Multivariable Calculus Cheat Sheet
by Druv Pai

Notation
iff: if and only if
f: scalar or scalar-valued function
$\mathbf{x}$: vector or vector-valued function
$\mathbf{a}\times\mathbf{b}, \mathbf{a}\cdot\mathbf{b}$: cross product, dot product
O: origin, point of $\mathbf{0}$
$\partial f/\partial x, f_x$: partial derivative
$\Gamma, \partial\Gamma = \gamma$: orientable manifold, boundary
$\int, \iint, \iiint$: multiple integral
$\oint, \oiint$: closed boundary integral

1 Parametrizations
Parametric functions: $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ that trace out graphs in $\mathbb{R}^m$ as the n parameters vary
In $\mathbb{R}^2$: $y'(x) = y'(t)/x'(t)$, $y''(x) = (y'(t))'/x'(t)$, $\int_a^b y(x)\,dx = \int_\alpha^\beta y(t)x'(t)\,dt$
Arc length: $s(t) = \int_a^b \sqrt{\sum_{k=1}^n (x_i'(t))^2}\,dt$
Surface area of solid of revolution: $S(t) = \int_a^b 2\pi y(t)\,ds(t)$ (or $x(t)$ depending on axis)
Polar coordinates ($\mathbb{R}^2$): $r = \sqrt{x^2+y^2}, \theta = \arctan(y/x), x = r\cos\theta, y = r\sin\theta$
Graphing polar coordinates: method is to find easy points and connect them
Derivative in polar coordinates: $y'(x) = y'(\theta)/x'(\theta) = (r'(\theta)\sin(\theta) + r\cos(\theta))/(r'(\theta)\cos(\theta) - r\sin(\theta))$
Area in polar coordinates: $A = \left(\int_a^b r^2(\theta)\,d\theta\right)/2$, between curves: find intersection points and subtract area integrals
Arc length in polar coordinates: $s(\theta) = \int_{\theta_0}^\theta \sqrt{r^2(\theta) + (r'(\theta)^2)}\,d\theta$
Conic section: (translate/permute for symmetry)
Parabola with focus $(0,p)$ and directerix $y = -p$: $x^2 = 4py$
Ellipse with foci on x axis: $x^2/a^2 + y^2/b^2 = 1$, foci at $(\pm\sqrt{a^2-b^2}, 0)$, vertices at $(\pm a, 0)$
Hyperbola with foci on x-axis: $x^2/a^2 - y^2/b^2 = 1$, foci at $(\pm\sqrt{a^2+b^2},0)$, vertices $(\pm a,0)$, asymptotes $y = \pm(b/a)x$
Polar equation $r = ed/(1 \pm (e\cos(\theta)$ or $e\sin(\theta)))$ is conic with eccentricity e, ellipse if $e<1$, parabola if $e=1$, hyperbola if $e>1$

2 Vectors, Geometry
Distance (metric) between points $\mathbf{x}$ and $\mathbf{y}$: $\overrightarrow{XY} = \|\mathbf{x}-\mathbf{y}\| = \sqrt{\sum_{k=1}^n (x_k-y_k)}$
Dot (inner) product on $\mathbb{R}^n$: $\mathbf{x}\cdot\mathbf{y} = \sum_{k=1}^n x_k y_k = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$ for $\theta = \angle XOY$
Orthogonality: $\mathbf{x},\mathbf{y}$ orthogonal iff $\mathbf{x}\cdot\mathbf{y}=0$
Scalar proj. of $\mathbf{x}$ onto $\mathbf{y}$: $\mathrm{comp}_\mathbf{y}\mathbf{x} = \mathbf{x}\cdot\mathbf{y}/\|\mathbf{y}\|$
Vector proj. of $\mathbf{x}$ onto $\mathbf{y}$: $\mathrm{proj}_\mathbf{y}\mathbf{x} = \mathbf{y}(\mathrm{comp}_\mathbf{y}\mathbf{x})/\|\mathbf{y}\| = \mathbf{y}(\mathbf{x}\cdot\mathbf{y}/\|\mathbf{y}\|^2)$
Cross (outer) product on $\mathbb{R}^3$: $\mathbf{x}\times\mathbf{y} = \det\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_1&x_2&x_3\\y_1&y_2&y_3\end{bmatrix}$, $\|\mathbf{x}\times\mathbf{y}\| = \|x\|\|y\|\sin\theta$
Vector $\mathbf{x}\times\mathbf{y}$ orthogonal to $\mathbf{x},\mathbf{y}$ (use right hand rule); $\mathbf{x},\mathbf{y}$ parallel iff $\mathbf{x}\times\mathbf{y}=\mathbf{0}$
Volume of generated parallelepiped (triple/scalar product): $V = |\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})|$
Equation of line/line segment: $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$, $\mathbf{x}(t) = \mathbf{x}_0 t + \mathbf{x}_1(1-t)$
Vector/scalar equation of hyperplane: $\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0) = 0$, $\sum_{k=1}^n a_k(x_k - x_{k0}) = 0$, $\mathbf{x}_0$ on plane
Distance from point to line: $d(\mathbf{x}, \mathbf{x}_0 + t\mathbf{v}) = \|\mathbf{v}(\mathbf{v}\cdot(\mathbf{x}-\mathbf{x}_0)) - (\mathbf{x}-\mathbf{x}_0)\| = \|(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}\|/\|\mathbf{v}\|$
Distance from point to hyperplane: $d(\mathbf{x}, \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)) = |\mathrm{comp}_\mathbf{n}\mathbf{x}|$
Distance between two parallel planes: pick point on one and compute distance
Distance between two skew lines: find parallel planes for them and compute distance
Quadric surfaces in $\mathbb{R}^3$:
– Sphere: $(x^2+y^2+z^2)/r^2 = 1$ or generally $\|\mathbf{x}-\mathbf{c}\| = r$
– Ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
– Elliptic paraboloid: $z/c = x^2/a^2 + y^2/b^2$
– Hyperbolic paraboloid: $z/c = x^2/a^2 - y^2/b^2$
– Cone: $z^2/c^2 = x^2/a^2 + y^2/b^2$
– Hyperboloid of one sheet: $z^2/c^2 + 1 = x^2/a^2 + y^2/b^2$
– Hyperboloid of two sheets: $z^2/c^2 - 1 = x^2/a^2 + y^2/b^2$

3 Vector Functions
Vector limits: $\lim_{t\to a}\mathbf{x}(t) = (\{\lim_{t\to a} x_k(t)\})$
Vector derivatives (by limit definition): $\mathbf{x}'(t) = (\{x_k'(t)\})$
Derivative rules:
– $(\mathbf{x}(t)+\mathbf{y}(t))' = \mathbf{x}'(t)+\mathbf{y}'(t)$
– $(c\mathbf{x}(t))' = c\mathbf{x}'(t)$
– $(f(t)\mathbf{x}(t))' = f'(t)\mathbf{x}(t) + f(t)\mathbf{x}'(t)$
– $(\mathbf{x}(t)\cdot\mathbf{y}(t))' = \mathbf{x}'(t)\cdot\mathbf{y}(t) + \mathbf{x}(t)\cdot\mathbf{y}'(t)$
– $(\mathbf{x}(t)\times\mathbf{y}(t))' = \mathbf{x}'(t)\times\mathbf{y}(t)+\mathbf{x}(t)\times\mathbf{y}(t)$
– $(\mathbf{u}(f(t)))' = f'(t)\mathbf{u}'(f(t))$
Vector integration (by limit definition): $\int\mathbf{x}(t)\,dt = (\{\int x(t)\,dt\})$
Vector definite integration: $\int_a^b \mathbf{x}(t)\,dt = (\{\int_a^b x_k(t)\,dt\})$
Vector FTC: $\int_a^b \mathbf{x}(t)\,dt = \mathbf{X}(b) - \mathbf{X}(a)$ for $\mathbf{X}(t) = \int\mathbf{x}(t)\,dt$
Arc length: $s(t) = \int_{t_0}^t \|\mathbf{x}'(t)\|\,dt$
Parameterizing in terms of arc length: writing $t(s)$ from $s(t)$ when convenient is good for describing curves independently of parameter
Tangent (unit) vector: $\mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$
Curvature: $\kappa(t) = \|\mathbf{T}'(s(t))\| = \|\mathbf{T}'(t)\|/\|\mathbf{x}'(t)\| = \|\mathbf{x}'(t)\times\mathbf{x}''(t)\|/\|\mathbf{x}'(t)\|^3$
Unit normal vector $\mathbf{n}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$; binormal vector $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{n}(t)$ (in $\mathbb{R}^3$)
Osculating plane: plane containing $\mathbf{n},\mathbf{B}$; osculating circle: circle on that plane with radius $1/\kappa$ touching point where curve has curvature κ
Acceleration formula: $\mathbf{x}''(t) = \|\mathbf{x}'(t)\|'\mathbf{T}(t) + \kappa(t)\|\mathbf{x}'(t)\|^2\mathbf{n}(t)$

4 Partial Derivatives
Level sets: sets of points $\mathbf{x}$ such that a function $f(\mathbf{x}) = k$ for constant k, can graph these to help graph 3d functions
Multivariable limits: $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{F}(\mathbf{x}) = \mathbf{L}$ iff limit holds along every path to $\mathbf{a}$; else DNE
Computing limits: try Squeeze Theorem, rational/polynomials continuous on domain; show limits DNE by picking good paths
Partial derivatives: $\partial f(\mathbf{x})/\partial x_i = f_{x_i} = (\lim_{\varepsilon\to0}(f(x_1,\dots,x_i+\varepsilon,\dots,x_n) - f(\mathbf{x}))/\varepsilon$
Computation of partials: differentiate holding everything but indicated variable constant
Higher partial derivatives: iteratively take partial derivatives w.r.t the indicated variables, ordered right to left in notation
Jacobian matrix: $J[\mathbf{F}(\mathbf{x})] = [\partial f_i/\partial x_j]$
Hessian tensor: $H[\mathbf{F}(\mathbf{x})] = [\partial^2 f_i/\partial x_j\partial x_k]$
Clairaut's theorem: f defined and $f_{x_ix_j}, f_{x_jx_i}$ continuous on open ball, then $f_{x_ix_j} = f_{x_jx_i}$
Differentiability: all partial derivatives of f exist/are continuous near $\mathbf{a}\to f$ differentiable at $\mathbf{a}$; alternatively Δf can be expressed in the form $\Delta f(\mathbf{x}) = (\sum_{k=1}^n (f_{x_k}(\mathbf{x}) + \varepsilon_k)\Delta x_i)$
Total differential: for function $f(\mathbf{x})$, total differential $\mathrm{d}f(\mathbf{x}) = \sum_{k=1}^n f_{x_k}(\mathbf{x})\,\mathrm{d}x_k$
Chain rule: $f(\mathbf{x}(\mathbf{t}))$ differentiable function of $\mathbf{x}(\mathbf{t})$: $f_{t_i}(\mathbf{x}(\mathbf{t})) = \sum_{k=1}^n (\partial f/\partial x_k)(\partial x_k/\partial t_i)$
Implicit differentiation: differentiate $F(\mathbf{x}) = 0$ and solve; $n=2 \to y'(x) = -F_x/F_y$
Implicit function theorem: if $F(\mathbf{x})$ defined on ball D with continuous partial derivatives, can write $x_k = f(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)$
Gradient operator: $\nabla f(\mathbf{x}) = \sum_{k=1}^n f_{x_k}(\mathbf{x})\mathbf{e}_k$, points towards fastest growth (rate $\|\nabla f\|$)
Directional derivative: $D_\mathbf{u} f(\mathbf{x}) = \lim_{\varepsilon\to0}(f(\mathbf{x}+\varepsilon\mathbf{u}) - f(\mathbf{x}))/\varepsilon = \nabla f(\mathbf{x})\cdot\mathbf{u}$
Tangent plane to f at $\mathbf{x}_0$: $\nabla f(\mathbf{x})\cdot(\mathbf{x}-\mathbf{x}_0) = 0$
Local max. of f at $\mathbf{x}$: exists neighborhood of $\mathbf{x}$ s.t. $f(\mathbf{x}) \ge f(\mathbf{a})$ for all $\mathbf{a}$ in neighborhood
Local min. of f at $\mathbf{x}$: opposite of local max.
Critical point: point $\mathbf{a}$ such that $\nabla f(\mathbf{a}) = 0$
Optima: at local min./max. $\mathbf{a}$, continuous first partial derivatives implies $\mathbf{a}$ critical point (but not all critical points are optima)
Absolute max./min.: local max./min. of f over the whole domain
Second derivative test: $f(\mathbf{a})$ critical point, then $H[f(\mathbf{a})]$ positive definite $\to$ **a** local min., negative definite $\to$ **a** local max., indefinite $\to$ **a** saddle point, semidefinite $\to$ no info.
Extreme value theorem: if f continous on closed ball B, then f absolute max. $f(\mathbf{x}_1)$ at some point $\mathbf{x}_1$ and absolute min. $f(\mathbf{x}_2)$ at some other point $\mathbf{x}_2$, for $\mathbf{x}_1,\mathbf{x}_2\in B$
Finding absolute optima: find optima inside the set using SDT, find optima on boundaries, compare to find absolute optima
Lagrange multipliers: optimizing function $f(\mathbf{x})$ w.r.t. n nonzero constraints $g_1(\mathbf{x}),\dots,g_n(\mathbf{x})$– find and compare all values $\mathbf{a}$ s.t. $\nabla f(\mathbf{a}) = \sum_{k=1}^n \lambda_k\nabla g_k(\mathbf{a})$

5 Multiple Integrals
Double integral: integral of f over region R - $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^n\sum_{j=1}^n f(x_i,y_j)\Delta A$
Midpoint rule: $\iint_R f(x,y)\,dA \approx \sum_{i=1}^m\sum_{j=1}^n f(\bar{x}_i,\bar{y}_j)\Delta A$
Iterated integral: integrate from the innermost integral to the outer integrals, only variable used in each step is the one used in integration
Fubini's theorem: f continuous on rectangle $abcd \to \int_a^b\int_c^d f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dA$
Factorization: $\int_a^b\int_c^d g(x)h(y)\,dx\,dy = (\int_c^d g(x)\,dx)(\int_a^b h(y)\,dy)$
General regions: f continuous on domain $a<x<b, k_1(x)<y<k_2(x) \to \iint_D f(x,y)\,dA = \int_a^b\int_{k_1(x)}^{k_2(x)} f(x,y)\,dA$ (same for x)
Union of regions: integral over union of regions is sum of integrals of regions
Area: $\iint_D dA = A(D)$
Change of coordinates in $\mathbb{R}^2$: $\iint_D f(\mathbf{x})\,dA = \iint_{\mathbf{t}(D)} f(\mathbf{x}(\mathbf{t}))|\det(J[\mathbf{x}[\mathbf{t}]])|\,dt_1\,dt_2$
Mass: $m = \int_D \rho(\mathbf{x})\bigwedge_{k=1}^n dx_k$
Moment: $M_{x_i} = \int_D x_i\rho(\mathbf{x})\bigwedge_{k=1}^n dx_k$
Center of mass: $\bar{x_i} = M_{x_i}/m$
Inertia: $I_{x_i} = \int_D\|\mathbf{x}\|^2\rho(\mathbf{x})\bigwedge_{k=1}^n dx_k$
Surface given by $f(\mathbf{x})$: $A(S) = \iint_D\sqrt{1+\sum_{k=1}^n (f_{x_k})^2}\,dA$
Triple integral over box B: $\iiint_B f(\mathbf{x})\,dV = \lim_{p,q,r\to\infty}\sum_{i=1}^p\sum_{j=1}^q\sum_{k=1}^r f(x_i,y_j,z_k)$ (extends to arbitrarily many dimensions)
Iteration: f continuous on box $abcdef \to \iiint_B f(\mathbf{x})\,dV = \int_e^f\int_c^d\int_a^b f(\mathbf{x})\,dx\,dy\,dz$
General regions: same procedure as before, but inner two integrals can be functions of the not-integrated-yet variables
Volume: $\iiint_E dV = V(E)$
General change of coordinates procedure ($\mathbb{R}^n$): $\int_D f(\mathbf{x})\bigwedge_{k=1}^n dx_k = \int_{\mathbf{t}(D)} f(\mathbf{x}(\mathbf{t}))|\det(J[\mathbf{x}(\mathbf{t})])|\bigwedge_{k=1}^n dt_k$
Polar/cylindrical Jacobian determinant: r, spherical Jacobian determinant: $\rho^2\sin(\phi)$

6 Vector Calculus
Vector field: function $\mathbf{F}\colon\mathbb{R}^n\to\mathbb{R}^n$
Gradient vector field: vector field $\mathbf{F}$ s.t. there exists f s.t. $\mathbf{F} = \nabla f$, f is potential function for $\mathbf{F}$, $\mathbf{F}$ is conservative
Line integral: f defined on C, then line integral along C is $\int_C f(\mathbf{x})\,d\mathbf{x} = \int_{t_0}^{t_1} f(\mathbf{C}(t))\,dt$
Vector line integral: $\mathbf{F}$ defined on C, then line integral along C is $\int_C \mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \int_{t_0}^{t_1}\mathbf{F}(\mathbf{C}(t))\cdot\mathbf{C}'(t)\,dt$
Fundamental theorem of line integrals: C smooth, f differentiable, then $\int_C\nabla f\cdot d\mathbf{x} = f(\mathbf{C}(t_1)) - f(\mathbf{C}(t_0))$
Path independence of function f: $\int_{C_1} f(\mathbf{x})\,d\mathbf{x} = \int_{C_2} f(\mathbf{x})\,d\mathbf{x}$ for any two paths C_1, C_2 with same endpoints
Line integrals of conservative vector fields are independent of path
$\int_C\mathbf{F}\cdot d\mathbf{x}$ is independent of path in D iff $\oint\mathbf{F}\cdot d\mathbf{x} = 0$ for all closed paths C
If $\mathbf{F}$ continuous on open simply connected region D, and $\int_C\mathbf{F}\cdot d\mathbf{x}$ path independent, then $\mathbf{F}$ conservative
If $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$, then $f(\mathbf{x}) = \int\mathbf{F}(\mathbf{x})\cdot d\mathbf{x}$
Divergence: $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$ - acts like "dot product" of gradient operator with $\mathbf{F}$
Curl ($\mathbb{R}^2$ or $\mathbb{R}^3$): $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}$ - acts like "cross product" of gradient operator with $\mathbf{F}$ in $\mathbb{R}^3$, the scalar $\mathbf{k}$-component of this cross product in $\mathbb{R}^2$
Laplacian: $\nabla\cdot\nabla f$, abbreviated as $\nabla^2 f$
Identities: $\nabla\times(\nabla f(\mathbf{x})) = 0, \nabla\cdot(\nabla\times\mathbf{F}) = 0$
Surface area of parameterization: $A(S) = \iint_D\|\mathbf{S}_{t_1}\times\mathbf{S}_{t_2}\|\,dA$
Unit normal vector of parameterization: $\mathbf{n} = (\mathbf{S}_{t_1}\times\mathbf{S}_{t_2})/\|\mathbf{S}_{t_1}\times\mathbf{S}_{t_2}\|$
Scalar surface integral: $\iint_{\mathbf{S}(D)} f(\mathbf{x})\,dS = \iint_D f(\mathbf{S}(\mathbf{t}))\|\mathbf{S}_{t_1}\times\mathbf{S}_{t_2}\|\,dA$
Vector surface integral: $\iint_{\mathbf{S}(D)}\mathbf{F}(\mathbf{x})\cdot d\mathbf{S} = \iint_{\mathbf{S}(D)}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D\mathbf{F}(\mathbf{x})\cdot(\mathbf{S}_{t_1}\times\mathbf{S}_{t_2})\,dA$
Orientation: curve/surface orientable if there exists a continuous mapping from point to normal vector
Positive orientation: for curve, counterclockwise; for surface, normal vector outwards
Green's theorem: D region in $\mathbb{R}^2$; $C = \partial D$ positively oriented, piecewise-smooth, closed; $\mathbf{F}$ continuous on $D \to \oint_C\mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \iint_D\nabla\times\mathbf{F}\,dA$
Stokes' theorem: S piecewise-smooth surface, $C = \partial S$ simple, closed piecewise-smooth boundary with positive orientation, $\mathbf{F}$ has continuous partial derivatives, then $\oint_C\mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}$
Divergence theorem: E solid region, $S = \partial E$ with positive orientation, $\mathbf{F}$ has continous partial derivatives, then $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iiint_E\nabla\cdot\mathbf{F}\,dV$