Linear Algebra & Differential Equations Cheat Sheet
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1 Vector Spaces

Vector space V: set with commutative, distributive, associative addition, scalar multiplication, additive/multiplicative identity

Vector subspace W: subset of vector space V with zero vector, closed under addition and scalar multiplication

Linear independence: $\sum_{k=1}^{n} a_k \mathbf{v}_k = 0 \to a_k = 0$; form matrix of $\mathbf{v}_k$ and show $\mathsf{N}(A) = \{\mathbf{0}\}$, for $A \in \mathsf{M}_{m\times n}$ (here and after)

Linear dependence: $\sum_{k=1}^{n} a_k \mathbf{v}_k = 0$ for a_k not all zero

$\mathsf{S}(A)$: set of linear combinations of the columns of A.

$\mathsf{N}(A)$: span of solutions of $A\mathbf{x} = \mathbf{0}$. Row reduce A.

$\mathsf{R}(A)$: space spanned by the rows of A. Row-reduce A and choose the rows that contain the pivots.

$\mathsf{C}(A)$: space spanned by the columns of A. Row-reduce A and choose the columns of A that contain the pivots.

$\mathsf{LN}(A)$: span of solutions of $\mathbf{x}^\mathsf{T} A = \mathbf{0}$. Find $\mathsf{N}(A^\mathsf{T})$.

Linearity of transform: $\mathscr{T}(\sum_{k=1}^{n} c_k \mathbf{v}_k) = \sum_{k=1}^{n} c_k \mathscr{T}(\mathbf{v}_k)$

Matrix of linear transform: $[\mathscr{T}(\mathbf{e}_1) \quad \ldots \quad \mathscr{T}(\mathbf{e}_n)]$

One-to-one: $\mathscr{T}(\mathbf{u}) = \mathbf{0} \to \mathbf{u} = \mathbf{0}$, onto: $\mathrm{Ran}(\mathscr{T}(\mathbf{x})) = \mathsf{R}^m$

Basis vectors: linearly independent set $\mathcal{B}$ s.t. $\mathsf{S}(\mathcal{B}) = \mathsf{V}$. To show something is a basis, show linear independence and full span.

Basis from set: find column space of matrix formed from the set.

Form a basis from a vector space: write any element of a vector space as a linear combination of a spanning set.

Dimension: number of elements in a basis.

$\mathrm{rank}(A) = \dim(\mathsf{C}(A))$ = number of pivots, $\mathrm{nullity}(A) = \dim(\mathsf{N}(A))$, **Rank-Nullity Theorem**: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

Theorem: if one basis of V has n vectors, then so does every basis.

Basis Theorem: if $\dim(\mathsf{V}) = n$, any linearly independent set has $\leq n$ vectors, any spanning set has $\geq n$ vectors, where equality occurs iff the set is a basis.

Basis isomorphism: for each vector in $\mathcal{B}$, write it as a linear combination of the vectors of $\mathcal{C}$. Put the weights in a vector as coordinates; put all the vectors as columns of a matrix to get $P_{\mathcal{C}\leftarrow\mathcal{B}}$ where $[\mathbf{x}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_\mathcal{B}$. A special case is where $\mathcal{C} = \mathcal{E}$ (standard basis) and $P_{\mathcal{E}\leftarrow\mathcal{B}} = P_\mathcal{B}$.

Matrix of linear transform in basis isomorphism: take $[\mathscr{T}(\mathbf{b}_k)]_\mathcal{C}$ instead of $[\mathbf{b}_k]_\mathcal{C}$ in the intermediate step of above process.

Change of basis: $[\mathbf{x}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_\mathcal{B} \to [\mathcal{C} \quad \mathcal{B}] \sim [I \quad P_{\mathcal{C}\leftarrow\mathcal{B}}]$

Invertible matrix theorem and extensions:

The solution (is)...	Exists	Unique
Pivot in every row	x	
Pivot in every column		x
$\mathsf{S}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \mathsf{R}^m$	x	
$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ linearly independent		x
$A\mathbf{x} = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$		x
$\mathsf{C}(A) = \mathsf{R}^m$	x	
$\mathsf{N}(A) = \mathbf{0}$		x
$\dim(\mathsf{C}(A)) = m$	x	
$\dim(\mathsf{N}(A)) = 0$		x
$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ basis for R^m $(m = n)$	x	x
$\mathrm{Ran}(\mathscr{T}(\mathbf{x})) = \mathsf{R}^m$	x	
$\mathrm{Ker}(\mathscr{T}(\mathbf{x})) = \{\mathbf{0}\}$		x
$\mathscr{T}(\mathbf{x})$ is onto (surjective)	x	
$\mathscr{T}(\mathbf{x})$ is one-to-one (injective)		x
$\mathsf{C}(A^\mathsf{T}) = \mathsf{R}^n$		x
$\mathsf{N}(A^\mathsf{T}) = \{\mathbf{0}\}$	x	
A^{-1} exists $(m = n)$	x	x
A is row equivalent to I $(m = n)$	x	x
A^T is invertible $(m = n)$	x	x
$CA = AC = I$ where C exists	x	x
$\det(A) \neq 0$ $(m = n)$	x	x
$\det(A^\mathsf{T}) \neq 0$ $(m = n)$	x	x
$\mathsf{R}(A) = \mathsf{R}^n$	x	
No zero eigenvectors of A $(m = n)$	x	x

Determinants: through cofactors - fix i or j (WLOG i), then $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{j=1}^{n} a_{ij} C_{ij}$; matrix of cofactors $C = [C_{ij}]$

Matrix inverse: General: $[A \quad I] \sim [I \quad A^{-1}] = \frac{1}{\det(A)} C^\mathsf{T}$, $A \in \mathsf{M}_{2\times 2}$: $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

2 Diagonalization

Characteristic polynomial: $\chi_A(\lambda) = \det(A - \lambda I)$; provides eigenvalues as solutions.

Eigenvalues, eigenvectors: $\mathbf{v}, \lambda$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Find by solving $\chi_A(\lambda) = 0$ and for each λ solving $\mathsf{N}(A - \lambda I)$

Multiplicity: algebraic multiplicity – number of times $(\lambda - \lambda_k)$ divides evenly $\chi_A(\lambda)$; geometric multiplicity – number of linearly independent eigenvectors associated with λ

Eigenspaces: E_{λ_k} – span of eigenvectors associated with λ_k, $\mathsf{E}_{\lambda_k}(A) = \mathsf{N}(A - \lambda_k I)$.

Generalized eigensystem of rank m: $\mathbf{v}, \lambda$ such that $(A - \lambda I)^m \mathbf{v} = \mathbf{0}$. Find by solving $(A - \lambda I)\mathbf{v} = \mathbf{x}$ for $\mathbf{x}$ a generalized eigenvector of rank $m - 1$.

Generalized eigenspaces: $\mathsf{E}^g_{\lambda_k}$ – span of generalized eigenvectors associated with λ_k, $\mathsf{E}^g_{\lambda_k}(A) = \bigcup_{m\in\mathbb{N}} \mathsf{N}((A - \lambda_k I)^m)$.

Jordan normal form: $J = \mathrm{diag}(J_1, \ldots, J_n)$ where $J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}$ is $m \times m$, where m is the algebraic multiplicity of λ_k with regards to the polynomial $\det(A - \lambda I)$.

Diagonalizablility: A is diagonalizable if $A = PDP^{-1}$ for some diagonal D and invertible P. We construct P, D by making the kth column of each the kth eigenvector/eigenvalue respectively.

Similarity: $A \sim B$ if $A = PBP^{-1}$ for some invertible P.

Theorem: A is diagonalizable iff A has n linearly independent eigenvectors.

Theorem: If A has n distinct eigenvalues, then A is diagonalizable.

Matrices can be *diagonalizable* but not *invertible*, and some matrices aren't diagonalizable.

Complex eigenvalues: If for some diagonalizable matrix A and integer k, $\lambda_k = a + bi$ (which has a conjugate pair), and $\mathbf{v}_k$ is its eigenvector, then $A = PDP^{-1}$, where $P = [\mathbf{v}_1 \quad \ldots \quad \mathrm{Re}(\mathbf{v}_k) \quad \mathrm{Im}(\mathbf{v}_k) \quad \ldots \quad \mathbf{v}_n]$, $D = \mathrm{diag}(\lambda_1, \ldots, C, \ldots, \lambda_n)$, $C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (note that we thus skip the conjugate pair of the eigenvalue/eigenvector entirely).

3 Orthogonality

Orthogonality: two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if $\langle \mathbf{u} | \mathbf{v} \rangle = 0$, the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is orthogonal if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$.

Orthonormality: the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is orthonormal if it is orthogonal and $\|\mathbf{u}_k\| = \sqrt{\langle \mathbf{u}_k | \mathbf{u}_k \rangle} = 0$ (**norm**) for all k.

Orthogonal subspace $\mathsf{W}^\perp$: set of $\mathbf{v}$ which are orthogonal to every $\mathbf{w} \in \mathsf{W}$. To find orthogonal complement, find a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of W, and put the vector $\mathbf{w}_k^\mathsf{T}$ as the kth row of A. Then $\mathsf{W}^\perp = \mathsf{N}(A)$.

Fundamental spaces: $(\mathsf{R}(A))^\perp = \mathsf{N}(A) = \mathsf{C}(A^\mathsf{T})$; $(\mathsf{C}(A))^\perp = \mathsf{LN}(A) = \mathsf{N}(A^\mathsf{T})$.

Coordinates in an orthogonal basis: if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis, then $\mathbf{y} = \sum_{k=1}^{n} c_k \mathbf{u}_k \to c_k = \frac{\langle \mathbf{y} | \mathbf{u}_k \rangle}{\langle \mathbf{u}_k | \mathbf{u}_k \rangle}$.

Orthogonal projection: if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for W, then the orthogonal projection of $\mathbf{y}$ on W is $\hat{\mathbf{y}} = \sum_{k=1}^{n} \frac{\langle \mathbf{y} | \mathbf{u}_k \rangle}{\langle \mathbf{u}_k | \mathbf{u}_k \rangle} \mathbf{u}_k$; thus, $\mathrm{proj}_{\mathsf{C}(A)} \mathbf{v} = A(A^\mathsf{T} A)^{-1} A^\mathsf{T} \mathbf{v}$.

Orthogonal matrix: a matrix (say, Q) with orthonormal columns; fulfills: $Q^\mathsf{T} Q = I$, QQ^T is the orthogonal projection matrix on $\mathsf{C}(Q)$, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, and $\langle Q\mathbf{x} | Q\mathbf{y} \rangle = \langle \mathbf{x} | \mathbf{y} \rangle$

Gram-Schmidt: start with $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Then for $1 \leq j \leq n$ we have $\mathbf{v}_j = \mathbf{u}_j - \sum_{k=1}^{j-1} \frac{\langle \mathbf{u}_j | \mathbf{v}_k \rangle}{\langle \mathbf{v}_k | \mathbf{v}_k \rangle} \mathbf{v}_k$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathsf{S}(\mathcal{B})$, and if $\mathbf{w}_k = \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|}$, then $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathsf{S}\{\mathcal{B}\}$.

QR-factorization: the columns $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ of Q are orthonormal basis for $\mathsf{C}(A)$ (find using Gram-Schmidt). Then $R = Q^\mathsf{T} A$; alternatively, $R_{ij} = \langle \mathbf{q}_i | \mathbf{a}_j \rangle$ for $i \leq j$, and 0 otherwise.

Least-squares solution: to solve an inconsistent system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense, solve it for $A^\mathsf{T} A\hat{\mathbf{x}} = A^\mathsf{T}\mathbf{b}$. It is guaranteed that $\hat{\mathbf{x}} = \mathrm{argmin}_\mathbf{x} \|A\mathbf{x} - \mathbf{b}\|$. Note that $\hat{\mathbf{x}}$ isn't in $\mathsf{C}(A)$ or $\mathsf{N}(A)$, but $A^\mathsf{T}\mathbf{b} \in \mathsf{C}(A^\mathsf{T} A)$. Alternatively, if $A = QR$, then $\hat{\mathbf{x}} = R^{-1} Q^\mathsf{T} \mathbf{b}$.

Inner product spaces: vector spaces with arbitrary inner products. All the techniques still work, except possibly with linear transformations instead of matrices.

Inequalities: Cauchy-Schwarz – $\langle \mathbf{u} | \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$; triangle – $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

4 Symmetric Matrices

Symmetric matrix: $A = A^\mathsf{T}$, has n real eigenvalues, always diagonalizable, orthogonally diagonalizable ($A = PDP^{-1}$, where P is an orthogonal matrix.) The same properties hold for Hermitian matrices over the complex numbers. (That is, $A = A^* = \overline{A^\mathsf{T}}$)

Theorem: if A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Orthogonal diagonalization: first diagonalize, then apply Gram-Schmidt on each eigenspace and normalize the orthogonal eigenvectors. Then P is the matrix of orthonormal eigenvectors, and D is the diagonal matrix of eigenvalues, and $A = PDP^\mathsf{T}$.

Quadratic forms: to find the matrix Q, put the x_i^2-coefficients on the diagonal, and evenly distribute the other terms between Q_{ij} and Q_{ji}. Then orthogonally diagonalize $Q = PDP^\mathsf{T}$. Then let $\mathbf{y} = P^\mathsf{T}\mathbf{x}$ (which is just coordinate isomorphism), then the quadratic form becomes $Q\mathbf{y} = \sum_{k=1}^{n} \lambda_k y_k^2$, where $\lambda_k = D_{kk}$ (the kth eigenvalue in the given diagonalization).

Theorem: if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of R^n consisting of eigenvectors of $A^\mathsf{T} A$ in order of the corresponding eigenvalues, and A has r nonzero singular values, then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is an orthogonal basis for $\mathsf{C}(A)$, and $\mathrm{rank}(A) = r$.

Singular values: positive square roots of the eigenvalues of $A^\mathsf{T} A$.

Singular Value Decomposition: $A = U\Sigma V^\mathsf{T}$, where A has rank r, Σ is an $m \times n$ matrix where $\Sigma_{kk} = \sigma_k$ for $1 \leq k \leq r$ and $\sigma_1 \geq \cdots \geq \sigma_n$ (for σ_k singular values) and zero elsewhere, and U is $m \times m$ orthogonal, and V is $n \times n$ orthogonal.

Bases: let $A = U\Sigma V^\mathsf{T}$ be an SVD, then orthonormal bases for – $\mathsf{C}(A)$: $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$; $\mathsf{LN}(A)$: $\{\mathbf{u}_{r+1}^\mathsf{T}, \ldots, \mathbf{u}_m^\mathsf{T}\}$; $\mathsf{R}(A)$: $\{\mathbf{v}_1^\mathsf{T}, \ldots, \mathbf{v}_r^\mathsf{T}\}$; $\mathsf{N}(A)$: $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$.

Computing SVD: find an orthogonal diagonalization of $A^\mathsf{T} A$. Arrange the eigenvalues of $A^\mathsf{T} A$ in decreasing order. Construct V as the set of corresponding unit eigenvectors $V = [\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_n]$. Take the square roots of the r nonzero singular values (in order) $\sigma_1, \ldots, \sigma_r$ and place them in Σ as the first r elements on the diagonal. The first r columns of U are the normalized vectors obtained from $A\mathbf{v}_k$; that is, $\mathbf{u}_k = \frac{1}{\sigma_k} A\mathbf{v}_k$ for $1 \leq k \leq r$. The rest are obtained by extending $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to a basis of R^m and performing Gram-Schmidt with normalizations to get $U = [\mathbf{u}_1 \quad \ldots \quad \mathbf{u}_m]$.

Spectral decomposition: $A = \sum_{k=1}^{n} \lambda_k \mathbf{u}_k \mathbf{u}_k^\mathsf{T}$, for $\mathbf{u}_k$ the kth unit eigenvector of A associated with eigenvalue λ_k.

5 Linear Ordinary Differential Equations

Principle of superposition: $y(t) = y_p(t) + y_h(t)$

Homogeneous solutions: auxiliary equation – replace equation by polynomial – $\sum_{k=0}^{n} a_k y^{(k)} \to \sum_{k=0}^{n} a_k y^k$. Solve polynomial. Simple zeros (multiplicity 1) yield linearly independent eigenfunctions $e^{\lambda t}$ for λ a zero. Multiplicity m zeros give $\sum_{k=0}^{m-1} c_k t^k e^{\lambda t}$ for λ the zero to maintain linear independence of eigenfunctions. Complex zeros $\lambda = a + bi$ give $c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$. Summing these where applicable gets $y_h(t)$.

Undetermined coefficients: if the inhomogeneous term is $Ct^m e^{rt}$, then $y_p(t) = t^s e^{rt} \sum_{k=0}^{m} a_k t^k$, where $s = m$ if r is a root of the auxiliary polynomial with multiplicity m, and 0 otherwise. If the inhomogeneous term is $Ct^m e^{\alpha t}\sin(\beta t)$ or $Ct^m e^{\alpha t}\cos(\beta t)$, then $y_p(t) = t^s e^{\alpha t}(\sin(\beta t)(\sum_{k=0}^{m} a_k t^k) + \cos(\beta t)(\sum_{k=0}^{m} b_k t^k))$, where $s = m$ if $\alpha + \beta i$ is a root of the auxiliary polynomial with multiplicity m, and 0 otherwise.

Wronskian: $W[y_1, \ldots, y_n] = \begin{bmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$

Variation of parameters: suppose $y_p(t) = \sum_{k=1}^{n} v_k(t) y_k(t)$, where each of the y_k are homogeneous solutions. Then $W[y_1, \ldots, y_n] \begin{bmatrix} v_1' \\ \vdots \\ v_{n-1}' \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$. Invert the Wronskian and solve for the v_k' (maybe by Cramer's rule), integrate to get the v_k, and finally use $y_p(t) = \sum_{k=1}^{n} v_k(t) y_k(t)$.

Linear independence: $y_1, \ldots, y_n$ are linearly independent if $\sum_{k=1}^{n} a_k(t) y_k(t) = 0 \to a_k = 0$ for all k. To show linear dependence, find coefficients directly. To show linear independence, form the Wronskian, pick t_0 such that $\det(W[y_1, \ldots, y_n](t_0))$ is easy to evaluate, and compute it. If the determinant is nonzero, then the functions are linearly independent.

Fundamental solution set: $y_1, \ldots, y_n$ are linearly independent solutions, then they form a fundamental solution set

Largest interval of existence: for each term in the differential equation, look at the domain and the part containing the initial condition, then intersect all valid regions and make the set open.

Reduction of order: for a differential equation $\sum_{k=0}^{n} a_k y^{(k)}(t) = f(t)$, we reduce it to a first order vector equation $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t)$, where $\mathbf{y} = (y, y', \ldots, y^{(n-1)})$ so $\mathbf{y}' = (y', y'', \ldots, y^{(n)})$, $\mathbf{f}(t) = (0, \ldots, 0, f(t))$, and $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$.

The auxiliary polynomial for the differential equation is the characteristic polynomial of the matrix A. Using other methods yields particular solutions.

6 System of Differential Equations

Homogeneous solution: $\mathbf{x}' = A\mathbf{x}$ solved by $\mathbf{x}_h(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t}\mathbf{v}_k$, where λ_k are eigenvalues of A corresponding to $\mathbf{v}_k$

Generalized eigenvectors: if we don't have enough eigenvectors for an eigenvalue λ_k, find the generalized eigensystem $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ where subscript indicates rank. Then $\mathbf{y}_p(t) = \sum_{i=1}^{m} c_i e^{\lambda_k t}\left(\sum_{j=1}^{i} t^{i-j+1}\mathbf{v}_j\right)$ (Ex: $Ae^{\lambda t}\mathbf{v}_1 + B(te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)$).

Complex eigenvalues: if $\lambda_k = \alpha + i\beta$ and $\mathbf{v}_k = \mathbf{a} + i\mathbf{b}$, then the corresponding term in the homogeneous solution is $e^{\alpha t}(c_1(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}) + c_2(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}))$.

Undetermined coefficients: single-variable undetermined coefficients, except coefficients are n-dimensional vectors; plug into the differential equation and solve for coefficients.

Fundamental solution matrix $X(t)$: matrix of homogeneous solutions as column vectors; set coefficients to 1 for simplicity. $y_h(t) = X(t)\mathbf{c}$ for $\mathbf{c}$ a constant vector.

Variation of parameters: $\mathbf{y}_p(t) = X(t)\int X^{-1}(t)\mathbf{f}(t)\,dt$.

Matrix exponential: $e^{At} = \sum_{k=0}^{n} \frac{A^n t^n}{k!}$. Diagonalization: $A = PDP^{-1} \to e^{At} = Pe^{Dt}P^{-1}$, where e^{Dt} is a diagonal (or Jordan) matrix with diagonal entries $e^{\lambda_k t}$. Note that $e^{At} = X(t)X(0)^{-1}$.

Jordan matrices are $J_i = e^{\lambda_i t}\sum_{k=0}^{m-1} \frac{(A-\lambda I)^k t^k}{k!}$ for m the algebraic multiplicity of λ. Solution of $\mathbf{x}' = A\mathbf{x}$ is then $\mathbf{x}(t) = e^{At}\mathbf{c}$ for $\mathbf{c}$ a constant vector.

7 Fourier Series

Fourier Series: f defined on $(-L, L)$ – $f(x) \sim \frac{f(x^+)+f(x^-)}{2} = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(\frac{\pi kx}{L}) + b_k\sin(\frac{\pi kx}{L}))$, $a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi kx}{L})\,dx$, $b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi kx}{L})\,dx$, $\frac{a_0}{2}$ is the average value of f on $(-L, L)$, and $\lim_{\epsilon\to 0^\times} f(x + \epsilon) = f(x^\times)$.

Cosine/Sine Series: f defined on $(0, L)$, g either cosine or sine – $f(x) \sim \sum_{k=0}^{\infty} a_k g(\frac{\pi kx}{L})$, $a_k = \frac{2}{L}\int_0^T f(x) g(\frac{\pi kx}{L})\,dx$

Orthogonality formulae: $\int_{-L}^{L}\cos(\frac{\pi mx}{L})\sin(\frac{\pi nx}{L})\,dx = 0$, $\int_{-L}^{L}\cos(\frac{\pi mx}{L})\cos(\frac{\pi nx}{L})\,dx = \int_{-L}^{L}\sin(\frac{\pi mx}{L})\sin(\frac{\pi nx}{L})\,dx = L\delta_{mn}$.