colon ]r, r[ \to M$ such that $\gamma(0) = x$, the corresponding derivation acts on $f \in \mathcal{C}^\infty(M)$ as

$$X(f) = \frac{d\,(f \circ \gamma)}{dt}(0).$$

Given a chart $(U, \varphi)$ containing $x$, we define the tangent vector $\left(\frac{\partial}{\partial x_i}\right)_x$ by

$$\left(\frac{\partial}{\partial x_i}\right)_x (f) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x_i}(\varphi(x))$$

for all $f \in \mathcal{C}^\infty(M)$. In fact, the $n$-tuple $\left\{\left(\frac{\partial}{\partial x_i}\right)_x\right\}_{1 \le i \le n}$ is a basis of $T_x M$, so for every vector $X \in T_x M$ there exists $n$-scalars $X_1, X_2, \ldots, X_n$ such that

$$X = \sum_{i=1}^{n} X_i \left(\frac{\partial}{\partial x_i}\right)_x,$$

and for a local chart $\left(U, \varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n)\right)$ we have

$$X_i = X(\varphi^i).$$

**Definition 3.1.5.** Let $f\colon M \to \mathbb{R}$ be a smooth function on the manifold $M$. For any point $x \in M$ we define the *differential* $df(x)$ by

$$df(x)\colon \begin{cases} T_x M \longrightarrow \mathbb{R} \\ X \longmapsto (df(x))\,X := X(f). \end{cases}$$

*Remark* 3.1.6. There exists another equivalent ways to define the tangent space (see [GHL]).

The (disjoint) union of the tangent spaces

$$TM := \coprod_{x \in M} T_x M$$

admit a structure of smooth $2n$-dimensional manifold, called the *tangent bundle* of $M$.

**Definition 3.1.7.** A *vector field* of a manifold $M$ is a *section* of the tangent bundle $TM$, i.e., a smooth map

$$X: \begin{cases} M \longrightarrow TM \\ x \longmapsto X_x \in T_x M. \end{cases}$$

We denote by $\Gamma(M)$ the set of vector fields on $M$. This set can be equipped with the *Lie bracket*: for $(X, Y) \in \Gamma(M)^2$, for all $f \in \mathcal{C}^\infty(M)$ and for all $x \in M$,

$$[X, Y]_x(f) := X_x(Y(f)) - Y_x(X(f))$$

where $Y(f): y \in M \mapsto Y_y(f) \in \mathbb{R}$ and $X(f): y \in M \mapsto X_y(f) \in \mathbb{R}$ are smooth functions.

Let $X$ be a smooth vector field of $M$ and $x$ a fixed point on $M$. By classical results on dynamical systems and ODE theory there exists an unique smooth curve $\gamma: ]r, r[\to M$ such that:

$$\begin{cases} \gamma'(t) = X(\gamma(t)) \\ \gamma(0) = x. \end{cases}$$

Hence the *local flow* of the vector field $X$ with origin $x$ at time $t$ is defined by

$$\varphi_t(x) = \gamma(t).$$

For a fixed point $x \in M$, the *cotangent space* of $M$ at $x$ is as defined as the (algebraic) dual space $T_x^\star M$ of $T_x M$, and the associated dual basis of $\left\{\left(\frac{\partial}{\partial x_i}\right)_x\right\}_{1\leq i\leq n}$ is denoted $\{(dx_i)_x\}_{1\leq i\leq n}$: namely,

$$(dx_i)_x\left(\left(\frac{\partial}{\partial x_i}\right)_x\right) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ otherwise.} \end{cases}$$

The (disjoint) union of the cotangent spaces

$$T^\star M := \coprod_{x\in M} T_x^\star M$$

also admits a structure of smooth $2n$-dimensional manifold, called the *cotangent bundle* of $M$.

Let $f\colon M \to N$ be a smooth function between two smooth manifolds. For a fixed point $x \in M$ we define the map $T_x f\colon T_x M \to T_{f(x)} N$ by the rule: for all $u \in T_x M$ and $g \in \mathcal{C}^\infty(N)$,

$$(T_x f(u))\,(g) := u(g \circ f).$$

It is easy to see that this map in linear between the spaces $T_x M$ and $T_{f(x)} N$. Consequently we can also define the *(linear) tangent map of* $f$ by

$$Tf\colon \begin{cases} TM \longrightarrow TN \\ (x,u) \longmapsto (f(x), (T_x f(u))) \,. \end{cases}$$

**Definition 3.1.8.** *A connection* on $M$ is a map

$$D\colon TM \times \Gamma(M) \longrightarrow TM$$

such that:

(i) for all $x \in M$, if $X \in T_x M$ and $Y \in \Gamma(M)$ then $D(X,Y) \in T_x M$,

(ii) for all $x \in M$, the map $D\colon T_x M \times \Gamma(M) \to T_x M$ is bilinear,

(iii) for all $x \in M$, all $X \in T_x M$, and all $Y \in \Gamma(M)$, if $f\colon M \to \mathbb{R}$ is differentiable, then

$$D(X, fY) = X(f)Y_x + f(x)D(X,Y).$$

If $D$ is a connection on $M$, we denote

$$D_X Y := D(X,Y).$$

The operator $D$ is a *local operator* in the sense that for any point $x \in M$ and for any $X, Y, Z \in \Gamma(M)$, if $Y = Z$ in a neighbourhood of $x$, then $D_X Y = D_X Z$.

In this book, for $x \in M$ and $k \in \mathbb{N}$ we denote by $\Lambda^k\left(T_x^\star M\right)$ the set of $k$-alternating forms on the vector space $T_x^\star M$. The (disjoint) union of these spaces is denoted by

$$\Lambda^k(M) := \coprod_{x \in M} \Lambda^k\left(T^\star M\right)$$

and admits a structure of fiber bundle. If $k = 1$, $\Lambda^1(M) = T^\star M$, and by convention for $k = 0$ we put $\Lambda^0(M) = \mathbb{R}$.

**Definition 3.1.9.** A *differential form of degree* $k$ (or a $k$-form) on $M$ is a smooth section of the bundle $\Lambda^k(M)$. We denote by $\Omega^k(M)$ the set of smooth sections of $\Lambda^k(M)$, with the convention $\Omega^0(M) = \mathcal{C}^\infty(M)$.

The set $\Omega^k(M)$ can be equipped with the *exterior derivative operator* $d$,

$$d\colon \begin{cases} \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \\ \qquad \alpha \longmapsto d\alpha. \end{cases}$$

Namely, if

$$\alpha = \sum_i \alpha_i \, dx_i,$$

then we put

$$d\alpha = \sum_{i,j} \left( \frac{\partial \alpha_i}{\partial x_j} \right) dx_j \wedge dx_i,$$

where $\wedge$ is the wedge (exterior) product on alternating forms: for an $n_1$-form $\omega_1$ and for an $n_2$-form $\omega_2$, the element $\omega_1 \wedge \omega_2$ is the $(n_1+n_2)$-form defined in local coordinates for all $\left(V_1, V_2, \dots, V_{n_1+n_2}\right) \in T_x^\star M$ by the formula

$$\begin{aligned} (\omega_1 \wedge \omega_2)\left(V_1, V_2, \dots, V_{n_1+n_2}\right) = \sum_{\sigma \in S_{n_1+n_2}} & \frac{(-1)^\sigma}{n_1! n_2!} \omega_1 \left(V_{\sigma(1)}, V_{\sigma(2)}, \dots, V_{\sigma(n_1)}\right) \\ & \cdot \omega_2 \left(V_{\sigma(n_1+1)}, V_{\sigma(n_1+2)}, \dots, V_{\sigma(n_1+n_2)}\right) \end{aligned}$$

where $S_{n_1+n_2}$ denotes the symmetric group with $n_1 + n_2$ elements

The operator $d$ has the fundamental property

$$d \circ d = 0.$$

If $k = 0$, the operator $d$ coincides with the usual differential operator on functions. For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^\ell(M)$ we have the relation:

$$d\,(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

A differential form $\alpha \in \Omega^k(M)$ is said to be *closed* if $d\alpha = 0$, and $\alpha$ is said to be *exact* if there exists a differential form $\beta \in \Omega^{k-1}(M)$ such that $\alpha = d\beta$. Since $d^2 = 0$, every exact form is closed. By the *Poincaré lemma*, on a contractible[1] domain every closed form is exact. The codifferential operator $\delta$ is the formal $L^2$-adjoint of $d$ on $\Omega^k(M)$.

On the set of differential forms we have for any vector field $X$ the *interior product operator* $i_X$, defined by

$$i_X\colon \begin{cases} \Omega^{k+1}(M) \longrightarrow \Omega^k(M) \\ \qquad \alpha \longmapsto i_X(\alpha), \end{cases}$$

[1] The fundamental group is trivial.

where for all $x \in M$ and for all $(V_1, V_2, \ldots, V_k) \in T_x^\star M$,

$$i_X(\alpha)(x)\,(V_1, V_2, \ldots, V_k) = \alpha(x)(X(x), V_1, V_2, \ldots, V_k).$$

For any smooth map $\varphi\colon M \to N$ between two manifolds $M, N$, *the pull-back* of $\varphi$ is defined by: for any $x \in M$, $\beta \in \Omega^k(M)$, and $(V_1, V_2, \ldots, V_k) \in T_x^\star M$,

$$\left(\varphi^\star \beta\right)(x)(V_1, V_2, \ldots, V_k) = \beta(\varphi(x))\,(T_x\varphi(V_1), T_x\varphi(V_2), \ldots, T_x\varphi(V_k))\,.$$

Finally, the *Lie derivative* associated to a smooth vector field $X$ is defined by the formula: for all $\beta \in \Omega^k(M)$,

$$\mathcal{L}_X(\beta) := \lim_{t\to 0} \frac{\varphi_t^\star \beta - \beta}{t},$$

where $\varphi_t$ is the local flow generated by $X$. We have the *Cartan formula*

$$\mathcal{L}_X(\beta) = i_X d\beta + d\,(i_X\beta)\,;$$

in particular, for a function $f$ (i.e., $f \in \Omega^0(M)$), we get

$$\mathcal{L}_X(f) = i_X df = df(X).$$

### 3.1.2 Riemannian manifolds and examples

**Definition 3.1.10.** *A Riemannian metric* $g$ on a manifold $M$ is a map $g$ which associates to each point $x$ of $M$ a scalar product $g(x)(\cdot,\cdot)$ on the vector space $T_xM$. We also require that $g$ is smooth: if $X$ and $Y$ are $\mathcal{C}^\infty$ vector fields, then the map $x \mapsto g(\cdot)(X, Y)$ is smooth on $M$. A *Riemannian manifold* is a manifold equipped with a Riemannian metric.

Given a manifold $M$, using a partition of unity, the existence of a Riemannian metric on $M$ is given by the following classical result:

**Theorem 3.1.11.** *On every manifold $M$ there exists at least one Riemannian metric.*

**Example 3.1.12.**

- The simplest example of Riemannian manifold is the Euclidean space $\mathbb{R}^n$ with the *canonical metric* "can", which is the usual scalar product on $\mathbb{R}^n$: for every $x \in \mathbb{R}^n$, and all $X, Y \in T_x\mathbb{R}^n$ with $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$

$$\mathrm{can}(x)(X, Y) := \langle X, Y\rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} x_i y_i\,.$$

- Another example of Riemannian metric on the Euclidean space $\mathbb{R}^n$ is obtained as follows: for a fixed positive definite $n \times n$ matrix $A$ we define the metric $g_A$ by: for every $x \in \mathbb{R}^n$, and all $X, Y \in T_x\mathbb{R}^n$,

$$g_A(x)(X,Y) := \langle AX, Y\rangle_{\mathbb{R}^n} .$$

Let us give an example on the sphere.

**Example 3.1.13.** The *Euclidean sphere*: the sphere $\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1};\ \|x\| = 1\right\}$ equipped with the metric it inherits as a subspace of $\mathbb{R}^{n+1}$ is a Riemannian manifold.

We finish with a hyperbolic example.

**Example 3.1.14.** The *hyperbolic space*: let $\mathbb{H}^n := \{x \in \mathbb{R}^n;\ \|x\| < 1\}$, the open unit ball in $\mathbb{R}^n$. The hyperbolic space $\mathbb{H}^n$ can be equipped with the *hyperbolic metric* $g$ defined for every point $x \in \mathbb{H}^n$ and for all $X, Y \in T_x\mathbb{H}^n$ by

$$g(x)(X,Y) := \frac{4\,\langle X, Y\rangle_{\mathbb{R}^n}}{\left(1 - \|x\|^2_{\mathbb{R}^n}\right)^2}.$$

In the case of dimension 2, $\mathbb{H}^2$ is also called the *Poincaré disk*.

Consider a Riemannian manifold $(M, g)$ of dimension $n$. Using the metric $g$ we can do differential calculus on $M$ via the Levi-Civita connection.

**Definition 3.1.15.** Let $(M, g)$ be a Riemannian manifold. The *Levi-Civita connection* is the unique connection $D$ on $TM$ such that

(i) $D$ is torsion free: for all smooth vector fields $X, Y$ of $M$,

$$D_X Y - D_X Y = [X, Y] .$$

(ii) $D$ is compatible with the metric $g$: for all smooth vector fields $X, Y, Z$ of $M$,

$$X(g(Y,Z)) = g\left(D_X Y, Z\right) + g\left(Y, D_X Z\right).$$

For a Riemannian manifold $(M, g)$ of dimension $n$ and for a local chart $\phi\colon U \subset M \to \mathbb{R}^n$ which coordinates are denoted by $\left(x^1, x^2, \dots, x^n\right)$, we denote by $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\right)$ the associated vector fields. Then we have the following local expression of the Levi-Civita connection:

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

where $\Gamma_{ij}^k$ denote the *Christoffel symbols*.

### 3.1.3 Metric structure and geodesics on a Riemannian manifold

On a Riemannian manifold we have a natural metric structure, indeed, for a curve $\gamma\colon [0,1] \to M$, the length of $\gamma$ is defined by

$$\mathcal{L}(\gamma) := \int_0^1 \sqrt{g(\gamma(t))\,(\dot\gamma(t), \dot\gamma(t))}\, dt,$$

where $\dot\gamma(t) \in T_{\gamma(t)}M$ is the velocity vector of the curve. We can now define a distance (called the *Riemannian distance*) on the manifold $(M, g)$: for any pair of points $(x, y)$ in $M$, the Riemannian distance between $x$ and $y$ is

$$d(x, y) := \inf_{\gamma \in \mathcal{C}(x \to y)} \mathcal{L}(\gamma),$$

where the infimum is taken over the set $\mathcal{C}(x \to y)$ of smooth curves on $M$ going from the point $x$ to the point $y$. The topology associated to this metric coincides with the natural topology of the manifold $M$. If the metric space $(M, d)$ is complete (i.e., every Cauchy sequence is convergent), then every pair of points $(x, y)$ on $M$ can be joined by a curve $\gamma$ and

$$d(x, y) = \mathcal{L}(\gamma).$$

This result is a consequence of the Hopf–Rinow theorem (see Theorem 3.1.19). The curve $\gamma$ going from the point $x$ to the point $y$ is a geodesic. *Geodesics* are curves which locally minimize length and realize the distance between two points on the manifold, i.e., for $\varepsilon$ small enough the curve $\gamma$ is the shortest path between the points $\gamma(t)$ and $\gamma(t+\varepsilon)$. Geodesic curves are curves which satisfy the non-linear second-order differential equation

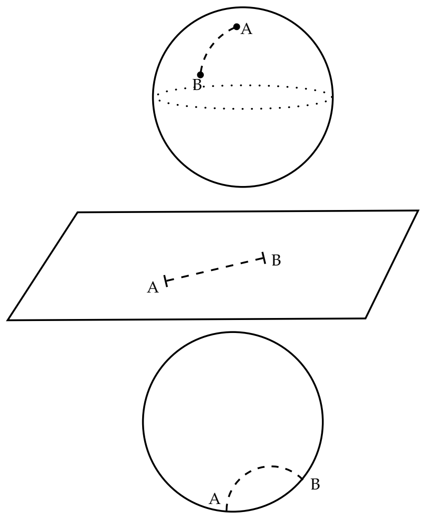
$$D_{\dot\gamma(t)}\dot\gamma(t) = 0,$$

in local coordinates,

$$\ddot\gamma_k(t) + \sum_{i,j}^{n} \Gamma_{ij}^k \dot\gamma_i(t)\dot\gamma_j(t) = 0.$$

By the standard existence and uniqueness results for ordinary differential equations, for every point $x \in M$ and for every vector $U \in T_xM$ there exist $\eta > 0$ and a unique geodesic $\gamma_U\colon ]-\eta, \eta[ \to M$ such that $\gamma_U(0) = x$ and $\dot\gamma_U(0) = U$. For example

- on the Euclidean plane $\mathbb{R}^2$ geodesics are straight lines,
- on the round sphere $\mathbb{S}^2$ geodesics are great circles,
- on the hyperbolic Poincaré disk geodesics are arcs of circles meeting the unit circle orthogonally, or segments through the origin.




The Hopf–Rinow theorem states that the mapping $t \mapsto \gamma_U(t)$ is well defined for all $t \in \mathbb{R}$ if and only if the metric space $(M, d)$ is complete.

**Definition 3.1.16.** We say a Riemannian manifold $(M, d)$ is *complete* if the metric space $(M, d)$ is complete.

We have also

**Definition 3.1.17.** Let $(M, d)$ be a complete Riemannian manifold. For every fixed point $x \in M$ we define the *exponential map* at the point $x$ by

$$\exp_x : \begin{cases} T_x M \longrightarrow M \\ U \longmapsto \exp_x U := \gamma_U(1), \end{cases}$$

where $t \mapsto \gamma_U(t)$ is the unique geodesic such that $\gamma_U(0) = x$ and $\dot{\gamma}_U(0) = U$.

Geometrically, the point $\exp_x U$ on $M$ is obtained by drawing the geodesic starting from the point $x$ and going in the direction of the vector $U$. The Riemannian distance between the point $x$ and $\exp_x U$ is exactly equal to $\|U\|$. If we denote by $B(x, r)$ the open ball of center $x$ and radius $r$ of $M$ for the Riemannian distance, and by $B_{T_x M}(0_x, r)$ the open ball of center $x$ and radius $r$ of $T_x M$ for the Euclidean norm, then

$$\exp_x(B_{T_x M}(0_x, r)) = B_M(x, r)$$

for all $r > 0$. The exponential map is well defined on the whole tangent space (since the manifold is complete), but is not a global diffeomorphism from $T_x M$

on $M$. By the local inversion theorem, there exists a radius $r > 0$ such that exponential map at the point $x$ is a local diffeomorphism from $B_{T_xM}(0_x, r)$ to $\exp_x(B_{T_xM}(0_x, r)) = B_M(x, r)$.

**Definition 3.1.18.** The *injectivity radius* of the manifold $(M, g)$ at the point $x \in M$ is the largest radius $r > 0$ such that the exponential map at the point $x$ is a local diffeomorphism from the ball of radius $r$ and center $0_x \in T_xM$. The *injectivity radius* of the manifold $(M, g)$ is the infimum of the injectivity radius over all points $x \in M$.

In other words, the injectivity radius of $M$ is the largest number $r > 0$ such that for every pair of points $(x, y) \in M^2$ with $d_M(x, y) < r$ there exists an unique geodesic curve which joins $x$ and $y$. If the manifold $(M, g)$ is compact, the injectivity radius is strictly positive. We have (see [GHL] for a proof):

**Theorem 3.1.19** (Hopf–Rinow Theorem). *Let $(M, g)$ be a Riemannian connected manifold. The following conditions are equivalent:*

(i) *The manifold $(M, g)$ is complete.*

(ii) *There exists $x \in M$ such that the map* $\exp_x$ *is defined on the whole tangent space $T_xM$.*

(iii) *The map* $\exp_x$ *is defined on the whole tangent space $T_xM$ for all $x \in M$.*

(iv) *Compact sets of $M$ are exactly closed and bounded sets of $M$.*

*Moreover, any of these conditions implies that every pair of points $(x, y) \in M^2$ can be joined by a geodesic curve.*

*From now on, all Riemannian manifolds are supposed to be complete.*

### 3.1.4 Curvatures on a Riemannian manifold

Now, let us recall the main notions of curvature in Riemannian geometry. We start with the tensorial point of view. Let $D$ be the Levi-Civita connection associated to the metric $g$. Let $X$ and $Y$ be smooth vector fields on $M$. The *curvature tensor* is defined by

$$R(X, Y) := [D_X, D_Y] - D_{[X,Y]}.$$

So, for two smooth vector fields $X$ and $Y$, $R(X, Y)$ is a map from the set of smooth vector fields on $M$ into itself; and for any smooth vector field $U$ on $M$ we have

$$R(X, Y)U = D_X(D_YU) - D_Y(D_XU) - D_{[X,Y]}U.$$

In fact, we can use $R$ to build a tensor of type $(0, 4)$. Namely, for four smooth vector fields $X, Y, Z, T$, we define the tensor $R(X, Y, Z, T)$ by

$$R(X, Y, Z, T) := g(R(X, Y)T, Z).$$

For a fixed point $x$ on the manifold $M$, the *sectional curvature* of a 2-plane $P \subset T_x M$ spanned by a basis $X_1, X_2$ is the number

$$K_x(P) := \frac{g\left(R\left(X_1, X_2\right) X_2, X_1\right)}{g\left(X_1 \wedge X_2, X_1 \wedge X_2\right)} = \frac{R(X_1, X_2, X_1, X_2)}{\|X_1 \wedge X_2\|^2},$$

where $\wedge$ is the wedge product.

The *Ricci curvature* tensor of $(M, g)$ is defined as follows: for any $x \in M$ and for any vector $v \in T_x M$,

$$\mathrm{Ric}_x(v, v) := \sum_{i=1}^{n} R\left(v, e_i, v, e_i\right)$$

where $(e_i)_{1 \le i \le n}$ is an orthonormal basis of the vector space $T_x M$.

Next, the *scalar curvature* of $(M, g)$ is defined by: for all $x \in M$

$$\mathrm{Scal}(x) := \sum_{i=1}^{n} \mathrm{Ric}_x\left(e_i, e_i\right) \in \mathbb{R},$$

where $(e_i)_{1 \le i \le n}$ is an orthonormal basis of $T_x M$. In Riemannian geometry, the scalar curvature at the point $x$ is also denoted by $R(x)$.

With the tensorial point of view the above notions of curvatures are not really intuitive. So let us give here their geometrical interpretation. Start by the simplest, the sectional curvature. Let $x$ be a fixed point on the manifold $M$, and $P \subset T_x M$ a 2-plane in the vector space $T_x M$. For $r > 0$, denote by $\mathcal{C}(0, r)$ the circle in $P$ of center 0 and radius $r$. Consider the image of $\mathcal{C}(0, r)$ under the exponential map $\exp_x$ at the point $x$. This image is a closed curve in the manifold $M$ whose length has the expansion (see for example [BGM]):

$$\mathcal{L}\left(\exp_x\left(\mathcal{C}(0, r)\right)\right) = 2\pi r \left(1 - \frac{r^2}{6} K_x(P) + O(r^3)\right)$$

as the radius $r$ goes to zero. In particular,

$$\mathcal{L}\left(\exp_x\left(\mathcal{C}(0, r)\right)\right) \underset{r \to 0^+}{\sim} 2\pi r.$$

Therefore, the sectional curvature $K_x(P)$ can be expressed as an infinitesimal deviation of circles tangent to $P$ from having Euclidean length.

*Remark* 3.1.20. In the case of surfaces embedded in $\mathbb{R}^3$, the notions of sectional curvature and the *Gaussian curvature* coincides. In fact, the sectional curvature is the generalization of the Gaussian curvature.

**Example 3.1.21.**

- In the case $M = \mathbb{R}^n$, $T_x M \simeq \mathbb{R}^n$ for all $x \in M$. For any 2-plane $P \subset T_x M$ and any radius $r > 0$ we have
$$\mathcal{L}\left(\exp_x\left(\mathcal{C}(0,r)\right)\right) = 2\pi r,$$
hence $K_x(P) = 0$. Therefore, all sectional curvatures on $\mathbb{R}^n$ are 0.
- The second usual example is the 2-sphere $M = \mathbb{S}^2$: in this case for all $x \in M$, $T_x M \simeq \mathbb{R}^2$, so for any radius $r > 0$ we have
$$\mathcal{L}\left(\exp_x\left(\mathcal{C}(0,r)\right)\right) = 2\pi \sin(r).$$
Since $\sin(r) = r - \frac{r^3}{6} + o(r^4)$, we have $K_x(P) = 1$. More generally, all sectional curvatures on $\mathbb{S}^n$ are $+1$.
- We finish with the hyperbolic space (and Poincaré disk), for which $K_x(P) = -1$ (see [GHL] for details); hence, all sectional curvatures on $\mathbb{H}^n$ are $-1$.

By the definition of Ricci curvature, for $x$ in $M$ and for $v$ a vector of $T_x M$
$$\mathrm{Ric}_x(v,v) = \sum_{j=1}^{n-1} K_x\left(P_{e_j,v}\right),$$
where $\left(e_j\right)_{1\leq j\leq n-1}$ is an orthonormal basis of the $(n-1)$-dimensional vector $v^\perp$, and $P_{e_j,v}$ denotes the 2-plane spanned by $e_j$ and $v$.

**Example 3.1.22.** In the case of spaces of constant curvature, the Ricci curvature is just a multiple of the metric:

- on $\mathbb{R}^n$, $\mathrm{Ric} = 0$;
- on $\mathbb{S}^n$, $\mathrm{Ric} = (n-1)g$;
- on $\mathbb{H}^n$, $\mathrm{Ric} = -(n-1)g$.

For finish observe that the scalar curvature is just the trace of the Ricci curvature on $T_x M$
$$\mathrm{Scal}(x) = \sum_{i=1}^{n} \mathrm{Ric}_x\left(e_i, e_i\right) = \sum_{i=1}^{n}\sum_{j\neq i} K_x\left(P_{e_j,e_i}\right),$$
where $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of $T_x M$. In other words, the scalar curvature is the sum of the eigenvalues of the Ricci curvature tensor.

**Example 3.1.23.** In the case of spaces of constant curvature, the scalar curvature is just a constant function:

- on $\mathbb{R}^n$, Scal $= 0$;
- on $\mathbb{S}^n$, Scal $= n(n-1)$;
- on $\mathbb{H}^n$, Scal $= -n(n-1)$.

Thus, the sectional curvatures determine the Ricci curvature, and the Ricci curvature determines the scalar curvature. Conversely, in the case of dimension 3 the Ricci curvature determines also all sectional curvatures, because we have the relation

$$K_x\left(P_{e_1,e_2}\right) = \frac{\mathrm{Ric}_x(e_1,e_1) + \mathrm{Ric}_x(e_2,e_2) - \mathrm{Ric}_x(e_3,e_3)}{2}.$$

### 3.1.5 Integration on a Riemannian manifold

Let us construct the Hilbert space $L^2(M,g)$ associated to a Riemannian manifold $(M,g)$. For this, we first need to define the canonical measure on a Riemannian manifold.

Consider a Riemannian manifold $(M,g)$ of dimension $n$ and a local chart $\phi\colon U \subset M \to \mathbb{R}^n$ with coordinates denoted by $\left(x^1, x^2, \cdots, x^n\right)$. Set $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, $g := \det(g_{ij})$ and $g^{jk} := (g_{jk})^{-1}$. The *canonical measure* of $(M,g)$ is

$$d\mathcal{V}_g := \sqrt{g}\, dx^1 dx^2 \dots dx^n$$

where $dx^1 dx^2 \dots dx^n$ is the standard Lebesgue measure of $\mathbb{R}^n$. The *volume form* of $(M,g)$ (here we suppose $M$ is oriented) is

$$\omega_g := \sqrt{g}\, dx^1 \wedge dx^2 \dots \wedge dx^n$$

Hence we have $\mathrm{Vol}(M) = \int_M \omega_g$.

For a local chart $\phi\colon U \subset M \to \mathbb{R}^n$ of the manifold $M$ and for a measurable function $f\colon U \to \mathbb{R}$ we define the *integral* of $f$ on $U \subset M$ (in the chart $\phi$) by

$$\int_U f\, d\mathcal{V}_g := \int_{\phi(U)} \left(f \circ \phi^{-1}\right) \sqrt{g \circ \phi^{-1}}\, dx^1 dx^2 \dots dx^n.$$

Using the change of variables formula we see that this integral does not depend on the chart $\phi\colon U \subset M \to \mathbb{R}^n$. Next, to define integration over the whole of $M$, we use a *partition of unity* on $M$, i.e., a finite family $(U_\alpha, \phi_\alpha)_\alpha$ where $(U_\alpha)_\alpha$ are open sets of $M$ and $(\phi_\alpha)_\alpha$ are smooth positives functions on $M$ which satisfy

$$\mathrm{supp}\,(\phi_\alpha) \subset U_\alpha$$

and
$$\sum_{\alpha} \phi_\alpha = 1.$$
So we can define the *integral* of $f$ on $M$ by the following formula (which does not depend on the choice of the partition):
$$\int_M f \, d\mathcal{V}_g := \sum_{\alpha} \int_{U_\alpha} f \phi_\alpha \, d\mathcal{V}_g .$$
Using the Riemannian metric $g$ we get a canonical isomorphism $\flat$ between the tangent bundle $TM$ and the cotangent bundle $T^\star M$:
$$\flat : \begin{cases} TM \longrightarrow T^\star M \\ \\ (x, V) \longmapsto \flat(V) \end{cases}$$
where $\flat(V)(U) := g(x)(V, U)$ with $V, U \in T_x M$. With this isomorphism we can define the notion of gradient of a function: for a function $f \in \mathcal{C}^1(M)$ we define the *gradient* by
$$\nabla f := \flat^{-1}(df);$$
in other words, for all $V \in TM$ we have $g(\nabla f, V) = (df)(V)$. For a local chart $\phi: U \subset M \to \mathbb{R}^n$ of $M$, we have the local formula
$$\nabla f = \sum_{j,k=1}^{n} \frac{\partial (f \circ \phi^{-1})}{\partial x^j} g^{jk} \frac{\partial}{\partial x^k}.$$

Now, let us introduce the divergence operator. Let $X$ be a $\mathcal{C}^1$ vector field on $M$. The *divergence* of $X$ is given by
$$\operatorname{div}(X) := \operatorname{tr}(Y \mapsto D_Y X).$$
For a local chart $\phi: U \subset M \to \mathbb{R}^n$ of $M$ and for a vector field
$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j},$$
we have
$$\operatorname{div}(X) = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial \left(\sqrt{g} X^j\right)}{\partial x^j}.$$
A classical result about this operator is the following: if $X$ is a vector field of $M$ with compact support, then *divergence formula*
$$\int_M \operatorname{div}(X) \, d\mathcal{V}_g = 0.$$

## 3.2 Analysis on manifolds

### 3.2.1 Distributions on a Riemannian manifold

For a complete introduction to the theory of L. Schwartz distributions on manifolds and more generally to the theory of currents (generalization for differential forms) see for example the book of G. de Rham [DRh].

Let us denote by $\mathcal{D}(M)$ the set of smooth functions with compact support on $M$. In the language of distributions, $\mathcal{D}(M)$ is called the set of *test functions*.

**Definition 3.2.1.** A *distribution* on $M$ is a linear form

$$T\colon \begin{cases} \mathcal{D}(M) \longrightarrow \mathbb{R} \\ \\ \varphi \longmapsto T\varphi, \end{cases}$$

which is continuous in the following sense: if $(\varphi_n)_n$ is a sequence in $\mathcal{D}(M)$ such that for any $n$ the support of $\varphi_n$ is contained in a single compact set which lies in the interior of a local coordinate system $\left(x^1, x^2, \cdots, x^n\right)$ and such that all the derivatives of $\varphi_n$ tend uniformly to zero as $n \to +\infty$, then $T\varphi_n$ tends in $\mathbb{R}$ to zero as $n \to +\infty$. The set of distributions on $M$ is denoted $\mathcal{D}'(M)$.

*Remark* 3.2.2. In fact, the space $\mathcal{D}(M)$ can be equipped with a topology and $\mathcal{D}'(M)$ is the topological dual space of $\mathcal{D}(M)$.

In this book we also use the following notation: for $T \in \mathcal{D}'(M)$ and for $\varphi \in \mathcal{D}(M)$,

$$\langle T, \varphi \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := T\varphi.$$

**Example 3.2.3.** Let $a$ be a point $M$. The *Dirac distribution* $\delta_a$ is defined by: for all $\varphi \in \mathcal{D}(M)$,

$$\langle \delta_a, \varphi \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := \varphi(a).$$

**Example 3.2.4.** A locally integrable function $f$ on $M$ defines a distribution, called the *regular distribution* associated to $f$, by: for all $\varphi \in \mathcal{D}(M)$

$$\left\langle T_f, \varphi \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := \int_M f\varphi \, d\mathcal{V}_g.$$

*Remark* 3.2.5. From now we do not distinguish between locally integrable functions and the associated regular distributions.

For a distribution $T \in \mathcal{D}'(M)$ and for a local coordinate system $(x^1, x^2, \cdots, x^n)$ we define the partial derivative $\frac{\partial T}{\partial x^i}$ of $T$ by: for all $\varphi \in \mathcal{D}(M)$

$$\left\langle \frac{\partial T}{\partial x^i}, \varphi \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} := -\left\langle T, \frac{\partial \varphi}{\partial x^i} \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}.$$

### 3.2.2 Sobolev spaces on a Riemannian manifold

First we define the *Lebesgue space* $L^2(M,g)$ (or simply $L^2(M)$) on the manifold $(M,g)$ by

$$L^2(M,g) := \left\{ f : M \to \mathbb{R} \text{ measurable such that } \int_M |f|^2 \, d\mathcal{V}_g < +\infty \right\}.$$

This space is a Hilbert space for the scalar product

$$\langle u, v \rangle_{L^2(M)} := \int_M uv \, d\mathcal{V}_g .$$

Next we define the *Sobolev space* $H^1(M,g)$ (or simply $H^1(M)$) by

$$H^1(M,g) := \overline{\mathcal{C}^\infty(M)},$$

where the closure is with respect to the norm $\|\cdot\|_{H^1}$ defined by

$$\|u\|_{H^1} := \sqrt{\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2}.$$

Another way to define the space $H^1(M)$ is

$$H^1(M) = \left\{ u \in L^2(M);\ \nabla u \in L^2(M) \right\}$$

where the derivation $\nabla$ is in the sense of distributions.

$H^1(M)$ is a Hilbert space for the scalar product

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2} .$$

Finally, we define the Sobolev space $H_0^1(M)$ (or simply $H_0^1(M)$) by

$$H_0^1(M,g) := \overline{\mathcal{D}(M)},$$

where the closure is taken with the respect to the norm $\|\cdot\|_{H^1}$.

So we have:

$$\mathcal{D}(M) \subset H_0^1(M) \subset H^1(M) \subset L^2(M).$$

Recall that for the norm $\|\cdot\|_{L^2}$ we have

$$\overline{\mathcal{D}(M)} = L^2(M).$$

To finish let us state two important theorems in Sobolev space theory. The first is the *Poincaré inequality* (see, for example, [Heb]):

**Theorem 3.2.6** (Poincaré theorem). *Suppose the manifold $(M, g)$ is compact. There exists a positive constant $C_M$ such that for any function $u \in H_0^1(M)$*

$$\|u\|_{L^2(M)} \leq C_M \|\nabla u\|_{L^2(M)}.$$

Consequently, the bilinear form

$$\langle \cdot, \cdot \rangle_{H_0^1} : \begin{cases} H_0^1(M) \times H_0^1(M) \longrightarrow \mathbb{R} \\ (u, v) \longmapsto \langle u, v \rangle_{H_0^1} := \langle \nabla u, \nabla v \rangle_{L^2} \end{cases}$$

is a scalar product on $H_0^1(M)$. Moreover, the associated norm $\|\cdot\|_{H_0^1}$ is equivalent to the norm $\|\cdot\|_{H^1}$. Therefore $\left(H_0^1(M), \langle \cdot, \cdot \rangle_{H_0^1}\right)$ is a Hilbert space.

An obvious consequence of the Poincaré inequality is:

**Corollary 3.2.7.** *Suppose the manifold $(M, g)$ is compact. The canonical embedding of $\left(H_0^1(M), \|\cdot\|_{H_0^1}\right)$ into $\left(L^2(M), \|\cdot\|_{L^2}\right)$ is bounded.*

Now, let us recall the *Rellich theorem* (see for example [Heb]).

**Theorem 3.2.8** (Rellich theorem). *Suppose the manifold $(M, g)$ is compact. The canonical embeddings*

$$\left(H^1(M), \|.\|_{H^1} \to L^2(M), \|.\|_{L^2}\right)$$

*and*

$$\left(H_0^1(M), \|.\|_{H_0^1} \to L^2(M), \|.\|_{L^2}\right)$$

*are compacts.*

### 3.2.3 The Laplacian operator and the Green formula

We define the *Laplace–Beltrami operator* (or simply *Laplacian*) on the Riemannian manifold $(M, g)$ by

$$\Delta_g : \begin{cases} \mathcal{C}^2(M) \longrightarrow \mathcal{C}^0(M) \\ f \longmapsto \Delta_g(f) := \operatorname{div}(\nabla f). \end{cases}$$

In local coordinates, for a $\mathcal{C}^2$ real valued function $f$ on $M$ and for a local chart $\phi : U \subset M \to \mathbb{R}$ of $M$, the Laplace–Beltrami operator is given by the local expression

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial (f \circ \phi^{-1})}{\partial x_k} \right),$$

where $g = \det(g_{ij})$ and $(g^{jk}) = (g_{jk})^{-1}$. The Laplacian operator is an unbounded linear operator which satisfies for any $\mathcal{C}^2$ pair of functions $(\varphi, \psi)$

$$\Delta_g(\varphi\psi) = \Delta_g\varphi\psi + 2g\,(\nabla\varphi, \nabla\psi) + \varphi\Delta_g\psi.$$

This formula in turn yields the famous *Green's formula*:

**Theorem 3.2.9** (Green's formula aka. Integration by parts formula). *Let $(M, g)$ be an oriented Riemannian manifold and $\Omega$ a subset of $M$ with a smooth boundary $\partial M$. Denote by $\nu$ the unit normal vector field to the boundary $\partial M$. For any $\varphi \in \mathcal{C}^2(M)$ and $\psi \in \mathcal{C}^1(M)$, at least one of which has a compact support, we have:*

$$\int_M \psi\Delta_g\varphi\, d\mathcal{V}_g = \int_{\partial M} \left(g\,(\nu, \nabla\varphi)\,\psi - g\,(\nu, \nabla\psi)\,\varphi\right)\, d\mathcal{A}_g + \int_M \varphi\Delta_g\psi\, d\mathcal{V}_g.$$

## 3.3 Exercises

**Exercise 3.3.1.** *If $f$ is a constant function on a manifold show that $df = 0$ everywhere.*

**Exercise 3.3.2.** *Prove that flat tori are Riemannian manifolds.*

**Exercise 3.3.3.** *Verify that the hyperbolic space $\mathbb{H}^n := \{x \in \mathbb{R}^n,\ \|x\| < 1\}$ equipped with the hyperbolic metric g is a Riemannian manifold.*

**Exercise 3.3.4.** *Let $(M, g)$ be a Riemannian manifold and $f\colon M \to \mathbb{R}$ a function such that $\|\nabla f\| = 1$. Show that the integral curves of $\nabla f$ are geodesics.*

**Exercise 3.3.5.** *Let $(\gamma_t)_t$ be a one-parameter family of closed geodesics on a Riemannian manifold $(M, g)$. Prove that for any $t, t'$ the length of $\gamma_t$ and $\gamma_{t'}$ are equals.*

**Exercise 3.3.6.** *Prove that a Riemannian manifold $M$ equipped with the Riemannian distance (see Section 3.1.3) is a metric space and the topology associated to this metric coincides with the natural topology of the manifold $M$.*

**Exercise 3.3.7.** *Verify that the volume of the sphere with the metric can is*

$$\mathrm{Vol}(\mathbb{S}^{2n}, can) = \frac{(4\pi)^n(n-1)!}{(2n-1)!}$$

*and*

$$\mathrm{Vol}(\mathbb{S}^{2n+1}, can) = 2\frac{\pi^{n+1}}{n!}.$$

**Exercise 3.3.8.** *Show that the Ricci tensor is symmetric.*

**Exercise 3.3.9.** *Show that:*

1. *on the Euclidean plane* $\mathbb{R}^2$ *geodesics are straight lines;*
2. *on the round sphere* $\mathbb{S}^2$ *geodesics are great circles;*
3. *on the hyperbolic Poincaré disk geodesics are arcs of circles meeting the unit circle orthogonally, or segments through the origin.*

**Exercise 3.3.10.** *Prove that* $H^1(M)$ *is a Hilbert space.*

# Chapter 4
# Spectrum of the Laplacian on a compact manifold

In this chapter our first main goal is to prove that the Laplace–Beltrami operator on a compact Riemannian manifold is self-adjoint with a discrete spectrum. Next, we will show that the spectrum is a nice geometric invariant. We also present the minimax principle for the Laplacian. At the end we also define the Schrödinger operator on a compact manifold.

*From now on, all manifolds are supposed to be compact and connected.*

## 4.1 Physical examples

### 4.1.1 The wave equation on a string

Consider an one-dimensional string of length $L > 0$. The vibrations of this string are governed by the *wave equation*

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

with boundary conditions

$$u(0,t) = u(L,t) = 0$$

and initial data

$$u(x,0) = u_0(x).$$

Here $u(x,t)$ is the amplitude of the point $x$ on the string at time $t$.

Using the theory of Fourier series (see Section 5.1.1 for more details), the family

$$\left(e_n(x) := \sin\left(\frac{n\pi x}{L}\right)\right)_{n\geq 1}$$

form a Hilbert basis of $H := \left\{f \in L^2([0,T])\,;\ f(0) = f(T) = 0\right\}$, and for any integer $n \geq 1$ we have the relation

$$-\frac{\partial^2 e_n}{\partial x^2}(x) = \frac{n^2\pi^2}{L^2} e_n(x).$$

In fact, the spectrum of the operator $-\frac{\partial^2}{\partial x^2}$ with the boundary conditions $u(0,t) = u(L,t) = 0$ is equal to

$$\left(\frac{n^2\pi^2}{L^2}\right)_{n\geq 1}.$$

So if $u(x,t)$ is a solution of the wave equation, there exists a sequence of real number $(a_n(t))_{n\geq 1}$ (which depends on the time $t$) such that

$$u(x,t) = \sum_{n=1}^{+\infty} a_n(t) e_n(x).$$

Plugging this expression into the wave equation we get

$$\sum_{n=1}^{+\infty} \left(a_n''(t) + \frac{n^2\pi^2}{L^2} a_n(t)\right) e_n(x) = 0,$$

and since $(e_n)_n$ is a basis, for any integer $n \geq 1$ we obtain the following ODE in the time variable

$$a_n''(t) + \frac{n^2\pi^2}{L^2} a_n(t) = 0.$$

Thus we get for any integer $n \geq 1$ and for all $t \geq 0$

$$a_n(t) = A_n \cos\left(\frac{n\pi}{L} t\right) + B_n \sin\left(\frac{n\pi}{L} t\right),$$

where $A_n, B_n$ are real constants.

Finally, we obtain the expression

$$u(x,t) = \sum_{n=1}^{+\infty} \left(A_n \cos\left(\frac{n\pi}{L} t\right) + B_n \sin\left(\frac{n\pi}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right),$$

the coefficients $A_n, B_n$ of which are calculated from the initial data $u(x,0) = u_0(x)$.

### 4.1.2 The heat equation

Consider here a domain $\Omega \subset \mathbb{R}^3$ with a boundary $\Gamma := \partial\Omega$. *The heat equation* on $\Omega$ describes the heat diffusion process on this domain. It is the linear equation

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t)$$

with boundary conditions

$$u(x,t) = u_\ell(x,t) \quad \text{for } x \in \Gamma$$

and initial data

$$u(x, 0) = u_0(x).$$

Here $u(x, t)$ is the temperature at the point $x \in \Omega$ at time $t$. For simplify, we suppose that $f = 0$ and $u = 0$. Using the separation variables technique, we look for a solution in the form $u(x, t) = \varphi(t)w(x)$. Plugging this into the heat equation we get

$$\varphi'(t)w(x) - \varphi'(t)\Delta w(x) = 0.$$

Formally, we have for all $x \in \Omega$, $t \geq 0$

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\Delta w(x)}{w(x)},$$

and since the variables $x$ and $t$ are independent, there exists a real constant $\lambda$ such that for all $x, t$ we have

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

In particular, $\varphi'(t) = -\lambda\varphi(t)$, hence

$$\varphi(t) = \varphi(0)e^{-\lambda t}.$$

Moreover,

$$-\Delta w(x) = \lambda w(x)$$

with $w(x) = 0$ for $x \in \Gamma$. In other words, the scalar $\lambda$ is an eigenvalue of $-\Delta$ with Dirichlet conditions, therefore if we have a basis $(e_k)_{k\geq 0}$ of eigenvectors,

$$-\Delta e_k = \lambda_k e_k,$$

and we get for all $x, t$

$$u(x, t) = \sum_{k=1}^{+\infty} c_k e^{-\lambda_k t} e_k(x).$$

### 4.1.3 The Schrödinger equation

In Quantum Mechanics, a physical particle on a manifold $M$ in a time interval $I$ is describe by a *wave function*, that is, a function

$$\psi : \begin{cases} M \times I \longrightarrow L^2(M) \\ (x, t) \longmapsto \psi(x, t), \end{cases}$$

where the quantity $\int_\Omega \psi(x,t)\,dx$ represents the probability to find the particle in the domain $\Omega \subset M$ at time $t$. In particular, this requires that

$$\|\psi\|^2_{L^2(M)} = \int_\Omega \psi(x,t)\,dx = 1.$$

The *Schrödinger operator* on a Riemannian manifold $(M, g)$ is the linear operator

$$P_\hbar := -\frac{h^2}{2}\Delta_g + V$$

with $V$ a function of $L^\infty_{\text{loc}}(M)$ such that $\lim_{|x|\to\infty} V(x) = +\infty$. This operator is self-adjoint and its spectrum is discrete. For an initial state $\psi_0 \in L^2(M)$ the quantum dynamics associated to $P_h$ is given by the formula (using the bounded functional calculus)

$$\psi(t) = U(t)\psi_0 \in L^2(M),$$

where

$$U(t) := e^{-i\frac{t}{h}P_h}.$$

The derivative of this equation is

$$\frac{\partial\psi(t)}{\partial t} = -\frac{i}{h}P_h\psi(t),$$

so we arrive at the famous *Schrödinger equation*

$$ih\frac{\partial\psi}{\partial t}(x,t) = -\Delta_g\psi(x,t) + V(x)\psi(x,t).$$

Now if we have a basis $(e_k)_{k\geq 0}$ of eigenvectors of $-\Delta$, using also the bounded functional calculus, we get for all $t$ and for any integer $n$

$$\left(e^{-i\frac{t}{h}P_h}\right)e_n = \left(e^{-i\frac{t}{h}\lambda_n}\right)e_n;$$

thus, if we write the initial state as $\psi_0 = \sum_{n=0}^{+\infty} a_n e_n$ with $a_n = \langle\psi_0, e_n\rangle_{L^2}$, we get for all $t$

$$\psi(t) = U(t)\psi_0 = \left(e^{-i\frac{t}{h}P_h}\right)\left(\sum_{n=0}^{+\infty} a_n e_n\right) = \sum_{n=0}^{+\infty} a_n e^{-i\frac{t}{h}\lambda_n} e_n.$$

Note that a good way to understand (in the semi-classical limit $h \to 0$: see Section 5.2.4) this quantum dynamics is to study the *quantum auto-correlation function*

$$\mathbf{a}(t) := |\mathbf{r}(t)| = |\langle\psi(t), \psi_0\rangle_{\mathcal{H}}|\,.$$

This function is a quantum analog of the *Poincaré's return function*: if the wave packet $\psi(t)$ is localised on the initial state $\psi_0$, then $\mathbf{a}(t)$ is close to 1 (because $\|\psi_0\|_{L^2} = 1$). The auto-correlation function is a fundamental tool in the dynamical study of quantum integrable systems (see for example [Co-Ro], [Robi1], [Robi2], [Rob2], [Rob3], [Lab1], [Lab2]). Indeed, in the case of semi-classical completely integrable systems and for a localized initial state, at the beginning the quantum dynamics is periodic with a period which corresponds to the *classical Hamiltonian flow period*. Next for a large time scale, a new period $T_{\text{rev}}$ of the quantum dynamics appears: this is the *revival phenomenon*. For $t = T_{\text{rev}}$ the initial wave packet reforms again at $t = T_{\text{rev}}$. We have also the phenomenon of *fractional revival* of initial wave packets for time $t = \frac{p}{q} T_{\text{rev}}$, with $\frac{p}{q} \in \mathbb{Q}$: there is a formation of a finite number of clones of the original wave packet $\psi_0$ with a constant amplitude and differing in the phase plane from the initial wave packet by fractions $\frac{p}{q} T_{\text{rev}}$.

## 4.2 A class of spectral problems

### 4.2.1 The closed eigenvalue problem

Let $(M, g)$ be a closed manifold, i.e., compact without boundary, for example the sphere, the torus... *The closed spectral* problem is: find all real numbers $\lambda$ such that there exists a function $u \in \mathcal{C}^\infty(M)$ with $u \neq 0$ for which

$$-\Delta_g u = \lambda u.$$

From a spectral point of view we want to find the (point) spectrum of $-\Delta_g$ on the domain $D = \mathcal{C}^\infty(M)$.

### 4.2.2 The Dirichlet eigenvalue problem

Let $(M, g)$ be a compact manifold with boundary. *The Dirichlet spectral problem* is: find all real numbers $\lambda$ such that there exists a function $u \in \mathcal{C}^\infty(M)$ with $u \neq 0$ for which

$$\begin{cases} -\Delta_g u = \lambda u \\ u = 0 \text{ on the boundary } \gamma \text{ of } M. \end{cases}$$

From a spectral point of view we want to find the (point) spectrum of $-\Delta_g$ on the domain $D = \mathcal{D}(M)$.

### 4.2.3 The Neumann eigenvalue problem

Let $(M, g)$ be a compact manifold with boundary. *The Neumann spectral problem* is: find all real numbers $\lambda$ such that there exists a function $u \in \mathcal{C}^\infty(M)$ with $u \neq 0$ for which

$$\begin{cases} -\Delta_g u = \lambda u \\ \\ \nabla u \cdot \gamma = 0 \text{ on the boundary } \gamma \text{ of } M, \end{cases}$$

where $\gamma$ is the unit inward normal vector field on $\Gamma$.

From a spectral point of view we want to find the (point) spectrum of $-\Delta_g$ on the domain $D = \{f \in \mathcal{C}^\infty(M);\ \nabla f \cdot \gamma = 0 \text{ on the boundary } \gamma \text{ of } M\}$.

### 4.2.4 Other problems

Obviously there exist many other problems, for example

- *Mixed problem*: suppose the boundary $\Gamma$ of $M$ can be written as $\Gamma = \Gamma_1 \coprod \Gamma_2$ the problem is: find all real numbers $\lambda$ such that there exists a function $u \in \mathcal{C}^\infty(M)$ with $u \neq 0$ for which

$$\begin{cases} -\Delta_g u = \lambda u \\ u = 0 \ \text{ on } \Gamma_1 \\ \nabla u \cdot \gamma = 0 \ \text{ on } \Gamma_2. \end{cases}$$

- *Steklov problem*: for a fixed function $\rho \in \mathcal{C}^0(\Gamma)$, find all real numbers $\lambda$ for which there exists a function $u \in \mathcal{C}^\infty(M)$ with $u \neq 0$ such that

$$\begin{cases} -\Delta_g u = 0 \text{ in } M \\ \\ \nabla u \cdot \gamma = \lambda \rho u \text{ on the boundary } \gamma \text{ on } M. \end{cases}$$

## 4.3 Spectral theorem for the Laplacian

For each of the three above cases (closed, Dirichlet, or Neumann problem), using Green's formula conclude that for all functions $\varphi, \psi$ in the domain

- $D = \mathcal{C}^\infty(M)$
- $D = \mathcal{D}(M)$
- $D = \{f \in \mathcal{C}^\infty(M);\ \nabla f \cdot \gamma = 0 \text{ on } \Gamma \text{ the boundary of } M\}$

we always have the relation

$$\int_\Omega \Delta_g \varphi \psi \, d\mathcal{V}_g = \int_\Omega \varphi \Delta_g \psi \, d\mathcal{V}_g.$$

Hence, the operator $-\Delta_g$ is symmetric. Moreover, for any functions $\varphi, \psi$ in the domain $D$ we have also

$$-\int_\Omega \varphi \Delta_g \varphi \, d\mathcal{V}_g = \int_\Omega |\nabla \varphi|^2 \, d\mathcal{V}_g \geq 0,$$

therefore the operator $-\Delta_g$ is positive.

Consequently, let $\lambda, \mu$ be two eigenvalues such that $\lambda \neq \mu$ and let $u, v$ be respective eigenfunctions (i.e., $-\Delta_g u = \lambda u$ and $-\Delta_g v = \mu v$). Since $-\Delta_g$ is symmetric, we have

$$\langle -\Delta_g u, v \rangle_{L^2} = \langle u, -\Delta_g v \rangle_{L^2},$$

i.e.,

$$(\lambda - \mu) \langle u, v \rangle_{L^2} = 0.$$

Therefore, if we denote by $E(\lambda)$ the eigenspace corresponding to the eigenvalue $\lambda$, the spaces $E(\lambda)$ and $E(\mu)$ are $L^2$-orthogonal.

Moreover, every eigenvalue is non-negative. Indeed, let $\lambda$ be an eigenvalue and let $u$ be an eigenfunction of $\lambda$. Then

$$\langle -\Delta_g u, u \rangle_{L^2} = \lambda \|u\|_{L^2} \geq 0,$$

so $\lambda \geq 0$.

In fact, we have the more general theorem concerning the spectral theory of the Laplacian (see also [Bér2]):

**Theorem 4.3.1.** *Let $(M, g)$ be a compact Riemannian manifold with boundary $\Gamma$ (possibly empty). Then the following assertions hold true for each of the above spectral problems.*

(i) *The spectrum and the point spectrum of $-\Delta_g$ coincide and consist of a real infinite sequence*

$$0 \leq \lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \cdots$$

*such that $\lambda_k(M) \to +\infty$ as $k \to +\infty$.*

(ii) *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are $L^2$-orthogonal, and if we denote by $E(\lambda_k)$ the eigenspace corresponding to the eigenvalue $\lambda_k$, then*

$$\overline{\bigoplus_{k \geq 1} E(\lambda_k)} = L^2(M)$$

*where the closure is taken for the norm $\|\cdot\|_{L^2}$.*

*Moreover, for the closed problem each eigenfunction belongs to the space $H^1(M)$, for the Dirichlet problem each eigenfunction belongs to the space $H_0^1(M)$, and for the Neumann problem the gradient of every eigenfunction vanishes on the boundary of $M$.*

(iii) *Each eigenfunction is analytic.*

Hence, in conclusion, for each Riemannian manifold there exists a unique sequence of real numbers $(\lambda_k(M))_{k\geq 0}$ such that

$$\mathrm{Spec}(-\Delta_g, D) = \{\lambda_k(M),\, k \geq 0\}\,.$$

Recall that the *multiplicity* of an eigenvalue $\lambda_k(M)$ is the dimension of the vector space $\ker\big(I + \Delta_g\big)$.

**Definition 4.3.2.** The sequence above is called the *spectrum of the manifold* $(M, g)$ and is denoted by $\mathrm{Spec}(M, g)$.

Spectral geometry studies the relationships between the geometry of $(M, g)$ and the spectrum $\mathrm{Spec}(M, g)$. A fundamental notion in spectral geometry is the following:

**Definition 4.3.3.** Two (compact) Riemannian manifolds $(M, g)$ and $(M', g')$ are *isospectral* if $\mathrm{Spec}(M, g) = \mathrm{Spec}(M', g')$. A *spectral invariant* on $(M, g)$ is a quantity which is completely determined by $\mathrm{Spec}(M, g)$.

Since the spectrum of a manifold depends only on the divergence and the gradient operators, and those operators depend only on the Riemannian structure on $(M, g)$, the spectrum depends only on the Riemannian structure. Consequently, we have

**Proposition 4.3.4.** *Any two isometric Riemannian manifold are isospectral.*

## 4.4 A detailed proof by a variational approach

In this subsection we give a detailed proof of the Theorem 4.3.1. The proof uses only the spectral theory of bounded operators. In the book [Bér8] by P. Bérard we can find another type of proof. Here we use the Lax–Milgram theorem.

### 4.4.1 Variational generic abstract eigenvalue problem

Consider two Hilbert spaces $H$ and $V$ such that $V \subset H$ and $V$ is dense in $H$ (in the norm of $H$). Let us denote by $I_{10}$ the canonical injection from $V$ into $H$ and suppose that $I_{10}\colon (V, \|\cdot\|_V) \to (H, \|\cdot\|_H)$ is bounded and compact. In particular, this means that there exists a constant $K \geq 0$ such that for all $u \in V$ we have $\|u\|_H \leq K\,\|u\|_V$. We also consider a bilinear symmetric form $a$ on $V^2$ which is *continuous* and *elliptic* (or coercive) (that is, there exists a constant $C \geq 0$ such that for all $(u, v) \in V^2$ we have $|a(u, v)| \leq C\,\|u\|_V\,\|v\|_V$, and there exists a constant $\alpha > 0$ such that for all $u \in V$ we have $|a(u, u)| \geq \alpha\,\|u\|_V^2$).

Now, for $f$ a fixed vector in $H$ we shall be interested in the following variational problem:

$$\begin{cases} \text{find } u \in V, \text{ such that for all } v \in V \\ \qquad a(u,v) = \langle f, v\rangle_H \,. \end{cases}$$

If we denote by $\ell$ the map

$$\ell\colon \begin{cases} V \longrightarrow \mathbb{R} \\ v \longmapsto \langle f, v\rangle_H \,, \end{cases}$$

it is clear that $\ell$ is linear and by the Cauchy–Schwarz inequality, for all $v \in V$

$$\begin{aligned} |\ell(v)| = |\langle f, v\rangle_H| &\le \|f\|_H \, \|v\|_H \\ &\le K \, \|f\|_H \, \|v\|_V \,, \end{aligned}$$

and so the operator $\ell\colon (V, \|\cdot\|_V) \to (\mathbb{R}, |\cdot|)$ is bounded. Therefore, by the Lax–Milgram theorem, there exists a unique vector $u \in V$ such that

$$a(u,v) = \ell(v)$$

for all $v \in V$, and moreover

$$\|u\|_V \le \frac{K}{\sqrt{\alpha}} \, \|f\|_H \,.$$

Since for every vector $f \in H$ the vector solution $u \in V$ of the variational problem is unique, we can define the following map

$$T\colon \begin{cases} H \longrightarrow V \\ f \longmapsto u, \end{cases}$$

where $u$ is the unique solution of the problem

$$\begin{cases} \text{find } u \in V, \text{ such that for all } v \in V \\ \qquad a(u,v) = \ell(v). \end{cases}$$

This mean that for all $v \in V$,

$$a\,(Tf, v) = \langle f, v\rangle_H \,.$$

It is clear (use the unicity of solution) that $T$ is a linear map between the Hilbert spaces $H$ and $V$. Moreover,

$$\|Tf\|_V = \|u\|_V \le \frac{K}{\sqrt{\alpha}} \, \|f\|_H$$

for all $f \in H$, and so the operator $T\colon (H, \|\cdot\|_H) \to (V, \|\cdot\|_V)$ is bounded.

Let us introduce the restriction $T_{\mathrm{r}}$ of the operator $T$ to the space Hilbert space $V \subset H$

$$T_{\mathrm{r}}\colon \begin{cases} V \longrightarrow V \\ f \longmapsto Tf. \end{cases}$$

Since

$$\|Tf\|_V \leq \frac{K}{\sqrt{\alpha}} \|f\|_H \leq \frac{K^2}{\sqrt{\alpha}} \|f\|_V$$

for all $f \in V$, we deduce that the operator $T_{\mathrm{r}}\colon (V, \|\cdot\|_V) \to (V, \|\cdot\|_V)$ is bounded. Clearly,

$$T_{\mathrm{r}} = T \circ I_{10}$$

and since the $I_{10}$ is bounded and compact, the operator $T_{\mathrm{r}}$ is also compact (the set of compact operator is an ideal of the bounded operators, see Theorem 2.5.3). Now let us verify that zero is not an eigenvalue of the operator $T_{\mathrm{r}}$. Suppose on the contrary, that there exists $u \in V$ with $u \neq 0$ such that $T_{\mathrm{r}}u = Tu = 0$. Since $a\,(T_{\mathrm{r}}u, u) = \langle u, u\rangle_H$, we get $0 = \|u\|_H$, which is a contradiction.

Next for all $u, v \in V$ we have

$$\begin{aligned} a\,(T_{\mathrm{r}}u, v) &= a\,(Tu, v) \\ &= \langle u, v\rangle_H = \langle v, u\rangle_H \\ &= a\,(T_{\mathrm{r}}v, u) = a\,(u, T_{\mathrm{r}}v)\,, \end{aligned}$$

so the bounded operator $T_{\mathrm{r}}$ is self-adjoint with respect to the scalar product $a\,(\cdot,\cdot)$ on the space Hilbert space $V$.

Thus, the operator $T_{\mathrm{r}}$ is bounded, compact, self-adjoint and 0 is not an eigenvalue, so using the fundamental theorem of the spectral theory of compact operators (see Theorem 2.5.6) the spectrum $\sigma(T_{\mathrm{r}})$ and the point spectrum $\mathrm{Spec}(T_{\mathrm{r}})$ coincide and this set is discrete: it consists of a sequence $(\mu_k)_k$ of real eigenvalues such that $\mu_k > 0$ and $\mu_k \to 0$ as $k \to +\infty$. For any integer $k \geq 0$ the dimension of $\ker\,(\mu_k I - T_{\mathrm{r}})$ is finite, i.e., each eigenvalue of $T_{\mathrm{r}}$ has finite multiplicity. Moreover, the corresponding eigenvectors $(v_k)_k$ of $(\mu_k)_k$ form a Hilbert basis of the space $V$. In particular for every integer $k, \ell$,

$$Tv_k = T_{\mathrm{r}}v_k = \mu_k v_k \quad \text{and} \quad a\,(v_k, v_\ell) = \delta_{k,\ell}.$$

To finish, observe that for every integer $k$ and for all $v \in V$,

$$a\,(T_{\mathrm{r}}v_k, v) = \langle v_k, v\rangle_H$$

and on the other hand we have also

$$a\,(T_{\mathrm{r}}v_k, v) = \mu_k a\,(v_k, v)\,.$$

Consequently,

$$a\left(v_k, v\right) = \frac{1}{\mu_k} \left\langle v_k, v\right\rangle_H .$$

So, if we denote for each integer $n \geq 0$

$$e_k := \frac{v_k}{\sqrt{\mu_k}}$$

we have for all $(k, \ell) \in \mathbb{N}^2$

$$\begin{aligned} \left\langle e_k, e_k\right\rangle_H &= \left\langle \frac{v_k}{\sqrt{\mu_k}}, \frac{v_k}{\sqrt{\mu_k}}\right\rangle_H = \frac{1}{\mu_k} \left\langle v_k, v_k\right\rangle_H \\ &= a\left(v_k, v_k\right) = 1, \end{aligned}$$

and if $k \neq \ell$

$$\begin{aligned} \left\langle e_k, e_\ell\right\rangle_H &= \left\langle \frac{v_k}{\sqrt{\mu_k}}, \frac{v_\ell}{\sqrt{\mu_\ell}}\right\rangle_H = \frac{1}{\sqrt{\mu_k}} \frac{1}{\sqrt{\mu_\ell}} \left\langle v_k, v_\ell\right\rangle_H \\ &= \frac{\sqrt{\mu_k}}{\sqrt{\mu_\ell}} a\left(v_k, v_\ell\right) = 0. \end{aligned}$$

Hence, the sequence $(e_k)_k$ is an $H$-orthonormal family in $V$, therefore by the density of $V$ in $H$, the family $(e_k)_k$ form a Hilbert basis of $H$ such that for every integer $k$ and for all $v \in V$ we have

$$a\left(e_k, v\right) = \frac{1}{\sqrt{\mu_k}} a\left(v_k, v\right) = \frac{1}{\sqrt{\mu_k}} \frac{1}{\mu_k} \left\langle v_k, v\right\rangle_H = \frac{1}{\mu_k} \left\langle e_k, v\right\rangle_H .$$

Now let us apply this theory to our spectral problems (closed, Dirichlet or Neumann problem).

### 4.4.2 The closed eigenvalue problem

For a fixed $f \in L^2(M)$ consider the following problem

$$\begin{cases} \text{find } u \in L^2(M), \text{ such that} \\ -\Delta_g u + u = f \quad \text{in } \mathcal{D}'(M). \end{cases}$$

The corresponding variational formulation reads

$$\begin{cases} \text{find } u \in H^1(M), \text{ such that for all } v \in H^1(M) \\ \qquad a(u, v) = \ell(v) \end{cases}$$

where for all $u, v \in H^1(M)$

$$a(u,v) := \langle u,v\rangle_{H^1} = \int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g$$

and for all $v \in H^1(M)$

$$\ell(v) := \int_M fv \, d\mathcal{V}_g.$$

The two previous formulations are equivalent. Indeed, suppose $u \in L^2(M)$ is a solution of $-\Delta_g u + u = f$ in $\mathcal{D}'(M)$. This means that for all $v \in \mathcal{D}(M)$ we have

$$\left\langle -\Delta_g u + u, v \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}.$$

Hence, for all $v \in \mathcal{D}(M)$,

$$\langle \nabla u, \nabla v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} + \langle u, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)},$$

so by definition of regular $L^2$-distributions, for all $v \in \mathcal{D}(M)$

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M fv \, d\mathcal{V}_g.$$

Therefore, since $\mathcal{D}(M)$ is dense in $H^1(M)$ in the $L^2$ norm,

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M fv \, d\mathcal{V}_g,$$

for all $v \in H^1(M)$, i.e., $a(u,v) = \ell(v)$. Conversely, suppose $u \in H^1(M)$ satisfies $a(u,v) = \ell(v)$ for all $v \in H^1(M)$, hence for all $v \in \mathcal{D}(M)$ we have

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M fv \, d\mathcal{V}_g.$$

Integrating by parts we get

$$-\int_M \Delta_g uv \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M fv \, d\mathcal{V}_g$$

for all $v \in \mathcal{D}(M)$. Consequently,

$$\left\langle -\Delta_g u + u, v \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}$$

for all $v \in \mathcal{D}(M)$, hence $-\Delta_g u + u = f$ in $\mathcal{D}'(M)$.

Now let us focus on the variational formulation. Here both $V := H^1(M)$ and $H := L^2(M)$ are Hilbert spaces and obviously $H^1(M) \subset L^2(M)$ and $H^1(M)$ is dense in $L^2(M)$ (in the norm of $L^2(M)$). Since for all $u \in H^1(M)$,

$\|u\|_{L^2} \leq \|u\|_{H^1}$ Rellich's theorem (Theorem 3.2.8) shows that the canonical injection $I_{10}$: $\left(H^1(M), \|\cdot\|_{H^1}\right) \to \left(L^2(M), \|\cdot\|_{L^2}\right)$ is bounded and compact. Moreover by the Cauchy–Schwarz inequality, we obtain for any $u, v \in H^1(M)$:

$$|a(u, v)| = |\langle u, v\rangle_{H^1}| \leq \|u\|_{H^1} \|v\|_{H^1}$$

and for any $u \in H^1(M)$ we have $a(u, u) := \|u\|_{H^1}^2$. In consequence the bilinear form $a(\cdot, \cdot)$ is bounded and elliptic on the Hilbert space $\left(H^1(M), \langle\cdot, \cdot\rangle_{H^1}\right)$. On the other hand, for any $v \in H^1(M)$

$$\begin{aligned} |\ell(v)| &= \left| \int_M f v \, d\mathcal{V}_g \right| \leq \|f\|_{L^2} \|v\|_{L^2} \\ &\leq \|f\|_{L^2} \|v\|_{H^1} . \end{aligned}$$

So the linear form $\ell$ is bounded on $H^1(M), \langle\cdot, \cdot\rangle_{H^1}$. It follows from the previous theory with $V := H^1(M)$ and $H := L^2(M)$ (in this case the operator $T$ is equal to $T = \left(-\Delta_g + I\right)^{-1}$, which means that $-1$ is in the resolvent set of $-\Delta_g$) that there exists a infinite sequence $(\mu_k)_k$ of real numbers such that every $\mu_k > 0$ and $\mu_k \to 0$ as $k \to +\infty$ (in fact $\mathrm{Spec}(T_{\mathrm{r}}) = \{(\mu_k)_k\}$) and there exists a Hilbert basis $(e_k)_k$ of $H$ such that for every integer $k$ and for all $v \in V$ we have

$$a(e_k, v) = \frac{1}{\mu_k} \langle e_k, v\rangle_{L^2} .$$

Hence, for every integer $k$ and for all $v \in H^1(M)$,

$$\int_M \nabla e_k \nabla v \, d\mathcal{V}_g + \int_M e_k v \, d\mathcal{V}_g = \frac{1}{\mu_k} \int_M e_k v \, d\mathcal{V}_g .$$

Integrating by parts (here $M$ is closed) we get for all $v \in H^1(M)$

$$-\int_M \Delta_g e_k v \, d\mathcal{V}_g + \int_M e_k v \, d\mathcal{V}_g = \frac{1}{\mu_k} \int_M e_k v \, d\mathcal{V}_g$$

i.e., for all $v \in H^1(M)$

$$\int_M \left(-\Delta_g + \left(1 - \frac{1}{\mu_k}\right) I\right) (e_k) \, v \, d\mathcal{V}_g = 0.$$

Using the density of $H^1(M)$ into $L^2(M)$ we obtain that for any integer $k$

$$\left(-\Delta_g + \left(1 - \frac{1}{\mu_k}\right) I\right) e_k = 0,$$

hence, for all integers $k$,

$$-\Delta_g e_k = \left(\frac{1}{\mu_k} - 1\right) e_k.$$

So, if we denote $\lambda_k := \frac{1}{\mu_k} - 1$ for any integer $k$, we have

$$-\Delta_g e_k = \lambda_k e_k.$$

Consequently, $(\lambda_k)_k \subset \operatorname{Spec}(-\Delta_g)$ and $(e_k)_k$ is a Hilbert eigenbasis of $-\Delta_g$ on the space $L^2(M)$.

Conversely let us verify the important fact that there are no other eigenvalues in $\operatorname{Spec}(-\Delta_g)$. Indeed, let $\lambda \in \operatorname{Spec}(-\Delta_g)$, so there exists $f \neq 0$ in $H^1(M)$ such that

$$-\Delta_g f = \lambda f.$$

Then

$$-\Delta_g f + f = (\lambda + 1) f,$$

i.e.,

$$\left(-\Delta_g + I\right) f = (\lambda + 1) f.$$

Since $-1$ is in the resolvent set of $-\Delta_g$ we have (here $T = \left(-\Delta_g + I\right)^{-1}$)

$$f = (\lambda + 1) T f.$$

Next, since $f \in V = H^1(M)$ we deduce that

$$f = (\lambda + 1) T_{\mathrm{r}} f$$

and since $\lambda \neq -1$ (because $-1$ is in the resolvent set of $-\Delta_g$) we get

$$T_{\mathrm{r}} f = \frac{1}{\lambda + 1} f.$$

Therefore $\frac{1}{\lambda+1} \in \operatorname{Spec}(T_{\mathrm{r}})$ hence there exists an integer $k$ such that $\frac{1}{\lambda+1} = \mu_k$. It follows that $\lambda = \frac{1}{\mu_k} - 1$, i.e., and so we have $(\lambda_k)_k = \operatorname{Spec}(-\Delta_g)$.

### 4.4.3 The Dirichlet eigenvalue problem

For a fixed $f \in L^2(M)$ consider the problem

$$\begin{cases} \text{find } u \in L^2(M), \text{ such that} \\ -\Delta_g u = f \quad \text{in } \mathcal{D}'(M) \\ u|_\Gamma = 0. \end{cases}$$

The variational formulation of this problem reads

$$\begin{cases} \text{find } u \in H_0^1(M), \text{ such that for all } v \in H_0^1(M) \\ \qquad\qquad a(u,v) = \ell(v) \end{cases}$$

where for all $u, v \in H_0^1(M)$,

$$a(u,v) := \langle u, v \rangle_{H_0^1} = \int_M \nabla u \nabla v \, d\mathcal{V}_g,$$

and for all $v \in H_0^1(M)$,

$$\ell(v) := \int_M f v \, d\mathcal{V}_g.$$

The two previous formulations are equivalent. Indeed, suppose $u \in L^2(M)$ is solution of $-\Delta_g u = f$ in $\mathcal{D}'(M)$, this means that for all $v \in \mathcal{D}(M)$ we have

$$\langle -\Delta_g u, v \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)},$$

hence for all $v \in \mathcal{D}(M)$

$$\langle \nabla u, \nabla v \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v \rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}.$$

By the definition of regular $L^2$-distributions, we get for all $v \in \mathcal{D}(M)$

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g,$$

whence, since $\mathcal{D}(M)$ is dense in $H_0^1(M)$ in the $L^2$ norm, we have for all $v \in H_0^1(M)$

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g,$$

i.e., $a(u,v) = \ell(v)$. Now suppose $u \in H_0^1(M)$ satisfies $a(u,v) = \ell(v)$ for all $v \in H_0^1(M)$, hence for all $v \in \mathcal{D}(M)$ we get

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g.$$

Integrating by parts (here $v$ vanishes on the boundary of $M$) for all $v \in \mathcal{D}(M)$ we have

$$-\int_M \Delta_g u v \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g.$$

Consequently,

$$\left\langle -\Delta_g u, v \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}$$

for all $v \in \mathcal{D}(M)$, hence $-\Delta_g u = f$ in $\mathcal{D}'(M)$.

Now let us focus on the variational formulation. Here both $V = H_0^1(M)$ and $H = L^2(M)$ are Hilbert spaces, $H_0^1(M) \subset L^2(M)$, and $H_0^1(M)$ is dense in $L^2(M)$ (in the norm of $L^2(M)$). From the Poincaré inequality (Theorem 3.2.6) and from the Rellich theorem (Theorem 3.2.8) we deduce that the canonical injection $I_{10}\colon \left(H_0^1(M), \|\cdot\|_{H_0^1}\right) \to \left(L^2(M), \|\cdot\|_{L^2}\right)$ is bounded and compact. Moreover, by the Cauchy–Schwarz inequality, for any $u, v \in H_0^1(M)$

$$|a(u,v)| = \left|\langle u, v\rangle_{H_0^1}\right| \le \|u\|_{H_0^1} \|v\|_{H_0^1},$$

and for any $u \in H_0^1(M)$, $a(u,u) := \|u\|_{H_0^1}^2$. Therefore, the bilinear form $a(\cdot,\cdot)$ is bounded and elliptic on the Hilbert space $\left(H_0^1(M), \langle\cdot,\cdot\rangle_{H_0^1}\right)$. On the other hand, the Poincaré inequality implies that for any $v \in H_0^1(M)$ we get

$$\begin{aligned}|\ell(v)| &= \left|\int_M f v \, d\mathcal{V}_g\right| \le \|f\|_{L^2} \|v\|_{L^2} \\ &\le C_M \|f\|_{L^2} \|\nabla v\|_{L^2} = C_M \|f\|_{L^2} \|v\|_{H_0^1},\end{aligned}$$

and so the linear form $\ell$ is bounded on the Hilbert space $\left(H_0^1(M), \langle\cdot,\cdot\rangle_{H_0^1}\right)$. It follows from the previous theory with $V := H_0^1(M)$ and $H := L^2(M)$ (in this case the operator $T$ is equal to $T = -\Delta_g^{-1}$, this meaning that 0 is in the resolvent set of $-\Delta_g$) that there exists a infinite sequence $(\mu_k)_k$ of real numbers such that every $\mu_k > 0$ and $\mu_k \to 0$ as $k \to +\infty$ (in fact $\mathrm{Spec}(T_\mathrm{r}) = \{(\mu_k)_k\}$); and there exists a Hilbert basis $(e_k)_k$ of $H$, such that for every integer $k$ and for all $v \in V$

$$a(e_k, v) = \frac{1}{\mu_k} \langle e_k, v\rangle_{L^2}.$$

Therefore, for every integer $k$ and for all $v \in H_0^1(M)$,

$$\int_M \nabla e_k \nabla v \, d\mathcal{V}_g = \frac{1}{\mu_k} \int_M e_k v \, d\mathcal{V}_g.$$

Integrating by parts (here $v$ vanishes on the boundary of $M$) we obtain for all $v \in H_0^1(M)$

$$-\int_M \Delta_g e_k v \, d\mathcal{V}_g = \frac{1}{\mu_k} \int_M e_k v \, d\mathcal{V}_g,$$

i.e.,

$$\int_M \left(-\Delta_g - \frac{1}{\mu_k} I\right)(e_k)\, v \, d\mathcal{V}_g = 0$$

for all $v \in H_0^1(M)$, so by the density of $H_0^1(M)$ in $L^2(M)$ we have for any $k$

$$\left(-\Delta_g - \frac{1}{\mu_k} I\right) e_k = 0,$$

i.e.,

$$-\Delta_g e_k = \frac{1}{\mu_k} e_k.$$

Thus, if we denote $\lambda_k := \frac{1}{\mu_k}$ for any $k$, we get

$$-\Delta_g e_k = \lambda_k e_k.$$

Hence, $(\lambda_k)_k \subset \mathrm{Spec}(-\Delta_g)$ and $(e_k)_k$ form a Hilbert eigenbasis of $-\Delta_g$ on the space $L^2(M)$.

Conversely, let $\lambda \in \mathrm{Spec}(-\Delta_g)$, so there exists $f \neq 0$ in $H_0^1(M)$ such that

$$-\Delta_g f = \lambda f.$$

Since 0 is in the resolvent set of $-\Delta_g$ we have (here $T = \left(-\Delta_g\right)^{-1}$)

$$f = \lambda T f$$

and since $v \in V = H_0^1(M)$

$$f = \lambda T_{\mathrm{r}} f.$$

To conclude, since $\lambda \neq 0$ (because 0 is in the resolvent set of $-\Delta_g$) we get

$$T_{\mathrm{r}} f = \frac{1}{\lambda} f,$$

thus $\frac{1}{\lambda} \in \mathrm{Spec}(T_{\mathrm{r}})$: there exists a $k$ such that $\frac{1}{\lambda} = \mu_k$, hence $\lambda = \frac{1}{\mu_k}$ and $\lambda = \lambda_k$. We have shown that $(\lambda_k)_k = \mathrm{Spec}(-\Delta_g)$.

### 4.4.4 The Neumann eigenvalue problem

To finish, we consider the Neumann eigenvalue problem: for a fixed $f \in L^2(M)$,

$$\begin{cases} \text{find } u \in L^2(M), \text{ such that} \\ -\Delta_g u + u = f \quad \text{in } \mathcal{D}'(M) \\ \quad \nabla u \cdot \gamma = 0 \quad \text{on } \Gamma. \end{cases}$$

The variational formulation of this problem reads

$$\begin{cases} \text{find } u \in H^1(M), \text{ such that for all } v \in H^1(M) \\ \qquad a(u,v) = \ell(v) \end{cases}$$

where for all $u, v \in H^1(M)$

$$a(u,v) := \langle u, v\rangle_{H^1} = \int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g$$

and for all $v \in H^1(M)$

$$\ell(v) := \int_M f v \, d\mathcal{V}_g.$$

The two previous formulations are equivalent. Indeed, suppose $u \in L^2(M)$ is a solution of $-\Delta_g u + u = f$ in $\mathcal{D}'(M)$ with $\nabla u \cdot \gamma = 0$ on $\Gamma$; this means that for all $v \in \mathcal{D}(M)$ we have

$$\left\langle -\Delta_g u + u, v\right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}.$$

Hence, for all $v \in \mathcal{D}(M)$

$$\langle \nabla u, \nabla v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} + \langle u, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)},$$

so by the definition of regular $L^2$-distributions, we get for all $v \in \mathcal{D}(M)$

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g.$$

Since $\mathcal{D}(M)$ is dense in $H^1(M)$ in the $L^2$ norm, for all $v \in H^1(M)$ we get

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g$$

i.e., $a(u,v) = \ell(v)$. Now, suppose $u \in H^1(M)$ satisfies $a(u,v) = \ell(v)$ for all $v \in H^1(M)$, hence for all $v \in \mathcal{D}(M)$

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g,$$

so integrating by parts we get for all $v \in \mathcal{D}(M)$

$$-\int_M \Delta_g uv \, d\mathcal{V}_g + \int_M uv \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g.$$

Consequently, for all $v \in \mathcal{D}(M)$

$$\left\langle -\Delta_g u + u, v \right\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)} = \langle f, v\rangle_{\mathcal{D}'(M)\times\mathcal{D}(M)}$$

hence $-\Delta_g u + u = f$ in $\mathcal{D}'(M)$. On the other hand, for all $v \in H^1(M)$ integrating by parts we get

$$\int_M \Delta_g u v \, d\mathcal{V}_g = \int_\Gamma \nabla u \cdot \gamma v \, d\gamma - \int_M \nabla u \nabla v \, d\mathcal{V}_g.$$

Therefore,

$$\int_M \nabla u \nabla v \, d\mathcal{V}_g - \int_\Gamma \nabla u \cdot \gamma v \, d\gamma + \int_M u v \, d\mathcal{V}_g = \int_M f v \, d\mathcal{V}_g,$$

Whence

$$\int_\Gamma \nabla u \cdot \gamma v \, d\gamma = 0$$

for all $v \in H^1(M)$. By density, we conclude that $\nabla u \cdot \gamma = 0$ on $\Gamma$.

From this point on, the study of the variational formulation is carried out in exactly the same way as in the case of closed problem.

### 4.4.5 A remark on the variational formulation

The variational formulation is also very important in numerical analysis, indeed, *the finite element method* is based on this formulation. To outline this method, consider the generic variational problem

$$\begin{cases} \text{find } u \in V, \text{ such that for all } v \in V \\ \qquad a(u, v) = \ell(v) \end{cases}$$

where $V$ is a Hilbert space, $a(\cdot,\cdot)$ is a bilinear symmetric form on $V^2$ continuous and elliptic and $\ell(\cdot)$ is a bounded linear form on $V$. The approach of the finite element method is to find a “nice” finite-dimensional subspace $V_h$ of $V$ and to solve the associated variational problem on $V_h$:

$$\begin{cases} \text{find } u_h \in V_h, \text{ such that for all } v_h \in V_h \\ \qquad a(u_h, v_h) = \ell(v_h). \end{cases}$$

If we have a basis $(e_i)_{1\le i\le n}$ of $V_h$, the previous problem is equivalent to

$$\begin{cases} \text{find } u_h \in V_h, \text{ such that for all } i \in \{1,\dots,n\} \\ \qquad a(u_h, e_i) = \ell(e_i). \end{cases}$$

Since every vector $u_h \in V_h$ is characterized by its coordinates $\left(u_h^j\right)_{1\le j\le n}$ in the basis $\left(e_j\right)_{1\le j\le n}$, namely

$$u_h = \sum_{j=1}^{n} u_h^j e_j,$$

the previous problem is equivalent to

$$\begin{cases} \text{find the scalars } \left(u_h^j\right)_{1\le j\le n} \in \mathbb{R}, \text{ such that for all } i \in \{1,\dots,n\} \\[2ex] \displaystyle\sum_{j=1}^{n} u_h^j a(e_j, e_i) = \ell(e_i). \end{cases}$$

In the matrix language we have to find a vector $U = \left(u_h^1 \; u_h^2 \; \dots \; u_h^n\right)^t$ such that

$$AU = L$$

where $A := \left(a(e_j, e_i)\right)_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{R})$ and $L := (\ell(e_1)\; \ell(e_2)\; \dots\; \ell(e_n))$.

## 4.5 The minimax principle and applications

### 4.5.1 A physical example

Let us consider a one-dimensional elastic beam of length $L > 0$. The energy of the deformation (under vibrations) for a homogeneous elastic beam is given by

$$\mathcal{E}_0 = \int_0^L \left|u''(x)\right|^2 \, dx.$$

Therefore, the main frequency of the vibrations is equal to

$$\mathcal{F}_0 = \min_{u\neq 0} \frac{\displaystyle\int_0^L \left|u''(x)\right|^2 \, dx}{\displaystyle\int_0^L |u(x)|^2 \, dx}.$$

In the case of a nonhomogeneous elastic beam the energy is

$$\mathcal{E} = \int_0^L \left|a(x)u''(x)\right|^2 \, dx$$

where $a$ is a non-negative function, hence in this case the main frequency of the vibrations for the nonhomogeneous elastic beam is equal to

$$\mathcal{F} := \min_{u \neq 0} \frac{\displaystyle\int_0^L \left|a(x)u''(x)\right|^2 \, dx}{\displaystyle\int_0^L |u(x)|^2 \, dx}.$$

## 4.5.2 The minimax theorem

Let $(M, g)$ be a compact Riemannian manifold (closed or with a boundary where Dirichlet or Neumann boundary conditions are imposed).

**Definition 4.5.1.** The *Rayleigh quotient* $R(\varphi)$ of a function $\varphi$, $\varphi \neq 0$ on $(M, g)$ is defined by

$$R(\varphi) := \frac{\displaystyle\int_M |\nabla\varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}.$$

*Remark* 4.5.2. In the case where $M$ is closed, for any function $\varphi \in \mathcal{C}^\infty(M)$ (or more generally for any function $\varphi \in H^1(M)$) we have the equality

$$R(\varphi) = \frac{\displaystyle\int_M |\nabla\varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g} = \frac{\displaystyle\int_M -\Delta_g\varphi\varphi \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}.$$

This also holds in the case of the Dirichlet problem for any function $\varphi \in \mathcal{D}(M)$ (or more generally for any function $\varphi \in H_0^1(M)$), and the same is true for the Neumann problem for any function $\varphi \in \mathcal{C}^\infty(M)$ such that the gradient vanishes on the boundary of $M$.

First, let us show

**Proposition 4.5.3.** *For each spectral problem, let* $\lambda$ *be an eigenvalue of* $(M, g)$ *and* $u \in \mathcal{C}^\infty(M)$, $u \neq 0$ *an associated eigenfunction. Then*

$$\lambda = R(u) = \frac{\displaystyle\int_M |\nabla u|^2 \, d\mathcal{V}_g}{\displaystyle\int_M u^2 \, d\mathcal{V}_g}.$$

*Proof.* Let $u \in \mathcal{C}^\infty(M)$, $u \neq 0$ be an eigenfunction:

$$-\Delta_g u = \lambda u.$$

Then, by multiplying by $u$ and integrating over the manifold $M$ we obtain

$$\int_M -\Delta_g uu \, d\mathcal{V}_g = \lambda \int_M u^2 \, d\mathcal{V}_g.$$

In integrating by parts[1] we have

$$\int_M -\Delta_g uu \, d\mathcal{V}_g = \int_M \nabla u \cdot \nabla u \, d\mathcal{V}_g = \int_M |\nabla u|^2 \, d\mathcal{V}_g,$$

so we finally get

$$\lambda = \frac{\displaystyle\int_M |\nabla u|^2 \, d\mathcal{V}_g}{\displaystyle\int_M u^2 \, d\mathcal{V}_g} = R(u). \qquad \square$$

**Proposition 4.5.4.** *For each spectral problem and for any $k \geq 1$, the following characterization holds true:*

$$\lambda_k(M) = \max_{\substack{\varphi \in V_k \\ \varphi \neq 0}} R(\varphi),$$

*where* $V_k :=$ span $(e_1, e_2, \ldots, e_k)$. *An alternative characterization is*

$$\lambda_1(M) = \min_{\substack{\varphi \in V \\ \varphi \neq 0}} R(\varphi)$$

*and for all $k \geq 2$*

$$\lambda_k(M) = \min_{\substack{\varphi \in V_{k-1}^{\perp} \\ \varphi \neq 0}} R(\varphi)$$

*where* $V_{k-1}^{\perp} =$ span $(e_k, e_{k+1}, \ldots)$.

*Proof.* For any integer $n$ we compute the Rayleigh quotient for the Hilbert eigenbasis $(e_n)_n$ and we get

$$R(e_n) = \lambda_n(M) = R(v_n).$$

[1] In the closed case this is clear, in the Dirichlet case the eigenfunction $u$ vanishes on the boundary of $M$, and in the Neumann case the gradient of the eigenfunction $u$ vanishes on the boundary of $M$.

Since $R(\alpha\varphi) = R(\varphi)$ for all scalar $\alpha$ and for any function $\varphi$, it is clear that

$$\lambda_1(M) = \max_{\substack{\varphi \in V_1 \\ \varphi \neq 0}} R(\varphi),$$

where $V_1 := \operatorname{span}(e_1)$.

Next for any function $\varphi$ such that

$$\varphi = \sum_{i=0}^{+\infty} \langle \varphi, e_i \rangle_{L^2} e_i,$$

we have

$$R(\varphi) = \frac{\displaystyle\int_M -\Delta_g \varphi \varphi \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g} = \frac{\displaystyle\sum_{i=0}^{+\infty} \lambda_i(M) \langle \varphi, e_i \rangle_{L^2}^2}{\displaystyle\sum_{i=0}^{+\infty} \langle \varphi, e_i \rangle_{L^2}^2}.$$

Proceeding by induction, consider the set $V_k := \operatorname{span}(e_1, e_2, \ldots, e_k)$. Then, for all $\varphi \in V_k$,

$$R(\varphi) = \frac{\displaystyle\sum_{i=0}^{k} \lambda_i(M) \langle \varphi, e_i \rangle_{L^2}^2}{\displaystyle\sum_{i=0}^{k} \langle \varphi, e_i \rangle_{L^2}^2} \leq \frac{\displaystyle\sum_{i=0}^{k} \lambda_k(M) \langle \varphi, e_i \rangle_{L^2}^2}{\displaystyle\sum_{i=0}^{k} \langle \varphi, e_i \rangle_{L^2}^2} = \lambda_k(M).$$

Hence, $\max_{\varphi \in V_k,\, \varphi \neq 0} R(\varphi) \leq \lambda_k(M)$ and since $R(e_k) = \lambda_k(M)$, we get

$$\lambda_k(M) = \max_{\substack{\varphi \in V_k \\ \varphi \neq 0}} R(\varphi).$$

For the other characterization, we have

$$\lambda_1(M) = \min_{\substack{\varphi \in V \\ \varphi \neq 0}} R(\varphi)$$

and for all $k \geq 2$

$$\lambda_k(M) = \min_{\substack{\varphi \in V_{k-1}^{\perp} \\ \varphi \neq 0}} R(\varphi),$$

with $V_{k-1}^{\perp} = \operatorname{span}(e_k, e_{k+1}, \ldots)$. Indeed, for all $\varphi \in V_{k-1}^{\perp}$

$$R(\varphi) = \frac{\displaystyle\sum_{i=k}^{+\infty} \lambda_i(M) \langle \varphi, e_i \rangle_{L^2}^2}{\displaystyle\sum_{i=k}^{+\infty} \langle \varphi, e_i \rangle_{L^2}^2} \geq \frac{\displaystyle\sum_{i=k}^{+\infty} \lambda_k(M) \langle \varphi, e_i \rangle_{L^2}^2}{\displaystyle\sum_{i=k}^{+\infty} \langle \varphi, e_i \rangle_{L^2}^2} = \lambda_k(M).$$

Hence, $\min_{\varphi\in V_{k-1}^{\perp},\,\varphi\neq 0} R(\varphi) \geq \lambda_k(M)$, and since $R(e_k) = \lambda_k(M)$ we get

$$\lambda_k(M) = \min_{\substack{\varphi\in V_{k-1}^{\perp}\\ \varphi\neq 0}} R(\varphi). \qquad \square$$

Finally we get the minimax variational characterization of eigenvalues:

**Theorem 4.5.5** (Minimax principle). *For each of the above spectral problems and for any $k \geq 1$, we have*

$$\lambda_k(M) = \min_{\substack{E\subset H^\star(M)\\ \dim(E)=k}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\displaystyle\int_M |\nabla\varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}$$

*where $H^\star(M) := H_0^1(M)$ in the case of Dirichlet problem and $H^\star(M) = H^1(M)$ in the case of Neumann or closed problem.*

*Proof.* For all $E \subset H^\star(M)$ such that $\dim(E) = k$, we have

$$\lambda_k(M) \leq \max_{\substack{\varphi\in E\\ \varphi\neq 0}} R(\varphi).$$

Indeed, for all nontrivial functions $\varphi \in E$ such that $\varphi \in V_{k-1}^{\perp}$ (here we have $E \cap V_{k-1}^{\perp} \neq \emptyset$ because $\dim(E) = k$), since $\lambda_k(M) = \min_{\varphi\in V_{k-1}^{\perp},\,\varphi\neq 0} R(\varphi)$ we obtain

$$\lambda_k(M) \leq R(\varphi).$$

Consequently, for all $E \subset H^\star(M)$ such that $\dim(E) = k$ we have $\lambda_k(M) \leq \max_{\varphi\in E} R(\varphi)$, hence

$$\lambda_k(M) \leq \min_{\substack{E\subset H^\star(M)\\ \dim(E)=k}} \max_{\varphi\in E} R(\varphi).$$

And if $E = V_k$ we have (Proposition 4.5.4)

$$\lambda_k(M) = \max_{\substack{\varphi\in V_k\\ \varphi\neq 0}} R(\varphi) = R(e_k)$$

so we deduce that:

$$\lambda_k(M) \geq \min_{\substack{E\subset H^\star(M)\\ \dim(E)=k}} \max_{\varphi\in E} R(\varphi). \qquad \square$$

### 4.5.3 Properties of the first eigenvalue

An obvious consequence of the minimax principle is the following result.

**Proposition 4.5.6.** *Let $(M, g)$ be a compact Riemannian manifold.*

(i) *In the cases of a closed manifold or of Neumann boundary conditions the first eigenvalue is*
$$\lambda_1(M) = 0$$
*and the corresponding eigenfunction is the constant function $M \ni x \mapsto 1$.*

(ii) *In the case of the Dirichlet boundary condition the first eigenvalue satisfies*
$$\lambda_1(M) > 0.$$

*Proof.*

(i) By the spectral theorem, $\lambda_1(M) \geq 0$. With the minimax principle we also get
$$\lambda_1(M) = \min_{\substack{\varphi \in \mathcal{C}^\infty(M) \\ \varphi \neq 0}} \frac{\displaystyle\int_M |\nabla \varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}.$$
Since the constant function $M \ni x \mapsto 1$ belongs to the space $H^1(M)$ and since
$$\frac{\displaystyle\int_M |\nabla 1|^2 \, d\mathcal{V}_g}{\displaystyle\int_M 1^2 \, d\mathcal{V}_g} = 0$$
we deduce that $\lambda_1(M) \leq 0$, therefore $\lambda_1(M) = 0$. Moreover, $-\Delta_g(1) = 0$.

(ii) We consider the Dirichlet problem and by contradiction assume that $\lambda_1(M) = 0$, there exists a non trivial function $u$ such that $-\Delta_g u = 0$ and $u$ vanishes on the boundary of $M$. Therefore,
$$\int_M -\Delta_g u \, u \, d\mathcal{V}_g = 0$$
hence integrating by parts (here $u$ vanishes on the boundary of $M$) we get
$$\int_M |\nabla u|^2 \, d\mathcal{V}_g = 0.$$
Thus $u$ is constant on $M$, and from the boundary conditions it follows that $u = 0$ on $M$, which is a contradiction. $\square$

*Remark* 4.5.7. In fact we have already seen in the proof of the spectral theorem 4.3.1 (Section 4.4.3: The Dirichlet eigenvalue problem) that the eigenvalues of the Laplacian for the Dirichlet problem satisfy $\lambda_k(M) := \frac{1}{\mu_k}$, where $\mu_k > 0$, thus $\lambda_1(M) > 0$.

More generally, we have also:

**Proposition 4.5.8.** *For each spectral problem above we have:*

(i) *The first eigenfunction $e_1$ of the manifold $(M, g)$ satisfy $e_1 > 0$ or $e_1 < 0$ in the interior of $M$.*

(ii) *The first eigenvalue $\lambda_1(M)$ is always simple.*

*Proof.*

(i) Suppose that the function $e_1$ changes sign in $M$. Since $e_1 \in H^1(M)$, the function $f := |e_1|$ belongs to $H^1(M)$ and $|df| = |de_1|$ (see for example [Gi-Tr]), hence
$$R(f) = R(e_1) = \lambda_1(M).$$
Therefore the function $f$ is also a first eigenfunction of $(M, g)$ which satisfies $f \geq 0$ on $M$. Moreover, $f$ vanishes in the interior of $M$ and
$$-\Delta_g f = \lambda_1(M) f \geq 0$$
on $M$. By the maximum principle [Pr-We], the function $f$ cannot attain its minimum in an interior point of the manifold $M$, hence $f$ does not vanish on $M$, which is a contradiction.

(ii) Suppose that the first eigenvalue $\lambda_1(M)$ is not simple: there exists two orthogonal non-trivial eigenfunctions $f_1, \varphi_1$ associated to $\lambda_1(M)$. Using the assertion (i) we may assume that $f_1, \varphi_1 > 0$ in the interior of $M$, thus
$$\langle f_1, \varphi_1 \rangle_{L^2} = \int_M f_1 \varphi_1 \, d\mathcal{V}_g > 0;$$
but this is absurd because $f_1, \varphi_1$ are orthogonal. □

Note that the assertions (i) and (ii) hold for a large class of operators.

**Proposition 4.5.9.** *In the cases of a closed manifold or the Neumann problems we have for any integer $k \geq 2$*
$$\int_M e_k \, d\mathcal{V}_g = 0$$
*where $e_k$ is an eigenvector of the $k$-th eigenvalue $\lambda_k(M)$.*

*Proof.* In the cases of a closed manifold or the Neumann problem the first eigenvalue $\lambda_1(M)$ is equal to zero and simple. Thus for all $k \geq 2$ we have

$$\lambda_k(M) \geq \lambda_2(M) > 0$$

and by definition $-\Delta_g e_k = \lambda_k(M) e_k$. Therefore,

$$\int_M -\Delta_g e_k \, d\mathcal{V}_g = \lambda_k(M) \int_M e_k \, d\mathcal{V}_g$$

and integrating by parts[2] we get

$$\int_M -\Delta_g e_k \, d\mathcal{V}_g = \int_M -\Delta_g e_k .1 \, d\mathcal{V}_g = \int_M \nabla e_k . \nabla 1 \, d\mathcal{V}_g = 0$$

so for all $k \geq 2$

$$0 = \lambda_k(M) \int_M e_k \, d\mathcal{V}_g .$$

Next, since for all $k \geq 2$ we have $\lambda_k(M) > 0$ we deduce that for all $k \geq 2$ $\displaystyle\int_M e_k \, d\mathcal{V}_g = 0$. □

## 4.5.4 Monotonicity domain principle

Another obvious classical application of minimax property is the monotonicity principle for Dirichlet problem:

**Proposition 4.5.10.** *Let $(M, g)$ be a Riemannian manifold and $A, B$ two submanifolds of $(M, g)$ of same dimension $n$ such that $A \subset B$ and $A, B$ are compact with boundary. Then for every integer $k \geq 1$ we have:*

$$\lambda_k(B) \leq \lambda_k(A).$$

*Proof.* For every integer $k \geq 1$ and for any $i \in \{1, \ldots, k\}$, let $f_k \in H_0^1(A)$ be an eigenfunction corresponding to the eigenvalue $\lambda_i(A)$. This eigenfunction may be extended by 0 on $B$, and this extension belong to the space $H_0^1(B)$. Hence we have

$$\min_{\substack{E \subset H_0^1(B) \\ \dim(E) = k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\displaystyle\int_M |\nabla \varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g} \leq \min_{\substack{E \subset H_0^1(A) \\ \dim(E) = k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\displaystyle\int_M |\nabla \varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g} . \qquad \square$$

[2] In the closed manifold case this is clear and in the Neumann case one uses the fact that the gradient of the eigenfunction $u$ vanishes on the boundary of $M$.

Let us finish this section by introducing a notation. For each spectral problem (closed, Neumann and Dirichlet) the spectrum of $-\Delta_g$ on $(M, g)$ can be rewritten as

$$\mathrm{Spec}(M, g) = \{0 \leq \lambda_1(M) < \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \cdots\}$$

or simply

$$\mathrm{Spec}(M, g) = \{0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots\}$$

In the specific cases of closed manifold or Neumann problems we have

$$\mathrm{Spec}(M, g) = \{0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots\},$$

whereas for the Dirichlet problem

$$\mathrm{Spec}(M, g) = \{0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots\}$$

The notation $0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ does not indicate the multiplicity of eigenvalues. In certain situations it is convenient to use the following alternative

*Notation* 4.5.11. Let us denote the distinct eigenvalues of $(M, g)$ by

$$0 \leq \left(\widetilde{\lambda_1}, m_1 = 1\right) < \left(\widetilde{\lambda_2}, m_2\right) < \cdots < \left(\widetilde{\lambda_k}, m_k\right) < \cdots,$$

where the integer $m_i$ denote the multiplicity of the eigenvalue $\widetilde{\lambda_i}$.

### 4.5.5 A perturbation of metric result

We present here a very simple result concerning a metric perturbation on the manifold $(M, g)$:

**Theorem 4.5.12.** *Let $M$ be a compact manifold equipped with a family of Riemannian metrics $(g_n)_n$. If the sequence of metrics $(g_n)_n$ converge as $n \to +\infty$ (in the Whitney topology) to a Riemannian metric $g_0$, then for every problem (Dirichlet, Neumann, or closed manifold), and for any fixed integer $k$, the eigenvalues $\lambda_k(M, g_n)$ converge as $n \to +\infty$ to $\lambda_k(M, g_0)$.*

The proof is a direct consequence of the following lemma.

**Lemma 4.5.13.** *Let $g$ and $g_0$ be two metrics on a compact Riemannian manifold such that $\alpha g_0 \leq g \leq \beta g_0$. Then for every problem (Dirichlet, Neumann, or closed manifold) and for any integer $k \geq 1$ we have*

$$\frac{\alpha^{\frac{n}{2}}}{\beta^{\frac{n}{2}+1}}\lambda_k(M, g_0) \leq \lambda_k(M, g) \leq \frac{\beta^{\frac{n}{2}}}{\alpha^{\frac{n}{2}+1}}\lambda_k(M, g_0).$$

*Proof.* Using the definition of the volume form $\mathcal{V}_g$, we have

$$\alpha^{\frac{n}{2}}\mathcal{V}_{g_0} \le \mathcal{V}_g \le \beta^{\frac{n}{2}}\mathcal{V}_{g_0}.$$

On the other hand, using the definition of the gradient we also have

$$\frac{1}{\beta}\left|\nabla_{g_0}\varphi\right|^2 \le \left|\nabla_g\varphi\right|^2 \le \frac{1}{\alpha}\left|\nabla_{g_0}\varphi\right|^2.$$

Therefore for any function $\varphi \in H^1(M)$ we get

$$\begin{aligned}\frac{\displaystyle\int_M |\nabla\varphi|^2\, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_g} &\le \frac{\frac{1}{\alpha}\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_g} \le \frac{\frac{\beta^{\frac{n}{2}}}{\alpha}\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_{g_0}}{\alpha^{\frac{n}{2}}\displaystyle\int_M \varphi^2\, d\mathcal{V}_{g_0}}\\ &= \frac{\beta^{\frac{n}{2}}}{\alpha^{\frac{n}{2}+1}}\frac{\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_{g_0}}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_{g_0}}.\end{aligned}$$

Consequently for any integer $k \ge 1$ we obtain:

$$\min_{\substack{E\subset H^\star(M)\\ \dim(E)=k}}\ \max_{\substack{\varphi\in E\\ \varphi\ne 0}} \frac{\displaystyle\int_M |\nabla\varphi|^2\, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_g} \le \frac{\beta^{\frac{n}{2}}}{\alpha^{\frac{n}{2}+1}}\ \min_{\substack{E\subset H^\star(M)\\ \dim(E)=k}}\ \max_{\substack{\varphi\in E\\ \varphi\ne 0}} \frac{\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_{g_0}}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_{g_0}}$$

where $H^\star(M) := H_0^1(M)$ in the case of the Dirichlet problem and $H^\star(M) = H^1(M)$ in the case of the Neumann or closed manifold problems. Hence we get for any integer $k \ge 1$

$$\lambda_k(M,g) \le \frac{\beta^{\frac{n}{2}}}{\alpha^{\frac{n}{2}+1}}\lambda_k(M,g_0).$$

Obviously we also have

$$\begin{aligned}\frac{\displaystyle\int_M |\nabla\varphi|^2\, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_g} &\ge \frac{\frac{1}{\beta}\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_g} \ge \frac{\frac{\alpha^{\frac{n}{2}}}{\beta}\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_{g_0}}{\beta^{\frac{n}{2}}\displaystyle\int_M \varphi^2\, d\mathcal{V}_{g_0}}\\ &= \frac{\alpha^{\frac{n}{2}}}{\beta^{\frac{n}{2}+1}}\frac{\displaystyle\int_M \left|\nabla_{g_0}\varphi\right|^2\, d\mathcal{V}_{g_0}}{\displaystyle\int_M \varphi^2\, d\mathcal{V}_{g_0}}.\end{aligned}$$

It follows that for any integer $k \geq 1$

$$\frac{\alpha^{\frac{n}{2}}}{\beta^{\frac{n}{2}+1}} \min_{\substack{E \subset H^\star(M) \\ \dim(E)=k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\int_M |\nabla\varphi|^2 \, d\mathcal{V}_{g_0}}{\int_M \varphi^2 \, d\mathcal{V}_{g_0}} \leq \min_{\substack{E \subset H^\star(M) \\ \dim(E)=k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\int_M |\nabla_g\varphi|^2 \, d\mathcal{V}_g}{\int_M \varphi^2 \, d\mathcal{V}_g},$$

i.e., for any integer $k \geq 1$

$$\frac{\alpha^{\frac{n}{2}}}{\beta^{\frac{n}{2}+1}} \lambda_k(M, g_0) \leq \lambda_k(M, g). \qquad \square$$

# 4.6 Complements: The Schrödinger operator and the Hodge–de Rham Laplacian

## 4.6.1 The Schrödinger operator

In this section $(M, g)$ is a complete connected Riemannian manifold of dimension $n \geq 1$, $h$ denotes the Planck constant, and $V$ is a *potential* on $M$: a function $V: M \to \mathbb{R}$.

### Definition and basic properties

**Definition 4.6.1.** The *Schrödinger operator* on $(M, g)$ with potential $V$ is the linear unbounded operator

$$H := \frac{-h^2}{2}\Delta_g + V$$

with domain $\mathcal{D}(M)$, in the sense that $H\varphi = \frac{-h^2}{2}\Delta_g + V\varphi$.

The Schrödinger operator plays a central role in Quantum Mechanics; in particular, we use it to determine the quantum dynamics of a particle via the famous Schrödinger equation: see Sections 2.6.3 and 4.1.3. Let us give here some examples of potentials:

- Free motion potential $V = 0$.
- The double slit experiment: $V \to +\infty$ on the plate, and $V = 0$ elsewhere.
- The hydrogen atom potential: $V = -\frac{k}{\|x\|}$.
- The harmonic oscillator: $V = \frac{\|x\|^2}{2}$.

If $V = 0$ the operator $H$ is just the Laplace–Beltrami operator, and we have seen in this case that the spectrum is discrete. In the general case the main questions about spectrum for Schrödinger operators are:

1. Is the operator $H$ self-adjoint (or essentially self-adjoint)?
2. Is the spectrum of $H$ discrete?

## Self-adjointness

Starting with the first question, recall that an unbounded linear operator $H$ is said to be essentially self-adjoint if its closure $\overline{H}$ is selfadjoint. In the case of $M = \mathbb{R}^n$ with standard metric, Carleman [Car] showed that if the function $V$ is locally bounded and if there exists $C$ such that $V \geq C$ on $M$, then the Schrödinger operator $H$ is essentially self-adjoint. Later T. Kato [Kat1] proved that it is possible to replace the hypothesis $V \in L^{\infty}_{\mathrm{loc}}(M)$ by $V \in L^{2}_{\mathrm{loc}}(M)$. Next, in the works of I.M. Oleinik [Ole1], [Ole2], [Ole3] we can find a general theorem with complex hypotheses on $V$. A very useful consequence of Oleinik's result is:

**Theorem 4.6.2.** *Let $(M, g)$ be a complete connected Riemannian manifold of dimension $n \geq 1$ and let $V \in L^{\infty}_{\mathrm{loc}}(M)$ be a potential such that for all $x \in M$, $V(x) \geq C$, where $C$ is a real constant. Then the operator $H = -\Delta_g + V$ is essentially self-adjoint.*

## Discreteness of the spectrum

As for the Laplace–Beltrami operator, if the manifold $(M, g)$ is compact, then the spectrum of the Schrödinger operator $H = -\Delta_g + V$ is discrete. In the non-compact case, the spectrum of $-\Delta_g$ is not discrete. Nevertheless, if $V$ is a *confining potential*, then the spectrum of $H = -\Delta_g + V$ is discrete.

### Compact setting

Here we suppose that the manifold $(M, g)$ is compact. The spectral problem associated to the operator $H$ is: find all pairs $(\lambda, u)$ with $\lambda \in \mathbb{R}$ and $u \in L^2(M)$ such that

$$-\Delta_g u + V u = \lambda u.$$

In the case of manifolds with boundary, we need boundary conditions on the functions $u$ (Dirichlet, Neumann, or closed manifold conditions).

Similarly to the Laplace–Beltrami operator case (Theorem 4.3.1), we have

**Theorem 4.6.3.** *Consider a compact connected Riemannian manifold $(M, g)$ of dimension $n \geq 1$, and a potential $V \in L^{\infty}_{\mathrm{loc}}(M)$. For the Dirichlet, Neumann, or* *closed manifold problems, the spectrum of* $H = -\Delta_g + V$ *is discrete. Specifically, it consists of an infinite increasing sequence of eigenvalues with finite multiplicity,*

$$\inf_{x \in M} V(x) \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \infty.$$

*Moreover, the associated eigenfunctions* $(e_k)_{k \geq 0}$ *form a Hilbert basis of the space* $L^2(M)$.

### Non-compact setting

Now suppose that the manifold $(M, g)$ is not compact.

**Definition 4.6.4.** Let $(M, g)$ be a smooth manifold and $V : M \to \mathbb{R}$. We said that the function $V$ is *confining* if $\lim_{|x| \to \infty} V(x) = +\infty$, in the sense that

$$\forall A > 0,\ \exists K \subset\subset M,\ \forall x \in M \setminus K,\ |f(x)| \geq A.$$

A general theorem concerning the spectrum of the Schrödinger operator on a manifold (with bounded geometry) is due to Kondrat'ev and Shubin [Ko-Sh1], [Ko-Sh2]:

**Theorem 4.6.5.** *Let* $(M, g)$ *be a complete connected Riemannian manifold of dimension* $n \geq 1$, *and let* $V \in L^\infty_{\text{loc}}(M)$ *be a potential such that* $\lim_{|x| \to \infty} V(x) = +\infty$. *Then the spectrum of* $H = -\Delta_g + V$ *is discrete. Specifically, it consists of an infinite increasing sequence of eigenvalues with finite multiplicity,*

$$\inf_{x \in M} V(x) \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \infty.$$

*Moreover, the associated eigenfunctions* $(e_k)_{k \geq 0}$ *form a Hilbert basis of the space* $L^2(M)$.

For an example of spectrum computation see Section 5.1.4.

## 4.6.2 The Hodge–de Rham Laplacian

Hodge theory links harmonic differential forms to cohomology, thus providing a bridge between algebraic topology and differential geometry. For details on the theory of Hodge–de-Rham Laplacian see the famous article of J. Dodziuk [Dod2], see also for example [Tay] or [War].

In this subsection let us consider $(M, g)$ a closed Riemannian manifold with dimension $n \geq 1$.

## The de Rham chain complex

Using the exterior derivative operator $d$ (see Section 3.1.1) we construct the so-called *de Rham chain complex*:

$$\Omega^0(M) \to \Omega^1(M) \to \cdots \to \Omega^n(M) \to 0$$

which is locally exact (by the Poincaré lemma).

A differential form $\eta \in \Omega^p(M)$ is called *co-closed* if $\delta\eta = 0$, and $\eta \in \Omega^p(M)$ is called *co-exact* if there exists $\beta \in \Omega^{p+1}(M)$ such that $\eta = \delta\beta$. Note that

$$d\Omega^{p-1}(M) = \operatorname{im} d^{p-1} := \left\{\eta \in \Omega^p(M);\ \exists\alpha \in \Omega^{p-1}(M),\ \eta = d\alpha\right\}$$

is the vector space of exact differential forms on $\Omega^p(M)$,

$$\delta\Omega^{p+1}(M) = \operatorname{im} \delta^{p+1} := \left\{\eta \in \Omega^p(M);\ \exists\gamma \in \Omega^{p+1}(M),\ \eta = \delta\gamma\right\}$$

is the vector space of co-exact differential forms on $\Omega^p(M)$,

$$\ker d^p := \{\eta \in \Omega^p(M);\ d\eta = 0\}$$

is the vector space of closed differential forms on $\Omega^p(M)$.

## Some definitions

**Definition 4.6.6.** The *Hodge–de Rham Laplacian* is the second-order elliptic operator acting on the set of differential forms $\Omega^*(M) := \bigcup_{k=0}^{+\infty} \Omega^k(M)$ by the formula

$$\Delta := d\delta + \delta d,$$

where $d$ is the exterior derivative operator and $\delta$ is the codifferential operator, i.e., the formal $L^2$-adjoint of $d$ on $\Omega^*(M)$.

For a fixed integer $p \geq 0$, the $p^{\text{th}}$*-Hodge–de Rham Laplacian*

$$\Delta^p := \Delta_{|\Omega^p(M)}$$

is the restriction of $\Delta$ to $\Omega^p(M)$. Note that the operator $\delta$ sends functions to zero, therefore the $0^{\text{th}}$-Hodge–de Rham operator is the classical Laplacian on functions. A form $\eta \in \Omega^p(M)$ is called *harmonic* if $\Delta^p\eta = 0$; using an integration by parts formula for differential forms one can show that a differential form $\eta \in \Omega^p(M)$ is harmonic if and only if $\eta$ is closed and co-closed.

Since $d^2 = d \circ d = 0$, the set $\operatorname{im} d^{p-1}$ is a subspace of $\ker d^p$. One of the main principal objects in Hodge theory is the $p^{\text{th}}$*-de Rham cohomology group*, defined as the quotient of vector spaces

$$H^p_{\mathrm{DR}}(M) := \ker d^p / \operatorname{im} d^{p-1}.$$

## Spectral theory of the Hodge–de Rham Laplacian

For any $p \geq 0$, the $p^{\text{th}}$-Hodge–de Rham Laplacian has the same spectral properties as the classical Laplacian on functions: this operator is self-adjoint, positive, its spectrum and point spectrum coincide and consist of an infinite sequence of real eigenvalues (with finite multiplicity)

$$\lambda_{p,1}(M) \leq \lambda_{p,2}(M) \leq \cdots \leq \lambda_{p,k}(M) \leq \cdots$$

such that $\lambda_{p,k}(M) \to +\infty$ as $k \to +\infty$.

## Hodge decomposition and applications

In Hodge theory we have the following main result

**Theorem 4.6.7** (Hodge decomposition). *For any fixed integer* $0 \leq p \leq n$*,*

$$\Omega^p(M) = \operatorname{im} d^{p-1} \oplus \ker \Delta^p \oplus \operatorname{im} \delta^{p+1},$$

*i.e., for any* $\eta \in \Omega^p(M)$ *there exists a unique triplet* $(\alpha, \beta, \gamma) \in \Omega^p(M) \times \Omega^{p-1}(M) \times \Omega^{p+1}(M)$ *such that*

$$\eta = \alpha + d\beta + \delta\gamma$$

*and* $\Delta^p \alpha = 0$.

*Remark* 4.6.8. In the case $p = 0$, for any function $\eta \in \mathcal{C}^\infty(M)$ there exists a unique couple $(\alpha, \gamma)$ with a function $\alpha \in \mathcal{C}^\infty(M)$ such that $\Delta_g \alpha = 0$ and $\alpha \in \Omega^1(M)$, such that $\eta = \alpha + \delta\gamma$.

Since $\Delta$ commutes with the operators $d$ and $\delta$, the previous decomposition is invariant under the action of $\Delta^p$. Moreover, since $d \colon \Omega^p(M) \to \Omega^{p+1}(M)$ and $\delta \colon \Omega^{p+1}(M) \to \Omega^p(M)$, if we denote by

$$\mu_{p,1}(M) \leq \mu_{p,2}(M) \leq \cdots \leq \mu_{p,k}(M) \leq \cdots$$

the spectrum of the Hodge–de Rham Laplacian restricted to $\delta\Omega^{p+1}(M)$, then we can write the spectrum of the Hodge–de Rham Laplacian restricted to $d\Omega^{p-1}(M)$:

$$\mu_{p-1,1}(M) \leq \mu_{p-1,2}(M) \leq \cdots \leq \mu_{p-1,k}(M) \leq \cdots,$$

and we have a natural vector space isomorphism between the associated eigenspaces.

One of the most important result of Hodge theory is:

**Theorem 4.6.9.** *The kernel of* $\Delta^p$ *is isomorphic to the* $p^{\text{th}}$*-de Rham cohomology group* $H^p_{\mathrm{DR}}(M)$.

*Proof.* Let $\eta$ be a closed form on $\Omega^p(M)$: $d\eta = 0$. Thanks to the Hodge decomposition, there exists a unique triplet $(a, b, c) \in (\Omega^p(M))^3$ with $\Delta^p a = 0$, and there exists $(\beta, \gamma) \in \Omega^{p-1}(M) \times \Omega^{p+1}(M)$ such that:

$$b = d\beta,\ c = \delta\gamma$$

and

$$\eta = a + b + c.$$

Then we have

$$d\eta = da + db + dc,$$

i.e.,

$$0 = dc$$

($da = 0$ because $a$ is harmonic, and hence closed; and $db = d^2\beta = 0$). On the other hand, we have also $\delta c = \delta^2\gamma = 0$. Consequently the differential form $c$ is harmonic. Using the unicity of the decomposition, we get $c = 0$ and finally we have $\eta = a + b$. In other words, we obtained the equality

$$\ker d^p = \operatorname{im} d^{p-1} \oplus \ker\, \Delta^p.$$

Therefore

$$\begin{aligned} H^p_{\mathrm{DR}}(M) = \ker\, d^p / \operatorname{im} d^{p-1} &= \left(\operatorname{im} d^{p-1} \oplus \ker\, \Delta^p\right) / \left(\operatorname{im} d^{p-1} \oplus 0\right) \\ &\simeq \ker \Delta^p. \end{aligned}$$ □

In algebraic topology the $p^{\text{th}}$ *Betti number* is defined as

$$b_p(M) := \dim H^p_{\mathrm{DR}}(M)$$

(these numbers are topological invariants). $b_p(M)$ is also the dimension of the kernel of $\Delta^p$. An example of application is the following result (see Theorem 4.7 in [GHL]):

**Theorem 4.6.10.** *Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 1$. Then:*

(i) *If the Ricci curvature of $M$ is strictly positive, $H^p_{\mathrm{DR}}(M) = \{0\}$.*

(ii) *If the Ricci curvature is non-negative, $0 \leq \dim H^p_{\mathrm{DR}}(M) \leq n$.*

(iii) *If the Ricci curvature is non-negative and* $\dim H^p_{\mathrm{DR}}(M) = n$, *the manifold $(M, g)$ is isometric to a flat torus (of dimension $n$).*

# Chapter 5
# Direct problems in spectral geometry

As we have seen in the Introduction, there are two main types of problems in spectral geometry: direct problems and inverse problems. This chapter is devoted to direct problems. The general statement of a direct problem is: given a Riemannian manifold $(M, g)$, can we compute or find some properties of the spectrum $(\lambda_k(M))_{k\geq 0}$ of $(M, g)$? We start this chapter with some examples of exact computation for compact manifolds, including an example of computation with a Schrödinger operator. We then present some classical examples of inequalities and asymptotic results concerning eigenvalues.

## 5.1 Explicit calculation of the spectrum

Unfortunately, the calculation of the spectrum is very difficult and in fact we have only a few examples where it can be done explicitly. Let us give some standard examples.

### 5.1.1 Flat tori

We begin with the one-dimensional case with the canonical metric: using the theory of Fourier series on the one-dimensional torus $\mathbb{T}_L$ of length $L > 0$, the spectrum of the Laplacian $-\frac{d^2}{dx^2}$ with closed manifold conditions is

$$\mathrm{Spec}(\mathbb{T}_L, \mathrm{can}) = \left\{\frac{4\pi^2 n^2}{L^2},\ n \in \mathbb{Z}\right\}$$

with associated eigenfunctions:

$$x \mapsto e_n(x) = e^{\frac{2i\pi n x}{L}},\ n \in \mathbb{Z}.$$

Indeed, if the scalar number $\lambda$ is in the spectrum of $-\frac{d^2}{dx^2}$, then there exists a non-trivial function $u \in L^2(\mathbb{T}_L)$ such that

$$-u'' = \lambda u.$$

It is a well known fact that (by the theory of $L^2$ Fourier series) the sequence of functions

$$\left(x \mapsto e_n(x) = e^{\frac{2i\pi n x}{L}}\right)_{n\in\mathbb{Z}}$$

form a Hilbert basis of $L^2(\mathbb{T}_L)$, hence there exists an unique sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ such that

$$u = \sum_{n\in\mathbb{Z}} a_n e_n.$$

Hence, we can rewrite the differential equation $-u'' - \lambda u = 0$ as

$$\sum_{n\in\mathbb{Z}} a_n \left(\frac{4\pi^2 n^2}{L^2} - \lambda\right) e_n = 0.$$

Since $u \neq 0$, there exists $n_0 \in \mathbb{Z}$ such that $a_{n_0} \neq 0$, hence because $(e_n)_{n\in\mathbb{Z}}$ is a basis we get

$$-u'' - \lambda u = 0 \Longrightarrow \lambda = \frac{4\pi^2 n_0^2}{L^2}.$$

Thus we have the inclusion $\mathrm{Spec}(\mathbb{T}_L, \mathrm{can}) \subset \left\{\frac{4\pi^2 n^2}{L^2},\ n \in \mathbb{Z}\right\}$.

Conversely, it is clear that for any integer $n \in \mathbb{Z}$ we have $-\frac{d^2}{dx^2} e_n = \frac{4\pi^2 n^2}{L^2} e_n$, so indeed

$$\mathrm{Spec}(\mathbb{T}_L, \mathrm{can}) = \left\{\frac{4\pi^2 n^2}{L^2},\ n \in \mathbb{Z}\right\}.$$

More generally, let $a := (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ be such that for all $i$, $a_i \neq 0$ and consider the lattice $\Gamma := a_1\mathbb{Z} + a_2\mathbb{Z} + \cdots + a_n\mathbb{Z}$ in $\mathbb{R}^n$ of rank $n \geq 1$. The associated flat torus is

$$\mathbb{T}_a^n := \mathbb{R}^n / \Gamma.$$

Let us also denote by $\Gamma^\star$ the dual lattice of $\Gamma$:

$$\Gamma^\star := \{x \in \mathbb{R}^n;\ \forall y \in \Gamma,\ \langle x, y\rangle \in \mathbb{Z}\}$$

Since $(a_1, a_2, \dots, a_n)$ is a $\mathbb{Z}$-basis of $\Gamma$, the dual lattice $\Gamma^\star$ is given by

$$\Gamma^\star = \left(A^{-1}\right)^T \mathbb{Z}^n,$$

where

$$A := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

thus $\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$ is $\mathbb{Z}$-basis of $\Gamma^\star$, and it follows that the volume of $\Gamma^\star$ is given by

$$\mathrm{Vol}(\Gamma^\star) = \frac{1}{\mathrm{Vol}(\Gamma)}.$$

Moreover, it is easy to show that

$$\Gamma^{\star\star} = \Gamma$$

but unfortunately we do not have $\Gamma^{\star} = \Gamma$.

Now, using the theory of the multidimensional Fourier series, the sequence of functions

$$\left(x \mapsto e^{2i\pi\langle\omega,x\rangle}\right)_{\omega\in\Gamma^{\star}} = \left((x_1,\ldots,x_n) \mapsto e^{\frac{2i\pi k_1 x_1}{a_1}+\cdots+\frac{2i\pi k_n x_n}{a_n}}\right)_{(k_1,\ldots,k_n)\in\mathbb{Z}^n}$$

form a Hilbert basis of $L^2\left(\mathbb{T}^n_a\right)$, so the same argument as in dimension one shows that

$$\operatorname{Spec}(\mathbb{T}^n_a, \mathrm{can}) = \left\{4\pi^2\left(\frac{k_1^2}{a_1^2}+\cdots+\frac{k_n^2}{a_n^2}\right),\ (k_1,\ldots,k_n)\in\mathbb{Z}^n\right\}.$$

In other words, the spectrum of the Laplacian (in the closed situation) on a flat torus $\Gamma \setminus \mathbb{R}^n$ is given by:

$$\operatorname{Spec}(\Gamma\setminus\mathbb{R}^n, \mathrm{can}) = \left\{4\pi^2\left\|\gamma^*\right\|_{\mathbb{R}^n},\ \gamma^*\in\Gamma^*\right\}.$$

### 5.1.2 Rectangular domains with boundary conditions

Consider the Dirichlet problem on a rectangular domain $[0,a]\times[0,b]$ in $\mathbb{R}^2$ with $a, b > 0$. Using the same argument as for the flat torus and the fact that the family

$$\left((x,y)\mapsto e_{n_1,n_2}(x,y) := \sin\left(\frac{n_1\pi x}{a}\right)\sin\left(\frac{n_2\pi y}{b}\right)\right)_{(n_1,n_2)\in\left(\mathbb{N}^{\star}\right)^2}$$

form a Hilbert basis of $H := \left\{f\in L^2\left([0,a]\times[0,b]\right);\ f(a)=f(b)=0\right\}$, it is easy to see that

$$\operatorname{Spec}_{\mathrm{Dirichlet}}([0,a]\times[0,b], \mathrm{can}) = \left\{\pi^2\left(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}\right),\ (n_1,n_2)\in\left(\mathbb{N}^{\star}\right)^2\right\},$$

with associated eigenfunctions $\left\{e_{n_1,n_2}, (n_1,n_2)\in\left(\mathbb{N}^{\star}\right)^2\right\}$.

### 5.1.3 Spheres

The one-dimensional case is trivial: since $\mathbb{S}^1 \simeq \mathbb{T}^1_1$, we have

$$\operatorname{Spec}(\mathbb{S}^1, \mathrm{can}) = \left\{4\pi^2 n^2,\ n\in\mathbb{Z}\right\}.$$

Consider the $n$-dimensional sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Using spherical coordinates $(r, \theta) \in \mathbb{R}_+^\star \times \mathbb{S}^n$, the Euclidean Laplacian may be expressed via the *spherical Laplacian* as

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta.$$

In fact, the homogeneous harmonic polynomials restricted to $\mathbb{S}^n$ are eigenfunctions of $\mathbb{S}^n$. Indeed, let $P$ be a degree $k$ homogeneous harmonic polynomial on $\mathbb{R}^{n+1}$, i.e., $P = r^k P|_{\mathbb{S}^n}$ and $\Delta_{\mathbb{R}^{n+1}} P = 0$. Using the previous expression for the Euclidean Laplacian we get

$$\begin{aligned} 0 &= \Delta_{\mathbb{R}^{n+1}} P \\ &= k(k-1)r^{k-2} P|_{\mathbb{S}^n} + nkr^{k-2} P|_{\mathbb{S}^n} - r^{k-2}\Delta_\theta \left(P|_{\mathbb{S}^n}\right), \end{aligned}$$

whence

$$-\Delta_\theta \left(P|_{\mathbb{S}^n}\right) = k(n+k-1) P|_{\mathbb{S}^n}.$$

Now, since the space of homogeneous harmonic polynomials restricted to $\mathbb{S}^n$ is dense in $L^2(\mathbb{S}^n)$ (see [GHL]), the homogeneous harmonic polynomials restricted to $\mathbb{S}^n$ are the only eigenfunctions of $\mathbb{S}^n$. Consequently,

$$\operatorname{Spec}(\mathbb{S}^n, \mathrm{can}) = \{k(k+n-1),\ k \in \mathbb{N}\}$$

and the multiplicity of $k(k+n-1)$ is equal to $\begin{pmatrix} n+k \\ k \end{pmatrix} - \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$.

### 5.1.4 Harmonic oscillator

This is a very famous example of explicit computation in Quantum Mechanics. The *one-dimensional harmonic oscillator* is the Schrödinger operator

$$-\frac{d^2}{dx^2} + x^2$$

on the manifold $\mathbb{R}$. Consider the set $Y := \{f \in H^1(\mathbb{R}),\ xf \in L^2(\mathbb{R})\}$. This set a subspace of $H^1(\mathbb{R})$, and equipped with the scalar product

$$\langle f, g\rangle_Y := \langle f, g\rangle_{L^2} + \left\langle f', g'\right\rangle_{L^2} + \langle xf, xg\rangle_{L^2}$$

$Y$ is a Hilbert space and is dense in $L^2(\mathbb{R})$. Moreover, by a classical argument of functional analysis, the canonical inclusion of $Y$ in $L^2(\mathbb{R})$ is compact.

Let us introduce the *annihilation operator* $A\colon D(A) \to L^2(\mathbb{R})$ with domain $D(A) = Y$, defined by

$$Af := f' + xf.$$

Using the integration by parts formula (and the fact that if $f \in Y$ then $\lim_{|x|\to+\infty} x|f(x)|^2 = 0$) we get for all $f \in Y$

$$\|Af\|_{L^2}^2 = -\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|xf\|_{L^2}^2,$$

for all $f \in Y$, thus we obtain

$$2\|f\|_{L^2}^2 + \|Af\|_{L^2}^2 = \|f\|_Y^2.$$

Consequently, the norms $\|\cdot\|_Y$ and $\|\cdot\|_A$ are equivalent on $Y$. Therefore the set $Y = D(A)$ is complete for the norm $\|\cdot\|_A$. It follows that the operator $A\colon D(A) \to L^2(\mathbb{R})$ is closed. The adjoint of $A$ is the operator $A^\star\colon D(A^\star) \to L^2(\mathbb{R})$ with domain $D(A^\star) = Y$ and defined by

$$A^\star f = -f' + xf.$$

This operator is called the *creation operator*. Now we consider the operator

$$H := A^\star A + I$$

with domain $D(H) := \{f \in Y,\ Af \in Y\}$. In fact we have

$$D(H) = \left\{f \in H^2(\mathbb{R}),\ x^2 f \in L^2(\mathbb{R})\right\}$$

and for all $f \in D(H)$

$$A^\star Af = -(Af)' + xA(f) = -f'' + x^2 f - f.$$

Hence,

$$Hf = -f'' + x^2 f$$

for all $f \in D(H)$. In particular, for all $f \in D(H)$ such that $\|f\|_{L^2}^2 = 1$ we have

$$\langle Hf, f\rangle_{L^2} = \left\langle A^\star Af, f\right\rangle_{L^2} + \|f\|_{L^2}^2 = \|Af\|_{L^2}^2 + 1 \geq 1,$$

therefore the closure of the numerical range $\Theta(H)$ is a subset of $[1, +\infty[$. We see that the set $\mathbb{C} - \overline{\Theta(H)}$ has just one connected component and the map

$$d\colon \begin{cases} \mathbb{C} \longrightarrow \mathbb{N} \cup \{+\infty\} \\ \lambda \longmapsto \dim\left(\ker\left(H^\star - \overline{\lambda} I\right)\right) \end{cases}$$

is constant on $\mathbb{C} - \overline{\Theta(H)}$, i.e., for all $\lambda \in \mathbb{C} - \overline{\Theta(H)}$,

$$d(\lambda) = d(i) = d(-i)$$

(because $i, -i \in \mathbb{C} - \overline{\Theta(H)}$). But since $H$ is self-adjoint

$$\ker\left(H^\star - \overline{\lambda} I\right) = \{0\}$$

thus $d(i) = d(-i) = 0$, and so $d(\lambda) = 0$ for all $\lambda \in \mathbb{C} - \overline{\Theta(H)}$. In other words, $\mathbb{C} - \overline{\Theta(H)} \subset \rho(H)$ so we get $\mathrm{Spec}(H) \subset \sigma(H) \subset \overline{\Theta(H)}$ because the spectrum of $H$ is discrete. We conclude that

$$\mathrm{Spec}(H) \subset [1, +\infty[.$$

By a simple computation

$$\left[A, A^\star\right] f = AA^\star f - A^\star A f = 2f$$

for all $f \in D(H)$. Next, if a vector $\varphi \in D(H)$ satisfies $H\varphi = \lambda\varphi$ (where $\lambda$ is a scalar), then $A\varphi \in D(H)$, $A^\star\varphi \in D(H)$, and we have

$$\begin{aligned} H(A\varphi) &= A^\star A A\varphi + A\varphi = AA^\star A\varphi - 2A\varphi + A\varphi \\ &= A\left(H\varphi - \varphi\right) - A\varphi = A\left(\lambda\varphi - \varphi\right) - A\varphi = (\lambda - 2)A\varphi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|A\varphi\|_{L^2}^2 &= \langle A\varphi, A\varphi\rangle_{L^2} = \left\langle A^\star A\varphi, \varphi\right\rangle_{L^2} \\ &= \langle H\varphi - \varphi, \varphi\rangle_{L^2} = \lambda \|\varphi\|_{L^2}^2 - \|\varphi\|_{L^2}^2 = (\lambda - 1)\|\varphi\|_{L^2}^2 . \end{aligned}$$

Similarly,

$$H(A^\star\varphi) = (\lambda + 2)A^\star\varphi, \qquad \left\|A^\star\varphi\right\|_{L^2}^2 = (\lambda + 1)\|\varphi\|_{L^2}^2 .$$

Now we want to show that $\mathrm{Spec}(H) \subset \{(2n+1),\, n \in \mathbb{N}\}$. For the moment we only know that $\mathrm{Spec}(H) \subset [1, +\infty[$. If $\lambda$ is an eigenvalue of $H$ in the interval $]1, 3[$, there exist $\varphi \neq 0$ in $D(H)$ such that $H\varphi = \lambda\varphi$, thus we have

$$H(A\varphi) = (\lambda - 2)A\varphi$$

and

$$\|A\varphi\|_{L^2}^2 = (\lambda - 1)\|\varphi\|_{L^2}^2 > 0$$

hence $A\varphi \neq 0$. Consequently, $\lambda - 2 \in \mathrm{Spec}(H)$ therefore $\lambda - 2 \in ]-1, 1[$, which is a contradiction. So

$$\mathrm{Spec}(H) \subset \{1\} \cup [3, +\infty[.$$

Using the same argument by induction we get easily that

$$\mathrm{Spec}(H) \subset \{(2n+1),\, n \in \mathbb{N}\},$$

as claimed. To finish, using the *Hermite family* $\{e_n\}_{n\in\mathbb{N}}$

$$e_n(x) = (2^n n!\sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x) \ \textit{with} \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

(which form a Hilbert basis of the space $L^2(\mathbb{R})$: for a proof of this fact see, for example [Tay] or the notes [Ev-Zw], and for a different method using the Bargmann transform see the original article of V. Bargmann [Bar] or also in [Tay]).

We have for any $n$

$$A^\star e_n = \sqrt{2n+2}e_{n+1}$$

and

$$Ae_n = \sqrt{2n}e_{n-1},$$

so for any $n$

$$He_n = (2n+1)e_n.$$

Consequently the spectrum of the operator $H$ is

$$\mathrm{Spec}(H) = \{(2n+1),\ n \in \mathbb{N}\}$$

and the associated eigenvectors are given the Hermite's family.

## 5.2 Qualitative properties of the spectrum

When it is impossible to compute the spectrum, we try to describe some of its properties, and two kinds of questions arise:

- Determine or estimate the bottom of the spectrum, i.e., look at the first eigenvalues $\lambda_1(M)$ and/or $\lambda_2(M)$.
- Determine the asymptotics of large eigenvalues (the *semi-classical paradigm*).

### 5.2.1 From the bottom of the spectrum …

We give here some examples concerning the bottom of the spectrum. For example, the first nonzero eigenvalue of a compact manifold is very important. We begin with the Lichnerowicz–Obata theorem:

**Theorem 5.2.1** (Lichnerowicz–Obata Theorem). *Let $(M, g)$ be a closed Riemannian manifold of dimension $n$ such that*

$$\mathrm{Ric} \geq \kappa(n-1) > 0$$

*(i.e., for any* $\zeta \in TM$ *we have* $\mathrm{Ric}(\zeta,\zeta) \geq \kappa(n-1)\,\|\zeta\|^2 > 0$*). Then the first nonzero eigenvalue* $\lambda_\star$ *of* $-\Delta_g$ *satisfies*

$$\lambda_\star \geq n\kappa.$$

*Moreover, we have equality if and only if* $M$ *is isometric to an Euclidean sphere of dimension* $n$*.*

*Proof.* The equality case is due to Obata. Here we just give a proof for the inequality. The proof is based on the *Bochner–Lichnerowicz formula* (see for example [BGM]): for any function $\varphi \in \mathcal{C}^\infty(M)$ it holds that

$$\frac{1}{2}\Delta_g\left(|\nabla f|^2\right) = |||\mathrm{Hess}\, f|||^2 + \nabla f.\nabla\Delta_g f + \mathrm{Ric}\left(\nabla f, \nabla f\right),$$

where Hess $f$ is the Hessian matrix of the function $f$ and $|||\cdot|||$ is the matrix–operator norm.

Now, let $\lambda_\star$ be an nonzero eigenvalue of the operator $-\Delta_g$: there exists a nontrivial function $u$ such that $-\Delta_g u = \lambda_\star u$; and we have (see Proposition 4.5.3)

$$\lambda_\star = \frac{\displaystyle\int_\Omega |\nabla u|^2\, d\mathcal{V}_g}{\displaystyle\int_\Omega u^2\, d\mathcal{V}_g} > 0.$$

Under the hypotheses of the theorem, the Bochner–Lichnerowicz formula yields

$$\begin{aligned}\frac{1}{2}\Delta_g\left(|\nabla u|^2\right) &= |||\mathrm{Hess}\, u|||^2 + \nabla u\cdot\nabla\Delta_g u + \mathrm{Ric}\left(\nabla u, \nabla u\right)\\ &= |||\mathrm{Hess}\, u|||^2 - \lambda_\star\nabla u\cdot\nabla u + \mathrm{Ric}\left(\nabla u, \nabla u\right)\\ &\geq |||\mathrm{Hess}\, u|||^2 - \lambda_\star|\nabla u|^2 + \kappa(n-1)|\nabla u|^2.\end{aligned}$$

By a classical result of the theory of finite-dimensional vector space (here Hess $u$ is a $n \times n$-matrix),

$$\frac{\mathrm{tr}^2(\mathrm{Hess}\, u)}{n} \leq |||\mathrm{Hess}\, u|||^2,$$

so in our context we get

$$\begin{aligned}\frac{1}{2}\Delta_g\left(|\nabla u|^2\right) &\geq \frac{\left(\Delta_g u\right)^2}{n} - \lambda_\star|\nabla u|^2 + \kappa(n-1)|\nabla u|^2\\ &= \frac{\lambda_\star^2 u^2}{n} + \left(\kappa(n-1) - \lambda_\star\right)|\nabla u|^2.\end{aligned}$$

Upon integrating over the manifold $(M, g)$ we obtain

$$\frac{1}{2}\int_M \Delta_g\left(|\nabla u|^2\right) d\mathcal{V}_g \geq \int_M \frac{\lambda_\star^2 u^2}{n} + (\kappa(n-1)-\lambda_\star)\,|\nabla u|^2\, d\mathcal{V}_g.$$

Recall that the manifold $(M, g)$ has no boundary, so using the integration by parts formula we obtain

$$\int_M \Delta_g\left(|\nabla u|^2\right) d\mathcal{V}_g = 0,$$

whence

$$0 \geq \frac{1}{n}\lambda_\star^2\,\|u\|^2_{L^2(M)} + (\kappa(n-1)-\lambda_\star)\int_M |\nabla u|^2\, d\mathcal{V}_g.$$

Since $u$ is not trivial we further have

$$0 \geq \frac{1}{n}\lambda_\star^2 + (\kappa(n-1)-\lambda_\star)\,\frac{\displaystyle\int_M |\nabla u|^2\, d\mathcal{V}_g}{\displaystyle\int_M u^2\, d\mathcal{V}_g},$$

hence

$$\begin{aligned} 0 &\geq \frac{1}{n}\lambda_\star^2 + (\kappa(n-1)-\lambda_\star)\,\lambda_\star \\ &= \frac{1}{n}\lambda_\star^2 + \kappa n\lambda_\star - \kappa\lambda_\star - \lambda_\star^2. \end{aligned}$$

To finish observe that

$$\frac{\lambda_\star(n-1)}{n}(\kappa n - \lambda_\star) = \frac{1}{n}\lambda_\star^2 + \kappa n\lambda_\star - \kappa\lambda_\star - \lambda_\star^2,$$

and so

$$\frac{\lambda_\star(n-1)}{n}(\kappa n - \lambda_\star) \leq 0,$$

hence $\kappa n - \lambda_\star \leq 0$. □

*Remark* 5.2.2. There exist analogous versions of this theorem for the case of compact manifolds with boundary (Dirichlet and Neumann problems).

We present here the *Cheeger inequality*. Let $(M, g)$ be a compact connected Riemannian manifold. For every domain $D$ ($D$ is a bounded regular subset) of $M$, consider the quantity

$$h(D, g) = \frac{\mathrm{Vol}(\partial D, g)}{\mathrm{Vol}(D, g)},$$

where $\partial D$ denotes the boundary of $D$ (thus $\mathrm{Vol}(\partial D, g)$ is the $(n-1)$-dimensional volume).

**Definition 5.2.3.** The *Cheeger's constant* on $(M, g)$ is the number

$$h(M, g) = \inf_{D \in X} h(D, g),$$

where $X$ is the set of domains $D \subset M$ such that $\mathrm{Vol}(D) \leq \frac{\mathrm{Vol}(M,g)}{2}$.

Using the co-area formula we obtain:

**Theorem 5.2.4** (Cheeger's inequality). *For each of the spectral problem (Dirichlet, Neumann or closed) the first nonzero eigenvalue $\lambda_\star$ of $-\Delta_g$ satisfies*

$$\lambda_\star \geq \frac{h(M, g)^2}{4}.$$

Another remarkable result is the Li–Yau Theorem: for a closed Riemannian manifold the first nonzero eigenvalue of $-\Delta_g$ is bounded (up to a constant) from below by the inverse of the diameter of the manifold

**Theorem 5.2.5** (Li–Yau Theorem). *Let $(M, g)$ be a closed Riemannian manifold of dimension $n$ with nonnegative Ricci curvature. Then the first nonzero eigenvalue $\lambda_\star$ of $-\Delta_g$ satisfies*

$$\lambda_\star \geq \frac{\pi^2}{4\mathrm{diam}^2(M, g)}$$

*where* $\mathrm{diam}(M, g) := \sup \{d(x, y),\ (x, y) \in M^2\}$.

Let us also gives examples of results about comparison between the first eigenvalues of two different domains.

**Theorem 5.2.6** (Faber–Krahn Theorem, 1953). *Let $M$ be a bounded domain of $\mathbb{R}^n$, and let us denote by $\lambda_1(M)$ the first non-null eigenvalue of $-\Delta$ with Dirichlet conditions. We have the inequality*

$$\lambda_1(M) \geq \lambda_1(B_M),$$

*where $B_M$ is the Euclidean ball with volume equal to* $\mathrm{Vol}(M)$*. Moreover, equality holds if and only if $M$ is isometric to $B_M$.*

In the same spirit, we have also

**Theorem 5.2.7** (Szegö–Weinberger Theorem, 1954). *Let $M$ a bounded domain of $\mathbb{R}^n$ and let us denote by $\mu_1(M)$ the first non-null eigenvalue of $-\Delta$ with Neumann boundary conditions. We have the inequality*

$$\mu_1(M) \leq \mu_1(B_M),$$

*where $B_M$ is the Euclidean ball with volume equal to* $\mathrm{Vol}(M)$*. Moreover, equality holds if and only if $M$ is isometric to $B_M$.*

### 5.2.2 ... to the large eigenvalues: the Weyl formula

The most popular result concerning large eigenvalues is the Weyl asymptotic formula. For the Laplacian in a rectangular domain $\Omega$ of $\mathbb{R}^2$ with Dirichlet boundary conditions, the physicist P. Debye arrived at the formula

$$\mathcal{N}(\lambda) \underset{\lambda\to+\infty}{\sim} \frac{\operatorname{Vol}(\Omega)}{4\pi}\lambda,$$

where $\mathcal{N}(\lambda) := \operatorname{Card}(\{k \in \mathbb{N},\ \lambda_k \leq \lambda\})$ and $\operatorname{Vol}(\Omega)$ is the area of the domain $\Omega$. In 1911, H. Weyl gave a proof of this conjecture. In the next subsection we explain the proof in the case of flat tori.

### 5.2.3 Weyl formula for a flat torus

Consider the flat torus of Section 5.1.1, i.e., $\mathbb{T}^n_a = \mathbb{R}^n/\Gamma$ with $\Gamma := a_1\mathbb{Z}+a_2\mathbb{Z}+\cdots+a_n\mathbb{Z}$ and $a := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. We have seen therein that

$$\operatorname{Spec}(\mathbb{T}^n_a, \text{can}) = \left\{4\pi^2\left(\frac{k_1^2}{a_1^2}+\frac{k_1^2}{a_1^2}+\cdots+\frac{k_n^2}{a_n^2}\right),\ (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n\right\}.$$

Here the eigenvalue counting function is

$$\begin{aligned}\mathcal{N}(\lambda) &= \operatorname{Card}\left(\left\{k = (k_1, k_2, \ldots, k_n)\,;\ \frac{k_1^2}{a_1^2}+\frac{k_2^2}{a_2^2}+\cdots+\frac{k_n^2}{a_n^2} \leq \frac{\lambda}{4\pi^2}\right\}\right)\\ &= \Gamma^\star \cap B,\end{aligned}$$

where $B := \overline{B_{\mathbb{R}^n}}\left(0, \frac{\sqrt{\lambda}}{2\pi}\right)$ denotes the closed ball of $\mathbb{R}^n$ centered in 0 and of radius $\frac{\sqrt{\lambda}}{2\pi}$. For all $\omega := (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^\star$ let us denote by $I_\omega$ the following compact set of $\mathbb{R}^n$:

$$I_\omega := \prod_{i=1}^n \left[\omega_i - \frac{1}{2a_{1i}}, \omega_i + \frac{1}{2a_i}\right]$$

and consider the union

$$P := \bigcup_{\omega\in\Gamma^\star\cap B} I_\omega.$$

Clearly the volume $\operatorname{Vol}(P)$ of $P$ is equal to

$$\operatorname{Vol}(P) = \mathcal{N}(\lambda)\frac{1}{a_1}\frac{1}{a_2}\cdots\frac{1}{a_n}.$$

Now we claim that the inclusions

$$\overline{B_{\mathbb{R}^n}}\left(0, \frac{\sqrt{\lambda}}{2\pi} - R\right) \subset P \subset \overline{B_{\mathbb{R}^n}}\left(0, \frac{\sqrt{\lambda}}{2\pi} + R\right)$$

hold with
$$R := \left\| \left( \frac{1}{2a_1}, \frac{1}{2a_2}, \dots, \frac{1}{2a_n} \right) \right\|_{\mathbb{R}^n}.$$
Indeed, using the triangle inequality for all $(x_1, x_2, \dots, x_n) \in B$, we have
$$\begin{aligned}\|(x_1, x_2, \dots, x_n)\|_{\mathbb{R}^n} - R &\leq \left\| \left( x_1 \pm \frac{1}{2a_1}, x_2 \pm \frac{1}{2a_2}, \dots, x_n \pm \frac{1}{2a_n} \right) \right\|_{\mathbb{R}^n} \\ &\leq \|(x_1, x_2, \dots, x_n)\|_{\mathbb{R}^n} + R.\end{aligned}$$

Consequently,
$$\mathrm{Vol}\left( \overline{B_{\mathbb{R}^n}} \left( 0, \frac{\sqrt{\lambda}}{2\pi} - R \right) \right) \leq \mathrm{Vol}(P) \leq \mathrm{Vol}\left( \overline{B_{\mathbb{R}^n}} \left( 0, \frac{\sqrt{\lambda}}{2\pi} + R \right) \right),$$
with
$$\mathrm{Vol}\left( \overline{B_{\mathbb{R}^n}} \left( 0, \frac{\sqrt{\lambda}}{2\pi} \pm R \right) \right) = B_n \left( \frac{\sqrt{\lambda} \pm R2\pi}{2\pi} \right)^n$$
where $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the closed unit ball in $\mathbb{R}^n$.

Moreover, since
$$\mathrm{Vol}(P) = \mathcal{N}(\lambda) \frac{1}{a_1} \frac{1}{a_2} \cdots \frac{1}{a_n} = \frac{\mathcal{N}(\lambda)}{\mathrm{Vol}(\mathbb{T}^n_a)},$$
we obtain
$$\mathcal{N}(\lambda) \underset{\lambda \to +\infty}{\sim} \frac{\mathrm{Vol}(\mathbb{T}^n_a) B_n}{(2\pi)^n} \lambda^{\frac{n}{2}}.$$

### 5.2.4 The semi-classical paradigm

*Passer de la mécanique de Newton à celle d'Einstein doit être un peu, pour le mathématicien, commme de passer du bon vieux dialecte provençal à l'argot parisien dernier cri. Par contre, passer à la mécanique quantique, j'imagine c'est passer au chinois.*

Alexandre Grothendieck [Grot], Récoltes et Semailles (1986).

The semi-classical analysis is a nice way to understand the relationship between classical Hamiltonian mechanics (symplectic[1] geometry) and quantum mechanics (operator algebras). The principle of semi-classical analysis is to extend the classical Hamiltonian formalism by a deformation, therefore we can see classical mechanics as a limit of quantum mechanics. For a complete introduction

[1] A symplectic manifold $(M, \omega)$ is a smooth differentiable manifold of dimension $m$ equipped with a non-degenerate closed 2-form $\omega$.

to semi-classical analysis see for example [Ba-We], [Di-Sj], [Ev-Zw], [Gr-Sj], [Mar], [Rob]. Here, we just explain the philosophy of semi-classical analysis on a concrete example.

Let $(M, g)$ be a compact Riemaniann manifold, and consider a (semi-classical) parameter[2] $h > 0$ and an another real number $E > 0$. The equation

$$-\frac{h^2}{2}\Delta_g \varphi = E\varphi$$

admits an eigenvector $\varphi_k$ as solution if and only if we have the scalar relation

$$-\frac{h^2}{2}\lambda_k = E.$$

Hence, if $h \to 0^+$ then $\lambda_k \to +\infty$. So there exists a correspondence between the semi-classical limit ($h \to 0^+$) and large eigenvalues of the Laplacian. The asymptotics of large eigenvalues for the Laplace–Beltrami operator $\Delta_g$ on a Riemaniann manifold $(M, g)$, or more generally for a pseudo-differential operator $P_h$, is linked to symplectic geometry, the phase space geometry. This is the same correspondence between quantum mechanics (spectrum, operator algebra) and classical mechanics (length of periodic geodesics, symplectic geometry). More precisely, for a pseudo-differential operator $P_h$ on $L^2(M)$ with a principal symbol $p \in \mathcal{C}^\infty(T^\star M)$, there exist a link between the geometry of the foliation $\left(p^{-1}(\lambda)\right)_{\lambda\in\mathbb{R}}$ and the spectrum of the operator $P_h$. Indeed, we have the famous result

$$(P_h - \lambda I)\, u_h = O(h^\infty)$$

then

$$MS(u_h) \subset p^{-1}(\lambda),$$

where $MS(u_h) \subset T^\star M$ denote the microsupport of the function $u_h \in L^2(M)$.

## 5.3 The spectral partition function $Z_M$

**Definition 5.3.1.** Let $(M, g)$ be a compact Riemannian manifold. The series of functions

$$Z_M(t) := \sum_{k=1}^{+\infty} e^{-t\lambda_k}$$

is called the *spectral partition function* of the manifold $(M, g)$.

[2] For example, the Planck constant.

Using the Weyl asymptotics, the series converge uniformly on $\mathbb{R}_+^\star$ (see Section 7.5.3). Using this uniform convergence it is clear that the function $t \mapsto Z_M(t)$ is continuous and non-increasing on $\mathbb{R}_+^\star$. We have also that

$$\lim_{t \to +\infty} Z_M(t) = \ell,$$

with $\ell = 0$ if $0 \in \operatorname{Spec}(M, g)$ and $\ell = 1$ otherwise.

With the notation (see 4.5.11)

$$0 \leq \left(\widetilde{\lambda_1}, m_1 = 1\right) < \left(\widetilde{\lambda_2}, m_2\right) < \cdots < \left(\widetilde{\lambda_k}, m_k\right) < \cdots$$

we have also the expression

$$Z_M(t) = \sum_{k=1}^{+\infty} m_k e^{-t\widetilde{\lambda_k}}.$$

The interest in the spectral partition function is justified by the following result

**Proposition 5.3.2.** *The function $Z_M$ determine the spectrum of the manifold $(M, g)$.*

*Proof.* Suppose for example, that $0 \in \operatorname{Spec}(M, g)$, hence $\widetilde{\lambda_1} = 0$. For $\mu > 0$ consider the function

$$e^{\mu t} Z_M(t) - e^{\mu t} = \sum_{k=1}^{+\infty} m_k e^{(\mu - \widetilde{\lambda_k})t} - e^{\mu t} = \sum_{k=2}^{+\infty} m_k e^{(\mu - \widetilde{\lambda_k})t}$$

If $\mu < \widetilde{\lambda_2}$, then

$$\lim_{t \to +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = 0;$$

if $\mu = \widetilde{\lambda_2}$, then

$$\lim_{t \to +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = m_2;$$

finally, if $\mu > \widetilde{\lambda_2}$, then

$$\lim_{t \to +\infty} e^{\mu t} Z_M(t) - e^{\mu t} = +\infty.$$

Thus $\widetilde{\lambda_2}$ is the unique real number $\mu > 0$ such that the function $t \mapsto e^{\mu t} Z_M(t) - e^{\mu t}$ has a finite limit as $t$ tends to infinity. Consequently, the function $Z_M$ determine the first non-null eigenvalue $\widetilde{\lambda_2}$. By induction, for any integer $i \geq 2$ considering the expression

$$e^{\mu t} Z_M(t) - \sum_{j=1}^{i-1} m_j e^{(\mu - \widetilde{\lambda_j})t} = \sum_{j=i}^{+\infty} m_j e^{(\mu - \widetilde{\lambda_j})t}$$

we get: if $\mu < \widetilde{\lambda_i}$, then

$$\lim_{t\to+\infty} e^{\mu t} Z_M(t) - \sum_{j=1}^{i-1} m_j e^{(\mu-\widetilde{\lambda_j})t} = 0;$$

if $\mu = \widetilde{\lambda_i}$, then

$$\lim_{t\to+\infty} e^{\mu t} Z_M(t) - \sum_{j=1}^{i-1} m_j e^{(\mu-\widetilde{\lambda_j})t} = m_i;$$

finally, if $\mu > \widetilde{\lambda_i}$, then

$$\lim_{t\to+\infty} e^{\mu t} Z_M(t) - \sum_{j=1}^{i-1} m_j e^{(\mu-\widetilde{\lambda_j})t} = +\infty.$$

Hence, for any integer $i$ the function $Z_M$ determines the eigenvalue $\widetilde{\lambda_i}$.

The case $0 \notin \mathrm{Spec}(M, g)$ is similar. □

## 5.4 Eigenvalues and eigenfunctions of surfaces

### 5.4.1 Spectrum of surfaces

In the case of dimension two (surfaces) there exists important topological restrictions on the multiplicities of the eigenvalues (see also Section 7.4). Here let us present quickly (without proofs) the main result concerning the relationship between the topology of a surface and the multiplicities of its eigenvalues. The following theorem is due to S. Y. Cheng [Che], G. Besson [Bes], Y. Colin de Verdière [Col4], N. Nadirashvili [Nad], and B. Sévennec [Sev].

**Theorem 5.4.1.** *For a connected complete surface $(M, g)$ we have*

(i) *If $X = \mathbb{S}^2$ or $\mathbb{R}^2$, then for any $k \geq 3$ we have $m_k \leq 2k - 3$.*

(ii) *If $X = \mathbb{P}^2(\mathbb{R})$ (the projective plane) or $K_2$ (the Klein Bottle), then for any $k \geq 1$ we have $m_k \leq 2k + 1$.*

(iii) *If $X = \mathbb{T}^2$ (the two-torus), then for any $k \geq 1$ we have $m_k \leq 2k + 2$.*

(iv) *If $\chi(M) < 0$ (where $\chi(M)$ denotes the Euler characteristic of the surface), then for any $k \geq 1$ we have $m_k \leq 2k - 2\chi(M)$.*

### 5.4.2 Eigenfunctions of surfaces and Courant's nodal theorem

After studying eigenvalues, an important further question is to understand the behaviour of the eigenfunctions. Let $(M, g)$ be a compact Riemannian surface with boundary. For every eigenfunction $\varphi$ of $-\Delta_g$ (with Dirichlet conditions) the *nodal set* (or set of nodal lines) associated to $\varphi$ is

$$N(\varphi) := \overline{\{x \in M;\ \varphi(x) = 0\}}.$$

Nodal sets appear explicitly in experimental physics: for example on a vibrating drum with dust, the dust accumulates along the nodal lines of the drum (this is *the Chladni's acoustic patterns* phenomenon, see for example in the classical [Co-Hi] and in [Bé-He] for recent pictures and discussion on this topic).

The connected components of the set $M - N(\varphi)$ are called the *nodal domains* of the eigenfunction $\varphi$. Here let us denote by $\mu(\varphi)$ the number of nodal domains of $\varphi$. We have the celebrated *Courant's nodal theorem* (see the original article [Coura] and also [Co-Hi], [Cha]):

**Theorem 5.4.2** (Courant's nodal Theorem). *Let $k \geq 1$. Then for every non-trivial eigenfunction $\varphi$ associated to the $k$-eigenvalue $\lambda_k(M)$ we have $\mu(\varphi) \leq k$.*

Consequently, the first eigenfunction never vanishes on $M$ and the multiplicity of the first eigenvalue is exactly equal to one (see also Proposition 4.5.8).

In 1956 A. Pleijel [Ple] showed that the case of equality in Courant's nodal theorem holds only for finitely many values of the integer $k$. For a recent discussion on the geometry of nodal domains for the Dirichlet problem on a square menbrane; see the recent article of P. Bérard and B. Helffer [Bé-He].

*Remark* 5.4.3. Courant's nodal theorem and Pleijel's theorem extends for compact manifolds (see [Cha] for a proof).

## 5.5 Exercises

**Exercise 5.5.1.** *Compute the spectrum of the Laplacian on a rectangular domain $[0, a] \times [0, b]$ in $\mathbb{R}^2$ with Neumann boundary conditions.*

**Exercise 5.5.2.** *Consider the operator:*

$$H = -\frac{h^2}{2m}\Delta - e^2 r^{-1}$$

*with domain $H^2(\mathbb{R}^3)$, where $m, e > 0$ and $r := \sqrt{x^2 + y^2 + z^2}$. Show that $H$ is self-adjoint and*

$$\operatorname{Spec}(H) = \left\{\frac{-me^4}{2h^2n^2},\ n \geq 1\right\}.$$

**Exercise 5.5.3.** *Prove that the Hermite functions*

$$e_n(x) = (2^n n!\sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x) \ \textit{with} \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

*form a Hilbert basis of* $L^2(\mathbb{R})$.

**Exercise 5.5.4.** *Compute the spectrum of a flat Klein bottle* (*see in* [BGM] *page 151*).

**Exercise 5.5.5.** *Prove the Weyl formula for the Laplacian on a rectangular domain.*

**Exercise 5.5.6.** *Let* $(M_1, g_1)$ *and* $(M_2, g_2)$ *be two closed Riemannian manifolds. Find the spectrum and eigenvectors of the product* $(M_1 \times M_2, g_1 \times g_2)$.

# Chapter 6

# Intermezzo: "Can one hear the holes of a drum?"

In this chapter we describe in details an application of the minimax principle. Here, we consider a closed Riemannian manifold $(M, g)$ and a subset $A$ of $M$. We are interested in comparing the eigenvalues $(\lambda_k(M))_{k\geq 1}$ of the manifold $(M, g)$ and the eigenvalues $(\lambda_k(M - A))_{k\geq 1}$ of the manifold $(M - A, g)$ with Dirichlet boundary conditions.

The behaviour of the spectrum of a Riemannian manifold $(M, g)$ under topological perturbation has been the subject of intensive research. The most famous example is the crushed ice problem [Kac2], see also [Ann]. This problem is to understand the behaviour of eigenvalues of the Laplacian with Dirichlet boundary conditions on a domain with small holes. This subject was first studied by M. Kac [Kac2] in 1974. Then, J. Rauch and M. Taylor [Ra-Ta] studied the case of Euclidean Laplacian in a compact set $M$ of $\mathbb{R}^n$; they showed that the spectrum of $\Delta_{\mathbb{R}^n}$ is invariant under topological excision of a compact subset $A$ with a Newtonian capacity zero. Later, S. Osawa, I. Chavel and E. Feldman [Ch-Fe1, Ch-Fe2] treated Riemannian manifold case. They used complex probabilistic techniques based on Brownian motion. In [Ge-Zh], F. Gesztesy and Z. Zhao investigated the case of a Schrödinger operator in $\mathbb{R}^n$ with Dirichlet boundary conditions, also using probabilistic tools. In 1995, in a nice article [Cou] G. Courtois studied the case of the Laplace–Beltrami operator on a closed Riemannian manifold. He used very simple techniques of analysis based on the minimax principle. In [Be-Co] J. Bertrand and B. Colbois treated also the case of the Laplace–Beltrami operator on a compact Riemannian manifold with boundary conditions, see also [Lab3] for the case of a Schrödinger operator $-\Delta_g + V$ treated on a closed Riemannian manifold.

In this section we present a part of the paper of G. Courtois [Cou].

## 6.1 The main result

Let $(M, g)$ be a closed Riemannian manifold. Following G. Courtois [Cou], we show that if we remove topologically a "small" part $A$ from the manifold, the spectrum of $M - A$ is close of the spectrum of $M$. More precisely, the "good" parameter for measuring the smallness of $A$ is a type of electrostatic capacity, defined by

$$\begin{aligned}\operatorname{cap}(A) := \inf\Big\{ \int_M |\nabla u|^2 \, d\mathcal{V}_g;\ & u \in H^1(M), \\ & \int_M u \, d\mathcal{V}_g = 0,\ u - 1 \in H_0^1(M - A) \Big\}\end{aligned}$$

where $H_0^1(M - A)$ is the *Sobolev space*

$$H_0^1(M - A) := \overline{\{g \in H^1(M);\ g = 0 \text{ on a open neighborhood of } A\}},$$

the closure is taken with respect to the norm $\|\cdot\|_{H^1(M)}$ and $H^1(M)$ is the usual Sobolev space on $M$ (see Section 3.2.2).

Indeed, the smaller $\operatorname{cap}(A)$ is, the closer the spectrum of $-\Delta_g$ on $M - A$ to the spectrum on $M$ in the following sense:

**Theorem 6.1.1** (Courtois, 1995). *Let $(M, g)$ be a closed Riemannian manifold. For any integer $k \geq 1$, there exists a constant $C_k$ depending on the manifold $(M, g)$ such that for any subset $A$ of $M$ we have*

$$0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{\operatorname{cap}(A)}.$$

## 6.2 Some useful spaces

The space $\mathcal{C}_c^\infty(M - A)$ is the set of smooth functions with compact support on $M - A$. For a compact subset $A$ of the manifold $M$ the usual Sobolev space $H_0^1(M - A)$ is defined as the closure of $\mathcal{C}_c^\infty(M - A)$ with respect to the norm $\|\cdot\|_{H^1(M)}$ (see Section 3.2.2):

$$H_0^1(M - A) := \overline{\mathcal{C}_c^\infty(M - A)}.$$

But what happens when the set $A$ is not compact? For example, if $A$ is a dense and countable subset of $M$, the space of test functions $\mathcal{C}_c^\infty(M - A)$ is reduced to $\{0\}$. Therefore we cannot define the space $H_0^1(M - A)$. This leads us to a more general definition of $H_0^1(M - A)$ which is valid for any subset $A$ of $M$.

**Definition 6.2.1.** We define the Sobolev spaces $\mathcal{H}_0^1(M - A)$ and $H_0^1(M - A)$ by

$$\mathcal{H}_0^1(M - A) := \big\{g \in H^1(M);\ g = 0 \text{ on a open neighborhood of } A\big\}$$

and

$$H_0^1(M - A) := \overline{\mathcal{H}_0^1(M - A)},$$

where the closure is taken with respect to the norm $\|\cdot\|_{H^1(M)}$.

We have the

**Proposition 6.2.2.** *If the set $A$ is compact, the previous definition of the space $H_0^1(M-A)$ coincides with the usual ones (see Section 3.2.2).*

*Proof.* Let $f \in H_0^1(M-A) := \overline{\mathcal{H}_0^1(M-A)}$. Then, by definition, for all $\varepsilon \geq 0$ there exists $g \in \mathcal{H}_0^1(M-A)$ such that $\|f-g\|_{H^1(M)} \leq \varepsilon$. Let us show that we can write $g$ as a limit of a sequence from the space $\mathcal{C}_c^\infty(M-A)$. Since $g \in \mathcal{H}_0^1(M-A)$, there exists an open set $U \supset A$ such that $g|_U = 0$. Consider two open sets $U_1$ and $U_2$ of the manifold $M$ such that

$$A \subset U_1,\ M-U \subset U_2,\ U_1 \cap U_2 = \emptyset;$$

and consider also a function $\varphi \in \mathcal{D}(M)$ such that

$$\varphi|_{U_1} = 0,\ \varphi|_{U_2} = 1.$$

Obviously, $\varphi$ belongs to the space $\mathcal{C}_c^\infty(M-A)$. Next, since $g \in \mathcal{H}_0^1(M-A) \subset H^1(M)$ and since the set of smooth functions $\mathcal{C}^\infty(M)$ is dense in $H^1(M)$, there exists a sequence $(g_n)_n$ in $\mathcal{C}^\infty(M)$ such that $\lim_{n\to+\infty} g_n = g$ in the norm $\|\cdot\|_{H^1(M)}$. We claim that $\lim_{n\to+\infty} \varphi g_n = g$ in the norm $\|\cdot\|_{H^1(M)}$. Indeed for any $n$ we have

$$\begin{aligned}\|\varphi g_n - g\|_{H^1(M)}^2 &\leq \|g_n - g\|_{H^1(M-U)}^2 + \|\varphi g_n - g\|_{H^1(U)}^2 \\ &\leq \|g_n - g\|_{H^1(M)}^2 + \|\varphi g_n - g\|_{H^1(U)}^2 .\end{aligned}$$

Next, note that for any $n$

$$\begin{aligned}\|\varphi g_n - g\|_{H^1(U)}^2 &= \|\varphi g_n\|_{H^1(U)}^2 \\ &= \int_U |\varphi g_n|^2\, d\mathcal{V}_g + \int_U |\nabla\varphi g_n + \varphi\nabla g_n|^2\, d\mathcal{V}_g \\ &\leq \int_U |\varphi g_n|^2\, d\mathcal{V}_g + \int_U |\nabla\varphi g_n|^2\, d\mathcal{V}_g + \int_U |\varphi\nabla g_n|^2\, d\mathcal{V}_g \\ &\quad + 2\int_U |\nabla\varphi g_n \varphi\nabla g_n|\, d\mathcal{V}_g \\ &\leq \|\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 + \|\nabla\varphi\|_{L^\infty(M)}^2 \|g_n\|_{L^2(U)}^2 \\ &\quad + \|\varphi\|_\infty^2 \|\nabla g_n\|_{L^2(U)}^2 + 2\|\nabla\varphi\|_\infty \|\varphi\|_\infty \int_U |g_n\nabla g_n|\, d\mathcal{V}_g \\ &\leq \|\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 + \|\nabla\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 \\ &\quad + \|\varphi\|_\infty^2 \|\nabla g_n\|_{L^2(U)}^2 \\ &\quad + 2\|\nabla\varphi\|_\infty \|\varphi\|_{L^\infty(M)} \|g_n\|_{L^2(U)} \|\nabla g_n\|_{L^2(U)}\end{aligned}$$

thanks to the Cauchy–Schwarz inequality.

Therefore,

$$\|\varphi g_n - g\|^2_{H^1(U)} \leq \|g_n\|^2_{H^1(U)} \left(2\,\|\varphi\|^2_\infty + \|\nabla\varphi\|^2_\infty + 2\,\|\nabla\varphi\|_\infty\,\|\varphi\|_\infty\right).$$

As a consequence, for any $n$

$$\|\varphi g_n - g\|^2_{H^1(M)} \leq \|g_n - g\|^2_{H^1(M-U)}$$

$$+ \|g_n\|^2_{H^1(U)} \left(2\,\|\varphi\|^2_\infty + \|\nabla\varphi\|^2_\infty + 2\,\|\nabla\varphi\|_\infty\,\|\varphi\|_\infty\right).$$

Now it suffices to note that

$$\|g_n\|^2_{H^1(U)} = \|g_n - g\|^2_{H^1(U)} \leq \|g_n - g\|^2_{H^1(M)}$$

(since $g = 0$ on the open set $U$) and we finally conclude that

$$\|\varphi g_n - g\|^2_{H^1(M)} \leq \|g_n - g\|^2_{H^1(M)} \left(1+2\,\|\varphi\|^2_\infty+\|\nabla\varphi\|^2_\infty+2\,\|\nabla\varphi\|_\infty\,\|\varphi\|_\infty\right).$$

The sequence $(\varphi g_n)_n$ belongs to $\mathcal{C}^\infty_c(M-A)^{\mathbb{N}}$ and since $\lim_{n\to+\infty} g_n = g$ in the norm $\|\cdot\|_{H^1(M)}$, the last inequality implies $\lim_{n\to+\infty}\varphi g_n = g$ in the norm $\|\cdot\|_{H^1(M)}$.

So we have show that every function $f \in H^1_0(M-A) := \overline{\mathcal{H}^1_0(M-A)}$ is the limit (in the norm $\|\cdot\|_{H^1(M)}$) of a sequence from $\mathcal{C}^\infty_c(M-A)$.

Conversely, since $\mathcal{C}^\infty_c(M-A) \subset \mathcal{H}^1_0(M-A)$, we get

$$H^1_0(M-A) := \overline{\mathcal{C}^\infty_c(M-A)} \subset H^1_0(M-A) := \overline{\mathcal{H}^1_0(M-A)}. \qquad \square$$

Let us also define the spaces $H^1_\star(M)$ and $S_A(M)$ by

$$H^1_\star(M) := \left\{ f \in H^1(M);\ \int_M f\, d\mathcal{V}_g = 0 \right\}$$

and

$$S_A(M) := \left\{ u \in H^1_\star(M);\ u - 1 \in H^1_0(M-A) \right\},$$

respectively. In the definition of the space $H^1_\star(M)$ the condition $\int_M f\, d\mathcal{V}_g = 0$ is analogous to a boundary condition.

Note that $H^1_\star(M)$ is a Hilbert space for the norm

$$\|u\|^2_\star := \|u\|^2_{H^1_0} = \int_M |\nabla u|^2\, d\mathcal{V}_g$$

and $S_A(M)$ is an affine closed subset of $H^1_\star(M)$.

## 6.3 Electrostatic capacity and the Poincaré inequality

Next, we introduce the notion of the electrostatic capacity of a set $A$.

**Definition 6.3.1.** The *electrostatic capacity* $\mathrm{cap}(A)$ of the set $A$ is defined by

$$\mathrm{cap}(A) := \inf\left\{\int_M |\nabla u|^2 \, d\mathcal{V}_g;\ u \in S_A(M)\right\}. \tag{6.1}$$

Let us remark that there exists a unique function $u_A \in S_A(M)$ such that

$$\mathrm{cap}(A) = \int_M |\nabla u_A|^2 \, d\mathcal{V}_g.$$

Indeed, here the capacity $\mathrm{cap}(A)$ is just the distance between the function 0 and the closed space $S_A(M)$. This distance is equal to $\|u_A\|_\star^2$, where $u_A$ is the orthogonal projection of 0 on $S_A(M)$:

$$\mathrm{cap}(A) = d_\star\left(0, S_A(M)\right) := \inf\left\{\|u\|_\star^2;\ u \in S_A(M)\right\} = \|u_A\|_\star^2.$$

The following lemma establishes the relationships between the capacity $\mathrm{cap}(A)$, the function $u_A$, and the Sobolev spaces $H_0^1(M - A),\ H^1(M)$.

**Lemma 6.3.2.** *For subset $A$ of the manifold $M$, the following properties are equivalent*

(i) $\mathrm{cap}(A) = 0$*;*

(ii) $u_A = 0$*;*

(iii) $1 \in H_0^1(M - A)$*;*

(iv) $H_0^1(M - A) = H^1(M)$.

*Proof.* It is clear from the formula (6.1) that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Next, suppose that (iii) holds. Hence, there exists a sequence $(v_n)_n \subset \mathcal{H}_0^1(M - A)^{\mathbb{N}}$ such that $\lim_{n\to+\infty} v_n = 1$ in the norm $\|\cdot\|_{H^1(M)}$. So, for any smooth function $\varphi \in \mathcal{C}^\infty(M)$ we have $\lim_{n\to+\infty} \varphi v_n = \varphi$ in the norm $\|\cdot\|_{H^1(M)}$; indeed, for any integer $n$

$$\|\varphi v_n - \varphi\|_{H^1(M)}^2 = \int_M |\varphi v_n - \varphi|^2 \, d\mathcal{V}_g + \int_M |\nabla(\varphi v_n) - \nabla\varphi|^2 \, d\mathcal{V}_g.$$

First, for any $n$, we have

$$\int_M |\varphi v_n - \varphi|^2 \, d\mathcal{V}_g = \int_M |\varphi(v_n - 1)|^2 \, d\mathcal{V}_g \le \|\varphi\|_\infty^2 \, \|v_n - 1\|_{L^2(M)}^2.$$

Since $\lim_{n\to+\infty} v_n = 1$ in the norm $\|\cdot\|_{H^1(M)}$, we get

$$\lim_{n\to+\infty} \int_M |\varphi v_n - \varphi|^2 \, d\mathcal{V}_g = 0.$$

On the other hand, for any $n$

$$\begin{aligned}\int_M |\nabla(\varphi v_n) - \nabla\varphi|^2 \, d\mathcal{V}_g &= \|\nabla\varphi v_n - \nabla\varphi + \varphi\nabla v_n\|^2_{L^2(M)} \\ &\le \left(\|\nabla\varphi v_n - \nabla\varphi\|_{L^2(M)} + \|\varphi\nabla v_n\|_{L^2(M)}\right)^2 \\ &\le \left(\|\nabla\varphi\|_\infty \|v_n - 1\|_{L^2(M)} + \|\varphi\|_\infty \|\nabla v_n\|_{L^2(M)}\right)^2.\end{aligned}$$

Since $\lim_{n\to+\infty} v_n = 1$ in the norm $\|\cdot\|_{H^1(M)}$, we deduce that

$$\lim_{n\to+\infty} \int_M |\nabla(\varphi v_n) - \nabla\varphi|^2 \, d\mathcal{V}_g = 0.$$

Therefore, for any $\varphi \in \mathcal{C}^\infty(M)$ we have $\lim_{n\to+\infty}\varphi v_n = \varphi$ in the norm $\|\cdot\|_{H^1(M)}$.

Next, since $\mathcal{C}^\infty(M)$ is dense in $H^1(M)$, for any function $f \in H^1(M)$ we have $\lim_{n\to+\infty} f v_n = f$. Since the sequence $(\varphi v_n)_n \subset \mathcal{H}^1_0(M-A)^{\mathbb{N}}$ we conclude that $f$ belongs to space $H^1_0(M-A)$.

Finally, it is easy to see that (iv) $\Rightarrow$ (iii). □

An obvious consequence of this lemma is the following result:

**Proposition 6.3.3.** *The spectra of $(M,g)$ and $(M-A,g)$ coincide if and only if* $\operatorname{cap}(A) = 0$.

Now, let us state the Poincaré inequality.

**Theorem 6.3.4** (Poincaré inequality). *If $\lambda_2(M)$ denotes the first non-null eigenvalue of the operator $-\Delta_g$ on the manifold $(M,g)$, then*

$$\|u_A\|^2_{L^2(M)} \le \frac{\operatorname{cap}(A)}{\lambda_2(M)}$$

*for any subset $A$ of $M$.*

*Proof.* The case $\operatorname{cap}(A) = 0$ is an obvious consequence of the previous lemma. Suppose now that $\operatorname{cap}(A) > 0$. Then $\|u_A\|_{L^2(M)} > 0$. By the minimax principle, the second eigenvalue $\lambda_2(M)$ of $(M,g)$ is given by:

$$\lambda_2(M) = \min_{\substack{E\subset H^1(M)\\ \dim(E)=2}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\displaystyle\int_M |\nabla\varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}.$$

Since $u_A$ belongs to the space $H^1(M)$, we get:

$$\lambda_2(M) \leq \max_{\substack{\varphi \in \mathrm{span}\{u_A, 1\} \\ \varphi \neq 0}} \frac{\displaystyle\int_M |\nabla\varphi|^2 \, d\mathcal{V}_g}{\displaystyle\int_M \varphi^2 \, d\mathcal{V}_g}$$

$$\leq \max_{\substack{(\alpha,\beta)\in\mathbb{R}^2 \\ (\alpha,\beta)\neq(0,0)}} \frac{\alpha^2 \displaystyle\int_M |\nabla u_A|^2 \, d\mathcal{V}_g}{\alpha^2 \displaystyle\int_M u_A^2 \, d\mathcal{V}_g + 2\alpha\beta \int_M u_A \, d\mathcal{V}_g + \beta^2 \mathrm{vol}(M)}.$$

Since $u_A \in S_A(M)$, we get $\displaystyle\int_M u_A \, d\mathcal{V}_g = 0$, whence

$$\lambda_2(M) \leq \max_{\substack{(\alpha,\beta)\in\mathbb{R}^2 \\ (\alpha,\beta)\neq(0,0)}} \frac{\alpha^2 \displaystyle\int_M |\nabla u_A|^2 \, d\mathcal{V}_g}{\alpha^2 \displaystyle\int_M u_A^2 \, d\mathcal{V}_g + \beta^2 \mathrm{vol}(M)}$$

$$\leq \frac{\displaystyle\int_M |\nabla u_A|^2 \, d\mathcal{V}_g}{\displaystyle\int_M u_A^2 \, d\mathcal{V}_g}.$$

And the statement of the theorem is established. □

## 6.4 A detailed proof of the main Theorem 6.1.1

Recall our main result of Chapter 6 (Theorem 6.1.1):

**Theorem** (Courtois, 1995). *Let $(M, g)$ be a closed Riemannian manifold. For any integer $k \geq 1$, there exists a constant $C_k$ depending on the manifold of $(M, g)$, such that for any subset $A$ of $M$ we have*

$$0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{\mathrm{cap}(A)}.$$

*Proof.* Let us denote by $(e_k)_{k\geq 1}$ an orthonormal eigenbasis (for the operator $-\Delta_g$ on the closed manifold $(M, g)$) of the space $L^2(M)$. For any integer $k \geq 1$, we consider the sets

$$F_k := \mathrm{span}\{e_1, e_2, \ldots, e_k\} \quad \text{and} \quad E_k := \{f\,(1 - u_A)\,,\; f \in F_k\}\,.$$

First, observe that $E_k \subset H_0^1(M - A)$. For all $j \in \{1, \ldots, k\}$ we introduce also the functions

$$\phi_j := e_j\,(1 - u_A) \in E_k.$$

• **Step 1**: we compute the $L^2$-inner product $\langle \phi_i, \phi_j \rangle_{L^2(M)}$ for any pair $(i, j) \in \{1, \ldots, k\}^2$:

$$\begin{aligned}\langle \phi_i, \phi_j \rangle_{L^2(M)} &= \int_M e_i e_j\,(1 - u_A)^2\, d\mathcal{V}_g \\ &= \delta_{i,j} - 2\int_M e_i e_j u_A\, d\mathcal{V}_g + \int_M e_i e_j u_A^2\, d\mathcal{V}_g.\end{aligned}$$

Thus, for any pair $(i, j) \in \{1, \ldots, k\}^2$,

$$\left|\langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j}\right| \leq 2\int_M \left|e_i e_j u_A\right|\, d\mathcal{V}_g + \int_M \left|e_i e_j u_A^2\right|\, d\mathcal{V}_g.$$

Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\left|\langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j}\right| &\leq 2 \max_{1\leq i,j\leq k} \left\|e_i e_j\right\|_\infty \|u_A\|_{L^2(M)} \\ &\quad + \max_{1\leq i,j\leq k} \left\|e_i e_j\right\|_\infty \|u_A\|_{L^2(M)}^2 \\ &\leq 2 \max_{1\leq i,j\leq k} \left\|e_i e_j\right\|_\infty \sqrt{\mathrm{Vol}(M)}\, \|u_A\|_{L^2(M)} \\ &\quad + \max_{1\leq i,j\leq k} \left\|e_i e_j\right\|_\infty \|u_A\|_{L^2(M)}^2,\end{aligned}$$

and with the Poincaré inequality we have

$$\left|\langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j}\right| \leq B_{k,M}\left(\sqrt{\mathrm{cap}(A)} + \mathrm{cap}(A)\right)$$

where $B_k = B_k\,(e_1, e_2, \ldots, e_k, \lambda_2(M), M) \geq 0$. Since the eigenfunctions $e_1, e_2, \ldots, e_k$ and the eigenvalue $\lambda_2(M)$ depend only on $(M, g)$, for any integer $k$ the constant $B_k$ depends only on $(M, g)$, i.e., $B_k = B_k\,(M)$.

Therefore, there exists $\varepsilon_k \in\, ]0, 1[$ (which depends on the constant $B_k$) such that for all $A \subset M$ we have

$$\begin{aligned}&\mathrm{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and} \\ &\forall j \in \{1, \ldots, k\},\ \left|\|\phi_j\|_{L^2(M)}^2 - 1\right| \leq D_k \sqrt{\mathrm{cap}(A)}\end{aligned}$$

where (and for the same reasons as for $B_k$) for all integer $k$, the constant $D_k$ depends only on $M$, i.e., $D_k = D_k\,(M)$.

• **Step 2**: Take $\phi = f\,(1 - u_A) \in E_k$, with $f \in F_k$. Without loss generality we can assume that $\|f\|_{L^2(M)} = 1$, indeed, we have $R(\phi) = R\left(\frac{\phi}{\|f\|_{L^2(M)}}\right)$ and in our context we are interested in the Rayleigh quotient of $\phi$ (see the end of the final step of the proof).

We have

$$\begin{aligned}
\int_M |\nabla \phi|^2 \, d\mathcal{V}_g &= \int_M |\nabla f - \nabla (f u_A)|^2 \, d\mathcal{V}_g \\
&= \int_M |\nabla f|^2 \, d\mathcal{V}_g + \int_M |\nabla f u_A + f \nabla u_A|^2 \, d\mathcal{V}_g \\
&\quad - 2 \int_M \nabla f \nabla (f u_A) \, d\mathcal{V}_g \\
&= \int_M |\nabla f|^2 \, d\mathcal{V}_g + \int_M |\nabla f u_A|^2 \, d\mathcal{V}_g + \int_M |f \nabla u_A|^2 \, d\mathcal{V}_g \\
&\quad + 2 \int_M \nabla f \nabla u_A f u_A \, d\mathcal{V}_g - 2 \int_M |\nabla f|^2 u_A \, d\mathcal{V}_g \\
&\quad - 2 \int_M \nabla f \nabla u_A f \, d\mathcal{V}_g \\
&= \int_M |\nabla f|^2 \, d\mathcal{V}_g + \int_M |\nabla f u_A|^2 \, d\mathcal{V}_g + \int_M |f d u_A|^2 \, d\mathcal{V}_g \\
&\quad - 2 \int_M |\nabla f|^2 u_A \, d\mathcal{V}_g - 2 \int_M \nabla f \nabla u_A f \, (1 - u_A) \, d\mathcal{V}_g .
\end{aligned}$$

Hence

$$\begin{aligned}
\int_M |\nabla \phi|^2 \, d\mathcal{V}_g &= \underbrace{\int_M |\nabla f|^2 \, d\mathcal{V}_g}_{:=A(f)} + \underbrace{\int_M |\nabla f u_A|^2 \, d\mathcal{V}_g}_{:=B(f)} + \underbrace{\int_M |f \nabla u_A|^2 \, d\mathcal{V}_g}_{:=C(f)} \\
&\quad - 2 \underbrace{\int_M |\nabla f|^2 u_A \, d\mathcal{V}_g}_{:=D(f)} - 2 \underbrace{\int_M \nabla f \nabla u_A f \, (1 - u_A) \, d\mathcal{V}_g}_{:=E(f)} .
\end{aligned}$$

♦ **Estimate of** $A(f) := \int_M |\nabla f|^2 \, d\mathcal{V}_g \geq 0$. Since $f \in F_k$ we can write $f = \sum_{i=1}^k \alpha_i e_i$ where $(\alpha_i)_{1 \leq i \leq k} \in \mathbb{R}^k$ with $\sum_{i=1}^k \alpha_i^2 = 1$ (since $\|f\|_{L^2(M)} = 1$), and so

$$\begin{aligned}
A(f) &= \left\langle \sum_{j=1}^k \alpha_j \nabla e_j , \sum_{i=1}^k \alpha_i \nabla e_i \right\rangle_{L^2(M)} = \sum_{i,j} \alpha_i \alpha_j \left\langle \nabla e_j , \nabla e_i \right\rangle_{L^2(M)} \\
&= \sum_{i,j} \alpha_i \alpha_j \left( - \left\langle e_j , \Delta_g e_i \right\rangle_{L^2(M)} \right) = \sum_{i,j} \alpha_i \alpha_j \left\langle e_j , -\Delta_g e_i \right\rangle_{L^2(M)} \\
&= \sum_{i,j} \alpha_i \alpha_j \lambda_i(M) \left\langle e_j , e_i \right\rangle_{L^2(M)} = \sum_{i=1}^k \alpha_i^2 \lambda_i(M) \leq \lambda_k(M).
\end{aligned}$$

Hence, for any integer $k$, and for any function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \leq A(f) \leq \lambda_k(M).$$

♦ **Estimate of** $B(f) := \int_M |\nabla f u_A|^2 \, d\mathcal{V}_g$. We get $B(f) \leq \|\nabla f\|_\infty^2 \, \|u_A\|_{L^2(M)}^2$ and, with the Poincaré inequality

$$\|u_A\|_{L^2(M)}^2 \leq \frac{\operatorname{cap}(A)}{\lambda_2(M)}$$

we deduce that for any $k$ and any function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \leq B(f) \leq E_k \operatorname{cap}(A)$$

where $E_k = E_k\,(e_2, \lambda_2(M)) > 0$. Moreover, since the eigenfunction $e_2$ and the eigenvalue $\lambda_2(M)$ depend only on $(M, g)$, for any integer $k$ the constant $E_k$ depend only on $(M, g)$, i.e., $E_k = E_k\,(M)$.

♦ **Estimate of** $C(f)$. The term $C(f)$ is equal to $\int_M |f \nabla u_A|^2 \, d\mathcal{V}_g$. We have

$$C(f) \leq \|f\|_\infty^2 \, \|\nabla u_A\|_{L^2(M)}^2$$

so, since $\|\nabla u_A\|_{L^2(M)}^2 \leq \operatorname{cap}(A)$, for any $k$ and any function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$, it holds that

$$0 \leq C(f) \leq F_k \operatorname{cap}(A),$$

where $F_k = F_k\,(f, e_2, \lambda_2(M)) > 0$. Here, for $k$ fixed, the constant $F_k$ depends also on $f$, and $f$ depends on the functions $f_1, f_2, \ldots, f_k$ (which depend only on $M$) and on the scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$; since $\sum_{i=1}^k \alpha_i^2 = 1$, all the $(\alpha_i)_{1 \leq i \leq k}$ are bounded in $\mathbb{R}$, so finally, for any $k$ the constant $F_k$ can be bounded by a constant (we denote it also by $F_k = F_k(M)$) which depends only on $M$.

♦ **Estimate of** $|D(f)|$: we have

$$\begin{aligned}
|D(f)| = \left| \int_M |\nabla f|^2 \, u_A \, d\mathcal{V}_g \right| &\leq \|\nabla f\|_\infty^2 \int_M |u_A| \, d\mathcal{V}_g \\
&\leq \|\nabla f\|_\infty^2 \sqrt{\operatorname{Vol}(M)} \, \|u_A\|_{L^2(M)} \\
&\leq \|\nabla f\|_\infty^2 \sqrt{\operatorname{Vol}(M)} \sqrt{\frac{\operatorname{cap}(A)}{\lambda_2(M)}}.
\end{aligned}$$

Hence, for any $k$ and any $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$

$$|D(f)| \leq G_k \sqrt{\operatorname{cap}(A)}$$

where (and for the same reasons as in the estimation of $C(f)$, see the constant $F_k$) for any $k$, the constant $G_k$ depends only on $M$, i.e., $G_k = G_k\,(M)$.

♦ **Estimate of** $|E(f)|$: recall that $E(f) = \int_M \nabla f \nabla u_A f\,(1-u_A)\,d\mathcal{V}_g$, hence

$$|E(f)| \leq \int_M |\nabla f \nabla u_A|\,|f|\,d\mathcal{V}_g + \int_M |\nabla f \nabla u_A|\,|f u_A|\,d\mathcal{V}_g.$$

For the first term we have

$$\int_M |\nabla f \nabla u_A|\,|f|\,d\mathcal{V}_g \leq \|f\|_\infty \|\nabla f\|_\infty \sqrt{\mathrm{Vol}(M)}\,\|\nabla u_A\|_{L^2(M)};$$

so if we denote $H_k := \|f\|_\infty \|\nabla f\|_\infty \sqrt{\mathrm{Vol}(M)}$, we get

$$\int_M |\nabla f \nabla u_A|\,|f|\,d\mathcal{V}_g \leq K\sqrt{\|\nabla u_A\|^2_{L^2(M)}} \leq H_k\sqrt{\mathrm{cap}(A)}$$

where (for the same reasons as above), for any integer $k$, the constant $H_k$ depends only on $M$, i.e., $H_k = H_k\,(M)$.

Next, for the second term we have

$$\begin{aligned}\int_M |\nabla f \nabla u_A|\,|f u_A|\,d\mathcal{V}_g &\leq \|\nabla f\|_\infty \|f\|_\infty \|\nabla u_A\|_{L^2(M)} \|u_A\|_{L^2(M)} \\ &\leq \|\nabla f\|_\infty \|f\|_\infty \sqrt{\frac{\mathrm{cap}(A)}{\lambda_2(M)}} H_k \sqrt{\mathrm{cap}(A)} \\ &\leq H'_{k,M}\mathrm{cap}(A),\end{aligned}$$

where (for the same reasons as above), for any integer $k$, the constant $H_k$ depends only on $M$, i.e., $H'_k = H_k\,(M)$.

So, for any integer $k$ we get

$$|E(f)| \leq H''_{k,M}\left(\sqrt{\mathrm{cap}(A)} + \mathrm{cap}(A)\right),$$

where $H''_k := H''_k\,(M)$.

Combining the estimates for $A(f), B(f), C(f), |D(f)|$ and $|E(f)|$, for any $k$ and any function $\phi = f\,(1-u_A) \in E_k$, with $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we obtain

$$\int_M |\nabla \phi|^2\,d\mathcal{V}_g \leq \lambda_k(M) + I_k\left(\sqrt{\mathrm{cap}(A)} + \mathrm{cap}(A)\right)$$

where, for any $k$, the constant $I_k$ depend only on $M$, i.e., $I_k = I_k\,(M, V)$.

• **Step 3**: Now we claim that for any $A \subset M$ such that $\mathrm{cap}(A) \leq \varepsilon_k$ and any function $\phi \in E_k$,

$$\|\phi\|^2_{L^2(M)} \geq 1 - J'_{k,M}\sqrt{\mathrm{cap}(A)},$$

where, for any $k$, the constant $J'_{k,M}$ depend only on $M$, i.e., $J'_{k,M} = J'_{k,M}(M)$.

Indeed, let $\phi \in E_k$. We have seen above in Step 1 that

$$\begin{aligned} &\mathrm{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and} \\ &\qquad\qquad \forall j \in \{1,\dots,k\},\ \left| \|\phi_j\|^2_{L^2(M)} - 1 \right| \leq D_k\sqrt{\mathrm{cap}(A)}. \end{aligned}$$

Hence, since $\phi \in E_k$, we can write $\phi = (1-u_A)f$ with $f = \sum_{i=1}^k \alpha_i e_i$, where $(\alpha_i)_{1\leq i\leq k} \in \mathbb{R}^k$. As in the Step 2, we can assume that $\|f\|_{L^2(M)} = 1$, and so $\sum_{i=1}^k \alpha_i^2 = 1$. Next, we compute $\|\phi\|^2_{L^2(M)}$:

$$\begin{aligned} \|\phi\|^2_{L^2(M)} &= \left\| \sum_{i=1}^k (1-u_A)\,\alpha_i e_i \right\|^2_{L^2(M)} = \left\| \sum_{i=1}^k \alpha_i \phi_i \right\|^2_{L^2(M)} \\ &= \sum_{i=1}^k \alpha_i^2 \|\phi_i\|^2_{L^2(M)} + \sum_{i,j=1, i\neq j}^k \alpha_i\alpha_j \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=1}^k \alpha_i^2 \|\phi_i\|^2_{L^2(M)} &= \sum_{i=1}^k \alpha_i^2 \left[ 1 - 2\int_M e_i^2 u_A \, d\mathcal{V}_g + \int_M e_i^2 u_A^2 \, d\mathcal{V}_g \right] \\ &= 1 - \sum_{i=1}^k \alpha_i^2 \left[ 2\int_M e_i^2 u_A \, d\mathcal{V}_g - \int_M e_i^2 u_A^2 \, d\mathcal{V}_g \right] \\ &= 1 - \sum_{i=1}^k \alpha_i^2 \int_M e_i^2 \left(2u_A - u_A^2\right) \, d\mathcal{V}_g, \end{aligned}$$

we obtain

$$\|\phi\|^2_{L^2(M)} = 1 - \sum_{i=1}^k \alpha_i^2 \int_M e_i^2 \left(2u_A - u_A^2\right) \, d\mathcal{V}_g + \sum_{i,j=1, i\neq j}^k \alpha_i\alpha_j \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)}.$$

We have seen in Step 1 that, for $\mathrm{cap}(A)$ small enough,

$$\left| \left\langle \phi_i, \phi_j \right\rangle_{L^2(M)} - \delta_{i,j} \right| \leq B_k \left( \sqrt{\mathrm{cap}(A)} + \mathrm{cap}(A) \right).$$

hence, since all the $(\alpha_i)_{1\leq i\leq k}$ are bounded in $\mathbb{R}$, for $\mathrm{cap}(A)$ small enough we can find a constant $B'_{k,M}$ which depends only on $M$, i.e., $B'_k = B'_k(M)$ such that, for $\mathrm{cap}(A)$ small enough

$$\left|\sum_{i,j=1,i\neq j}^{k} \alpha_i\alpha_j \left\langle \phi_i, \phi_j\right\rangle_{L^2(M)}\right| \leq B'_k\sqrt{\mathrm{cap}(A)}.$$

Finally, in the same spirit as in the estimations in Step 2, there exists a constant $B''_{k,M}$ which depend only on $M$, i.e., $B''_k = B''_k(M)$ such that, for $\mathrm{cap}(A)$ small enough

$$\left|\sum_{i=1}^{k}\alpha_i^2\int_M e_i^2\left(2u_A - u_A^2\right)\,d\mathcal{V}_g\right| \leq B''_k\sqrt{\mathrm{cap}(A)}.$$

It follows that,

$$\|\phi\|^2_{L^2(M)} \geq 1 - B'''_k\sqrt{\mathrm{cap}(A)},$$

where the constant $B'''_k$ depend only on $M$, i.e., $B'''_k := B'''_k(M)$.

• **Final step**: By Steps 2 and 3, for any function $\phi \in E_k$ we get

$$\frac{\displaystyle\int_M |\nabla\phi|^2\,d\mathcal{V}_g}{\displaystyle\int_M \phi^2\,d\mathcal{V}_g} \leq \frac{\lambda_k(M) + I_k\left(\mathrm{cap}(A) + \sqrt{\mathrm{cap}(A)}\right)}{1 - B'''_k\sqrt{\mathrm{cap}(A)}}.$$

Hence, for $\mathrm{cap}(A)$ small enough (i.e., $\mathrm{cap}(A) \leq \varepsilon_k$),

$$\frac{\displaystyle\int_M |\nabla\phi|^2\,d\mathcal{V}_g}{\displaystyle\int_M \phi^2\,d\mathcal{V}_g} \leq \lambda_k(M) + L_k\sqrt{\mathrm{cap}(A)},$$

where $L_k := L_k(M)$. Next, since for any $k \geq 1$

$$\lambda_k(M - A) = \min_{\substack{E\subset H_0^1(M-A)\\ \dim(E)=k}}\ \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\displaystyle\int_M |\nabla\varphi|^2\,d\mathcal{V}_g}{\displaystyle\int_M \varphi^2\,d\mathcal{V}_g}$$

and since $\phi \in H_0^1(M - A)$, we get for all $k \geq 1$

$$\lambda_k(M - A) \leq \frac{\displaystyle\int_M |\nabla\phi|^2\,d\mathcal{V}_g}{\displaystyle\int_M \phi^2\,d\mathcal{V}_g} \leq \lambda_k(M) + C_k\sqrt{\mathrm{cap}(A)}$$

This completes the proof of the theorem. □

# Chapter 7
# Inverse problems in spectral geometry

This chapter is devoted to inverse problems. First we present a brief historical review of inverse problems, next we give a proof of Milnor's counter-example, and then we explain the notion of heat kernel and its applications to spectral inverse problems. We also provide a detailed example of isospectral but non-isometric domains in the plane $\mathbb{R}^2$. We conclude this chapter with some elements of conformal geometry, including some recent results in Riemannian geometry.

## 7.1 Can one hear the shape of a drum?

In 1966, M. Kac [Kac1] in his famous article "Can one hear the shape of a drum?" investigated the following question: given a Riemannian manifold $(M, g)$ (the membrane of a drumhead), does the spectrum of $\Delta_g$ (the harmonics of the drumhead) determine geometrically, up to an isometry, the manifold $(M, g)$? So, one of the most fundamental questions in spectral geometry is: if two manifolds are isospectral, are they isometric?

Before 1964, this problem remained quite mysterious. In 1964 J. Milnor [Mil] gave the first counter-example: a pair of 16-dimensional flat tori which are isospectral and nonisometric (see Section 7.3 for details). Subsequently, many other counter-examples have been constructed: see the papers of T. Sunada in 1985 [Sun], the papers [Go-Wi] of C. Gordon and E.N. Wilson in 1984, and also P. Bérard's method of isospectral construction by transplantation [Bér2], [Bér3]. In 1992, C. Gordon, D. Webb, and S. Wolpert [GWW1], produced the first planar counter-example. For more details see [GWW1], [GWW2] and the survey of P. Bérard [Bér5], [Bér6], [Bér7]. Let us also mention the article of S. Zelditch [Zel] in 2000, in which he showed that for a simply connected domain in $\mathbb{R}^2$ with analytic boundary and with two orthogonal axes of symmetry, the spectrum determines completely the geometry of the domain.

## 7.2 Length spectrum and trace formulas

The spectrum of the Laplacian determines some important geometric invariants: the dimension, the volume of the manifold, and the integral of the scalar curvature $\mathrm{Scal}_g$ over the manifold, (for the details see the next section). In fact, the spectrum of the Laplacian determines some other invariants, as for example the length spectrum of the manifold. The length spectrum of a Riemannian manifold $(M, g)$ is the set of lengths of closed geodesics[1] on $(M, g)$ counted with multiplicities (the multiplicity of a length is the number of free homotopy classes of closed curves containing a geodesic of the given length). In 1973 Y. Colin de Verdière [Col1], [Col2] showed that, in the compact case, up to a generic hypothesis (which it always satisfied in the case of negative sectional curvature), the spectrum of the Laplacian determines the length spectrum. Colin de Verdière's proof is based on trace formulas. More generally, trace formulas are useful tools in the analysis of PDE, in spectral theory, mathematical physics, etc. ...

The formal principle of trace formulas is the following. Consider an unbounded linear operator $H$ on a Hilbert space and suppose the spectrum of $H$ is discrete:

$$\mathrm{Spec}(H) = \{\lambda_k,\, k \geq 1\}\,.$$

Let $f$ be a "nice" function. The principle is to compute the trace of the operator $f(H)$ in two ways:

- the first: using the eigenvalues of the linear operator $f(H)$:

$$\mathrm{trace}(f(H)) = \sum_{k \geq 1} f(\lambda_k),$$

- the second: using the integral kernel of $f(H)$: if

$$f(H)\varphi(x) = \int_M K_f(x, y)\varphi(y)\, dy$$

  for all $x \in M$, then

$$\mathrm{trace}(f(H)) = \int_M K_f(x, x)\, dx.$$

We then get the following equality:

$$\sum_{k \geq 1} f(\lambda_k) = \int_M K_f(x, x)\, dx.$$

However, the major difficulty with this method is to find the "good" choice for the function $f$. Some of the usual choices are

- $f(x) = e^{-xt}$, where $t \geq 0$ (heat function);
- $f(x) = \frac{1}{x^s}$, where $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ (Riemann Zêta function);
- $f(x) = e^{-\frac{itx}{h}}$, where $t \geq 0$ (Schrödinger function).

[1] Note that geodesics are trajectories associated to the dynamics of classical Hamiltonian flow of $H = -\Delta_g$ (geodesic flow) in the phase plane, identified with the cotangent bundle $T^\star M$.

The simplest example of exact trace formula is the Poisson summation formula for a lattice $\Gamma$ of $\mathbb{R}^n$.

Recall that for any $f \in \mathcal{S}(\mathbb{R}^n)$ the *Poisson summation formula* reads

$$\sum_{k\in\Gamma} f(k) = \frac{1}{\mathrm{Vol}\,(\Gamma)} \sum_{\ell\in\Gamma^\star} \widehat{f}(\ell),$$

where

$$\widehat{f}(y) := \int_{\mathbb{R}^n} f(t) e^{-2i\pi\langle y,t\rangle}\, dt$$

is the *Fourier transform* of the function $f$. A classical calculation shows that the Fourier transform of the function

$$f(x) = e^{-\alpha\|x\|^2}$$

for $x \in \mathbb{R}^n$, where $\alpha > 0$, is given for all $y \in \mathbb{R}^n$ by

$$\widehat{f}(y) = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{\pi^2\|y\|^2}{\alpha}}.$$

Now, applying Poisson's formula to $f(x) = e^{-4\pi^2 t\|x\|^2}$, where $t > 0$ we get

$$\sum_{x\in\Gamma} e^{-4\pi^2 t\|x\|^2} = \frac{1}{\mathrm{Vol}(\Gamma)(4\pi t)^{\frac{n}{2}}} \sum_{y\in\Gamma^\star} e^{-\frac{\|y\|^2}{4t}}.$$

Replacing the lattice $\Gamma$ by $\Gamma^\star$ we obtain

$$\sum_{x\in\Gamma^\star} e^{-4\pi^2 t\|x\|^2} = \frac{\mathrm{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{y\in\Gamma} e^{-\frac{\|y\|^2}{4t}},$$

because $\Gamma^{\star\star} = \Gamma$ and $\mathrm{Vol}(\Gamma^\star) = \frac{1}{\mathrm{Vol}(\Gamma)}$.

Since the spectrum of the Laplacian (with closed manifold conditions) on the flat torus $\Gamma$ is given by (see Section 5.1.1):

$$\mathrm{Spec}(\Gamma, \mathrm{can}) = \left\{4\pi^2 \|x\|_{\mathbb{R}^n}\, ,\ x \in \Gamma^*\right\}$$

we obtain for all $t \geq 0$

$$Z_\Gamma(t) = \sum_{\lambda\in\mathrm{Spec}(\Gamma,\mathrm{can})} e^{-\lambda t} = \frac{\mathrm{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{y\in\Gamma} e^{-\frac{\|y\|^2}{4t}}. \tag{7.1}$$

Further, since the distance between the origin and a point of the lattice $\Gamma$ is exactly the length of a periodic geodesic on the flat torus, we get the equality

$$\sum_{\lambda\in\mathrm{Spec}(\Gamma,\mathrm{can})} e^{-\lambda t} = \frac{\mathrm{Vol}(\Gamma)}{(4\pi t)^{\frac{n}{2}}} \sum_{\ell\in\Sigma} e^{-\frac{\ell^2}{4t}},$$

where $\Sigma$ is the length spectrum. In this last equality on the right-hand side we have the geometric information (dimension, volume, ...) and on the left-hand side we have spectral information. For a recent reference on trace formulas, see the paper of Y. Colin de Verdière [Col3].

## 7.3 Milnor's counterexample

In this section we present a counterexample due to J. Milnor [Mil] in 1964.

**Theorem 7.3.1** (Milnor, 1964). *There exist two lattices $\Gamma$ and $\Gamma'$ of $\mathbb{R}^{16}$ such that the associated tori $\mathbb{T}^{16}(\Gamma)$ and $\mathbb{T}^{16}(\Gamma')$ are isospectral, but not isometric.*

To prove this theorem we consider for any integer $n \in 8\mathbb{N}$ the following sublattice of $\mathbb{Z}^n$ of index 2:

$$\Gamma_2 := \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n;\ \sum_{j=1}^{n} x_j \in 2\mathbb{Z} \right\}$$

and consider also the vector

$$\omega_n := \left( \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2} \right) \in \mathbb{R}^n.$$

Denote by $\Gamma(n)$ the lattice generated by $\Gamma_2$ and $\omega_n$.
Since $2\omega_n = (1, 1, \dots, 1) \in \Gamma_2$ the lattice $\Gamma_2$ is of index 2 in $\Gamma(n)$, thus we have

$$\begin{aligned} \mathrm{Vol}(\Gamma(n)) &= \frac{1}{2}\mathrm{Vol}(\Gamma_2) \\ &= \mathrm{Vol}(\mathbb{Z}^n) = 1. \end{aligned}$$

The following two lemmas are obvious.

**Lemma.** *For any $n \in 8\mathbb{Z}$, if $y \in \Gamma(n)$, then $\|y\|^2 \in 2\mathbb{Z}$.*

*Proof.* If $x = \sum_{j=1}^{n} x_j \in \Gamma_2$, then

$$\|x\|^2 = \sum_{j=1}^{n} x_j^2 = \left( \sum_{j=1}^{n} x_j \right)^2 - 2 \sum_{i<j} x_i x_j,$$

thus $\|x\|^2 \in 2\mathbb{Z}$.

If $x \in \omega_n \mathbb{Z}$, then there exists $\alpha \in \mathbb{Z}$ such that $x = \alpha\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$, whence

$$\|x\|^2 = \alpha^2 \sum_{j=1}^{n} \frac{1}{4} = \frac{\alpha^2 n}{4} \in 2\mathbb{Z}$$

because $n \in 8\mathbb{Z}$.
Finally, if $y = x + \alpha\omega_n$ with $\alpha \in \mathbb{Z}$, $x = (x_1, x_2, \dots, x_n) \in \Gamma_2$

$$\langle x, \omega_n \rangle = \frac{1}{2} \sum_{j=1}^{n} x_i \in \mathbb{Z}$$

thus

$$\|y\|^2 = \|x\|^2 + \alpha^2 \|\omega_n\|^2 + 2\alpha \langle x, \omega_n \rangle \in 2\mathbb{Z}. \qquad \square$$

The second lemma is

**Lemma 7.3.2.** *For any $n \in 8\mathbb{Z}$, we have $\Gamma^\star(n) = \Gamma(n)$.*

*Proof.* First consider $(y, y') \in \Gamma(n)^2$ such that $y = x + k\omega_n$ and $y' = x' + k'\omega_n$ with $(k, k') \in \mathbb{Z}^2$, $x = (x_1, x_2, \dots, x_n) \in \Gamma_2$, $x' = \left(x'_1, x'_2, \dots, x'_n\right) \in \Gamma_2$. We claim that

$$\left\langle y, y' \right\rangle \in \mathbb{Z},$$

and consequently $y' \in \Gamma^\star(n)$. Indeed,

$$\left\langle y, y' \right\rangle = \left\langle x, x' \right\rangle + kk' \langle \omega_n, \omega_n \rangle + k' \langle x, \omega_n \rangle + k \left\langle x', \omega_n \right\rangle.$$

Here it is clear that $\langle x, x' \rangle \in \mathbb{Z}$, and

$$\langle \omega_n, \omega_n \rangle = \|\omega_n\|^2 = \sum_{j=1}^{n} \frac{1}{4} = \frac{n}{4} \in \mathbb{Z}$$

(because $n \in 8\mathbb{N}$). Moreover,

$$k' \langle x, \omega_n \rangle = k' \sum_{j=1}^{n} \frac{x_j}{2} = \frac{k'}{2} \sum_{j=1}^{n} x_j \in \mathbb{Z}$$

and obviously for the same reason we have $k \langle x', \omega_n \rangle \in \mathbb{Z}$. We obtain that $\langle y, y' \rangle \in \mathbb{Z}$ and $y' \in \Gamma^\star(n)$. Therefore, we have

$$\Gamma(n) \subset \Gamma^\star(n).$$

Since

$$\mathrm{Vol}(\Gamma^\star(n)) = \frac{1}{\mathrm{Vol}(\Gamma(n))} = 1,$$

the lattices $\Gamma(n)$ and $\Gamma^\star(n)$ have the same volume, and so $\Gamma^\star(n) = \Gamma(n)$. $\square$

From the two previous lemmas we get

**Proposition 7.3.3.** *The lattices* $\Gamma(16)$ *and* $\Gamma(8) \oplus \Gamma(8)$ *are non-isometric, therefore the associated tori are also non-isometric.*

*Proof.* Let us denote by $e_1, e_2, \ldots, e_8$ the canonical basis of $\mathbb{R}^8$. It is clear that the family

$$\mathcal{G} := \{e_1 - e_8, e_2 - e_8, \ldots, e_7 - e_8, 2e_8\}$$

generates the lattice $\Gamma_2$. Moreover, since

$$2e_8 = 2\omega_8 - (e_1 + e_2) - (e_3 + e_4) - (e_5 + e_6) - (e_7 - e_8)$$

the family

$$\mathcal{G} := \{e_1 - e_8, e_2 - e_8, \ldots, e_7 - e_8, e_1 + e_2, e_3 + e_4, e_5 + e_6, \omega_8\}$$

generates the lattice $\Gamma(8)$. Here the norm of every vectors from $\mathcal{G}$ is equal to $\sqrt{2}$.

On the other hand, the lattice $\Gamma(16)$ cannot be generated by vectors with norm equal to $\sqrt{2}$; indeed, every element of $\Gamma(16)$ can be expressed as $\sum_{j=1}^{16} a_i e_i$ or $\sum_{j=1}^{16} \left(a_i + \frac{1}{2}\right) e_i$, with $(a_1, a_2, \ldots, a_n) \in \Gamma_2$. So a generator system of $\Gamma(16)$ admits necessarily elements of second type, and for $x := \sum_{j=1}^{16} \left(a_i + \frac{1}{2}\right) e_i$ such a vector we have

$$\|x\|^2 = \frac{1}{4} \sum_{j=1}^{16} (2a_i + 1)^2 \geq \frac{16}{4} = 4,$$

hence $\|x\| \geq 2$. In $\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$ the lattice $\Gamma(8) \oplus \Gamma(8)$ is generated by elements with norm equal to $\sqrt{2}$ and the lattice $\Gamma(16)$ is not, consequently $\Gamma(8) \oplus \Gamma(8)$ and $\Gamma(16)$ cannot be isometric. $\square$

To finish, prove

**Proposition 7.3.4.** *The tori associated to the lattices* $\Gamma(16)$ *and* $\Gamma(8) \oplus \Gamma(8)$ *are isospectral.*

*Proof.* The proof is based on the theta function

$$\theta_\Gamma(t) := \sum_{x \in \Gamma} e^{-\pi t \|x\|^2}$$

defined for $t > 0$. Note that for all $t > 0$, the formula (7.1) yields

$$Z_\Gamma\left(\frac{t}{4\pi}\right) = \sum_{\lambda \in \mathrm{Spec}(\Gamma, \mathrm{can})} e^{-\lambda \frac{t}{4\pi}} = \sum_{x \in \Gamma^\star} e^{-\pi t \|x\|^2} = \theta_\Gamma(t)$$

thus the theta function determines the spectrum of $\Gamma$ (because the spectral partition function $Z_\Gamma$ determine the spectrum of $\Gamma$, see Proposition 5.3.2).
Next, the trick here is to consider the function

$$f(y) = e^{-\pi\|y\|^2}.$$

Then, for all $x$ we get

$$\widehat{f}(x) = e^{-\pi\|x\|^2}.$$

Consider the lattice $\Gamma_t := \sqrt{t}\Gamma$, where $t > 0$ and $\Gamma$ is a lattice of $\mathbb{R}^n$ such that $\Gamma^\star = \Gamma$ (hence $\mathrm{Vol}(\Gamma) = 1$) and such that for all $x \in \Gamma$ we have $\|x\|^2 \in 2\mathbb{Z}$. Applying the Poisson summation formula[2] to $f$ and the lattice $\Gamma_t$, we get for all $t > 0$

$$\sum_{x\in\Gamma_t} e^{-\pi\|x\|^2} = \frac{1}{\mathrm{Vol}\,(\Gamma_t)} \sum_{\ell\in\Gamma_t^\star} e^{-\pi\|\ell\|^2}.$$

Since $\Gamma_t^\star = \frac{1}{\sqrt{t}}\Gamma^\star$, we further have

$$\sum_{x\in\Gamma} e^{-\pi t\|x\|^2} = \frac{1}{t^{\frac{n}{2}}\mathrm{Vol}(\Gamma)} \sum_{\ell\in\Gamma^\star} e^{-\frac{\pi\|\ell\|^2}{t}}.$$

Therefore,

$$\sum_{x\in\Gamma} e^{-\pi t\|x\|^2} = \frac{1}{t^{\frac{n}{2}}} \sum_{\ell\in\Gamma} e^{-\frac{\pi\|\ell\|^2}{t}},$$

i.e.,

$$\theta_\Gamma(t) = \frac{1}{t^{\frac{n}{2}}}\theta_\Gamma\left(\frac{1}{t}\right),$$

where the function $\theta_\Gamma$ is defined for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$ by the same formula

$$\theta_\Gamma(z) := \sum_{x\in\Gamma} e^{-\pi z\|x\|^2}$$

$\theta_\Gamma$ admits a holomorphic extension to $\mathcal{H} := \{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$, the half-plane $\mathcal{H}$ is invariant under the mapping $z \mapsto \frac{1}{z}$ and the function

$$\varphi\colon z \mapsto \theta_\Gamma(z) - \frac{1}{z^{\frac{n}{2}}}\theta_\Gamma\left(\frac{1}{z}\right)$$

is holomorphic on $\mathcal{H}$ (here $n \in 8\mathbb{N}$). Next, since $\varphi$ vanishes on the real positive axis $\mathbb{R}_+^\star$, the function $\varphi$ vanishes on the whole $\mathcal{H}$, so for all $z \in \mathcal{H}$ we have

$$\theta_\Gamma(z) = \frac{1}{z^{\frac{n}{2}}}\theta_\Gamma\left(\frac{1}{z}\right).$$

[2] Based on the theory of Fourier series; for a proof see for example [BGM].

Moreover, for all $z \in \mathcal{H}$

$$\theta_\Gamma(z+i) = \sum_{x\in\Gamma} e^{-\pi t z\|x\|^2} e^{-\pi i\|x\|^2} = \sum_{x\in\Gamma} e^{-\pi t z\|x\|^2} = \theta_\Gamma(z),$$

because $\|x\|^2 \in 2\mathbb{Z}$.

Thus the theta function $\theta_\Gamma$ is a modular function, in fact even a modular form, because if $z = u + iv$ we have $\theta_\Gamma(z) \to 1$ for $v \to +\infty$ uniformly in $u$. For $n = 16$ the weight of the modular form $\theta_\Gamma$ is $\frac{n}{4} = 4$ and the space of modular forms with weight equal to 4 is a space of dimension one (see for example [Ser]). Consequently, since $\theta_\Gamma(\infty) = \theta_{\Gamma'}(\infty) = 1$, we get

$$\theta_\Gamma = \theta_{\Gamma'}$$

therefore the tori $\Gamma(16)$ and $\Gamma(8)\oplus\Gamma(8)$ are isospectral (because the theta function determines the spectrum, see Proposition 5.3.2). □

In dimension two and three the spectrum of the Laplacian characterizes a flat torus. This is no longer true for dimension $n \geq 4$: there exists isospectral but non-isometric tori. In the general case, for a given flat torus the number of flat tori with the same spectrum is finite and an upper bound of this number is known.

## 7.4 Prescribing the spectrum on a manifold

As discussed into the Introduction, for a fixed topological manifold $M$ and a finite increasing sequence of real numbers $0 < a_1 \leq a_2 \leq \cdots \leq a_N$ there exists a metric $g$ such that for any integer $1 \leq k \leq N$ we have $\lambda_k(M,g) = a_k$. Indeed, in 1987, Y. Colin de Verdière [Col4] proved the following amazing theorem:

**Theorem 7.4.1** (Colin de Verdière, 1987). *Let $M$ be a compact manifold of dimension $n \geq 3$. For every finite increasing sequence $0 < a_1 \leq a_2 \leq \cdots \leq a_N$ there exists a metric $g$ on $M$ such that for any $k \leq N$ we have $\lambda_k(M,g) = a_k$.*

This theorem implies that for a compact manifold of dimension $n \geq 3$, there are no restrictions on the multiplicities of the (non-null) eigenvalues. The hypothesis on the dimension is fundamental, since we have seen in Section 5.4 that in dimension two there exists topological restrictions on the multiplicities of the eigenvalues.

For the proof of this theorem Y. Colin de Verdière considered a complete graph with $N+1$ vertices and constructed a Laplacian operator on this graph with the prescribing spectrum (see the book of Y. Colin de Verdière for a complete introduction about Laplacians on graphs). Next the trick of the proof is to embed the graph into the manifold and tending the $N$ first eigenvalues of the manifold to the spectrum of the graph; the final step if based on the *Arnold's transversality argument*.

## 7.5 Heat kernel and spectral geometry

In this section $(M, g)$ is a closed Riemannian manifold and $(\lambda_k)_{k\geq 0}$ denotes the spectrum of $(M, g)$.

### 7.5.1 The heat equation

**Definition 7.5.1.** The Cauchy problem for the *heat equation* on a closed Riemannian manifold $(M, g)$ is the following: find a function $u\colon M \times \mathbb{R}_+^* \to \mathbb{R}$ such that:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta_g u(x,t) \\ \\ u(x,0) = u_0(x) \quad \forall x \in M. \end{cases}$$

where $u_0$ is a given smooth function on $M$.

Physically, for every point $x \in M$ the number $u_0(x)$ is an initial temperature on the manifold $M$ and the solution $(x,t) \mapsto u(x,t)$ describes the evolution of the temperature at point $x \in M$ with the time $t$.

The first way to solve this equation is to use the spectral theory of $\Delta_g$ and the bounded functional calculus (see Section 2.6.2). Indeed, we can consider the bounded one-parameter semi-group $\{U(t)\}_{t\geq 0}$ defined for all $t \geq 0$ by

$$U(t) = e^{t\Delta_g}.$$

The generator of this semi-group is the operator $\Delta_g$. Consider the function

$$u(x,t) := U(t)u_0(x).$$

Then

$$\frac{du}{dt} = \Delta_g u(t)$$

and

$$u(x,0) = U(0)u_0(x) = u_0(x).$$

Finally, the function $u(x,t) = e^{t\Delta_g}u_0(x)$ is a solution of the heat equation. By the bounded functional calculus, for any $t \geq 0$ and for any integer $k$ we also have

$$\left(e^{t\Delta_g}\right) e_k = \left(e^{-t\lambda_k}\right) e_k,$$

where $\lambda_k = \lambda_k(M)$. So for any initial condition $u_0 \in L^2(M)$, if we denote by $(a_n)_{n\in\mathbb{N}}$ the sequence of $\ell^2(\mathbb{N})$ defined by $(a_n)_{n\in\mathbb{N}} := \pi(\psi_0)$, where $\pi$ is the following unitary projection operator:

$$\pi : \begin{cases} L^2(M) \longrightarrow \ell^2(\mathbb{N}) \\ \\ \varphi \longmapsto \langle \varphi, e_k \rangle_{L^2}, \end{cases}$$

then for all $t \geq 0$

$$\begin{aligned} u(x,t) = U(t)u_0(x) &= \left(e^{t\Delta_g}\right)\left(\sum_{k=0}^{+\infty} a_k e_k(x)\right) \\ &= \sum_{k=0}^{+\infty} a_k e^{-t\lambda_k} e_k(x). \end{aligned}$$

In conclusion, for all $x \in M$ and for all $t \geq 0$,

$$u(x,t) = \sum_{k=0}^{+\infty} \langle u_0, e_k \rangle_{L^2} e^{-t\lambda_k} e_k(x).$$

### 7.5.2 The heat kernel

Another way to solve the heat equation is to use the *heat kernel* of the manifold. The theory of heat kernels on Riemannian manifolds is deep and fundamental, providing powerful tools for understanding the relationship between geometry and diffusion processes. The diffusion operator related to the Laplacian on a compact Riemannian manifold is the heat operator. This operator acts on smooth functions and admits a fundamental solution which is called the heat kernel. Indeed, the heat kernel is the fundamental solution of the heat process diffusion equation (see also Sections 4.1.2 and 7.5.1)

$$\frac{\partial}{\partial t} u = \Delta u.$$

The heat kernel (see Definition 7.5.2 below) is a function

$$E : (x, y, t) \in M \times M \times \mathbb{R}_+^* \longmapsto E(x, y, t)$$

such that for all $y \in M$ the function $(x,t) \mapsto E(x,y,t)$ is a solution of the heat equation and for all $x \in M$ and every test function $\varphi \in \mathcal{D}(M)$ we have the initial data condition

$$\lim_{t\to 0^+} \int_M E(x,y,t)\varphi(y)\, d\mathcal{V}_g(y) = \varphi(x).$$

The theory of the heat kernel has many applications, for example

- in physics: from a physical point of view, the heat kernel is very important for understanding the space–time behaviour of the heat conduction in a material. Indeed, for an initial heat data at time $t = 0$ given by a Dirac distribution at the point $y$, the real number $E(x, y, t)$ is the temperature after time $t > 0$ at the point $x$.
- In analysis: for example, the heat kernel is an analytic tool which is very useful in the theory of function spaces (Hölder spaces, BMO spaces ...). In particular, the heat kernel is used for approximation of functions (see for example [Bu-Be], [Tri]).
- In spectral theory: the heat kernel can be used to build approximations of spectral projectors (see [Dav], [Dav2], [Kat], [Re-Si]).
- In geometry: one example of interaction between the heat kernel and the geometry of a compact manifold $(M, g)$ is the famous Varadhan's formula [Var]: for $x, y$ close enough,

$$\lim_{t\to 0^+} t \log E(x, y, t) = -\frac{d_M^2(x, y)}{4},$$

 where $E(x, y, t)$ is the heat kernel of $(M, g)$ and $d_M(x, y)$ is the Riemannian distance between $x$ and $y$. Let us give another example in geometry. In the article [BBG] P. Bérard, G. Besson and S. Gallot use the heat kernel and a finite number of eigenfunctions of $\Delta_g$ in order to embed[3] every compact Riemannian manifold in the usual following Hilbert space

$$\ell^2(\mathbb{R}) := \left\{ (a_n)_n \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{+\infty} |a_n|^2 < +\infty \right\}.$$

 In the next section we present in detail another major application of heat kernel in geometry and topology: the Minakshisundaram–Pleijel expansion (see Section 7.6).
- In applied mathematics and computer science: in 2006, R. Coifman and S. Lafon [Co-La] gave an algorithm for embedding a data set (graph or subset of $\mathbb{R}^n$) into a Euclidean space via the "diffusion map". In this space the data points are reorganized in such a way that the Euclidean distance corresponds to the "diffusion metric distance"; the "diffusion metric distance" describes the connectivity rate between two points of the set.
- In probability: the heat kernel is the distribution of a Brownian motion generated by the Laplacian on the manifold (see for example [SaC]).
- etc. ...

[3] In fact this embedding is asymptotically isometric (see [BBG]).

For a more complete introduction to heat kernel theory, see for example [Dav], [Ya-Sc], [BGV], [Gri] and for some application see also the article [SaC].

**Definition 7.5.2.** The *heat kernel* (or *fundamental solution*) of the heat equation on a closed Riemannian manifold $(M, g)$ is a function:

$$E : \begin{cases} M \times M \times \mathbb{R}_+^* \longrightarrow \mathbb{R} \\ (x, y, t) \longmapsto E(x, y, t) \end{cases}$$

such that:

(i) $E$ is $\mathcal{C}^0$ in the three variables $(x, y, t)$; $E$ is $\mathcal{C}^2$ in the second variable $y$ and $\mathcal{C}^1$ in the third variable $t$.

(ii) For all $(x, y, t) \in M^2 \times \mathbb{R}_+^*$,

$$\frac{\partial E}{\partial t}(x, y, t) = \Delta_{g,y} E(x, y, t),$$

where $\Delta_{g,y}$ is the Laplacian for the second variable $y$; in other words, for all $y \in M$ the function $(x, t) \mapsto E(x, y, t)$ is a solution of the heat equation.

(iii) For all $x \in M$,

$$E(x, \cdot, t) \rightharpoonup \delta_x$$

in $\mathcal{D}'(M)$ when $t \to 0^+$.

*Remark* 7.5.3. Item (iii) of this definition means that for all $x \in M$ and for all $\varphi \in \mathcal{D}(M)$ we have

$$\lim_{t \to 0^+} \langle E(x, \cdot, t), \varphi \rangle_{\mathcal{D}'(M) \times \mathcal{D}(M)} = \langle \delta_x, \varphi \rangle_{\mathcal{D}'(M) \times \mathcal{D}(M)},$$

i.e., for all $x \in M$ and for all $\varphi \in \mathcal{D}(M)$

$$\lim_{t \to 0^+} \int_M E(x, y, t) \varphi(y)\, d\mathcal{V}_g(y) = \varphi(x).$$

Since the manifold is compact, for every fixed $y \in M$ we have existence and uniqueness of a such function $E$ (for the construction see [BGM] or [Cha2]). We call this solution the heat kernel of $M$.

**Example 7.5.4.** In the Euclidean case $M = \mathbb{R}^n$ with $g = \text{can}$, the heat kernel is given by the expression

$$E(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d^2(x,y)}{4t}},$$

where $d$ is the usual Euclidean distance on $\mathbb{R}^n$. This example is very important because it can be used to it for construct the heat kernel in the general case.

Now, let us focus on the link between the heat kernel and the spectrum of the Laplacian. First, formally[4], consider the function $\mathcal{E}\colon M^2 \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\mathcal{E}(x, y, t) := \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) e_k(y).$$

We have

$$\Delta_{g,y}\mathcal{E}(x, y, t) = \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \Delta_{g,y} e_k(y) = \sum_{k=0}^{+\infty} -\lambda_k e^{-t\lambda_k} e_k(x) e_k(y).$$

On the other hand,

$$\frac{\partial \mathcal{E}}{\partial t}(x, y, t) = \sum_{k=0}^{+\infty} -\lambda_k e^{-t\lambda_k} e_k(x) e_k(y)$$

hence for all $(x, y, t) \in M^2 \times \mathbb{R}_+^*$ we obtain the equality

$$\frac{\partial \mathcal{E}}{\partial t}(x, y, t) = \Delta_{g,y}\mathcal{E}(x, y, t).$$

Moreover, for all $\varphi \in \mathcal{D}(M)$ we have

$$\begin{aligned} \int_M \mathcal{E}(x, y, t)\varphi(y)\, d\mathcal{V}_g(y) &= \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \int_M e_k(y)\varphi(y)\, d\mathcal{V}_g(y) \\ &= \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) \langle \varphi, e_k \rangle_{L^2}, \end{aligned}$$

so we get

$$\lim_{t \to 0^+} \int_M \mathcal{E}(x, y, t)\varphi(y)\, d\mathcal{V}_g(y) = \sum_{k=0}^{+\infty} e_k(x) \langle \varphi, e_k \rangle_{L^2} = \varphi(x).$$

Hence, formally the function $\mathcal{E}$ is a fundamental solution of heat equation which can be expressed as

$$\mathcal{E}(x, y, t) := \sum_{k=0}^{+\infty} e^{-t\lambda_k} e_k(x) e_k(y).$$

[4] Here, we suppose that all objects are well defined.

The rigorous argument for the previous calculation is provided by

**Theorem 7.5.5.** *If we denote by* $E$ *the heat kernel of* $(M, g)$*, then the series of functions*

$$(x, y, t) \longmapsto \sum_{k \geq 0} e^{-t\lambda_k} e_k(x) e_k(y)$$

*converges uniformly and absolutely on* $\overline{M} \times \overline{M} \times [\varepsilon, +\infty[$ *and the limit is equal to* $E(x, y, t)$.

*Proof.* Following J. Dodziuk [Dod1], let us introduce the one-parameter family of integral operators

$$P_t : \begin{cases} L^2(M) \longrightarrow L^2(M) \\ \varphi \longmapsto \int_M E(x, y, t)\varphi(y)\, d\mathcal{V}_g(y), \end{cases}$$

where $t > 0$ is the time parameter. In other words, for all $x \in M$,

$$P_t(\varphi)(x) = \int_M E(x, y, t)\varphi(y)\, d\mathcal{V}_g(y).$$

For all $t > 0$, the associated integral kernel is $(x, y) \mapsto E(x, y, t)$. Since for all $t > 0$ the function $(x, y) \mapsto E(x, y, t)$ is continuous on the compact manifold $(M, g)$, it belongs to the space $L^2(M \times M)$, so for all $t > 0$ the map $P_t$ is well defined and is a compact operator.

An important fact is that for all $\varphi \in L^2(M)$, the function $P_t(\varphi)$ is a solution of the heat equation.

Next, observe that for all $(\varphi, \psi) \in L^2(M)$ and for all $t > 0$ we have

$$|E(x, y, t)\varphi(y)\psi(x)| \leq \sup_{(x,y)\in M\times M} |E(x, y, t)|\, |\varphi(y)\psi(x)| \in L^2(M \times M),$$

so we can apply Fubini's theorem and get that for all $t > 0$

$$\begin{aligned}
\langle P_t\varphi, \psi\rangle_{L^2} &= \int_M P_t\varphi(x)\psi(x)\, d\mathcal{V}_g(x) \\
&= \int_M \left(\int_M E(x, y, t)\varphi(y)\, d\mathcal{V}_g(y)\right)\psi(x)\, d\mathcal{V}_g(x) \\
&= \int_M \int_M E(x, y, t)\varphi(y)\psi(x)\, \mathcal{V}_g(y) d\mathcal{V}_g(x)
\end{aligned}$$

$$= \int_M \left( \int_M E(x, y, t) \psi(x) \, d\mathcal{V}_g(x) \right) \varphi(y) \, d\mathcal{V}_g(y)$$
$$= \int_M P_t \psi(y) \varphi(y) \, d\mathcal{V}_g(y) = \langle \varphi, P_t \psi \rangle_{L^2} .$$

Therefore, for all $t > 0$ the bounded operator $P_t$ is self-adjoint.

To finish, note that by Duhamel's principle (see [Cha2])

$$E(x, z, t + s) = \int_M E(x, y, t) E(y, z, s) \, d\mathcal{V}_g(y),$$

and so

$$\begin{aligned}
(P_t \circ P_s) \varphi &= \int_M E(x, y, t) \, (P_s \varphi) \, (y) \, d\mathcal{V}_g(y) \\
&= \int_M E(x, y, t) \left( \int_M E(y, z, s) \varphi(z) \, d\mathcal{V}_g(z) \right) \, d\mathcal{V}_g(y) \\
&= \int_M \int_M E(x, y, t) E(y, z, s) \varphi(z) \, d\mathcal{V}_g(z) \, d\mathcal{V}_g(y) \\
&= \int_M \int_M E(x, y, t) E(y, z, s) \varphi(z) \, d\mathcal{V}_g(y) \, d\mathcal{V}_g(z) \\
&= \int_M \left( \int_M E(x, y, t) E(y, z, s) \, d\mathcal{V}_g(y) \right) \varphi(z) \, d\mathcal{V}_g(z) \\
&= \int_M E(x, z, t + s) \varphi(z) \, d\mathcal{V}_g(z) \\
&= P_{t+s}(\varphi)(x).
\end{aligned}$$

Therefore, the family $\{P_t\}_{t \geq 0}$ is a semi-group of bounded operator.

An obvious consequence of the semi-group property is that the operator $P_t$ is positive; indeed, for all $t > 0$

$$\begin{aligned}
\langle P_t \varphi, \varphi \rangle_{L^2} &= \left\langle \left( P_{t/2} \circ P_{t/2} \right) \varphi, \varphi \right\rangle_{L^2} \\
&= \left\langle P_{t/2} \varphi, P_{t/2} \varphi \right\rangle_{L^2} = \left\| P_{t/2} \varphi \right\|_{L^2} \geq 0.
\end{aligned}$$

Thus, for all $t > 0$, $P_t$ is bounded, compact, self-adjoint and positive operator. Hence there exists a Hilbert basis $(\phi_k(t))_{k \geq 0}$ of $L^2(M)$ consisting of eigenvectors of $P_t$ with corresponding eigenvalues $\mu_k(t)$ (see Theorem 2.5.6), i.e.,

$$P_t \phi_k(t) = \mu_k(t) \phi_k(t).$$

Moreover, for all $t > 0$ we have

$$\mu_0(t) \geq \mu_1(t) \geq \cdots \geq \mu_k(t) \geq 0$$

and $\mu_k(t) \to 0$ as $k \to +\infty$.

On the other hand, for all $\varphi \in \mathcal{D}(M)$ it holds that

$$\begin{aligned}
\frac{d}{dt}\|P_t\varphi\|_{L^2}^2 &= \frac{d}{dt}\langle P_t\varphi, P_t\varphi\rangle_{L^2} \\
&= 2\left\langle \frac{d}{dt}P_t\varphi, P_t\varphi\right\rangle_{L^2} = 2\langle \Delta_g P_t\varphi, P_t\varphi\rangle_{L^2} \\
&= 2\int_M \Delta_g(P_t\varphi)(x)(P_t\varphi)(x)\, d\mathcal{V}_g(x) \\
&= -2\int_M \nabla(P_t\varphi)(x)\nabla(P_t\varphi)(x)\, d\mathcal{V}_g(x) \\
&= -2\,\|\nabla(P_t\varphi)\|_{L^2} \leq 0.
\end{aligned}$$

Hence, the real valued function $t \mapsto \|P_t\varphi\|_{L^2}^2$ is non-increasing on $]0, +\infty[$. Since for all $t > 0$ and for all $\varphi \in \mathcal{D}(M)$

$$P_t(\varphi)(x) = \int_M E(x, y, t)\varphi(y)\, d\mathcal{V}_g(y),$$

we have that

$$\lim_{t\to 0^+} P_t(\varphi)(x) = \lim_{t\to 0^+}\int_M E(x, y, t)\varphi(y)\, d\mathcal{V}_g(y) = \varphi(x)$$

for all $x \in M$ and for all $\varphi \in \mathcal{D}(M)$, and it follows that

$$\|P_t\varphi\|_{L^2} < \|\varphi\|_{L^2}$$

for all $\varphi \in \mathcal{D}(M)$ and for all $t > 0$. Consequently, by the density of $\mathcal{D}(M)$ in $L^2(M)$, for all $\varphi \in L^2(M)$, we have

$$\lim_{t\to 0^+} P_t\varphi = \varphi \ \text{ and } \ \|P_t\varphi\|_{L^2} < \|\varphi\|_{L^2}.$$

Now, for any integer $k \geq 0$, let us consider the functions $\phi_k := \phi_k(1)$ and $\mu_k := \mu_k(1)$, so we have

$$P_1\phi_k = \mu_k\phi_k.$$

Consequently, for any integer $n$ we have

$$P_n\phi_k = P_{1+1+\cdots+1}\phi_k = (P_1)^n\,\phi_k = \mu_k^n\phi_k.$$

This implies that $P_r\phi_k = \mu_k^r\phi_k$ for any rational number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and since the functions $t \mapsto P_t\phi_k$ and $t \mapsto \mu_k^t\phi_k$ are continuous on $]0, +\infty[$ we get by density that for all $t > 0$

$$P_t\phi_k = \mu_k^t\phi_k = e^{t\ln\mu_k}\phi_k.$$

Here, for any $k \geq 0$ we have $\mu_k > 0$. Indeed, suppose $\mu_k = 0$. Since the function $\phi_k$ is non-trivial and since

$$\lim_{t \to 0^+} P_t \phi_k = \phi_k$$

and

$$\lim_{t \to 0^+} P_t \phi_k = \lim_{t \to 0^+} \mu_k^t \phi_k$$

we get a contradiction, so $\mu_k > 0$.

If we denote $\alpha_k := -\ln \mu_k$, i.e., $\mu_k := e^{-\alpha_k}$, then

$$P_t \phi_k = \mu_k^t \phi_k = e^{-\alpha_k t} \phi_k,$$

so we get $\|P_t \phi_k\|_{L^2} = \left\| e^{-\alpha_k t} \phi_k \right\|_{L^2} = e^{-\alpha_k t} \|\phi_k\|_{L^2}$. Since $\|P_t \phi_k\|_{L^2} < \|\phi_k\|_{L^2}$ we deduce that $\alpha_k > 0$ for any $k$.

Now, for any fixed $\varepsilon > 0$ we can apply Mercer's theorem (see, for example, [Hoc]) to the operator $P_1$ to conclude that the series of functions

$$(x, y, t) \longmapsto \sum_{k \geq 0} e^{-t\alpha_k} \phi_k(x) \phi_k(y)$$

converges uniformly and absolutely on $\overline{M} \times \overline{M} \times [\varepsilon, +\infty[$ and the limit is equal to $E$.

To complete the proof, we show that for any $k$, $\alpha_k = \lambda_k$ and $\phi_k = e_k$. Indeed, for any $k$

$$P_t \phi_k = e^{-\alpha_k t} \phi_k$$

and since the function $P_t \phi_k$ is a solution of the heat equation we get

$$\frac{\partial}{\partial t} \left( e^{-\alpha_k t} \phi_k \right)(x) = \Delta_g \left( e^{-\alpha_k t} \phi_k \right)(x),$$

i.e.,

$$-\alpha_k e^{-\alpha_k t} \phi_k(x) = e^{-\alpha_k t} \Delta_g (\phi_k)(x),$$

whence

$$-\Delta_g \phi_k = \alpha_k \phi_k. \qquad \square$$

*Remark* 7.5.6. In fact the family $\{P_t\}_{t \geq 0}$ is a non-negative semi-group of bounded operators and coincides with $U(t) := e^{t\Delta_g}$.

To finish, we sketch the construction of the heat kernel (for technical details see [BGM], [Cha2]). It is based on the heat kernel $G$ of $\mathbb{R}^n$ with the canonical metric can:

$$G(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\|x-y\|^2}{4t}}.$$

In fact, we regard $G$ as the first-order approximation for the heat kernel of a Riemannian manifold $M$. Now, we want to construct the second-order approximation of $E$. We achieve this by multiplying $G$ by a finite series of differentiable functions on a certain domain. More precisely, if we denote by $r$ the injectivity radius of $M$, this domain is

$$U_r := \{(x,y) \in M \times M;\ d(x,y) < r\}.$$

For any integer $N > 0$, let us introduce the function $H_N$ defined by

$$H_N(x,y,t) := G(x,y,t) \sum_{k=0}^{N} t^k u_k(x,y),$$

where for any integer $0 \leq k \leq N$, $u_k \in \mathcal{C}^\infty(U_r)$ and is such that for all $x \in M$

$$u_0(x,x) = 1$$

and

$$L_y H_N(x,y,t) = G(x,y,t) t^N \Delta_{g,y} u_N(x,y),$$

where $L_y := \Delta_{g,y} - \frac{\partial}{\partial t}$. This function is an approximation of the heat kernel, but is defined only on the open set $U_r$. We want to extend this solution to $M \times M$. To this end we introduce a smooth radial function $\eta\colon M \times M \to \mathbb{R}$ such that

$$\eta(x,y) = \left| \begin{array}{l} 1 \ \text{ if } d(x,y) < \frac{r}{4} \\[2ex] 0 \ \text{ if } d(x,y) > \frac{r}{2}. \end{array} \right.$$

Thus for $N$ large enough (namely, $N > \frac{n}{2} + 1$), the function $S_N := \eta H_N$ is a *parametrix*[5] of the heat kernel. Next, using geodesic spherical coordinates, one can compute the $u_k$ for any $k$ such that $0 \leq k \leq N$

$$\begin{aligned} u_k(x,y) = {} & \frac{1}{\sqrt{\theta(x,y)}} \frac{1}{d(x,y)^k} \\ & \cdot \int_0^1 t^{k-1} \theta\left(x, \exp_x(t \exp_x^{-1} y)\right) \left(\Delta_{g,y} u_{k-1}\right) \left(x, \exp_x(t \exp_x^{-1} y)\right) dt, \end{aligned}$$

where

$$\theta(x,y) := \frac{\sqrt{\det g}}{d(x,y)^{n-1}}.$$

[5]A parametrix of a function $L_y = \Delta_{g,y} - \frac{\partial}{\partial t}$ is a function $H$ such that (see [BGM]):

(i) $H$ is continuous on $M \times M \times ]0,+\infty[$;

(ii) $L_y H$ admit a continuous continuation to $M \times M \times [0,+\infty[$;

(iii) $\lim_{t \to 0^+} H(x,\cdot,t) = \delta_x$ for all $x \in M$.

In particular for $k = 0$ we get

$$u_0(x, y) = \frac{1}{\sqrt{\theta(x, y)}},$$

and for $k = 1$ we have

$$u_1(x, y) = \frac{1}{6}\mathrm{Scal}(x).$$

To finish, using the parametrix, we construct the heat kernel of $(M, g)$ in the form

$$E(x, y, t) = S_N + S_N \star H,$$

where $H$ is continuous on $M \times M \times [0, +\infty[$. Here $\star$ is the *space–time convolution product*[6], namely, for two continuous functions $H$ and $J$ on $M \times M \times [0, +\infty[$,

$$(H \star J)(x, y, t) := \int_0^t \int_M H(x, z, s) J(z, y, t - s)\, d\mathcal{V}_g(z)\, ds.$$

Since (by a computation)

$$L_y(S_N \star H) = L_y S_N \star H - H,$$

we get

$$0 = L_y K = L_y S_N + \big(L_y S_N\big) \star H - H.$$

We can solve this equation and find (note the analogy with the real equation $x + xy - y = 0$, with solution $y = \frac{x}{1-x} = \sum_{k=1}^{+\infty} x^k$)

$$H = \sum_{k=1}^{+\infty} \big(L_y S_N\big)^{\star k}.$$

Therefore,

$$E(x, y, t) = S_N + S_N \star \sum_{k=1}^{+\infty} \big(L_y S_N\big)^{\star k}.$$

Moreover, the function $\sum_{k=1}^{+\infty} \big(L_y S_N\big)^{\star k}$ is $m + \ell$ times differentiable on $M \times M \times ]0, +\infty[$, where $2\ell + m < N - \frac{n}{2}$. More precisely, the series of functions

$$\sum_{k=1}^{+\infty} \frac{\partial^\ell}{\partial t^\ell} \partial_y^m \big(L_y S_N\big)^{\star k}$$

(where $\partial_y^m = \partial_y \circ \partial_y \circ \cdots \circ \partial_y$ is the derivative with respect to the variable $y$ in a local system of coordinates) converges uniformly and absolutely on every compact subset of $M \times M \times ]0, +\infty[$.

[6] This convolution product is associative.

Finally,

$$\frac{\partial^\ell}{\partial t^\ell}\partial_y^m \sum_{k=1}^{+\infty}\left(L_y S_N\right)^{\star k} = \sum_{k=1}^{+\infty}\frac{\partial^\ell}{\partial t^\ell}\partial_y^m \left(L_y S_N\right)^{\star k} = O\left(t^{N-2n-2\ell-m+1}\right),$$

if we denote $Q_N(x,y,t) := S_N \star \sum_{k=1}^{+\infty}\left(L_y S_N\right)^{\star k}$ we get

$$E(x,y,t) = G(x,y,t)\sum_{k=0}^{N} t^k \eta(x,y) u_k(x,y) + Q_N(x,y,t),$$

with

$$Q_N(x,y,t) = O\left(t^{N-\frac{n}{2}-1}\right)$$

uniformly in $(x,y) \in M \times M$.

### 7.5.3 The spectral partition function $Z_M$ as a trace

Next, let us examine what happens with the heat kernel on the diagonal $y = x$. For a fixed $\varepsilon > 0$, for all $t \geq \varepsilon$ and for all $x \in M$, the series

$$(x,t) \longmapsto \sum_{k\geq 1} e^{-t\lambda_k} e_k^2(x)$$

is convergent and its limit is equal to $E(x,x,t)$. Consequently, for all $t \in [\varepsilon, +\infty[$

$$\begin{aligned}\int_M E(x,x,t)\,d\mathcal{V}_g(x) &= \int_M \sum_{k=1}^{+\infty} e^{-t\lambda_k} e_k^2(x)\,d\mathcal{V}_g(x)\\ &= \sum_{k=1}^{+\infty} e^{-t\lambda_k}\int_M e_k^2(x)\,d\mathcal{V}_g(x),\\ \sum_{k=1}^{+\infty} e^{-t\lambda_k}\,\|e_k\|_{L^2}^2 &= \sum_{k=1}^{+\infty} e^{-t\lambda_k}.\end{aligned}$$

So, we can define the trace of the operator $e^{t\Delta_g}$ by

$$Z_M(t) := \operatorname{trace}\left(e^{t\Delta_g}\right) = \operatorname{trace}(P_t) = \int_M E(x,x,t)\,d\mathcal{V}_g(x) = \sum_{k=1}^{+\infty} e^{-t\lambda_k}.$$

We conclude that the series of functions $t \mapsto \sum_{k=1}^{+\infty} e^{-t\lambda_k}$ is uniformly and absolutely convergent on $[\varepsilon, +\infty[$.

## 7.6 The Minakshisundaram–Pleijel expansion and the Weyl formula

A very important corollary of the heat kernel construction is the *Minakshisundaram–Pleijel expansion*:

**Theorem 7.6.1** (Minakshisundaram–Pleijel expansion). *Let $(M, g)$ be a compact Riemannian manifold of dimension $n$, and let us denote by $E$ the heat kernel of $(M, g)$. We have the expansion*

$$E(x, x, t) \underset{t \to 0^+}{\sim} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left( u_0(x, x) + u_1(x, x)t + \cdots + u_k(x, x)t^k + \ldots \right),$$

*where the functions $x \mapsto u_k(x, x)$ are smooth on $M$ and depend only on the curvature tensor and its covariant derivatives.*

Hence, if for any integer $k$ we denote

$$a_k := \int_M u_k(x, x)\, d\mathcal{V}_g(x),$$

then

$$Z_M(t) = \sum_{k=1}^{+\infty} e^{-t\lambda_k} \underset{t \to 0^+}{\sim} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left( a_0 + a_1 t + \cdots + a_k t^k + \cdots \right),$$

i.e., for any integer $N > 0$ and for all $t > 0$,

$$Z_M(t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{k=0}^{N} a_k t^k + O\left( t^{N - \frac{n}{2} + 1} \right).$$

The computation of the coefficients $a_k$ is very difficult. Nevertheless, we have (see [BGM]):

$$a_0 = \mathrm{Vol}(M, g)$$

and

$$a_1 = \frac{1}{6} \int_M \mathrm{Scal}_g \, d\mathcal{V}_g.$$

In particular, if $\dim(M) = 2$, the Gauss–Bonnet formula yields

$$a_1 = \frac{\pi}{3} \chi(M),$$

where $\chi(M)$ is the Euler–Poincaré characteristic of the surface $M$.

The expression of $a_2$ is also known:

$$a_2 = \frac{1}{360} \int_M \left(2\,|R|^2 - 2\,|\mathrm{Ric}|^2 + 5\mathrm{Scal}_g^2\right) d\mathcal{V}_g,$$

where $R$ is the Riemann curvature tensor.

*Remark* 7.6.2. In particular, for a closed surface $M$, we have:

$$Z_M(t) = \frac{\mathrm{Vol}(M,g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60} \int_M K^2\, d\mathcal{V}_g + P(t)t^2,$$

where $t \mapsto P(t)$ is a bounded function on $M$.

A fundamental corollary of the Minakshisundaram–Pleijel expansion is the following fact: if we know the spectrum of $(M,g)$, then in particular we know

- the dimension of the manifold;
- the volume of the manifold;
- the integral of the scalar curvature $\mathrm{Scal}_g$ over the manifold.

Hence we deduce

**Corollary 7.6.3.** *If two Riemannian manifolds $(M,g)$ and $(M',g')$ are isospectral, then:*

(i) *the dimensions of $M$ and $M'$ are equal.*

(ii) *The volumes of $(M,g)$ and $(M',g')$ are equal.*

(iii) *The integrals of the scalar curvature over $(M,g)$ and $(M',g')$ are equal. In particular, if $M$ and $M'$ are surfaces, then $\chi(M) = \chi(M')$.*

In the case of surfaces, the dimension and the Euler–Poincaré characteristic are topological invariants.

The coefficients $a_k$ for $k \geq 2$ are very complicated; more details can be found in the books [BGM] and [Bér8].

To finish, we note that the Karamata Tauberian theorem allows us to deduce from Minakshisundaram–Pleijel expansion the famous Weyl asymptotic formula:

**Theorem 7.6.4** (Weyl asymptotic formula). *Let $(M,g)$ be a compact Riemannian manifold of dimension $n$, and let $\lambda_k$ denote the eigenvalues of $-\Delta_g$ on $(M,g)$. Then*

$$\mathrm{Card}\left(\{k \in \mathbb{N},\ \lambda_k \leq \lambda\}\right) \underset{\lambda \to +\infty}{\sim} \frac{B_n \mathrm{Vol}(M,g)}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

*where $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the closed unit ball in $(\mathbb{R}^n, can)$.*

An obvious consequence of this asymptotics is the equivalent one

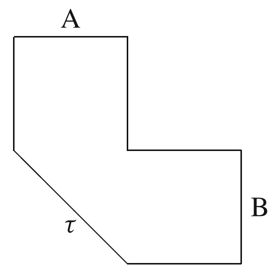
$$\lambda_k \underset{k\to+\infty}{\sim} \left(\frac{(2\pi)^n}{B_n \mathrm{Vol}(M,g)}\right)^{\frac{2}{n}} k^{\frac{2}{n}}.$$

## 7.7 Two planar isospectral nonisometric domains

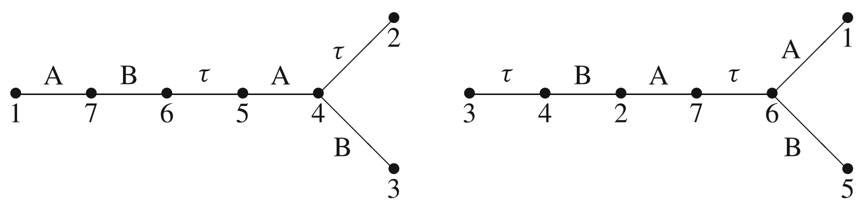
In this subsection we present a very simple counter-example for the isospectrality problem. This example is based on P. Bérard's transplantation method [Bér2], [Bér3]. We largely follow [Bér1].

### 7.7.1 Construction of the domains $D_1$ and $D_2$

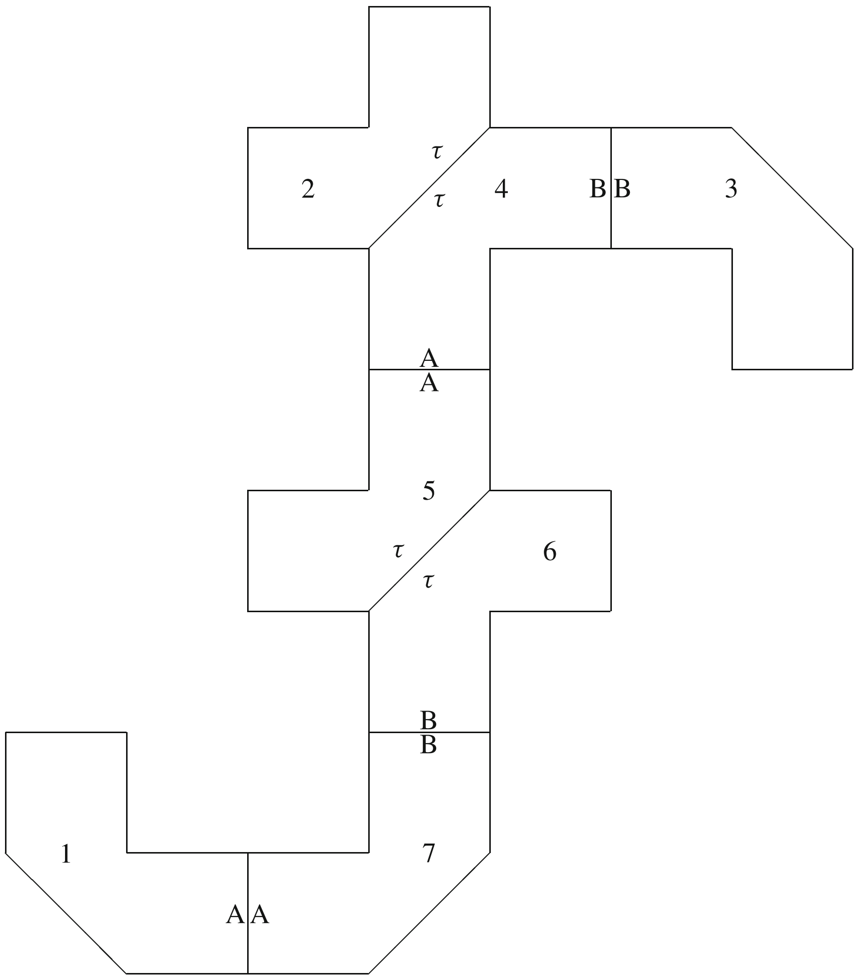
Let us consider the following elementary domain, called a *brick*:



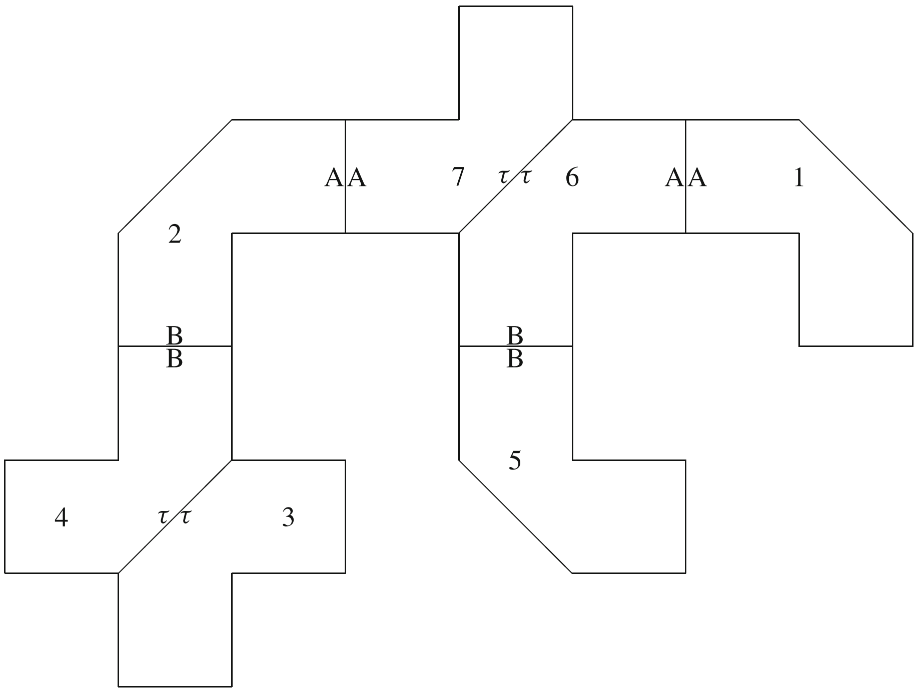

Here $A$, $B$, and $\tau$ are labels for pieces of the boundary. Consider also the following graphs:




where the integers $\{1, 2, \ldots, 7\}$ on the vertices are labels for bricks and $A, B, \tau$ are labels for edges. Using seven bricks $P_i$, $i \in \{1, 2, \ldots, 7\}$ and following combinatorial the first graph, we glue the bricks along the labeled edges. For example: we glue bricks $P_1$ and $P_7$ along the boundary $A$, we glue bricks $P_7$ and $P_6$ along the boundary $B$, etc. … Using this we obtain the following planar domain $D_1$:




Carrying out the same type of construction with the second graph, we get another planar domain $D_2$.




Obviously, for the metric can on $\mathbb{R}^2$ we have

**Proposition 7.7.1.** *The domains $D_1$ and $D_2$ are non-isometric.*

### 7.7.2 Isospectrality of the domains $D_1$ and $D_2$

Now let us consider the Laplacian $\Delta$ on the domains $D_1$ and $D_2$ with, for example, Neumann conditions (the Dirichlet case is very similar) and show that $D_1$ and $D_2$ are isospectral. For this let us introduce some notations and facts. First we define the *transplantation matrix*

$$T := \begin{pmatrix} a & a & a & a & b & b & b \\ a & b & a & b & a & a & b \\ a & a & b & b & b & a & a \\ a & b & b & a & a & b & a \\ b & a & b & a & a & a & b \\ b & a & a & b & a & b & a \\ b & b & a & a & b & a & a \end{pmatrix},$$

where $a, b \in \mathbb{R}$ are such that

$$\begin{cases} 4a^2 + 3b^2 = 1 \\ 2a^2 + 4ab + b^2 = 0 \\ 4a + 3b = 1. \end{cases}$$

An elementary computation establishes that the matrix $T$ is orthogonal (i.e., $T^\star T = I_7$).

Next consider the sets $X_j$, $j \in \{1, 2\}$, defined by

$$X_j := \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_7 \end{pmatrix}; \ \forall i \in \{1, 2, \ldots, 7\} \ x_i \in L^2(P_i) \ \text{and} \ (\star)_j \right\},$$

where

$$(\star)_1 \ : \ \begin{cases} x_1|_A = x_7|_A \\ x_7|_B = x_6|_B \\ x_6|_\tau = x_5|_\tau \\ x_5|_A = x_4|_A \\ x_4|_\tau = x_2|_\tau \\ x_4|_B = x_3|_B \end{cases} \quad \text{and} \quad (\star)_2 \ : \ \begin{cases} x_4|_\tau = x_3|_\tau \\ x_4|_B = x_2|_B \\ x_2|_A = x_7|_A \\ x_7|_\tau = x_6|_\tau \\ x_6|_B = x_5|_B \\ x_6|_A = x_1|_A \end{cases}$$

and $x_i|_Z$ denotes the restriction of the function $x_i$ to the domain $Z \in \{A, B, \tau\}$.

Now consider the maps $\pi_j$, $j \in \{1, 2\}$:

$$\pi_j \ : \ \begin{cases} L^2(D_j) \longrightarrow X_j \\ \varphi \longmapsto \pi_j(\varphi) := \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_7 \end{pmatrix} := \begin{pmatrix} \varphi_{|P_1} \\ \vdots \\ \varphi_{|P_7} \end{pmatrix}. \end{cases}$$

It is clear that $\pi_1$ and $\pi_2$ are isomorphisms.

**Definition 7.7.2.** The *transplantation map* between domains $D_1$ and $D_2$ is defined by

$$\mathfrak{T} \ : \ \begin{cases} L^2(D_1) \longrightarrow L^2(D_2) \\ \varphi \longmapsto \pi_2^{-1}(T.\pi_1(\varphi)). \end{cases}$$

Let us verify that the map $\mathfrak{T}$ is well defined. Given any function $\varphi \in L^2(D_1)$, consider the vector

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_7 \end{pmatrix} := T \cdot \pi(\varphi),$$

so we have $a\psi_1 = a\varphi_1 + a\varphi_2 + a\varphi_3 + a\varphi_4 + b\varphi_5 + b\varphi_6 + b\varphi_7$, etc.

Since $\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_7 \end{pmatrix}$ belongs to $X_1$ one can easily verify that $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_7 \end{pmatrix}$ belongs to $X_2$.

**Proposition 7.7.3.** *The transplantation map is unitary (i.e., a surjective isometry) from $L^2(D_1)$ onto $L^2(D_2)$. In particular, $H^1(D_1)$ is isometric to $H^1(D_2)$.*

*Proof.* Since $\pi_2, T, \pi_1$ are isomorphisms, the map $\mathfrak{T}$ is an isomorphism from $L^2(D_1)$ onto $L^2(D_2)$. Next for all $\varphi \in L^2(D_1)$ using the fact that the matrix $T$ is orthogonal we get

$$\begin{aligned} \|\mathfrak{T}\varphi\|^2_{L^2(D_2)} &= \sum_{k=1}^{7} \|\mathfrak{T}\varphi|_{P_k}\|^2_{L^2(P_k)} = \sum_{k=1}^{7} \|\psi_k\|^2_{L^2(P_k)} \\ &= \int_{D_1} \|T\pi_1(\varphi)\|^2_{\mathbb{R}^7}\, d\mathcal{V}_g = \int_{D_1} \|\pi_1(\varphi)\|^2_{\mathbb{R}^7}\, d\mathcal{V}_g = \|\varphi\|^2_{L^2(D_1)}. \end{aligned}$$

Similarly,

$$\|\nabla\mathfrak{T}\varphi\|^2_{L^2(D_2)} = \|\nabla\varphi\|^2_{L^2(D_1)}$$

for all $\varphi \in H^1(D_1)$; in particular,

$$\|\mathfrak{T}\varphi\|^2_{H^1(D_2)} = \|\varphi\|^2_{H^1(D_1)}$$

for all $\varphi \in H^1(D_1)$. To finish one can easily verify the equivalence

$$\varphi \in H^1(D_1) \Longleftrightarrow \mathfrak{T}\varphi \in H^1(D_2).$$

Therefore, the map $\mathfrak{T}$ induces an isometry from $H^1(D_1)$ onto $H^1(D_2)$. □

**Corollary 7.7.4.** *The domains $D_1$ and $D_2$ are isospectral.*

*Proof.* Since the map $\mathfrak{T}$ is bijective and

$$\|\mathfrak{T}\varphi\|^2_{L^2(D_2)} = \|\varphi\|^2_{L^2(D_1)}$$

and

$$\|\nabla\mathfrak{T}\varphi\|^2_{L^2(D_2)} = \|\nabla\varphi\|^2_{L^2(D_1)}$$

for all $\varphi \in H^1(D_1)$ (see the previous proof), using the minimax principle we get for each integer $k$

$$\begin{aligned}\lambda_k(D_2) &= \min_{\substack{E\subset H^1(D_2)\\ \dim(E)=k}} \max_{\substack{\psi\in E\\ \psi\neq 0}} \frac{\|\nabla\psi\|^2_{L^2(D_2)}}{\|\psi\|^2_{L^2(D_2)}} \\ &= \min_{\substack{E\subset H^1(D_1)\\ \dim(E)=k}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\|\nabla\mathfrak{T}\varphi\|^2_{L^2(D_2)}}{\|\mathfrak{T}\varphi\|^2_{L^2(D_2)}} \\ &= \min_{\substack{E\subset H^1(D_1)\\ \dim(E)=k}} \max_{\substack{\varphi\in E\\ \varphi\neq 0}} \frac{\|\nabla\varphi\|^2_{L^2(D_1)}}{\|\varphi\|^2_{L^2(D_1)}} = \lambda_k(D_1).\end{aligned}$$ □

## 7.8 Few words about Laplacian and conformal geometry

In this last Section we explain the relationship between the eigenvalues of a closed surface and its associated conformal geometry. For more details, see, for example [Cha] or [Gur].

### 7.8.1 Conformal geometry on surfaces

We begin with some basic facts on the conformal geometry on surfaces.

**Definition 7.8.1.** Let $(M, g_0)$ be a surface. A metric $h$ on $M$ is said *conformal* to $g_0$ if there exists a smooth function $u$ on $M$ such that $h = e^{2u}g_0$.

We define the *conformal class* $[g_0]$ of the metric $g_0$ on $M$ by

$$[g_0] := \left\{\widetilde{g} = e^{2u}g_0;\ u \in \mathcal{C}^\infty(M)\right\}.$$

For all vectors $X, Y \in T_xM$ ($\measuredangle_g$ denotes the angle for the metric $g$) we have

$$\begin{aligned}\cos\left(\measuredangle_{g_0}X, Y\right) &= \frac{g_0(X,Y)}{\sqrt{g_0(X,X)}\sqrt{g_0(Y,Y)}} \\ &= \frac{e^{-2u}\widetilde{g}(X,Y)}{\sqrt{e^{-2u}\widetilde{g}(X,X)}\sqrt{e^{-2u}\widetilde{g}(Y,Y)}} \\ &= \frac{\widetilde{g}(X,Y)}{\sqrt{\widetilde{g}(X,X)}\sqrt{\widetilde{g}(Y,Y)}} \\ &= \cos\left(\measuredangle_{\widetilde{g}}X, Y\right).\end{aligned}$$

Therefore, angles are conformally invariant: if we change the metric $g_0$ on a surface to a conformal metric $\widetilde{g} = e^{2u} g_0$, the length of vectors and curves changes, but the angle between two vectors is unchanged:

$$\measuredangle_{g_0} X, Y = \measuredangle_{\widetilde{g}} X, Y$$

for all $x \in M$ and for all vectors $X, Y \in T_x M$. In fact the geometries of $(M, g_0)$ and $(M, \widetilde{g})$ are the sames up to a change of scale.

When we change the initial metric $g_0$ to another conformal metric $g$ the curvature changes according to the *Gaussian curvature equation*:

**Proposition 7.8.2.** *For a conformal metric $\widetilde{g} = e^{2u} g_0$ on $M$ we have the local relation*

$$-\Delta_{g_0} u + K_{g_0} = K_{\widetilde{g}} e^{2u}.$$

One of the main questions in conformal geometry is: given a closed surface $M$ with a fixed metric $g_0$ and for a smooth function $K\colon M \to \mathbb{R}$, does there exist a conformal metric $\widetilde{g}$ such that $K_{\widetilde{g}} = K$? In other words, does there exist a smooth function $u$ such that

$$-\Delta_{g_0} u + K_{g_0} = K e^{2u}?$$

The answer is yes: using complex analysis, A. Weil solved this problem for Riemann surfaces.

### 7.8.2 Spectral zeta function and uniformization theorem on surfaces

Now, let us define the spectral zeta function associate to a surface $(M, g_0)$ by the formula

$$\zeta_{g_0}(s) := \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s},$$

where $s \in \mathbb{C}$ is such that $\mathrm{Re}(s) > 1$. Then we have

$$\zeta_{g_0}(s) = \sum_{n=1}^{+\infty} \lambda_n^{-s} = \sum_{n=1}^{+\infty} e^{-s \ln(\lambda_n)}.$$

Using the Weyl formula, i.e.,

$$\lambda_n \underset{n \to +\infty}{\sim} C n,$$

where

$$C := \frac{4\pi^2}{B_2 \mathrm{Vol}(M, g)} > 0,$$

we see that the series $\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$ and $\sum_{n=1}^{+\infty} \frac{1}{n^s}$ have the same nature. The series $\sum_{n=1}^{+\infty} \frac{1}{n^s}$ is the usual Riemann function which is convergent for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$. So the series $\zeta_{g_0}(s) := \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$ is well defined and convergent for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$.

Now we shall prove that the function $\zeta_{g_0}$ admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. First recall that $\Gamma$ the Euler Gamma function

$$\Gamma(s) := \int_0^{+\infty} e^{-t} t^{s-1}\, dt$$

converges for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$. Moreover, for all $s \in \mathbb{C}$ in the indicated range,

$$\frac{1}{\Gamma(s)} = \lim_{n \to +\infty} \frac{n! n^s}{s(s+1)(s+2)\cdots(s+n)}.$$

The function $\Gamma$ admits a meromorphic continuation to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ with simple poles at $s = 0, -1, -2, \ldots$. By changing variables we have for all $x > 0$

$$\frac{1}{x^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} t^{s-1}\, dt.$$

Therefore, for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ we get

$$\begin{aligned}
\zeta_{g_0}(s) &= \sum_{n=1}^{+\infty} \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-\lambda_n t} t^{s-1}\, dt \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} \left( \sum_{n=1}^{+\infty} e^{-\lambda_n t} \right) t^{s-1}\, dt \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} (Z_M(t) - 1)\, t^{s-1}\, dt
\end{aligned}$$

because $\lambda_0 = 0$. Thus,

$$\zeta_{g_0}(s) = \frac{1}{\Gamma(s)} \int_0^1 (Z_M(t) - 1)\, t^{s-1}\, dt + \frac{1}{\Gamma(s)} \int_1^{+\infty} (Z_M(t) - 1)\, t^{s-1}\, dt.$$

We start by studying the first integral. Using Remark 7.6.2, we have for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$

$$\begin{aligned}
&\frac{1}{\Gamma(s)} \int_0^1 (Z_M(t) - 1)\, t^{s-1}\, dt \\
&= \frac{1}{\Gamma(s)} \int_0^1 \left( \frac{\mathrm{Vol}(M, g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60} \int_M K^2\, d\mathcal{V}_g + P(t) t^2 - 1 \right) t^{s-1}\, dt
\end{aligned}$$

$$= \frac{1}{\Gamma(s)}\int_0^1 \left(\frac{\mathrm{Vol}(M,g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2\, d\mathcal{V}_g - 1\right) t^{s-1}\, dt$$
$$+ \frac{1}{\Gamma(s)}\int_0^1 P(t)t^{s+1}\, dt.$$

The function $s \mapsto \frac{1}{\Gamma(s)}\int_0^1 P(t)t^{s+1}\, dt$ is holomorphic for $\mathrm{Re}(s) > -2$ and

$$\frac{1}{\Gamma(s)}\int_0^1 \left(\frac{\mathrm{Vol}(M,g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2\, d\mathcal{V}_g - 1\right) t^{s-1}\, dt$$
$$= \frac{1}{\Gamma(s)}\left[\frac{t^{s-1}}{s-1}\frac{\mathrm{Vol}(M,g)}{4\pi} + \frac{\chi(M)t^s}{6s} + \frac{\pi t^{s+1}}{60(s+1)}\int_M K^2\, d\mathcal{V}_g - \frac{t^s}{s}\right]_{t=0}^{t=1}.$$

This quantity converges for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$ with a simple pole in $s = 1$ (this follows from the relation $\Gamma(s) = \lim_{n\to+\infty}\frac{s(s+1)(s+2)\cdots(s+n)}{n!n^s}$).

To finish, let us study the second integral. For all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$,

$$\frac{1}{\Gamma(s)}\int_1^{+\infty} (Z_M(t) - 1)\, t^{s-1}\, dt = \frac{1}{\Gamma(s)}\int_1^{+\infty}\left(\sum_{n=1}^{+\infty} e^{-\lambda_n t}\right) t^{s-1}\, dt,$$

and for all $t \geq 1$, $e^{-\lambda_n t} = e^{-\lambda_1 t}e^{(\lambda_1-\lambda_n)t}$, so since the eigenvalues $\lambda_n$ grows like $n$ as $n$ tends to infinity (Weyl asymptotic formula) and since $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$, it follows that for $t$ large enough

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} \leq Ce^{-\lambda_1 t}.$$

Hence, using the holomorphic theorem under integral sign we get easily that the function $s \mapsto \frac{1}{\Gamma(s)}\int_1^{+\infty}\left(\sum_{n=1}^{+\infty} e^{-\lambda_n t}\right) t^{s-1}\, dt$ is holomorphic on $\mathbb{C}$. Finally, the function $\zeta_{g_0}$ has a meromorphic extension to $\mathbb{C}$ with a simple pole in $s = 1$. In particular, $\zeta_{g_0}$ is analytic in 0, hence the value $\zeta'_{g_0}(0)$ exists and

$$\zeta'_{g_0}(0) = \lim_{s\to 0}\frac{\zeta_{g_0}(s) - \zeta_{g_0}(0)}{s-0}.$$

Next, the derivative of $\zeta_{g_0}$ is

$$\zeta'_{g_0}(s) = \sum_{i=1}^{+\infty} - \ln(\lambda_i)\frac{1}{\lambda_i^s},$$

so for $s = 0$ we get

$$\zeta'_{g_0}(0) = -\sum_{i=1}^{+\infty} \ln(\lambda_i) = -\ln\left(\prod_{i=1}^{n} \lambda_i\right).$$

Therefore,

$$e^{-\zeta'_{g_0}(0)} = \prod_{i=1}^{+\infty} \lambda_i = \det\left(-\Delta_{g_0}\right).$$

Thus we can define the determinant of $\Delta_{g_0}$ by the formula :

$$\det\left(-\Delta_{g_0}\right) = e^{-\zeta'_{g_0}(0)}.$$

Now, let $g \in [g_0]$, so there exists $u \in \mathcal{C}^\infty(M)$ such that $g = e^{2u} g_0$. The *Polyakov formula* (see for example [Cha]) says that

$$\ln\left(\frac{\det\left(-\Delta_g\right)}{\det\left(-\Delta_{g_0}\right)}\right) = -\frac{1}{12\pi}\int_M \left(\|\nabla u\|^2 + K_{g_0} u\right) d\mathcal{V}_{g_0}.$$

We can consider the functional

$$F : \begin{cases} \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} \\ u \longmapsto \ln\left(\frac{\det\left(-\Delta_g\right)}{\det\left(-\Delta_{g_0}\right)}\right). \end{cases}$$

Since the determinant is not a scale invariant we consider the *normalized functional determinant*

$$S : \begin{cases} \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} \\ u \longmapsto -12\pi \ln\left(\frac{\det\left(-\Delta_g\right)}{\det\left(-\Delta_{g_0}\right)}\right) + 2\pi\chi(M) \ln\left(\mathrm{Vol}(M, g)\right), \end{cases}$$

where $\chi(M)$ is the Euler characteristic of $M$. The critical functions of $S$ are functions $u$ such that

$$-\Delta_{g_0} u + K_{g_0} = c e^{2u},$$

where $c$ is a real constant. Using the Gaussian curvature equation we see that $u$ is a critical function of $S$ if and only if the surface $M$ with the conformal metric $g := e^{2u} g_0$ has a constant Gaussian curvature $K_g$. In 1998, B. Osgood, R. Phillips, and P. Sarnak [OPS1], [OPS2] proved the following theorem:

**Theorem 7.8.3** (Osgood, Philipps, Sarnak, 1998). *The supremum* $\sup_{u \in \mathcal{C}^\infty(M)} S(u)$ *of the above functional exists and it is a maximum: there exists* $u_0 \in \mathcal{C}^\infty(M)$ *such that Gaussian curvature* $K_g$ *(where* $g := e^{2u} g_0$*) is constant on* $M$.

It is turn out that each surface can be endowed with a metric conformally equivalent to a metric with constant Gaussian curvature. A main consequence is the famous *Poincaré–Klein–Koebe uniformization theorem.*

**Theorem 7.8.4** (Uniformization Theorem). *Let* $(M, g_0)$ *be a closed surface with a Riemannian metric* $g_0$*. Then there exists an unique metric*[7] $g$ *conformal to* $g_0$ *on* $M$ *with a constant Gaussian curvature* $K_g \in \{+1, 0, -1\}$.

The values of the constant $K_g \in \{+1, 0, -1\}$ depends on the topology of the surface $M$. Recall that the Gauss–Bonnet formula for a closed surface reads

$$\int_M K \, d\mathcal{V}_g = 2\pi \chi(M),$$

and for a surface with boundary

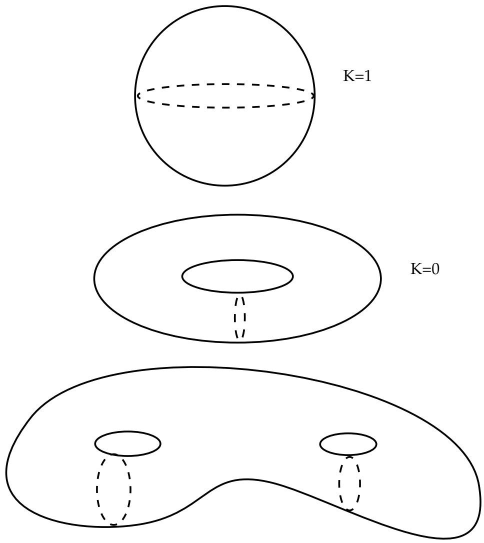
$$\int_M K \, d\mathcal{V}_g + \int_M K_g \, dS_g = 2\pi \chi(M),$$

where $\chi(M)$ is the Euler–Poincaré characteristic of $M$ and $K_g$ is the geodesic curvature on the boundary. According to the Gauss–Bonnet formula, the sign of the curvature is determined by the Euler–Poincaré characteristic[8] $\chi(M)$ of $M$. Therefore, the universal covering of every closed surface can be isometrically embedded in

1. the round sphere $\mathbb{S}^2$ for genus zero with $\chi(M) = 2 > 0$ (in this case $K_g = +1$).
2. the Euclidean plane $\mathbb{R}^2$ for genus one with $\chi(M) = 0$ (in this case $K_g = 0$).
3. the hyperbolic space $\mathbb{H}^2$ for genus $g \geq 2$ with $\chi(M) = -2 < 0$ (in this case $K_g = -1$).

[7] Called a uniformization metric.

[8] For $M$ a closed orientable surface $\chi(M) = 2 - 2g$, where $g$ is the genus of the surface.




### 7.8.3 Ricci flow on surfaces

At the beginning of the 1980's R. Hamilton [Ham] introduced a nice tool for studying topological spaces in low dimensions. This geometric tool is called the *Ricci flow*. In the case of dimension 2 (surface) the Ricci flow on a fixed surface $(M, g_0)$ is given by the non-linear PDE

$$\frac{\partial g}{\partial t} = -2K_{g(t)}g(t),$$

with $g(0) = g_0$.

R. Hamilton also introduced a variant of this equation, that for the so-called *normalized Ricci flow*:

$$\frac{\partial g}{\partial t} = \left(\frac{4\pi\chi(M)}{\mathrm{Vol}(M, g_0)} - 2K_{g(t)}\right) g(t)$$

with $g(0) = g_0$.

This version of Hamilton's surface Ricci flow preserves the total area, moreover, the Ricci flow converges to a metric with constant Gaussian curvature and for all $t$ the metric $g(t)$ is conformal to the initial metric. Indeed, we have

**Theorem 7.8.5** (Hamilton, 1988). *For a closed two-dimensional Riemannian manifold* $(M, g_0)$ *such that* $\chi(M) \leq 0$ *there exists a unique solution* $g(t)$ *of the normalized Ricci flow and this solution converge (in the Whitney topology) to a metric of constant Gaussian curvature and conformal to* $g_0$.

In the case of positive Euler–Poincaré characteristic we have:

**Theorem 7.8.6** (Chow, 1991). *For a closed two-dimensional Riemannian manifold* $(M, g_0)$ *such that* $\chi(M) \geq 0$ *there exists a unique solution* $g(t)$ *of the normalized Ricci flow and this solution converge (in the Whitney topology) to a metric of constant Gaussian curvature and conformal to* $g_0$.

These theorems are fundamental, but they are not a proof of the uniformization theorem using Ricci flow. In fact the arguments for convergence in the proofs of R. Hamilton and B. Chow use the uniformization theorem. However in 2006, X. Chen, P. Lu and G. Tian gave a proof of the uniformization theorem based on Hamilton's surface Ricci flow.

*Remark* 7.8.7. The Ricci flow on surfaces has many applications in computer vision (shape matching, medical imaging, visualization and design in computer games). For an introduction to the Ricci flow on surfaces see Chapter 5 in the book of B. Chow [Ch-Kn]. Nice applications of Ricci flow on surfaces can be also found in the book of X. D Gu and S. T Yau [Gu-Ya].

### 7.8.4 What about three manifolds?

In dimension $n = 2$ the situation is very simple: indeed, by the Uniformization Theorem 7.8.4, every manifold admits a metric of constant sectional curvature. Thus in dimension $n = 2$ every *geometry*[9] is *isotropic*[10]. The situation for dimension 3 (and beyond) is more complex. The geometric classification of three-dimensional manifolds was conjectured by W. Thurston in the end of 70's. *Thurston's Geometrization Conjecture* asserts that every three-dimensional manifold admits a canonical topological decomposition into elementary "geometric" pieces[11]. "Geometric" pieces mean here that every pieces can be endowed with one of the eight homogeneous *Thurston geometries*:

[9] A geometry $X$ is a *homogeneous unimodular* complete and simply-connected Riemannian manifold. Homogeneous means that the isometry group $\mathrm{Isom}(X)$ acts transitively on $X$. Homogeneous means also that the local geometry of the manifold is the same at all points. Unimodular means that the quotient $X/\mathrm{Isom}(X)$ admits a finite volume.

[10] A manifold $X$ is isotropic if the isometry group $\mathrm{Isom}(X)$ acts transitively on the unit tangent bundle. Isotropic means that the local geometry of the manifold is the same in all directions. A geometry $X$ is isotropic if and only if its sectional curvature is constant.

[11] The existence of this decomposition is based on the *Kneser sphere decompostion* and on the *Jaco–Shalen–Johannson torus decomposition.*

1. *Spherical isotropic geometry*, e.g., $\mathbb{S}^3$, $\mathbb{PR}^3$, Poincaré's sphere, lens space ...
2. *Euclidean isotropic geometry*, e.g., Euclidean space $\mathbb{E}^3$, three-torus $\mathbb{T}^3$ ...
3. *Hyperbolic isotropic geometry*, e.g., Seifert–Weber dodecahedral space ...
4. *Trivial product* $\mathbb{S}^2 \times \mathbb{E}^1$ *geometry*, e.g., $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{S}^2 \times I$ (with $I$ an interval) ...
5. *Trivial product* $\mathbb{H}^2 \times \mathbb{E}^1$ *geometry*, e.g., $\mathbb{H}^2 \times \mathbb{S}^1$ ...
6. *Nil geometry.*
7. $\widetilde{SL_2(\mathbb{R})}$ *geometry.*
8. *Sol geometry.*

**Conjecture 7.8.8** (Thurston's Geometrization Conjecture[12]). *The interior of every oriented compact 3-manifold can be split (along 2-spheres and 2-tori) into a finite number of pieces, such that each piece is geometric.*

An important corollary of the Geometrization Conjecture is the so-called *Elliptisation Conjecture*, which asserts that every compact orientable 3-manifold with a finite fundamental group has a Riemannian metric of constant positive sectional curvature:

**Conjecture 7.8.9** (Elliptisation Conjecture). *Every oriented closed 3-manifold with a finite fundamental group is spherical*[13].

In particular the elliptisation conjecture is true for any compact and simply connected three-manifold. A consequence is the famous *Poincaré Conjecture* (1904) [Poi]:

**Conjecture 7.8.10** (Poincaré Conjecture, 1904). *If $M$ is a compact and simply connected three-manifold, then $M$ is homeomorphic to the round sphere $\mathbb{S}^3$.*

In fact, W. Thurston proved the Geometrization Conjecture for a specific class of manifolds[14]. For more details on 3-manifolds and geometrization see for example the notes of W. Thurston [Thu], the book of J. Weeks [Wee] and the book of M. Boileau, S. Maillot and J. Porti [BMP].

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[12] We present here a simplified version of the conjecture; for details see [BBBMP].

[13] A manifold $M$ is called *spherical* if $M$ admits a $\mathbb{S}^n$- geometric model structure, i.e., $M$ is diffeomorphic to the quotient of $\mathbb{S}^n$ by a discrete sub-group acting freely on $\mathbb{S}^n$.

[14] Haken 3-manifolds.

Hamilton's Ricci flow on a fixed 3-manifold $(M, g_0)$ is the one-parameter family of smooth Riemannian metrics on $(M, g_0)$ satisfying the non-linear PDE

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}$$

with $g(0) = g_0$. Note that if we use harmonic local coordinates the Laplace–Beltrami operator appears in the Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = \Delta_{g(t)} g_{ij} + Q_{ij}(g, \partial g),$$

where $Q$ is a quadratic expression.

In fact the Ricci flow is very similar to the heat equation, but here we are not interested in the diffusion of heat, but in the diffusion of curvature. The Ricci flow process is a sort of uniformization process of curvature on the manifold. R. Hamilton proved that every compact 3-manifold with strictly positive Ricci curvature has a Riemannian metric of constant strictly positive sectional curvature. Hence, if the manifold is simply connected, then $M$ is diffeomorphic to the round sphere $\mathbb{S}^3$. Without the Ricci strict positivity hypothesis, the behaviour of the Ricci flow is more complex and singularities appear during the Ricci flow evolution, so the idea of R. Hamilton is to stop the flow, make surgery around the singularities, and restart the flow after the surgery (this is called the *Ricci flow with surgery*). But unfortunately this method requires to control the number of surgeries and understand precisely the nature of these singularities.

In 2002–2003, G. Perelman [Per1], [Per2], [Per3] completely solved this problem. More precisely, he proved that it is possible to make a finite number of surgeries in a finite time interval, and also that if the fundamental group $\pi_1(M)$ is finite (as is the case for the Poincaré conjecture), then the Ricci flow with surgery extincts after a finite time (i.e., the manifold shrinks to a point). In the case where $\pi_1(M)$ is infinite, the Ricci flow is defined on the interval $[0, +\infty[$ and G. Perelman analyzed the behaviour of the Ricci flow with surgery for $t \to +\infty$ (he used a *thin–thick decomposition*). He was then able to prove the Geometrization Conjecture and the Poincaré Conjecture.

For more details on this amazing story using Perelman's arguments see for example the surveys [Bes1], [Bes2], the article by B. Kleiner and J. Lott [Kl-Lo1], the book by J. Morgan and G. Tian [Mo-Ti], the article by H.D. Cao and X.P. Zhu [Ca-Zh] and the book by B. Chow and D. Knopf [Ch-Kn]. There exist alternatives approaches to some of Perelman's arguments, see for example the articles by T. Colding and W. Minicozzi [Co-Mi], T. Shioya and T. Yamaguchi [Sh-Ya], J. Cao and J. Ge [Ca-Ge], B. Kleiner and J. Lott [Kl-Lo2]. See also the recent book by L. Bessières, G. Besson, M. Boileau, S. Maillot and J. Porti [BBBMP]. In that monograph the authors give a complete proof of the Geometrisation Conjecture with an alternative approach to Perelman: they replace Perelman's Ricci flow by a variant called *Ricci flow with bubbling-off*. Perelman's Ricci flow with surgery and the Ricci flow with bubbling-off are similar, but in the case of Ricci flow with bubbling-off the metric evolves on a fixed manifold rather than an evolving manifold. One other major difference is that in the Ricci flow with bubbling-off process the surgery occurs before the flow becomes singular, rather than exactly at the time the singularity is formed (see [BBBMP]).

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