Mathematics 53

Multivariable Calculus

Lecture Notes

Textbook: James Stewart's *Calculus with Early Transcendentals*

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GSI: TBD

Contents

1 August 23, 2018

Today we'll talk about course policies and then move into the material. The primary method of communication is bCourses. TAs will be in charge of sections, quizzes, homeworks, and grading tests. The two lowest quiz grades will be dropped. The grading is as follows:

Event	Percent
Homework/Quiz	20%
Midterm 1	20%
Midterm 2	20%
Final Exam	40%

You can miss one midterm; if it is the first midterm, the second midterm counts for double; if it is the second midterm, then the final counts for 60%. If one misses the final exam, then they fail the course.

Everything is done by curves; the letter grades for each event are averaged. There is a quiz every week in recitation, which is unskippable.

The office hours are Tuesdays from 1 to 3:30 in 841 Evans.

Now to the material.

We will start talking about curves, both in R^2 and in R^3. First we discuss curves in the plane.

We can describe curves in the plane by:

- functional notation: $y = f(x)$, for example $y = x^2$.
- algebraic equation: $P(x, y) = c$, for example the unit circle $x^2 + y^2 = 1$. The upper half is given by $y = \sqrt{1 - x^2}$ and the lower half is given by $y = -\sqrt{1 - x^2}$.
- parametric equations: $(x, y) = (f(t), g(t))$ or equivalently $[a, b] \ni t \to (x(t), y(t)) \in \mathsf{R}^2$

Some motivation for parametric equations: think about a particle moving in a two dimensional plane. The obvious way to describe the curve of its movement is to show where the particle is at any given point in time.

Example 1.1

Convert $y = x^2$ to parametric form.

Solution Set $x = t$ and $y = t^2$, and we are done. Note that there are different parameterizations i.e. $x = t^3$ then $y = t^6$. □

The usage of multiple parameterizations is to account for the different speeds and forms of the individual functions of t. If a particle moves at t^3 in the x-dimension and t^6 in the y-dimension, it's the same in the xy-plane as moving at t in the x-dimension and t^2 in the y-dimension. To move in the other direction, we could just let $x = -t$ while $y = t^2$.

Now to the circle. We have $x^2 + y^2 = 1$. Let θ be the angle parameter; then $x = \cos(\theta)$ and $y = \sin(\theta)$. Note that we can choose the range of θ; choosing $0 \le \theta \le 2\pi$ traces the circle; choosing $0 \le \theta \le 4\pi$ traces out the circle twice; choosing $0 \le \theta \le \pi$ traces out the top half of the circle. To a more generalized example:

Example 1.2

Parameterize a circle of radius r which is centered at some point (a, b).

Solution We have $x = r\cos(\theta)$ for a circle of radius r at the origin, then translated a units, and a similar argument for b. Letting $0 \leq \theta \leq 2\pi$ to trace out the curve gives the solution set:

$$\begin{cases} x = r\cos(\theta) + a \\ y = r\sin(\theta) + b \\ 0 \leq \theta \leq 2\pi \end{cases}$$

□

Example 1.3

Parameterize an infinity sign at the origin.

Solution We start with

$$\begin{cases} x = \sin(t) \\ y = \sin(2t) \\ 0 \leq \theta \leq 2\pi \end{cases}$$

The motivation is that the circle is folded into itself, so we get the curve moving twice as fast in the y direction than the x direction. (If we did similarly for x instead of y, we would get an "8"). Finally, to finalize, let $0 \leq t \leq 2\pi$ to trace out the whole shape. □

Example 1.4

Parameterize a cycloid.

Solution Assume we are moving at speed 1 and the tire has radius 1. At time t, the tire will still touch the pavement. The length of the movement of the point on the bottom is just t, so the angle subtending the former bottom point and the current bottom point is also just t. We have $x(t) = t - \sin(t)$ because of how we move a distance to the right of t by translation and then move to the left by $\sin(t)$ by rotation. By a similar argument, we have $y(t) = 1 - \cos(t)$ because the radius is 1, and we move up or down by $\cos(t)$. Of course, t is in positive real numbers.

$$\begin{cases} x = t - \sin(t) \\ y = 1 - \cos(t) \\ t \in \mathbb{R}^+ \end{cases}$$

□

Say we have the curve in R^2 given by

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

The first property we are to calculate is the slope of the curve in the xy-plane. If it were given by $y = f(x)$, then the slope would be given, obviously, by $\mathrm{d}f/\mathrm{d}x$. If we have

$$\begin{cases} y = y(x) \\ x = x(t) \end{cases}$$

then we can obviously get the derivative by chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \to \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

If $\mathrm{d}x/\mathrm{d}t = 0$ then the equation has a vertical tangent.

Example 1.5

Compute the slopes of the infinity sign at $(x, y) = (0, 0)$.

Solution We see that since the infinity sign is given by

$$\begin{cases} x = \sin(t) \\ y = \sin(2t) \\ 0 \leq t \leq 2\pi \end{cases}$$

the particle is at the origin at $t = 0$ and $t = \pi$. At $t = 0$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos(2t)}{\cos(t)}|_{t=0} = 2$$

At $t = \pi$ we have $\mathrm{d}y/\mathrm{d}x = -2$ by symmetry. □

If we have a single variable function $y = f(x)$ then $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ gives the concavity of the function; if this quantity is less than zero the function is concave, and if it is greater than zero the function is convex.

By the chain rule we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right) = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right)$$

Example 1.6

Compute the concavity of the infinity curve.

Solution We see that since the infinity sign is given by

$$\begin{cases} x = \sin(t) \\ y = \sin(2t) \\ 0 \leq t \leq 2\pi \end{cases}$$

we have the quantities

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\cos(2t), \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \cos(t) \to \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos(2t)}{\cos(t)}$$

Therefore, by plugging into the equation, we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{\cos(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\cos(2t)}{\cos(t)}\right)$$

At this point it is horrifically annoying to finish, but note that the concavity changes every $\pi/2$ lengths of t. □

Now to arc lengths. Pick equidistant points along a curve, and enumerate them by the parameters used to generate them: $t = t_0, t = t_1, \ldots, t = t_n$. We have

$$\Delta x_j = x(t_{j+1}) - x(t_j), \quad j = 0, 1, 2, \ldots, n$$

$$\Delta y_j = y(t_{j+1}) - y(t_j), \quad j = 0, 1, 2, \ldots, n$$

Then the approximate length, by Pythagoras, is

$$S = \sum \sqrt{(\Delta x_j)^2 + (\Delta y_j)^2}$$

This looks like a Riemann sum, so we'll force it. We have

$$\Delta x_j \approx x'(t_j)\Delta t_j$$

$$\Delta y_j \approx y'(t_j)\Delta t_j$$

Therefore our approximate length turns into

$$S = \sum \sqrt{(x'(t_j))^2 + (y'(t_j))^2}\Delta t_j$$

which is an actual Riemann sum. Therefore we get, by conversion,

$$S = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$$

where the term of the square root is the "velocity" of the particle. Note that the length will not depend on the parameterization.

Example 1.7

Find the arc length of the circle given by

$$\begin{cases} x = a + r\cos(t) \\ y = b + r\sin(t) \\ 0 \le t \le 2\pi \end{cases}$$

Solution We have that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -r\sin(t)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\cos(t)$$

Thus, the velocity is $\sqrt{r^2(\sin^2(t) + \cos^2(t))} = r$. The arc length integral is

$$S = \int_0^{2\pi} r\,\mathrm{d}t = 2\pi r$$

□

Example 1.8

Compute the arc length of the cycloid given by the parameterization

$$\begin{cases} x - t - \sin(t) \\ y(t) = 1 - \cos(t) \\ 0 \leq t \leq 2\pi \end{cases}$$

Solution We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos(t)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin(t)$$

Therefore,

$$\begin{aligned} v^2 &= x'^2 + y'^2 \\ &= 1 - 2\cos(t) + \cos^2(t) + \sin^2(t) \\ &= 2(1 - \cos(t)) \end{aligned}$$

Thus, we have

$$S = \int_0^{2\pi} \sqrt{2(1 - \cos(t))}\,\mathrm{d}t = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) = 8$$

by the identity that $1 - \cos(t) = 2\sin^2(t/2)$. $\square$

2 August 24, 2018

The GSI is Danny Chupin, who has office hours in 826 Evans on Monday, Wednesday, Friday, at 11:30-12:30.

The discussion outline for non-quiz days (M,F) is: the first 5-10 minutes of review, 20 minutes of working on problems, and 20 minutes for discussion.

The discussion outline for quiz days (W) is: the first 20-25 minutes is a quiz (solutions uploaded in the afternoon), then whatever time remains is used for more discussion.

The homework assigned on Thursday is due Wednesday of the next week; the homework assigned on Tuesday is due on Friday of the same week. The quiz covers the material from the past week.

Make up quizzes can only be offered before the actual quiz date.

The extra credit policy for the section is one point to total quiz score, added per office hours visit, up to 4.

Lots of review of what a parametric curve is, what a cycloid is, and computing area between parametric curves – that we did yesterday.

A parametric curve is two functions $(x(t), y(t))$ with the same domain $t \in [a, b]$ which can cross itself and have "cusps".

For a given underlying geometric curve (a set of points in the plane) there are infinitely many ways of generating a parametric curve that traces out the geometric curve.

Example 2.1

Let the equation matching the curve we are generating we are generating be $y = x$. Then two parameterizations are

$$\begin{cases} x(t) = t \\ y(t) = t \end{cases}$$

and

$$\begin{cases} x(t) = t^3 \\ y(t) = t^3 \end{cases}$$

Note that the second equation traces out the curve much more slowly near the origin and much faster at points outside of the unit ball.

Why is this relevant? If we have a curve that doesn't "play nice", we can find a local parameterization that allows us to compute:

- Slope:
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$
- Arc length:
$$S = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$
- Area (signed area between a curve and an axis):
$$A = \int_{x=a}^{x=b} y(t)\,\mathrm{d}x(t) \quad \text{or} \quad A = \int_{y=a}^{y=b} x(t)\,\mathrm{d}y(t)$$

The cycloid is brought up.

3 August 27, 2018

Updated information: Danny's email is `daniel_chupin@berkeley.edu`, and the office hours are Monday from 11:30-12:30, Wednesday from 3:30-4:30, and Friday from 5:00-6:00.

We continue with the cycloid.

We have a wheel of radius R rolling to the right. We parameterize the motion by the angle θ (in radians) that has been rotated through at time t. Say a point on the curve started at the origin; then after the wheel moves θ radians, the horizontal distance covered by the center of the wheel is $R\theta$. We can write a vector from the origin to the center of the axle $\mathbf{r}_h(\theta)$ to keep track of the movement of the axle. We have

$$\mathbf{r}_h(\theta) = (x_h(\theta), y_h(\theta)) = (R\theta, R)$$

We create another vector $\mathbf{r}_{\text{rot}}(\theta)$ to track the position of the point on the wheel, with the understanding that $\mathbf{r}(\theta)$ (the absolute position of the curve after the wheel has rolled through θ radians) is given by

$$\mathbf{r}(\theta) = \mathbf{r}_h(\theta) + \mathbf{r}_{\text{rot}}(\theta)$$

We have that $\mathbf{r}_{\text{rot}}(0) = (0, -R)$ and obviously both parts are trigonometric functions. We claim that $\mathbf{r}_{\text{rot}}(\theta) = R(-\sin(\theta), -\cos(\theta))$ To check that this is the right combination of cos and sin, we check that $\mathbf{r}_{\text{rot}} = (0, -R)$ and $\mathbf{r}_{\text{rot}} = (-R, 0)$, which ends up being true.

We have that

$$\begin{aligned}
\mathbf{r}(\theta) &= \mathbf{r}_h(\theta) + \mathbf{r}_{\text{rot}}(\theta) \\
&= (R\theta, R) + (-R\sin(\theta), -R\cos(\theta)) \\
&= (R\theta - R\sin(\theta), R - R\cos(\theta)) \\
&= R(\theta - \sin(\theta), 1 - \cos(\theta))
\end{aligned}$$

What does the curve look like? We first need to figure out where the curve is vertical and where it is horizontal i.e. points of horizontal and vertical tangency. Note that the cycloid isn't really a solution to any one equation in the plane, which shows the power of parameterization. We have that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{R\sin(\theta)}{R(1-\cos(\theta))} = \frac{\sin(\theta)}{1-\cos(\theta)}$$

For horizontal tangency, find where $\sin(\theta) = 0$ while $1 - \cos(\theta) \neq 0$. It is easy to see that at all odd multiples of π, $\mathbf{r}(\theta)$ has a horizontal tangent. At all others ($\theta = 0, 2\pi$) we have

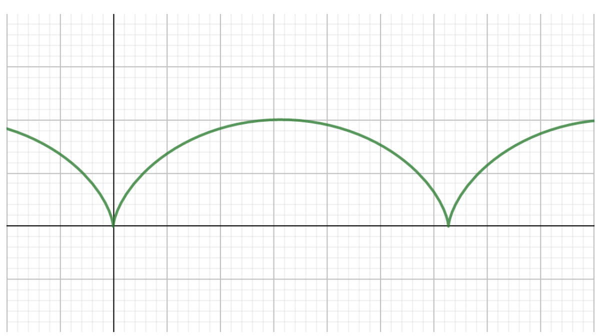
$$\left[\frac{\sin(\theta)}{1-\cos(\theta)}\right|_0 = \frac{0}{0}$$

To evaluate this correctly we need L'Hopital's rule. We have

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta\to 0^+} \frac{\cos(\theta)}{\sin(\theta)} = +\infty$$

Likewise, from the negative direction, the limit is $-\infty$.

We can thus trace out the cycloid:



Let's find the area under one arch of the general cycloid. We have

$$A = \int_{x(a)}^{x(b)} y(t)\,\mathrm{d}x(t) = \int_a^b y(x(t))\frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\mathrm{d}t$$

For us, $t = \theta$, $a = 0$, $b = 2\pi$. We have

$$\begin{aligned} A &= \int_0^{2\pi} (R - R\cos(\theta))(R - R\cos(\theta))\,\mathrm{d}\theta \\ &= R^2 \int_0^{2\pi} (1 - 2\cos(\theta) + \cos^2(\theta))\,\mathrm{d}\theta \\ &= R^2(2\pi + 0 + \pi) = 3\pi R^2 \end{aligned}$$

because of the fact that $\int_0^{2\pi} \cos^2(\theta)\,\mathrm{d}\theta = \pi$. This is proven by noticing that $\int_0^{2\pi} \cos^2(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \sin^2(\theta)\,\mathrm{d}\theta$ and the identity that $\sin^2(\theta) + \cos^2(\theta) = 1$. Integrating $\int_0^{2\pi} 1\,\mathrm{d}\theta$ gives 2π, so each of the integrals of $\sin^2(\theta)\,\mathrm{d}\theta$ and $\cos^2(\theta)\,\mathrm{d}\theta$ must give half, or π.

4 August 28, 2018

Our starting topic is to briefly cover polar coordinates. In the last lecture, when we talked about parametric curves in the plane, we described them in terms of Cartesian coordinates, where for each point in the plane, we looked at their x and y coordinates, where the x coordinate is the distance from the projection onto the y axis and vice versa.

We will now look at polar coordinates, where one coordinate is the line connecting the origin to the point - the **radius** r - and the other coordinate is the angle θ of this line with respect to the x axis and going counterclockwise. We have

$$r \in [0, \infty), \quad \theta \in [0, 2\pi]$$

We can have θ in any angle interval of measure 2π - so that it describes a full revolution.

Now we explore the conversion between (x, y) and (r, θ). We have

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}, \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \end{cases}$$

Now we discuss the properties of curves in polar coordinates.

Example 4.1

Describe the polar curve $r = 4$.

Solution The curve is a circle centered at the origin with radius 4. □

Example 4.2

Describe the polar curve $r = 2\sin(\theta)$.

Solution Multiplying both sides by r, we get

$$r^2 = 2r\sin(\theta)$$

Substituting in for r^2 and $r\sin(\theta)$, we have

$$x^2 + y^2 = 2y$$

Completing the square, we have

$$x^2 + (y-1)^2 = 1$$

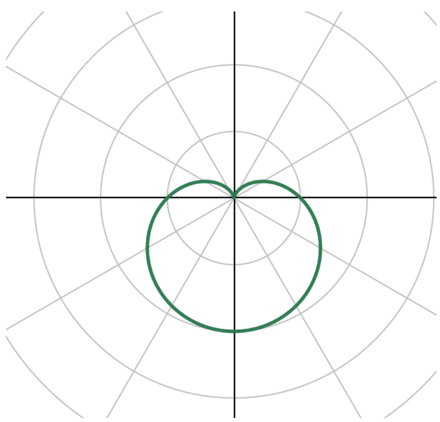
We have that this curve is a circle centered at $(0, 1)$ with radius 1. $\square$

Example 4.3

Describe the polar curve $r = 1 - \sin(\theta)$.

Solution The first step is to describe r and θ as if they were unbounded variables in the Cartesian plane. We have

Graphing such a curve in the polar plane by observing key values yields



For this parametric curve, we have, by substituting in r and using θ as a parameter,

$$x = \cos(\theta)(1 - \sin(\theta))y = \sin(\theta)(1 - \sin(\theta))$$

This curve is called a **cardioid** because it looks like a heart. $\square$

We can treat polar coordinates as parametric curves with parameter θ, We thus have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}\right)$$

and all the results from last week's discussion hold.

In particular, let us examine the area function in polar coordinates. Let $r = r(\theta)$. We pick radial infinitesimals. Consider the partition $[\theta, \theta + \Delta\theta]$. In this interval, the sector area ΔA can be approximated by the area of circles.

$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta \to \mathrm{d}A$$

We have

$$A = \sum \Delta A = \sum \frac{1}{2} r^2 \,\mathrm{d}\theta$$

and so, making the partition smaller and smaller, in the limit we have

$$\mathrm{d}A = \frac{1}{2} r^2 \,\mathrm{d}\theta$$

and so the total area is given by

$$A = \int \mathrm{d}A = \frac{1}{2} \int r^2 \,\mathrm{d}\theta$$

Example 4.4

Find the area of the cardioid given above.

Solution

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (1 - \sin(\theta))^2 \,\mathrm{d}\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 + \sin^2(\theta) - 2\sin(\theta)) \,\mathrm{d}\theta \\ &= \frac{\pi}{2} \end{aligned}$$

□

Now we consider arc length. For parametric curves, in general, we have

$$L = \int_a^b \underbrace{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t}_{\mathrm{d}s}$$

Now instead of t we have θ as the parameter. Recall that we have

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

So by product rule

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos(\theta) - r\sin(\theta)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta) + r\cos(\theta)$$

In particular, we have

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \cos^2(\theta) - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta)\cos(\theta) + r^2\sin^2(\theta)\\ &+ \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \sin^2(\theta) + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta)\cos(\theta) + r^2\cos^2(\theta)\\ &= \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\end{aligned}$$

Thus, for polar curves, we have

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2}\,\mathrm{d}\theta$$

For the cardioid, we have $r = 1 - \sin(\theta)$, and $\mathrm{d}r/\mathrm{d}\theta = \cos(\theta)$. Then $\mathrm{d}s = \sqrt{2 - 2\sin(\theta)}\,\mathrm{d}\theta = \sqrt{2}\sin(\theta/2)\,\mathrm{d}\theta$ which is trivial to integrate.

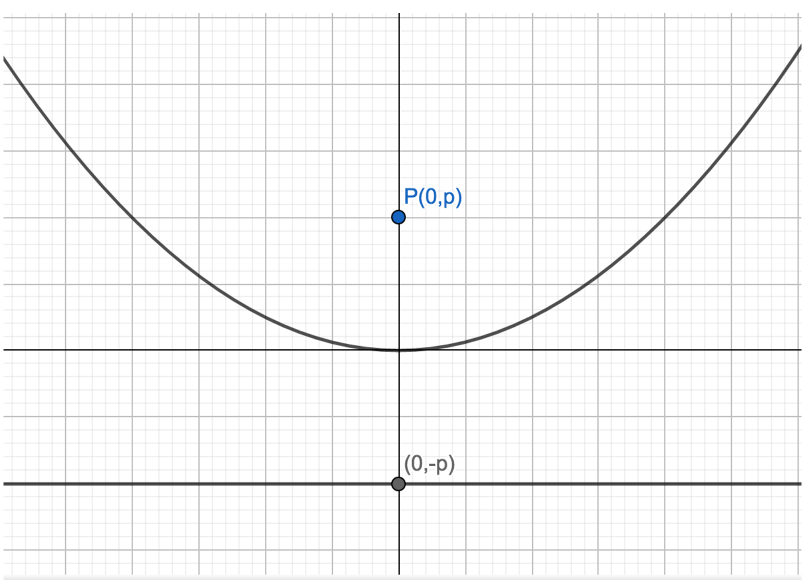
We can also write θ as a function of r i.e. $\theta = r/2$. We will mostly have to graph these problems.

The polar coordinate system is always orthogonal - the isoradial lines are always perpendicular at any point to the isoangular lines. The conversion between them is a **conformal** transformation – it preserves angles.

We now discuss conics in the plane. We have two simple examples of lines and circles.

But we're going to discuss parabolas today. A parabola is the set of points at equal distance from a point in the plane P – the **focus** – and a line – the **directrix**. If the focus is at $P(p, 0)$ and the directrix is a horizontal line through $(-p, 0)$,

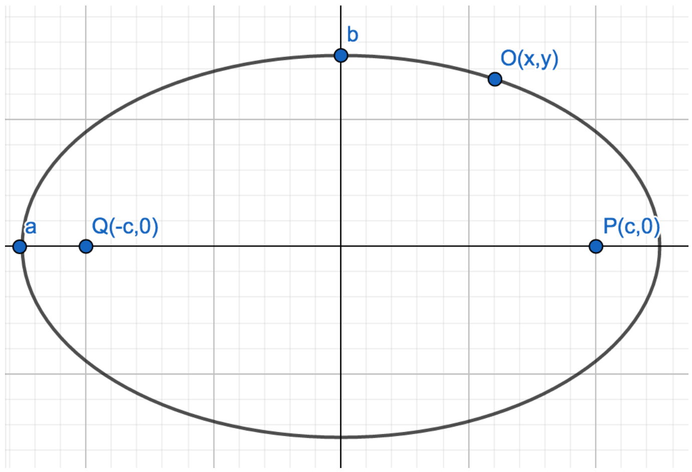
$$\begin{aligned} d^2((x, y), P) &= (x-0)^2 + (y-p)^2\\ d^2((x, y), L) &= (y-(-p))^2\\ x^2 + (y-p)^2 &= (y+p)^2\\ x^2 &= 4py\end{aligned}$$



We are also going to discuss ellipses, which are the set of points for which the sum of the distances from two points P and Q – the focuses – is constant; that is

$$OP + OQ = 2a$$

Let $P(c, 0)$ and $Q(-c, 0)$. By the triangle inequality, $a \geq c$, with equality occurring when the points are collinear (at the endpoints of the major axis for the ellipse).



Let O have coordinates $O(x, y)$. Then

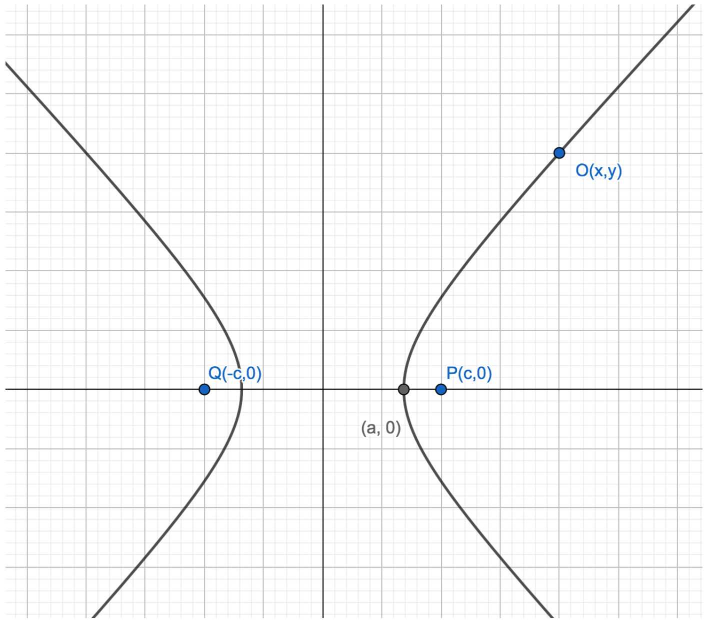
$$\begin{aligned}
OP^2 &= (x-c)^2 + y^2 \\
OQ^2 &= (x+c)^2 + y^2 \\
\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} &= 2a \\
\sqrt{(x-c)^2 + y^2} &= 2a - \sqrt{(x+c)^2 + y^2} \\
(x-c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2 + y^2 \\
4a\sqrt{(x+c)^2 + y^2} &= 4a^2 + 4xc \\
a^2\left((x+c)^2 + y^2\right) &= (a^2 + xc)^2 \\
a^2(x^2 + 2xc + c^2 + y^2) &= a^4 + 2a^2xc + x^2c^2 \\
a^2x^2 + a^2c^2 + a^2y^2 &= a^4 + x^2c^2 \\
\left(a^2 - c^2\right)x^2 + a^2y^2 &= a^2\left(a^2 - c^2\right) \\
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} &= 1
\end{aligned}$$

Letting $b^2 = a^2 - c^2$, we have our canonical equation for an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It is easy to read off the intercepts and endpoints. Also note that a circle is a special case of an ellipse when $a = b$.

Finally we cover hyperbolas. Take the point $Q(c, 0)$ and the symmetric point $P(-c, 0)$. Take a point $O(x, y)$ in the plane; we are asking for the differences in the pairwise distances to be equal to a constant. The set of points O for which this is true generate a hyperbola.



We are looking for, in particular,

$$OQ - OP = 2a$$

By the triangle inequality, $a < c$. We start in the same way until the fourth step.

$$\begin{aligned}
\sqrt{(x^2+y^2)+y^2} &= 2a + \sqrt{(x-c)^2+y^2} \\
(x-c)^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \\
-4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4xc \\
a^2\left((x-c)^2+y^2\right) &= (a^2-xc)^2 \\
a^2x^2+a^2c^2+a^2y^2 &= a^4+x^2c^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c)^2
\end{aligned}$$

Letting $a^2 - c^2 = -b^2$ and dividing by a^2b^2, we have

$$\frac{x^2}{a^2} - \frac{y^2}{a^2-c^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

5 August 29, 2018

After taking the quiz, we do some problems from a discussion section problem set. Nothing really to note here.

Quiz review: let $x^2 + y = \cdots = x^2 + y^2$. Oops! (9/10)

6 August 30, 2018

Today we will talk about R^3 (the book calls it V^3). In regular R^2 we have two coordinates, x, and y, which describe the projection of a point onto the x-axis and y-axis, respectively. In R^3, we have three axes and the coordinates are derived from the projections onto a plane composed of two axes, themselves projected onto the two axes that the plane is identified by.

In two dimensions if we want to compute the distance between two points we do, by Pythagorean theorem,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If we look at the three dimensional space we can basically do the same thing by using Pythagorean theorem twice, once for each degree of freedom orthogonal to their collinear line:

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This generalizes very well for R^n.

When we worked in R^2, we looked at curves. In R^3, we look at surfaces. They, like curves, have more than one way to describe them.

Example 6.1

Find equations for the xy, xz, and yz plane.

Solution For the xy plane, $z = 0$. For the xz plane, $y = 0$. For the yz plane, $x = 0$. □

We can also talk about surfaces as graphs of functions. We write $z = f(x, y)$ to write z as a function of x and y. We cannot represent some surfaces as functions, the same as curves in R^2. In particular, if the curve is self-intersecting or vertical at any point, we cannot define it as a function.

Example 6.2

Find the equation for the sphere centered at $P(x_0, y_0, z_0)$ with radius $r > 0$.

Solution We know that for any point Q on the sphere, we have $PQ = r$. Squaring both sides, we have

$$r^2 = PQ^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

Note that if you remove the final term the sphere equation reduces to that of a circle. □

Example 6.3

Find the characteristics of the surface represented by

$$x^2 + y^2 + z^2 - 2x + 2y = 0$$

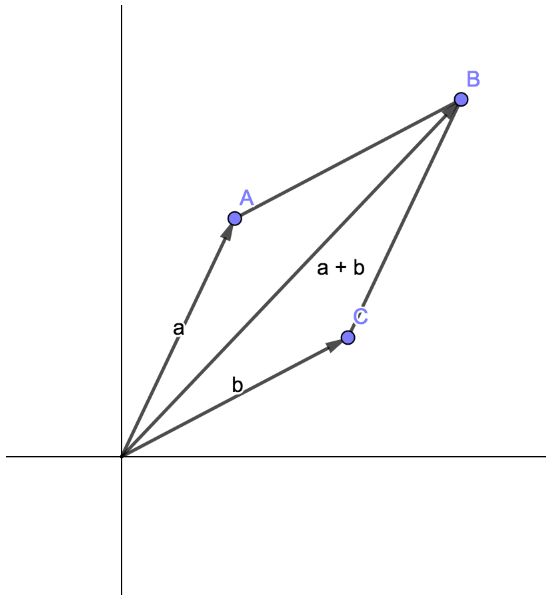
Solution By completing the square we factor the equation as

$$(x - 1)^2 + (y + 1)^2 + z^2 = 2$$

Thus the surface is a sphere with center $(1, -1, 0)$ and radius $\sqrt{2}$. □

Vectors arise when one needs to describe something with a magnitude and direction. These are the only properties of vectors; starting location is irrelevant. Thus two equally long vectors which are parallel are equivalent.

Suppose we want to add two vectors. Think of a particle moving along the first vector, then moving according to the second vector ("tip-to-tail"):



Another operation we can do with vectors is to multiply them by real numbers. This scales their length by the real number's magnitude and turns them according to the real number's sign.

The set of all vectors in $\mathsf{R}^3 = \mathsf{V}^3$ form a vector space, which has the following operations:

- $\mathsf{V} \ni \mathbf{u}, \mathbf{v} \to \mathbf{u} + \mathbf{v}$
- $\mathbb{R} \ni c, \mathbf{u} \to c\mathbf{u}$

From this we can define several properties:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} \to -\mathbf{u} = (-1)\mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $0\mathbf{u} = \mathbf{0}$
- $1\mathbf{u} = \mathbf{u}$

We can talk about vectors in any dimension, but if we specialize to three dimensions, then some fun stuff happens. We talked about vectors being an equivalence class – every two vectors with the same magnitude and direction are equivalent – but a special vector is that which starts at the origin $O(0, 0, 0)$. Let $P(x, y, z)$ be a point in R^3. Then the vector given by $\overrightarrow{OP}$ has components (x, y, z) and is known as the position vector of P.

Algebraically, we define the operations here. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and

$$c\mathbf{u} = (cu_1, cu_2, cu_3)$$

Another interesting quantity is the length or norm of a vector $\|\mathbf{u}\|$. This is defined by the norm on R^3 as

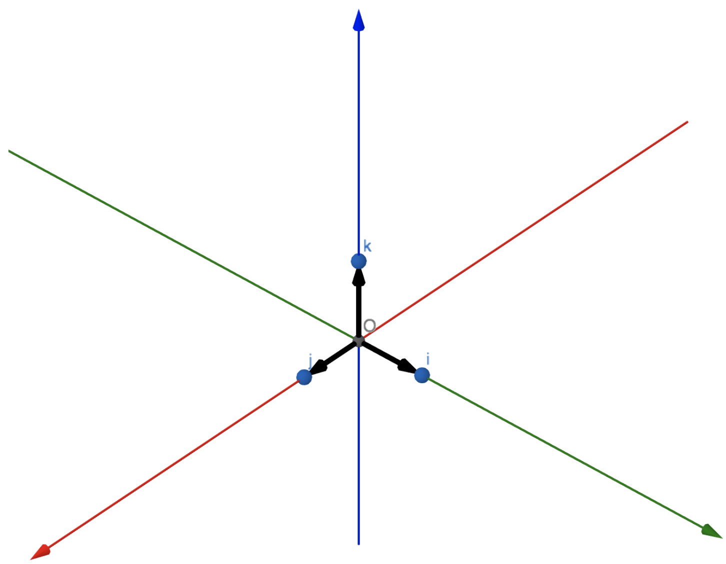
$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Now we return back to three-dimensional space, and discuss some special vectors - namely the unit vectors in the directions of the coordinate axes. They are

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

which is denoted as the standard basis in R^3, which means that all other vectors in R^3 can be composed of a linear combination of these vectors:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$



Some properties of the norm are given by the simple ideas

$$\|\mathbf{u}\| \geq 0 \quad \text{and} \quad \|\mathbf{u}\| = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}$$

and

$$\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$$

Fact 6.1

The triangle inequality states that, for any two vectors $\mathbf{u}$ and $\mathbf{v}$, we have

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

We now introduce the dot product.

Definition 6.1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors with the standard components. Then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

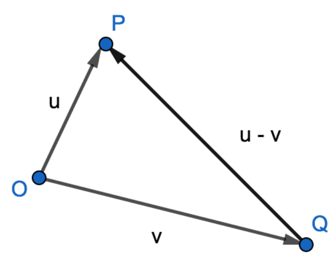
The dot product always produces a real number. It is useful because it is linear in $\mathbf{u}$ and $\mathbf{v}$, so

- $(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{v} = \mathbf{u}_1 \cdot \mathbf{v} + \mathbf{u}_2 \cdot \mathbf{v}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} + \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \|\mathbf{u}\|^2$

We use the last idea a lot:

$$\boxed{\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2}$$

Say we have the following configuration, where $\theta = \measuredangle QOP$:



By law of cosines we have $PQ^2 = OP^2 + OQ^2 - 2 \cdot OP \cdot OQ \cdot \cos(\theta)$.

Expanding, we have

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) \\
(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) \\
\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)
\end{aligned}$$

We thus have two properties:

Lemma 6.1

For all $\mathbf{u}$ and $\mathbf{v}$ in R^n, let θ be the angle between them on their shared plane. Then

$$\mathbf{u} \cdot [v] = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$$

and

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{0} \quad \text{iff} \quad \mathbf{u} \perp \mathbf{v}$$

We can also work on projections. The projection is required to be a vector. The projection from $\mathbf{u}$ onto $\mathbf{v}$ is denoted by $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

We have to first define the notion of a unit vector $\hat{\mathbf{v}}$ as defined by $\mathbf{v}/\|\mathbf{v}\|$; it is the vector with unit length pointing in the same direction of $\mathbf{v}$.

We thus have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

We can solve for the length of this projection by simple geometry, and then attach the vectorial form to it.

We define the cross product to answer the following question: given $\mathbf{u}, \mathbf{v} \in \mathsf{R}^3$, find $\mathbf{w}$ such that $\mathbf{w} \perp \mathbf{u}, \mathbf{w} \perp \mathbf{v}$.

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be a_i, b_i, and c_i respectively. Then we have the system of equations, by computing dot products to be zero:

$$\begin{array}{ccccccc} a_1c_1 & + & a_2c_2 & + & a_3c_3 & = & 0 \\ b_1c_1 & + & b_2c_2 & + & b_3c_3 & = & 0 \end{array}$$

Eliminating c_3 from the equation, we have

$$(a_3b_3 - b_1a_3c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

and repeating the computation, we have

$$\begin{cases} c_1 = a_2b_3 - a_3b_2 \\ c_2 = a_3b_1 - a_3b_3 \\ c_3 = a_1b_2 - a_2b_1 \end{cases}$$

Thus we have our cross product:

Definition 6.2. We have the cross product $\mathbf{u} \times \mathbf{v}$ as

$$\mathbf{u} \times \mathbf{v} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

We can also do a cyclical substitution to retrieve the other two terms from the first, in that $1 \to 2 \to 3 \to 1$.

There is some stuff about determinants and cofactor expansions which are well known. In particular

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The geometric interpretation of the cross product is

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The sin term indicates an area. Indeed, the quantity $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram generated by $\mathbf{u}, \mathbf{v}$. Moreover, the direction of the cross product gives us a perpendicular vector from the parallelogram.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathsf{R}^3$. Then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

This is easy if you check the determinant definition of the cross product. This expression is known as the triple product of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. One reason we care about triple products is that they give the signed volume of the parallelepiped generated by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

7 August 31, 2018

We do some of the problems on the discussion sheet for like 20 minutes.

Let's do a quick recap of cross and dot products.

The dot product of $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{a} \cdot \mathbf{b}$ and is defined

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i b_i$$

The dot product measures the "distance from orthogonality", that is, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a} \perp \mathbf{b}$.

The set of all vectors perpendicular to a given vector in R^2 make a line. Let $\mathbf{a} = (a_1, a_2)$; then the set of all vectors $\mathbf{b} = (x, y)$ which are perpendicular to $\mathbf{a}$ are given by

$$a_1 x + a_2 y = 0$$

which is the equation of a line. Similarly, in R^3, we have $\mathbf{a} = (a_1, a_2, a_3)$; then the set of all vectors $\mathbf{b} = (x, y, z)$ which are perpendicular to $\mathbf{a}$ are given by

$$a_1 x + a_2 y + a_3 z = 0$$

which is the equation of a plane.

8 September 4, 2018

Last time we talked about dot and cross products, and distance and length. Today we will talk about simple geometries in 3 dimensions. We begin with lines and planes.

In two dimensions, one can talk about lines as algebraic equations:

$$ax + by + c = 0$$

parametric representations:

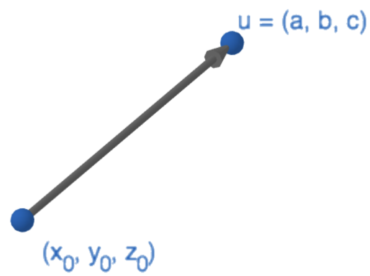
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}$$

or we can eliminate the parameter and write

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

Note that when we impose one condition we lose one dimension. This is entirely obvious but also less trivial when you get into three dimensions.

Now we cover lines in 3 dimensions. Remember we have two constraints.



Thus we can write the plane as

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + bt \end{cases}$$

or, eliminating the parameter, we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Now we move on to planes in 3 dimensions. We can describe htem through linear equations

$$ax + by + cz + d = 0$$

or the geometric interpretation of a point $P(x_0, y_0, z_0)$ on the plane perpendicular to a normal vector $\mathbf{v} = (a, b, c)$. Our equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

For the plane $ax + by + cz + d = 0$, the vector $\mathbf{v} = (a, b, c)$ is perpendicular to the plane.

The distances in R^3 are between a point and a plane, a point and a line, and a two lines.

Example 8.1

Let Π be a plane given by $ax + by + cz + d = 0$, $P_0(x_0, y_0, z_0)$ be a point on Π, and $P(x, y, z)$ be a point not on Π. Find the distance between P and Π.

Solution Let Q be a point on the plane directly underneath P (i.e. Q is the projection from P to Π). We have that the normal vector $\mathbf{u} = (a, b, c)$ is parallel to PQ. We have that $PQ \perp \Pi$ and thus $P_0Q \perp PQ$.

We have

$$d(P, \Pi) = PQ = \operatorname{comp}_{\mathbf{u}} \overrightarrow{P_0P} = \operatorname{comp}_{\mathbf{u}}(x - x_0, y - y_0, z - z_0) = \frac{(x - x_0, y - y_0, z - z_0) \cdot (a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

We thus have

$$d(P, \Pi) = \frac{a(x - x_0) + b(y - y_0) + c(z - z_0)}{\sqrt{a^2 + b^2 + c^2}}$$

with the constraint that

$$ax_0 + by_0 + cz_0 + d = 0$$

which signifies $P_0 \in \Pi$. Reworking the numerator, we have

$$d(P, \Pi) = \frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}}$$

□

Now we pivot to quadrics, which are surfaces described by quadratic equations. The general form of a quadratic equation is

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$$

The simplest quadric is a plane, in which there are no quadratic terms. The next is a cylinder of the form $x^2 + y^2 = 1$, with z free. The next is a parabola of the form $z = y^2$, which is a parabola in the zy-plane translated in the x direction.

Note that all these shapes are unions of straight lines. Furthermore, they are all on the canonical axes. Now we will cover the truly three-dimensional objects, the ellipsoid, the paraboloid, and the hyperboloid.

The ellipsoid is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

One way to graph objects in three dimensions is to look at the principal planes. We start with the xy plane, where $z = 0$. We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and similarly for the xz and yz planes. Since at all principal planes the figure is an ellipse, we conclude that however we section this object, we get an elliptical cross-section. We can conclude that our ellipsoid is a stretched sphere; indeed, the case where $a = b = c$ turns the ellipsoid into a sphere, while $a = b$ or $b = c$ generates rotational symmetry around one axis.

Now we move onto paraboloids. We have, to start,

$$z = ax^2 + by^2$$

There are two cases:

1. $\operatorname{sgn}(a) = \operatorname{sgn}(b)$ – elliptic paraboloids, have ellipses as cross sections
2. $\operatorname{sgn}(a) = -\operatorname{sgn}(b)$ – hyperbolic paraboloids, have hyperboloids as cross sections

We start with elliptic paraboloids. We have $z = x^2 + y^2$. Iin the yz plane, we have $z = y^2$. Now let $z = z_0 > 0$. Then $z_0 = x^2 + y^2$ represents a circle. Thus the parabola will have circles as cross sections.

The hyperbolic paraboloid is given by $z = y^2 - x^2$. In the yz plane we have $z = y^2$, which creates a parabola, same as the elliptic paraboloid. Now look at the xz plane, where we get $z = -x^2$. Taking different $x = x_0$ and $z = z_0$ planes, we see that the resulting figure is like a saddle.

We now consider the hyperboloid. We start with the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

Choose $+1$, we get the one-sheeted hyperboloid. In the xy plane we have $x^2/a^2 + y^2/b^2 = 1$. Let $z = z_0$. Then we have $x^2/a^2 + y^2/b^2 = 1 + z^2/c^2$. As z_0 becomes closer to zero, the ellipse cross section grows smaller; as z_0 increases in magnitude the ellipse cross section grows larger.

Now choose -1, we get the two-sheeted hyperboloid. At the xy plane we get nothing. We see that z takes a minimum at $\pm c$. Apart from that we have what we have before.

9 September 5, 2018

Quiz + review problems.

10 September 6, 2018

We first discuss vector-valued functions.

A function associates an object of a subset of R to another object in R. We call these functions scalar functions. We have

$$f\colon \mathsf{R} \to \mathsf{R} \mid t \to f(t)$$

A vector-valued function associates an object of a subset of R to an object in R^n. We have

$$\mathbf{u}\colon \mathsf{R} \to \mathsf{R}^n \mid t \to \mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$$

where each u_i is a component function. In R^2 and R^3 the natural interpretation of a vector valued function is a position vector as a function of time. The obvious analogy is to a parametric representation of a curve.

Example 10.1

Suppose we take the following vector-valued function:

$$\mathbf{u}(t) = (1 + t, 2t, -t)$$

We want to find the line through $(1, 0, 0)$ with a direction parallel to the vector $(1, 2, -1)$.

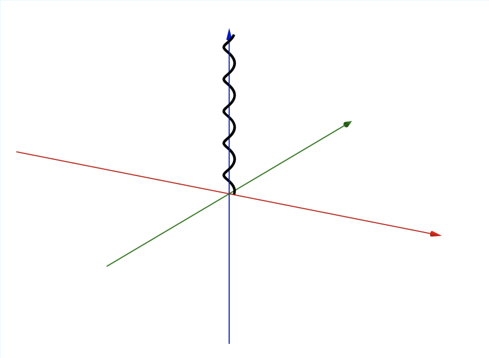
This is very simple.

Example 10.2

Suppose we want to draw the following vector-valued function:

$$\mathbf{u}(t) = (\cos(t), \sin(t), t)$$

To draw this curve, we look at its representation on the plane $z = 0$, and in this facsimile of R^2 we see that the curve traces out a circle moving counterclockwise.



We say that $f\colon \mathsf{R} \to \mathsf{R}$ is continuous if for each t_0, $\lim_{t\to t_0} f(t) = f(t_0)$.

Definition 10.1. We say that $\mathbf{u}\colon \mathsf{R} \to \mathsf{R}^n$ is continuous if each of its component functions are continuous. In particular,

$$\lim_{t\to t_0} \mathbf{u}(t) = \left(\lim_{t\to t_0} u_1(t), \lim_{t\to t_0} u_2(t), \ldots, \lim_{t\to t_0} u_n(t)\right)$$

and for $\mathbf{u}$ to be continuous, for each t_0 we have $\lim_{t\to t_0} \mathbf{u}(t) = u(t_0)$.

Definition 10.2. We can do differentiation in the usual way:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= \lim_{t\to t_0} \frac{\mathbf{u}(t) - \mathbf{u}(t_0)}{t - t_0} \\ &= \left(\lim_{t\to t_0} \frac{u_1(t) - u_1(t_0)}{t - t_0}, \lim_{t\to t_0} \frac{u_2(t) - u_2(t_0)}{t - t_0}, \ldots, \lim_{t\to t_0} \frac{u_n(t) - u_n(t_0)}{t - t_0}\right) \\ &= \left(\frac{\mathrm{d}u_1}{\mathrm{d}t}, \frac{\mathrm{d}u_2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}u_n}{\mathrm{d}t}\right)\end{aligned}$$

In this sense, $\mathbf{u}$ is differentiable if and only if $u_1, u_2, \ldots, u_n$ is differentiable.

We see that at all points where it exists $\mathrm{d}u/\mathrm{d}t$ is tangent to the curve. This can be seen to be the velocity of the curve.

There are several differentiation rules:

1. $\mathrm{d}(\mathbf{u} + \mathbf{v})/\mathrm{d}t = \mathrm{d}\mathbf{u}/\mathrm{d}t + \mathrm{d}\mathbf{v}/\mathrm{d}t$
2. $\mathrm{d}(f\mathbf{u})/\mathrm{d}t = \mathbf{u}\,\mathrm{d}f/\mathrm{d}t + f\,\mathrm{d}\mathbf{u}/\mathrm{d}t$
3. $\mathrm{d}(\mathbf{u} \cdot \mathbf{v})/\mathrm{d}t = \mathrm{d}\mathbf{u}/\mathrm{d}t \cdot \mathbf{v} + \mathbf{u} \cdot \mathrm{d}\mathbf{v}/\mathrm{d}t$
4. $\mathrm{d}(\mathbf{u} \times \mathbf{v})/\mathrm{d}t = \mathrm{d}\mathbf{u}/\mathrm{d}t \times \mathbf{v} + \mathbf{u} \times \mathrm{d}\mathbf{v}/\mathrm{d}t$
5. $\mathrm{d}\mathbf{u}(f(t))/\mathrm{d}t = (\mathrm{d}\mathbf{u}/\mathrm{d}f)(\mathrm{d}f/\mathrm{d}t)$

The chain rule is of specific interest in the reparameterization of curves.

Integration is defined likewise. Let $\mathbf{u}\colon \mathsf{R} \to \mathsf{R}^n$; then

$$\int_a^b \mathbf{u}(t)\,\mathrm{d}t = \left(\int_a^b u_1(t)\,\mathrm{d}t\,, \int_a^b u_2(t)\,\mathrm{d}t\,, \ldots, \int_a^b u_n(t)\,\mathrm{d}t\right)$$

The indefinite integral works the same way, except the constant at the end is a vector in R^n.

Say we are given a curve $\mathbf{u}$ in R^n. The arc length over $t \in [a, b]$ of $\mathbf{u}$ is given by

$$S = \int_a^b \mathrm{d}s = \int_a^b \sqrt{\sum_{i=1}^n \left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Note that $\mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathrm{d}u_1/\mathrm{d}t\,, \mathrm{d}u_2/\mathrm{d}t\,, \ldots, \mathrm{d}u_n/\mathrm{d}t)$. Thus, the norm of $\mathbf{u}$ gives a simpler expression in arbitrary dimensions:

$$S = \int_a^b \mathrm{d}s = \int_a^b \left\|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right\|\mathrm{d}t$$

A smooth curve is a curve that is differentiable infinitely many times and does not stop (that is, $\mathrm{d}\mathbf{u}/\mathrm{d}t \neq 0$ for any t).

We define the tangent vector

$$\mathbf{T}(t) = \frac{\mathrm{d}\mathbf{u}/\mathrm{d}t}{\|\mathrm{d}\mathbf{u}/\mathrm{d}t\|}$$

Note that $\mathbf{T}$ is a unit vector since it divides a vector by its length. Thus $\|\mathbf{T}\| = \|\mathbf{T}\|^2 = \mathbf{T}\cdot\mathbf{T} = 1$. We differentiate this last equality and get $2\mathbf{T}\cdot\mathrm{d}\mathbf{T}/\mathrm{d}t = 0$. This means that $\mathrm{d}\mathbf{T}/\mathrm{d}t$ is orthogonal to $\mathbf{T}$.

We now attempt to parameterize $\mathbf{T}$ in terms of s. Recall that $S(t) = \int_a^t \mathrm{d}s$. By chain rule,

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t}$$

If $\mathrm{d}\mathbf{T}/\mathrm{d}s = 0$, then $\mathbf{T}$ must be constant, so the trajectory of the curve is a line. On the other hand, if $\mathrm{d}\mathbf{T}/\mathrm{d}s \neq 0$, then the curve is – well, curved. In some sense $\mathrm{d}\mathbf{T}/\mathrm{d}s$ represents how not-straight the curve is. We define this quantity as the curvature κ.

$$\kappa = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|$$

Example 10.3

Compute the curvature of the circle given by

$$x^2 + y^2 = r^2$$

Solution Parameterizing this gives $\mathbf{u}(t) = (r\cos(t), r\sin(t))$. We have $\mathrm{d}\mathbf{u}/\mathrm{d}t = (-r\sin(t), r\cos(t))$ and $\|\mathrm{d}\mathbf{u}/\mathrm{d}t\| = 1$. Thus our vector $\mathbf{T} = \frac{\mathrm{d}\mathbf{u}/\mathrm{d}t}{\|\mathrm{d}\mathbf{u}/\mathrm{d}t\|} = (-\sin(t), \cos(t))$. We also compute $\mathrm{d}s = \|\mathrm{d}\mathbf{u}/\mathrm{d}t\|\,\mathrm{d}t = r\,\mathrm{d}t$ so $\mathrm{d}s/\mathrm{d}t = r$. Thus we get $\mathrm{d}\mathbf{T}/\mathrm{d}s = (-\cos(t), -\sin(t))/r$. Taking the norm of this yields that $\kappa = 1/r$. □

Suppose we have a non-circular curve. At some point on the curve, we draw the best-matched circle to the curve, one that matches the point, the first derivative, and the second derivative. Then $\kappa = 1/r$ where r is the radius of that circle.

A lot of stuff we noted is true in three dimensions, but we continue with more properties. Suppose we want the normal vector $\mathbf{N}$ to the curve. We define this quantity as

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{\|\mathrm{d}\mathbf{T}/\mathrm{d}s\|}$$

The normal vector $\mathbf{N}$ and the tangent vector $\mathbf{T}$ generate a plane which is the best match for the curve at a point on the curve, which is called the osculating plane. In particular $\mathsf{S}\{\mathbf{T}, \mathbf{N}\}$ generates the osculating plane.

It is easy to find normal vectors for the plane given by $\mathsf{S}\{\mathbf{T}, \mathbf{N}\}$. In particular we define the binormal vector $\mathbf{B}$ by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

It has length 1 because both $\mathbf{T}$ and $\mathbf{N}$ are normal vectors, and $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin(\theta)$ and $\theta = \pi/2$. The triplet $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is an orthonormal moving frame in the style of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

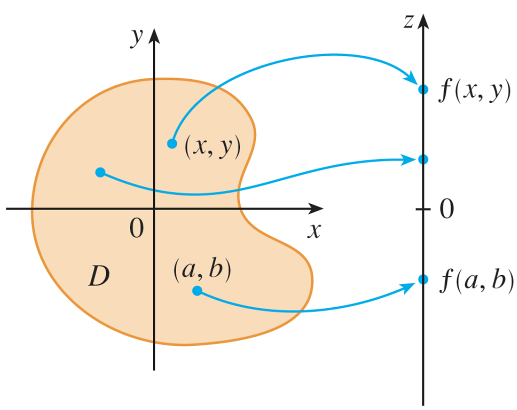
The plane generated by $\mathsf{S}\{\mathbf{B}, \mathbf{N}\}$ is called the normal or orthogonal plane, because it is orthogonal to the curve at that point.

11 September 11, 2018

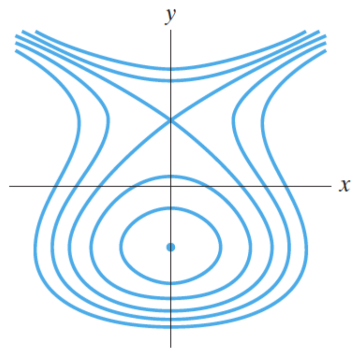
Today we discuss functions of multiple variables, limits, and partial derivatives.

Visualizations of multidimensional curves:

- "Arrow diagrams" emphasizing domain and range (useful in higher dimensions)

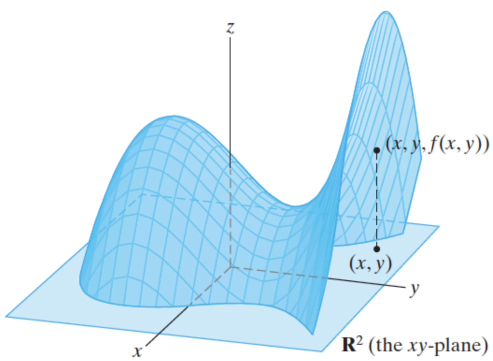


- Level curves (the level curve of f with value k is the set of tuples $(\mathbf{x})$ such that $f(\mathbf{x}) = k$).



A symmetric function will have symmetric level curves. Note that level curves are subsets of the domain; they are parameterized by values of the range.

- Graphs (the graph of f is the set of points $(\mathbf{x}, f(\mathbf{x}))$ in R^{n+1} with $\mathbf{x}$ in the domain of f.)



Definition 11.1. The limit of $\mathbf{f}$ as $\mathbf{x}$ approaches $\mathbf{a}$ is denoted

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$$

We say this equals $\mathbf{L}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{x}$ is in the domain of $\mathbf{f}$, and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ then $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \epsilon$.

Note that the squeeze theorem, sum rules, product rules, and so on hold.

Example 11.1

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

Solution We note that

$$\frac{x^2y}{x^2y^2} \leq \frac{x^2|y|}{x^2+y^2} = \frac{x^2}{x^2+y^2}|y| \leq |y|$$

So if the limit is to exist,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} |y| = 0$$

For a lower bound, we use

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \geq \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^2+y^2} \geq \lim_{(x,y)\to(0,0)} -|y|$$

Thus, by the same argument,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \geq \lim_{(x,y)\to(0,0)} -|y| = 0$$

By the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

□

Example 11.2
Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^4}$$

does not exist.

Solution Choose a path of approach. Let's try $y = 0$; then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Let's try $x = 0$ and approach from the positive direction; then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^4} = \lim_{y=0^+} \frac{1}{y^4} = \infty$$

Let's try $x = y^2$; then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

Since these paths do not agree on the limit, the limit does not exist. □

Definition 11.2. The partial derivative of $f(x_1, x_2, \ldots, x_n)$ with respect to x_i is denoted $\frac{\partial f}{\partial x_i}$ and is obtained by treating all variables other than x_i constant.

12 September 13, 2018

We formally define the partial derivative $\partial f/\partial x_i$ of a function $f(x_1, x_2, \ldots, x_n)$ as

$$\frac{\partial f}{\partial x_i} = \lim_{\epsilon\to 0} \frac{f(x_1, \ldots, x_i + \epsilon, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\epsilon}$$

Basically we fix all variables but x_i and take the single variable derivative. Note that the partial derivative is another function of n variables. Some notations for the partial derivative are:

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} f = \partial_{x_i} f = \partial_i f = f_{x_i} = D_{x_i} f$$

Since differentiation is a function of n variables, the operation $\partial/\partial x_j\ (\partial f/\partial x_i)$ makes sense (the second partial derivatives). Some notation for this is

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial f}{\partial x_i x_j} = \partial_j(\partial_i f) = (f_{x_i})_{x_j} = f_{x_i x_j}$$

We technically need to memorize the order that the differentiation occurs in, but under reasonable conditions the order of differentiation doesn't matter.

Theorem 12.1 (Clairaut's Theorem)

Let $f\colon X \subset \mathsf{R}^n \to \mathsf{R}$ be a scalar-valued function of class C^k (at least on an open disk). Then the order in which we calculate any k-th order partial derivative does not matter. If $(i_1, \ldots, i_k)$ are any k integers less than n (not necessarily distinct), and if $(j_1, \ldots, j_k)$ is any permutation of these integers, then

$$\frac{\partial^k f}{\prod_l \partial x_{i_l}} = \frac{\partial^k f}{\prod_l \partial x_{j_l}}$$

Definition 12.1. We define the gradient vector of $f(x_1, \ldots, x_n)$ as the vector

$$\boldsymbol{\nabla} f(x_1, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

The gradient has a lot of interesting applications. For instance, write $x_{n+1} = f(\mathbf{x}) = f(x_1, \ldots, x_n)$ as a hypersurface in R^{n+1}. If f is differentiable at $\mathbf{a} = (a_1, \ldots, a_n)$, then the hypersurface has a tangent hyperplane at $(\mathbf{a}, f(\mathbf{a}))$ given by the equation

$$x_{n+1} = f(\mathbf{a}) + \boldsymbol{\nabla} f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

Let X be open in R^n and let $\mathbf{f}\colon X \to \mathsf{R}^m$ be a vector-valued function of n variables. We define the matrix of partial derivatives of $\mathbf{f}$, denoted $D\mathbf{f}$, to be the $m \times n$ matrix whose ijth entry is $\partial f_i/\partial x_j$, where $f_i\colon X \subseteq \mathsf{R}^n \to \mathsf{R}$ is the ith component function of $\mathbf{f}$. That is,

$$D\mathbf{f}(\mathbf{x}) = D\mathbf{f}(x_1, \ldots, x_n) = \begin{bmatrix} (\boldsymbol{\nabla} f_1)^T \\ \vdots \\ (\boldsymbol{\nabla} f_m)^T \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_j} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_j} & \cdots & \frac{\partial f_i}{\partial x_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_j} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Theorem 12.2

Suppose $X \subseteq \mathsf{R}^m$ and $T \subseteq \mathsf{R}^n$ are open and $\mathbf{f}\colon X \to \mathsf{R}^p$ and $\mathbf{x}\colon T \to \mathsf{R}^m$ are defined so that $\operatorname{range}\mathbf{x} \subset X$. If $\mathbf{x}$ is differentiable at $\mathbf{t}_0 \in T$ and $\mathbf{f}$ is differentiable at $\mathbf{x}_0 = \mathbf{x}(\mathbf{t}_0)$, then the composite $\mathbf{f} \circ \mathbf{x}$ is differentiable at $\mathbf{t}_0$, and we have

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0)D\mathbf{x}(\mathbf{t}_0)$$

This can be done arbitrarily many times using a recursion.

We now do some examples.

Example 12.1

Prove the product rule.

Solution Let $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$. Then we have

$$\frac{\partial f}{\partial x_j} = \prod_{\substack{i=1 \\ i \neq j}}^{n} x_i$$

By the chain rule, we have

$$Df(\mathbf{x}) = \sum_j \frac{\partial f}{\partial x_j} = \sum_j \prod_{i \neq j} x_i$$

So we are done by factorization. □

Example 12.2

Find a formula for implicit differentiation given a function $f(x_1, \ldots, x_n)$.

Solution By the chain rule we have

$$\frac{\partial x_j}{\partial x_i} = -\frac{\partial f/\partial x_i}{\partial f/\partial x_j}$$

□

13 September 18, 2018

Reminder: the chain rule, in a general form, is

$$\frac{\partial f(\mathbf{x}(\mathbf{t}))}{\partial t_j} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}$$

The Implicit Function Theorem guarantees that we can write one variable as a function of the other variables for any neighborhood around a given point. This can be used in implicit differentiation. Suppose we can solve $x_{n+1} = f(\mathbf{x})$. Then there is a function such that $F(\mathbf{x}, x_{n+1}(\mathbf{x}))$. Taking the derivative with respect to $\mathbf{x}$ yields

$$\frac{\partial}{\partial x_j} F(\mathbf{x}, x_{n+1}(\mathbf{x})) = \sum_i \frac{\partial x_{n+1}}{\partial x_i} \frac{\partial x_i}{\partial x_j}$$

For the two variable case this reduces down to the case that $\frac{\partial y}{\partial x} = -F_x/F_y$. This is easily extendable to a more general case.

More specifically:

Theorem 13.1 (Implicit Function Theorem)
Suppose $F(\mathbf{x}_0) = c$ and $\partial f/\partial x_i \neq 0$ for some i. Then for $\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$ the function $F(\mathbf{x})$ can also be represented by a continuous and differentiable function $x_i = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. In particular

$$\frac{\partial x_i}{\partial x_j} = -\frac{g_{x_j}}{g_{x_i}}$$

for all j.

Let $\mathbf{u}$ be a unit vector. How does a function $f(\mathbf{x})$ change in the direction of $\mathbf{u}$? This is easy to compute using vectors:

$$D_{\mathbf{u}} f = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{u}) - f(\mathbf{x})}{\epsilon} = \boldsymbol{\nabla} f \cdot \mathbf{u}$$

This is called the directional derivative of f in the direction of $\mathbf{u}$. The directional derivative is maximized in the direction of the gradient and minimized in the negative direction of the gradient. Note that gradients are perpendicular to level sets.

We have that the tangent plane to $f(\mathbf{x}) = h$ at $\mathbf{a}$ is just $\boldsymbol{\nabla} f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.

14 September 20, 2018

It is implied that a local extremum at $\mathbf{a}$ requires that $Df(\mathbf{a})$ vanishes. However, the converse is not true.

Definition 14.1. If $Df(\mathbf{a}) = \mathbf{0}$ then $\mathbf{a}$ is a critical point.

Definition 14.2. We say that f has a local maximum at $\mathbf{a}$ if $f(\mathbf{a}) > f(\mathbf{b})$ for $\mathbf{b}$ in a neighborhood of $\mathbf{a}$. We define the local minimum similarly.

Definition 14.3. We say that $f\colon \mathsf{R}^n \to \mathsf{R}$ is convex at $\mathbf{a}$ if and only if the domain is convex and the Hessian H

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_i \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_n} \end{bmatrix}$$

is positive definite (as in, the eigenvalues are all positive) at $\mathbf{a}$. If H is indefinite (has both positive and negative eigenvalues) then there is a saddle point; otherwise the test is inconclusive. Alternatively, prove that f has only zero or one local minima.

If $\mathbf{f}\colon \mathsf{R}^n \to \mathsf{R}^m$, the Hessian is defined as

$$H = (H(f_1), \ldots, H(f_m))$$

We apply the tests from the single variable second derivative tests in this way to achieve our minima and maxima.

15 September 25, 2018

Today we examine Lagrange multipliers, which are a form of constrained optimization techniques.

Example 15.1

Let's maximize $f(\mathbf{x})$ on the set $g(\mathbf{x}) = 0$. Previously we would solve for a single variable in g and substitute into f. This is mostly intractable because it is impossible or extremely cumbersome.

At the extreme point, $f(\mathbf{x}) = f_{\text{ext}}$ and $g(\mathbf{x}) = 0$ are tangent, because if they are not then the function has not reached its extrema. Note: the gradient of a function determines the normals to its level sets. Their normals are parallel, so $\nabla f = \lambda \nabla g$ at the extreme point. Thus our optimization procedure is, in $\mathbf{x}$,

$$\begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases}$$

where λ is a constant called the Lagrange Multiplier.

Example 15.2

Maximize xy on the set $\{x^2 + y^2 = 1\}$.

Solution We have

$$\nabla f = (y, x)$$

and

$$\nabla g = (2x, 2y)$$

We write

$$\nabla f = \lambda \nabla g \to \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

Now we just need to solve our system. To do this we add the equations and get

$$x + y = 2\lambda(x + y) \to x + y = 0 \quad \text{or} \quad \lambda = \frac{1}{2}$$

so we either have $x = -y$ or $x = y$ (by checking this with the first two equations). Then since $x = \pm y$ and $x^2 + y^2 = 1$ then $x^2 = y^2 = 1/2$so $x = \pm y = \pm 1/\sqrt{2}$. This generates four possible points. The points where the signs of x and y are the same generate the maximum $1/2$ while different signs yield the minimum $-1/2$. □

Example 15.3

Minimize $x^2 + y^2 + z^2$ on the set $\{xyz = 1 \mid x, y, z > 0\}$.

Solution Note that solving equations in terms of variables and substituting is bad because it breaks inherent symmetry. Thus we choose to use Lagrange Multipliers. We have the system

$$\nabla f = \lambda \nabla g \to \begin{cases} 2x = \lambda yz \\ 2y = \lambda xz \\ 2z = \lambda xy \\ xyz = 1 \end{cases}$$

Dividing in the first equation, we have $2x = \lambda/x$ since $xyz = 1$. Thus $x^2 = \lambda/2$ and similarly for the other variables. Since $x, y, z > 0$we can take square roots and show that they are all equal. Thus $x = y = z = 1$. We have $f(1, 1, 1) = 3$ which is our minimum. □

Upon being given the setup with multiple constraints we can generalize Langrange multipliers to optimize functions with n-dimensional domains under m constraints.

Theorem 15.1

Let $f(\mathbf{x})$ be of class C^1 on R^n and $c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_m(\mathbf{x})$. be constraint equations. Then under the constraint $c_1(\mathbf{x}) = k_1, c_2(\mathbf{x}) = k_2, \ldots, c_m(\mathbf{x}) = k_m$, we have the system

$$\begin{cases} \nabla f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \nabla c_i(\mathbf{x}) \\ c_1(\mathbf{x}) = k_1 \\ c_2(\mathbf{x}) = k_2 \\ \qquad \vdots \\ c_m(\mathbf{x}) = k_m \end{cases}$$

Solving this system for $\mathbf{x}$ gets the extrema of f on the constraint hypersurfaces. If there are saddle points, use the bounded Hessian to compute optima.

Example 15.4

Find the highest point on the section $x + y + z = 1$ of the cylinder $x^2 + y^2 = 1$.

Solution We express our solution as

$$\max z \text{ on } \begin{cases} x + y + z = 1 \\ x^2 + y^2 = 1 \end{cases}$$

We have $f = z$, $g = x + y + z$, $k = x^2 + y^2$. We use Lagrange multipliers to get the system

$$\begin{cases} 0 = \lambda + 2\mu x \\ 0 = \lambda + 2\mu y \\ 1 = \lambda \end{cases}$$

Hence $\lambda = 1$, $2\mu x = -1$, and $2\mu y = -1$ so $2\mu x = 2\mu y$ and $x = y$ (assuming $\mu \neq 0$). Thus we have $x = y = 1/\sqrt{2}$ or $z = y = -1/\sqrt{2}$, so $z = 1 \pm \sqrt{2}$. □

16 October 2, 2018

We cover the one-dimensional integral.

Let $f\colon [a, b] \to \mathbb{R}$ be an integrable function. Create n equally sized partitions across the interval $[a, b]$, demarcated by x_i. such that $x_1 = a$ and $x_n = b$. Then for each partition pick a point x_i^*. Our area is approximated by

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$. In the limit this becomes

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n} = \int_a^b f(x)\,\mathrm{d}x$$

Now we discuss two-dimensional integrals. Let $X = [a, b] \times [c, d] \subseteq \mathsf{R}^2$ be a rectangular region in R^2, and $f\colon X \to \mathbb{R}$ be an integrable function. The goal is to find the volume V below the graph of f. To do this we split the rectangle into mn partitions, demarcated by x_i and y_i respectively, and integrate the same way. Our area is approximated by

$$A \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_i^*)\Delta x_i \Delta y_j$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$. In the limit this becomes

$$A = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i^*, y_i^*)\Delta x_i \Delta y_j = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i^*, y_i^*)\frac{b-a}{n}\frac{d-c}{m} = \int_a^b \int_c^d f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

If we are lazy then we can express

$$\int_a^b \int_c^d f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_X f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Functions which are Riemann integrable in multiple dimensions are continuous functions and functions which have jumps on a curve.

The way to approach iterated integrals is to consider the following:

$$\int_c^d \int_a^b f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_c^d \left(\int_a^b f(x, y) \, \mathrm{d}x \right) \mathrm{d}y$$

Evaluating the inner integral yields a function of y, and integrating that gets a number.

Example 16.1

Compute

$$\int_0^2 \int_0^1 x\mathrm{e}^y \, \mathrm{d}x \, \mathrm{d}y$$

Solution We have

$$\int_0^2 \int_0^1 x\mathrm{e}^y \, \mathrm{d}x \, \mathrm{d}y = \int_0^2 \left[\frac{x^2}{2} \mathrm{e}^y \Big|_{x=0}^{x=1} \right] \mathrm{d}y = \int_0^2 \frac{1}{2} \mathrm{e}^y \, \mathrm{d}y = \left[\frac{1}{2} \mathrm{e}^y \Big|_0^2 \right] = \frac{e^2 - 1}{2}$$

□

Theorem 16.1 (Fubini)

The order of integration in an iterated integral does not matter.

We want to think of more general regions than rectangles to integrate over. For example, we can (in theory) integrate over any region in R^n.

Let X be a region in R^2 and Y be a rectangle in R^2 which fully encloses X. Let $f^{\text{ext}}(x, y) = f(x, y)$ if $(x, y) \in X$, and 0 otherwise. Then

$$\iint\limits_X f(x, y) \, \mathrm{d}A = \iint\limits_Y f^{\text{ext}}(x, y) \, \mathrm{d}A$$

More concretely, if the bounds on y can be expressed in terms of functions of x (or vice versa), we have

$$\iint\limits_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x \quad \text{or} \quad \iint\limits_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

To deal with more complicated domains, we divide the domain up into multiple elementary parts and hope for the best.

We compute the area of a tetrahedron.

Example 16.2

Compute the volume of a tetrahedron in the first octant with intercepts at $x = a$, $y = b$, $z = c$.

Solution Write

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

so

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

Our domain in the xy plane is given by the triangle cut off by the line $x/a + y/b = 1$, so$0 \leq x \leq a$ and $0 \leq y \leq b(1 - x/a)$ since we are choosing to operate on it as a type 2 region. Thus our integral is

$$\begin{aligned}
V &= \int_0^a \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^a \left[y\left(1 - \frac{x}{a}\right) - \frac{cy^2}{2b} \right|_0^{b(1-x/a)} \mathrm{d}x \\
&= \int_0^a \left(cb\left(1 - \frac{x}{a}\right)^2 - \frac{cb}{2}\left(1 - \frac{x}{a}\right)^2 \right) \mathrm{d}x \\
&= \left[\frac{-abc}{3}\left(1 - \frac{x}{a}\right)^3 \right|_0^a \\
&= \frac{1}{6}abc
\end{aligned}$$

□

17 October 4, 2018

In Cartesian double integrals, we have

$$\iint_D f(x,y) \, \mathrm{d}A = \iint_D f(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

where we integrate with respect to the area element $\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y$. This relationship indicates that we are really summing up over a bunch of small rectangles with sides $\mathrm{d}x$ and $\mathrm{d}y$.

Now we consider polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Our job is to investigate the double integral in polar coordinates.

When we derived the double integral in Cartesian coordinates, it was necessary to split up the domain into many uniform rectangles, where the vertical lines are where x is constant, and the horizontal lines are where y is constant.

Now take the simplest polar construction, a circle. It is easy to see that circles are where the radius r is constant and the radial lines are where the angle θ are constant. Thus for a given radial line and radius, the area swept out by infinitesimal perturbations in r and θ is the area element $\mathrm{d}A$. The sector swept out actually depends on r; if the infinitesimal sector is further from the origin, it will have a larger area. Indeed, a first approximation gets us

$$\Delta A = \frac{1}{2}\Delta\theta\left((r+\Delta r)^2 - r^2\right) \approx \frac{1}{2}\Delta\theta(2r\Delta r) = r\Delta r\Delta\theta$$

In the limit, the approximation becomes exact, so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. Another way we can look at this is that for small $\Delta\theta$, the swept out sector looks approximately like a rectangle, with sides Δr and $r\Delta\theta$. Again, this becomes precise in the limit, so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and the two formulas agree.

Example 17.1

Compute the volume of a sphere using polar coordinates.

Solution We first write it down in Cartesian coordinates:

$$x^2 + y^2 + z^2 = R^2$$

Consider the upper half of the sphere, where $z = \sqrt{R^2 - x^2 - y^2}$. In polar coordinates, this becomes $z = \sqrt{R^2 - r^2}$, where $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$. Then we have

$$\begin{aligned}
\frac{1}{2}V &= \iint\limits_D \sqrt{R^2 - x^2 - y^2}\,\mathrm{d}A \\
&= \iint\limits_D \sqrt{R^2 - r^2}\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^R r\sqrt{R^2 - r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\int_0^R r\sqrt{R^2 - r^2}\,\mathrm{d}r \\
&= \pi\int_0^{R^2}\sqrt{r}\,\mathrm{d}r \\
&= \pi\left[\frac{2}{3}r^{3/2}\right|_0^{R^2} = \frac{2\pi}{3}R^3
\end{aligned}$$

Since this is $V/2$, we get that the full radius is $V = 4\pi R^3/3$, which is true. □

This suggests an overall strategy for dealing with polar integrals: examine the domain and see if it's possible to use polar coordinates, change variables in the function, and replace the area element.

Example 17.2

Compute the area of one of the two parts of $r = \cos^2\theta$.

Solution We have (from the midterm) that $-\pi/2 \leq \theta \leq \pi/2$. The way to compute area of a region is $A = \iint_D \mathrm{d}A$. We switch to polar coordinates:

$$A = \iint\limits_D 1\,\mathrm{d}A = \iint\limits_D r\,\mathrm{d}r\,\mathrm{d}\theta$$

We want to represent the domain D in polar coordinates. We have $-\pi/2 \leq \theta \leq \pi/2$ and $0 \leq r \leq \cos^2\theta$. Hence we have

$$A = \int_{-\pi/2}^{\pi/2} \int_0^{\cos^2\theta} r \, \mathrm{d}r \, \mathrm{d}\theta = \int_{-\pi/2}^{\pi/2} \frac{(\cos^2\theta)^2}{2} \, \mathrm{d}\theta = \frac{3\pi}{8}$$

$\square$

We compute the volume under bidimensional Gaussian. Boring.

18 October 9, 2018

Today we talk about $(n-1)$-measures of surfaces. Say we have $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))$. We defined $\mathrm{d}s = \|\mathrm{d}\mathbf{r}/\mathrm{d}t\| \, \mathrm{d}r$. Thus the arc length is

$$S = \int_a^b \mathrm{d}s = \int_a^b \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t$$

Take the two dimensional case. We can parameterize with respect to x, so we can write $x = (x, y(x))$. Then $\mathrm{d}\mathbf{r}/\mathrm{d}x = (1, \mathrm{d}y(x)/\mathrm{d}x)$. Then $\mathrm{d}s = \sqrt{1 + (\mathrm{d}y(x)/\mathrm{d}x)^2} \, \mathrm{d}x$.

Now we move to multiple variables. Say we have $\Gamma\colon x_{n+1} = f(x_1, \ldots, x_n)$, with $f\colon D \subseteq \mathsf{R}^n \to \mathsf{R}$. The goal is to find the n-measure of the surface.

To do this we parameterize the domain into unit n-hyperrectangles with n-measure $\Delta\zeta$. As $\Delta\zeta \to 0$, the surface approximates an $(n+1)$-hyperplane, so the projection of D onto Γ is a paralleletope of dimension n. Of course, this paralleletope can be computed through the exterior product of n vectors $(x_1, x_2, \ldots, x_n, x_n(x_1, \ldots, x_n))$, $(x_1 + \Delta x_1, x_2, \ldots, x_n, x_{n+1}(x_1 + \Delta x_1, \ldots, x_n))$, and so on until $(x_1, x_2, \ldots, x_n + \Delta x_n, x_{n+1}(x_1, \ldots, x_n + \Delta x_n))$. We get that the difference of the ith point with the source point (with no Δs involved) zeros out all terms but the ith term (leaving Δx_i in that place), and the last coordinate, which is $(\partial x_{n+1}/\partial x_i)\Delta x_i$. This shows that we have a good linear approximation. Our exterior products are thus relatively trivial to compute.

To find the measure ζ, we have

$$\zeta = \int_D \sqrt{1 + \sum_{i=1}^n \left(\frac{\partial x_{n+1}}{\partial x_i}\right)^2} \, \mathrm{d}\zeta$$

In the two-dimensional case this results in

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, \mathrm{d}A$$

We now discuss a motivating example.

Example 18.1

Say we have a surface given by

$$S = \begin{cases} z = x^2 + y^2 \\ x^2 + y^2 \leq 1 \end{cases}$$

Compute the surface area of S.

Solution We note that the surface area is the 2-measure of 3-space, so our idea still applies. We first compute the area element; note that $f(x, y) = x^2 + y^2$, and $f_x = 2x$ and $f_y = 2y$:

$$\mathrm{d}s = \sqrt{1 + 4x^2 + 4y^2}\,\mathrm{d}x\,\mathrm{d}y \to S = \iint_D \sqrt{1 + 4x^2 + 4y^2}\,\mathrm{d}x\,\mathrm{d}y$$

We use polar coordinates.

$$S = \int_0^{2\pi} \int_0^1 r\sqrt{1 + 4r^2}\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \int_0^1 r\sqrt{1 + 4r^2}\,\mathrm{d}r = \frac{\pi}{4} \int_1^5 \sqrt{r}\,\mathrm{d}r = \left[\frac{\pi}{6} r^{3/2}\right]_1^5 = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

□

Let's see how we can change variables arbitrarily in double integrals. Suppose we have a function $f\colon D \subseteq \mathsf{R}^n \to \mathsf{R}$, where points in D are mapped onto (in a surjection) by another set $C \subseteq \mathsf{R}^m$. We can write

$$f(\mathbf{x}) = f(\mathbf{x}(\mathbf{t})), \quad \mathbf{x} \in D, \quad \mathbf{t} \in C$$

We aim to find the measure-scaling element of the transformation $\mathbf{x}\colon C \to D$ as a function of either $\mathbf{x}$ or $\mathbf{t}$. In general we form an m- hyperrectangle in C and apply the transformation to each of the vertices, getting an n-paralleletope. We then compute the n-measure of this paralleletope by taking exterior products, giving us the same linear approximation as above.

If $m = n$, then the Jacobian determinant

$$J = \left|\frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_n)}\right| = \left|\det\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial t_1} & \cdots & \frac{\partial \mathbf{x}}{\partial t_j} & \cdots & \frac{\partial \mathbf{x}}{\partial t_n}\end{bmatrix}\right| = \left|\det\begin{bmatrix} \partial x_1/\partial t_1 & \ldots & \partial x_1/\partial t_j & \ldots & \partial x_1/\partial t_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial x_i/\partial t_1 & \ldots & \partial x_i/\partial t_j & \ldots & \partial x_i/\partial t_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial x_n/\partial t_1 & \ldots & \partial x_n/\partial t_j & \ldots & \partial x_n/\partial t_n \end{bmatrix}\right|$$

can be used to find this n-measure-scaling-form. Computing the case for $m \neq n$ is harder; one needs to go through all of the steps.

Example 18.2

Find the Jacobian of the polar coordinate transformation.

Solution We write $x = r\cos\theta$ and $y = r\sin\theta$, so

$$J = \left|\frac{\partial(x, y)}{\partial r\theta}\right| = \left|\det\begin{bmatrix}\cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{bmatrix}\right| = r$$

so $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, as previously known. □

Example 18.3

Compute

$$\iint_D \sin\left((x + y)^2\right)\mathrm{d}x\,\mathrm{d}y$$

over the domain $D = \{(x, y)\colon 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$.

Solution We use the change of variables $u = x + y$ and $v = x - y$. We have a system $(u, v) = f(x, y)$, but we want $(x, y) = g(u, v)$. So a bit of algebra gets us $x = (u + v)/2$ and $y = (u - v)/2$.

We compute the Jacobian:

$$J = \left| \det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \right| = \frac{1}{2}$$

so $\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v$. In addition, our domain is transformed into $D = \{0 \le u \le 1, -u \le v \le u\}$. We get

$$I = \frac{1}{2}\int_0^1 \int_{-u}^{u} \sin\left(u^2\right) \mathrm{d}v\,\mathrm{d}u = \int_0^1 u \sin\left(u^2\right) \mathrm{d}u = \left[-\cos\left(u^2\right)\right|_0^1 = \frac{1}{2}(1 - \cos(1))$$

□

19 October 11, 2018

Today we cover triple integrals, which are basically the same as double integrals with a three-dimensional domain. The theory is basically the same. Say we have a closed figure $D \in \mathsf{R}^3$ and some function f that we wish to integrate over it, so D is our domain. Then our triple integral is

$$I = \iiint\limits_D f(x, y, z)\,\mathrm{d}V$$

Introduce a 3-dimensional partition (naturally done using parallelepipeds) on D and pick a point in each partition. Then the sum of these images of these points becomes the integral of the function in the limit as the partition volume goes to zero. Note that this can be done in R^n for a function $f(\mathbf{x})$ using the same principle.

Example 19.1

Compute the volume of a parallelepiped $P = [a, b] \times [c, d] \times [r, s]$.

Solution We have

$$I = \iiint\limits_P f(x, y, z)\,\mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\,\mathrm{d}V$$

Here because of our partitions being parallelepipeds the volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in that order (note the limits of integration). So we have

$$I = \int_r^s \int_c^d \int_a^b f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

□

Note that if f is separable this just becomes the product of three single-dimensional integrals, or one two-dimensional integral and two one-dimensional integrals. Fubini's theorem still applies.

We can integrate over more domains than just rectangles. For instance, take $R = \{(x, y) \in D \mid \alpha(x, y) \le z \le \beta(x, y)\}$. Then

$$\iiint\limits_R f(x, y, z)\,\mathrm{d}V = \iint\limits_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x, y, z)\,\mathrm{d}z \right) \mathrm{d}A$$

We can do similar with x and y to integrate over arbitrary 2-dimensional domains.

Example 19.2

We have a region R bounded by the first quadrant and the plane $x+y+z=2$. Compute

$$\iiint\limits_R xyz \,\mathrm{d}V$$

Solution Our domain is $R = \{(x,y) \in D \mid 0 \le z \le 2-x-y\}$. Thus our integral becomes

$$I = \iiint\limits_R xyz \,\mathrm{d}V = \iint\limits_D \left(\int_0^{2-x-y} xyz \,\mathrm{d}z\right) \mathrm{d}A = \iint\limits_D \left[\frac{xyz^2}{2}\Bigg|_{z=0}^{z=2-x-y}\right] \mathrm{d}A = \frac{1}{2}\iint\limits_D \left(xy(2-x-y)^2\right) \mathrm{d}A$$

Now we write the domain $D = \{0 \le x \le 2 \mid 0 \le y \le 2-x\}$. So our integral is

$$I = \frac{1}{2}\iint\limits_D \left(xy(2-x-y)^2\right) \mathrm{d}A = \frac{1}{2}\int_0^2 \int_0^{2-x} \left(xy(2-x-y)^2\right) \mathrm{d}y \,\mathrm{d}x = \frac{4}{45}$$

by Mathematica. □

Suppose we have a solid $\Gamma \in \mathsf{R}^3$ with a mass density $\rho(x,y,z)$. We have that the mass of Γ is given by

$$m = \iiint\limits_\Gamma \rho(x,y,z) \,\mathrm{d}V$$

We can discuss the three first moments M_x, M_y, M_z:

$$M_x = \iiint\limits_\Gamma x\rho(x,y,z) \,\mathrm{d}V\,, \quad M_y = \iiint\limits_\Gamma y\rho(x,y,z) \,\mathrm{d}V\,, \quad M_z = \iiint\limits_\Gamma z\rho(x,y,z) \,\mathrm{d}V$$

We can find the center of mass $\overline{\mathbf{r}} = (\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_x}{m}, \quad \overline{y} = \frac{M_y}{m}, \quad \overline{z} = \frac{M_z}{m}$$

We can put the moments of inertia I_x, I_y, I_z as

$$I_x = \iiint\limits_\Gamma \left(y^2+z^2\right) \rho(x,y,z) \,\mathrm{d}V\,, \quad I_y = \iiint\limits_\Gamma \left(x^2+z^2\right) \rho(x,y,z) \,\mathrm{d}V\,, \quad I_z = \iiint\limits_\Gamma \left(x^2+y^2\right) \rho(x,y,z) \,\mathrm{d}V$$

20 October 16, 2018

Today we will explore coordinate systems in 3 dimensions. We begin with cylindrical coordinates.

Cylindrical coordinates map (x,y) to (r,θ), but leave z the same. Thus $(x,y,z) \mapsto (r,\theta,z)$. We have

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

and the inverse:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \\ z = z \end{cases}$$

Now we deal with integration in cylindrical coordinates. We have $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Since z is the same, $\mathrm{d}z$ is the same. However, we can convert $\mathrm{d}x\,\mathrm{d}y$ to $r\,\mathrm{d}r\,\mathrm{d}\theta$ since it is the same transformation as in polar coordinates. Thus the volume element in cylindrical coordinates is $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$. Thus our conversion is

$$\iiint\limits_D f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_D f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

We now consider the case of spherical coordinates. These map $(x, y, z) \mapsto (\rho, \theta, \phi)$, where θ is the angle on the xy plane and ϕ is the angle between the z axis and the xy plane. We can express these in terms of cylindrical coordinates:

$$\begin{cases} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases}$$

and from this the direct conversion from Cartesian coordinates follows:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

and the inverse transformation is given by

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan(y/x) \\ \phi = \arctan\left(z/\sqrt{x^2 + y^2}\right) \end{cases}$$

We now consider integration in spherical coordinates. We have the volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. By converting to cylindrical coordinates and doing some algebraic conversions we have $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$.

The range of the angles θ and ϕ are different. That is, $0 \leq \theta \leq 2\pi$, as expected. But $0 \leq \phi \leq \pi$, since we only need to consider half the sphere; the other half is already counted by the varying θ.

21 October 23, 2018

We already covered how to write single, double, and triple integrals. We have also talked about curves and surfaces. We can think about curves as intervals that are not necessarily straight. Now we will cover the formulation of integrals over a curve:

$$\int\limits_\gamma f(\mathbf{x})\,\mathrm{d}s$$

Later, we will cover the formulation of integrals over a surface:

$$\iint\limits_\zeta f(\mathbf{x})\,\mathrm{d}A$$

This all ties into the fundamental theorem of calculus. We know the single variable analogue; the value of the integrated function at the endpoints of the interval is equal to the integral over the interval of the integrand.

The largest multivariate generalization is Green's theorem, where we consider the perimeter of a surface to be the surface's endpoints.

Theorem 21.1 (Green's Theorem)

For a given orientable manifold Ω with boundary $\partial\Omega$, the integral of a differential form ζ over the whole manifold is equal to the integral of its exterior derivative $d\zeta$ over the whole of Ω, i.e.

$$\int_{\partial\Omega} \zeta = \int_{\Omega} d\zeta$$

Now we discuss line integrals. We start with the theory of vector fields. Consider a field $\mathbf{V}(\mathbf{x})$ in R^n. Then we can also write this in the form

$$\mathbf{V}(\mathbf{x}) = \sum_{i=1}^{n} V_i(\mathbf{x})\mathbf{e}_i$$

The way we represent vector fields is by arrows (usually in R^2 or R^3).

Some vector fields we use in real life are the gravitational force field, the electric force fields, fluid velocity fields, and so on.

Let $f\colon \mathsf{R}^n \to \mathsf{R}$ map $\mathbf{x} \mapsto f(\mathbf{x})$. Then we see that $\boldsymbol{\nabla} f$ is a vector field:

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

We call such vector fields conservative. In particular, a vector field is conservative if it can be written as the gradient of a function.

Not all vector fields are conservative. For example, $\mathbf{V}(x, y) = -y\mathbf{e}_1 + x\mathbf{e}_2$ is not conservative. We can show this easily.

Proof If $-y\mathbf{e}_1 + x\mathbf{e}_2$ is a vector field, then $(-y, x) = (f_x, f_y)$ for some $\mathbf{f}$. We know that by Clairaut's theorem, $f_{xy} = f_{yx}$. Thus we differentiate by y on the left side and x on the right side, so we get $-1 = 1$, obviously false. ■

The gravitational vector field $\mathbf{F} = k\mathbf{r}/\|\mathbf{r}\|^3$ is conservative, by the same argument.

Now we move onto line integrals. Let γ be a curve in R^n with length $L = \int_\gamma \mathrm{d}s$, and parameterized by $\mathbf{r}(t)$ for t in some real interval I. Then $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t$. Then

$$L = \int\limits_{\gamma} \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int\limits_{\gamma} \sqrt{\sum_{i=1}^{n}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2}$$

In a sense, this is the most obvious line integral. In general,

$$\int\limits_{\gamma} f(\mathbf{x})\,\mathrm{d}s = \int_I f(\mathbf{x}(\mathbf{t}))\|\mathbf{r}'(t)\|\,\mathrm{d}t$$

There is another type of line integral. Let $i \in [1, n]$. Then:

$$\int\limits_{\gamma} f(\mathbf{x})\,\mathrm{d}x_i = \int\limits_{I} f(\mathbf{x}) x_i'(t)\,\mathrm{d}t$$

More generally, say we are computing the integral

$$I = \int\limits_{\gamma} \sum_{i=1}^{n} V_i(\mathbf{x})\,\mathrm{d}x_i$$

Write $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ as a vector field. Then

$$I = \int\limits_{\gamma} \mathbf{V}(\mathbf{x}) \cdot \mathrm{d}\mathbf{r}$$

For example, if we have a force field and want to compute the work done by it, we can do

$$W = \int\limits_{\gamma} \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{r}$$

Note that the orientation of the curve matters i.e. reversing orientation turns the result to its negative.

22 October 25, 2018

Last time we learned how to compute two types of integrals: arc length integrals

$$\int\limits_{C} f(\mathbf{x})\,\mathrm{d}s$$

and line integrals

$$\int\limits_{C} \mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$$

Our discussion today will match the fundamental theorem of calculus. Let $I \subseteq \mathsf{R}$ be an interval whose endpoints are in the real numbers, and $F\colon I \to \mathbb{R}$ send each point in I to a real number. Let f be any antiderivative of F; then we have

$$\int_a^b F(x)\,\mathrm{d}x = f(b) - f(a)$$

There is one class of vector fields for which we can apply this same analysis. Recall that a vector field $\mathbf{F}$ is conservative if $\mathbf{F}$ is the gradient of a function i.e. $\mathbf{F} = \boldsymbol{\nabla} f$. For these vector fields, we have a direct analogue to the fundamental theorem of calculus.

Theorem 22.1 (Green's Theorem)
Let $\mathbf{F} = \boldsymbol{\nabla} f$ be a conservative vector field. Then over the path $C \subset \mathsf{R}^n$ which has endpoints $\mathbf{A}$ and $\mathbf{B}$, we have

$$\int\limits_{C} \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = f(\mathbf{B}) - f(\mathbf{A})$$

Proof Let C be parameterized by $\mathbf{x}(t)$, where $t \in [a, b]$. Then

$$\begin{aligned}
\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} &= \int\limits_C \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \\
&= \int_a^b \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) \, \mathrm{d}t \\
&= \left[f(\mathbf{x}(t)) \right|_{t=a}^{t=b} = f(\mathbf{B}) - f(\mathbf{A})
\end{aligned}$$

□

One consequence of this is shown.

Corollary 22.1

We use the same notation as above. Let C_1 and C_2 be any two curves in R^n. If $\mathbf{F} = \boldsymbol{\nabla} f$ is conservative, then

$$\int\limits_{C_1} \mathbf{F} \cdot \mathrm{d}x = \int\limits_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

This is called path independence. That is, if $\mathbf{F}$ is conservative, then its line integral is path independent (only depends on the endpoints).

The converse question is if the line integral is path independent, is the vector field necessarily conservative? It turns out that the answer is yes. The solution is to draw two curves C_1 and C_2 whose union is a closed curve C. Then

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int\limits_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} - \int\limits_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

The line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is path independent if and only if $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$ for every closed curve C. Now define f such that $F_i = \partial f / \partial x_i$, and so the partials of F_i are mostly mixed for all i. Set $f(\mathbf{0}) = \mathbf{K}$ and define

$$f(\mathbf{B}) = \int_{\mathbf{0}}^{\mathbf{A}} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

For this choice,

$$\int_{\mathbf{0}}^{\mathbf{B}} f(\mathbf{B}) - f(\mathbf{0})$$

Now define a new point $\mathbf{A}$. We have

$$\int_{\mathbf{0}}^{\mathbf{B}} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{\mathbf{0}}^{\mathbf{A}} \mathbf{F} \cdot \mathrm{d}\mathbf{x} + \int_{\mathbf{A}}^{\mathbf{B}} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = f(\mathbf{B}) - f(\mathbf{0}) - f(\mathbf{A}) + f(\mathbf{0}) = f(\mathbf{B}) - f(\mathbf{A})$$

We thus get

$$\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = f(\mathbf{B}) - f(\mathbf{A})$$

Now use a horizontal line C as our path. Then

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int\limits_C \left(\sum_{i=1}^{n} F_i \, \mathrm{d}x_i \right) = \int_{x_0}^{x_1} F_1 \, \mathrm{d}x = f(x_1, y_1) - f(x_0, x_0)$$

By the fundamental theorem of calculus, we have $F_i(\mathbf{x}) = \partial f(\mathbf{x})/\partial x_i$. Thus $\mathbf{F} = \boldsymbol{\nabla} f$ and we're done.

In R^2, it's easy to see that $\mathbf{F} = (P, Q)$ is conservative if and only if $\partial P/\partial y = \partial Q/\partial x$. This is due to Clairaut's theorem.

In all cases, our domains are strictly open, since we define partial derivatives on open disks.

We define a connected space as a set which cannot be represented as the union of two or more disjoint nonempty open subsets. In the context of this course, if we can travel between any two points without leaving the domain, the domain is connected.

We define a simply connected space as a connected space with genus zero. If we can define a C^1 diffeomorphism between the space and a single point that preserves continuity, the space is simply connected. More intuitively, the space cannot have holes.

Theorem 22.2

We deal now with R^2. If $\mathbf{F} = (P, Q)$ is defined on $D \subset \mathsf{R}^2$, where D is simply connected, then $P_y = Q_x$ implies that $\mathbf{F}$ is conservative.

23 November 8, 2018

Skipped lectures on Green's theorem. For completeness, it is

Theorem 23.1

Let $\mathbf{F} \colon \mathsf{R}^2 \to \mathsf{R}^2$ be a vector field and Γ be a surface in R^2. Then

$$\oint\limits_{\partial\Gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \iint\limits_{\Gamma} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}$$

Today we discuss parametric surfaces. Recall that a parametric curve $\mathbf{r} = \mathbf{r}(t)$ is a vector function $\mathsf{R} \to \mathsf{R}^n$. In this case, $t \in [a, b] \subseteq \mathsf{R}$ is a parameter.

A parametric surface $\mathbf{r}(u, v)$ is a vector function $\mathsf{R}^n \to \mathsf{R}^n$. We have that $u \in [a, b] \subseteq \mathsf{R}$ and $v \in [c, d] \subseteq \mathsf{R}$ are parameters.

Note that while the codomain of $\mathbf{r}$ is R^n, the range of $\mathbf{r}$ is k-dimensional, where k is the number of parameters $\mathbf{r}$ takes.

Example 23.1

THe plane through $\mathbf{r}_0$ extending in the directions $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

It is important that $\mathbf{a}$ and $\mathbf{b}$ be linearly independent.

Example 23.2

The parameterization of a cylinder lying along the x-axis with radius a is given by $\mathbf{r}(x, \theta) = (x, a\cos\theta, a\sin\theta)$.

Example 23.3

Parameterize the surface given by $z = f(x, y)$.

Solution Choose $\mathbf{r}(x, y) = (x, y, f(x, y))$. □

Example 23.4

Parameterize a sphere of radius a.

Solution If we use spherical coordinates, we have $\mathbf{r}(\phi, \theta) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$. We can interchange rows and so on as desired. □

If a parameterization obeys the following qualities, we call it *good*:

- $\mathbf{r} = \mathbf{r}(u, v)$ (can be written as a bivariate function)
- $\mathbf{r}(u, v)$ is differentiable
- $\mathbf{r}_u, \mathbf{r}_v$ are never 0
- $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ (that is, $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent.)

The derivatives $\mathbf{r}_u, \mathbf{r}_v$ describe the changes along the surface Γ according to some peturbation in the domain. These derivatives describe small shifts along Γ. In particular $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to Γ; their span is the tangent plane to Γ at a particular point.

Example 23.5

Find the tangent plane to the sphere with radius a centered at the origin.

Solution We compute

$$\mathbf{r}(\phi, \theta) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$$

We compute

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0)$$

and

$$\frac{\partial \mathbf{r}}{\partial \phi} = (a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi)$$

We can represent the tangent plane ω thusly either by writing

$$\omega = \left\{\mathbf{x} \mid \mathbf{x} \in \mathsf{S}\left\{\frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi}\right\}\right\}$$

or also by finding the normal vector $\mathbf{n}$

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}$$

Another way to solve this is to write

$$S_2 = \{\mathbf{x} \mid \|\mathbf{x}\| = a^2\}$$

which describes a function $f(\mathbf{x})$. Then the normal vector $\mathbf{n} = \boldsymbol{\nabla} f(\mathbf{x})$. $\square$

We now consider integrals over parametric surfaces.

Question How do we compute the area of a parametric surface?

Answer Imagine drawing a small rectangle in the uv-plane with side lengths $\mathrm{d}u$ and $\mathrm{d}v$. When the diffeomorphism $\mathbf{r}$ projects this rectangle onto the surface parameterized by $\mathbf{r}$, it will still – at least locally – have the properties of a rectangle. Its side lengths will now be $\mathbf{r}_u \,\mathrm{d}u$ and $\mathbf{r}_v \,\mathrm{d}v$. Call the area of this projected rectangle $\mathrm{d}A$. The area vector of this rectangle is

$$\mathrm{d}\mathbf{A} = (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v$$

with the scalar area defined similarly:

$$\mathrm{d}A = \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}u\,\mathrm{d}v$$

We are now finally ready to discuss the area of a parametric surface. We have

$$A = \iint_D \mathrm{d}A = \iint_{(u,v)\in D} \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}u\,\mathrm{d}v$$

We could write the area of a surface defined by a function $z = f(x, y)$ as

$$A = \iint_D \sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

as shown some time back. We attempt to unify them. Use the parameterization $\mathbf{r}(x, y) = (x, y, f(x, y))$. Then $\mathbf{r}_x = (1, 0, f_x)$ and $\mathbf{r}_y = (0, 1, f_y)$. We have

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix} = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

and so

$$\|\mathbf{r}_x \times \mathbf{r}_y\| = \left\| \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Hence the two areas are the same.

Example 23.6

Compute the surface area of the sphere of radius a.

Solution We use the parameterization as above. We have

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{bmatrix} -a \sin\theta \sin\phi \\ a \cos\theta \sin\phi \\ 0 \end{bmatrix} \times \begin{bmatrix} a \cos\theta \cos\phi \\ a \sin\theta \cos\phi \\ -a \sin\phi \end{bmatrix} = -a^2 \begin{bmatrix} \cos\theta \sin^2\phi \\ \sin\theta \sin^2\phi \\ \cos\phi \sin\phi \end{bmatrix}$$

and so

$$\left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \right\| = a^2 \sin\phi$$

Then

$$A = \iint_D \mathrm{d}A = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = 4\pi a^2$$

□

Example 23.7

Compute the surface area of the paraboloid

$$z = x^2 + y^2, \quad x^2 + y^2 \le 1$$

Solution We use the parameterization $\mathbf{s}(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. We compute $\mathbf{s}_r = (\cos\theta, \sin\theta, 2r)$, and $\mathbf{s}_\theta = (-r\sin\theta, r\cos\theta, 0)$. Hence

$$\frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 2r \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{bmatrix}$$

so

$$\left\| \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} \right\| = \sqrt{4r^4 + r^2}$$

and

$$A = \iint\limits_D \mathrm{d}A = \int_0^{2\pi} \int_0^1 \sqrt{4r^4 + r^2}\, \mathrm{d}r\, \mathrm{d}\theta$$

□

24 November 13, 2018

Today we are going to begin surface integration.

When we did line integrals, we discussed two types of line integrals: arclength integrals:

$$\int\limits_C f(x, y, z)\, \mathrm{d}s\,, \quad \mathrm{d}s = \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t$$

and vector line integrals:

$$\int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}\,, \quad \mathrm{d}\mathbf{s} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t$$

where the second integral is taken counterclockwise.

Let Γ be a surface in R^3. Then we compute

$$\iint\limits_\Gamma f(x, y, z)\, \mathrm{d}A$$

Last time we saw the area can be computed by

$$A = \iint\limits_\Gamma \mathrm{d}A$$

where we do the actual computation using a parameterization φ sending two parameters $(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$. We discussed that $\mathbf{r}_u$ and $\mathbf{r}_v$ are two tangent vectors to Γ and $\mathbf{r}_u \times \mathbf{r}_v$ is the normal vector to Γ. The differential element is given by

$$\mathrm{d}A = \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| \mathrm{d}u\,\mathrm{d}v = \|\mathbf{n}\|\,\mathrm{d}u\,\mathrm{d}v$$

Now we can compute surface integrals. Our earlier surface integral is given by

$$\iint\limits_{\Gamma} f(x, y, z)\,\mathrm{d}A = \iint\limits_{(u,v)\in D} f(x(u, v), y(u, v), z(u, v))\,\mathrm{d}u\,\mathrm{d}v$$

A simple case is when Γ is the graph of a function $z(x, y)$. Then we parameterize $(x, y) \mapsto (x, y, z(x, y))$. We have $\mathbf{r}_x = (1, 0, z_x)$ and $\mathbf{r}_y = (0, 1, z_y)$. Their cross product is $(-z_x, -z_y, 1)$, which has norm $\sqrt{1 + z_x^2 + z_y^2}$.

These are direct analogues of the first kind of line integrals.

Now we want to figure out what is the second kind of line integrals. To do this, we want to talk about the orientation of surfaces. This is a little more complex because some surfaces are not orientable. In general, if we can pick a consistent "up" or "down" – specifically, if the function mapping a point on the surface to a normal vector on the surface is continuous.

> **Definition 24.1.** A surface is orientable if and only if there exists a continuous normal vector field sending a point on the surface to an appropriate normal vector.

A mobius strip is non orientable, but a cylinder is orientable.

After determining whether a surface is orientable, we still need to pick an orientation.

Say we have a surface Γ given by $\Gamma = \{(x, y, z) \mid z = f(x, y)\}$. Then the standard parameterization $\mathbf{r}_x \times \mathbf{r}_y = (-f_x, -f_y, 1)$. Consider the orientation corresponding to the parameterization $(x, y) \mapsto (y, x)$. Then $\mathbf{r}_y \times \mathbf{r}_x = (f_x, f_y, -1)$. So, orientation matters.

Another example is the surface Γ given by $\Gamma = \{\mathbf{r} \mid \mathbf{r} = \mathbf{r}(u, v), (u, v) \in D\}$. If we pick good parameters, we have $\mathbf{r}_u, \mathbf{r}_v \neq 0$, so they are linearly independent. Then $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ is the normal vector to the surface at a given point.

It is helpful to now consider the unit normal vector:

$$\boldsymbol{\nu} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

The good orientation is one that points to the outside.

Let Γ be a manifold in R^{n+1} and $\partial\Gamma$ its boundary. The idea is that we assign an orientation recursively to $\partial\Gamma$, and call that orientation the positive orientation. Then we assign Γ an orientation pointing to its exterior, past the boundary. This is only possible if $\partial\Gamma$ admits an orientation.

Now we are ready to tackle the case of vector surface integrals:

$$I = \iint\limits_{\Gamma} \mathbf{F} \cdot \boldsymbol{\nu}\,\mathrm{d}A = \iint\limits_{\Gamma} \left(\sum_{k=1}^{n} F_i \frac{\partial \boldsymbol{\nu}}{\partial x_i}\right) \mathrm{d}A$$

To get the first form from the second form, note that

$$\iint\limits_{\Gamma} \mathbf{F} \cdot \boldsymbol{\nu} \, \mathrm{d}A = \iint\limits_{\Gamma} \sum_{\text{sym}} F_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

This can be found by computing the normal vector directly. In particular

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} = \frac{\partial(y,z)}{\partial(u,v)}\mathbf{i} - \frac{\partial(x,z)}{\partial(u,v)}\mathbf{j} + \frac{\partial(x,y)}{\partial(u,v)}\mathbf{k}$$

giving the symmetric sum as claimed.

Now we discuss the flux. Consider the velocity vector field $\mathbf{v}(\mathbf{x})$ of some fluid, and $\rho(\mathbf{x})$ be the density of the fluid. We define the flux $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{x}) = \iint\limits_{\partial\Gamma} \rho(\mathbf{x})\mathbf{v} \cdot \boldsymbol{\nu} \, \mathrm{d}A$$

This quantity is useful in fluid dynamics, thermodynamics, and electromagnetism. As an example, the heat flux is given by

$$\mathbf{F} = -\iint\limits_{\partial\Gamma} c\boldsymbol{\nabla}\mathbf{u} \cdot \boldsymbol{\nu} \, \mathrm{d}A$$

25 November 27, 2018

We have covered Green's and Stokes' theorems, a form of the generalized Stokes' theorem. Let Γ be an orientable manifold in R^n with boundary $\gamma = \partial\Gamma$. Let D be a subset of R^n and $\mathbf{F} \colon D \to \mathsf{R}^n$ be a smooth vector field with exterior derivative $d\mathbf{F}$ (in two or three dimensions, this would be the curl) defined in a neighborhood containing Γ. Also, let $\mathrm{d}\boldsymbol{\gamma}$ be the differential vector element corresponding to γ, and $\mathrm{d}\gamma$ the differential scalar element corresponding to the $(n-1)$-measure. Define $\mathrm{d}\boldsymbol{\Gamma}$ and $\mathrm{d}\Gamma$ the same way for the n-measure.

Theorem 25.1 (Green's Theorem)

Use the above notation, and let $n = 2$ with Γ embedded in R^2. Then

$$\oint\limits_{\gamma} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\gamma} = \iint\limits_{\Gamma} \boldsymbol{\nabla} \times \mathbf{F} \, \mathrm{d}A$$

We can use a change of variables here. Let $(\mathrm{d}x\,, \mathrm{d}y) \mapsto (-\,\mathrm{d}y\,, \mathrm{d}x)$. Then by componentwise analysis,

$$\oint\limits_{\gamma} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\gamma = \iint\limits_{\Gamma} \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}A$$

The generalization of the first form takes place in Stokes' Theorem:

Theorem 25.2 (Stokes' Theorem)
Use the above notation, and let $n = 2$ with Γ embedded in R^3. Then

$$\oint_{\Gamma} \mathbf{F} \cdot \mathrm{d}\gamma = \iint_{\Gamma} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

This generalizes in a straightforward way:

Theorem 25.3 (Generalized Stokes' Theorem)
Use the above notation.

$$\oint_{\gamma} \mathbf{F} \cdot \mathrm{d}\gamma = \int_{\Gamma} d\mathbf{F} \cdot \mathrm{d}\mathbf{\Gamma}$$

There's also a possible direct generalization of Green's theorem to multiple dimensions using the second formulation, with minimum $n = 3$.

Theorem 25.4 (Divergence Theorem)
Use the above notation.

$$\oiint_{\gamma} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\gamma = \iiint_{\Gamma} (\boldsymbol{\nabla} \cdot \mathbf{F}) \, \mathrm{d}\Gamma$$

Remark 1 One physical interpretation of the divergence theorem is as follows.

We work in R^n. Suppose we have a velocity vector field $\mathbf{v}(\mathbf{x})$ and density $\rho(\mathbf{x})$. Let $D \subset \mathsf{R}^n$ be our region Γ. We want to know the rate of change of the quantity within Γ. This is called the flux of the quantity, and is denoted by ϕ. It is given by the vector equation

$$\phi = \oint_{\gamma} \rho \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\gamma$$

By the divergence theorem,

$$\phi = \iint_{\Gamma} \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) \, \mathrm{d}\Gamma$$

A word on approach: the theorems are all going to be on exams. The important thing to do is consider what the manifold in question looks like.

- Line integral over non-closed curve: parameterize to convert it to a single-variable regular integral.
- Line integral over closed curve: use Green's theorem to convert it to an area integral over the manifold enclosed by the curve.
- Surface integral over non-closed surface with closed boundary: use Stokes' theorem to convert it to a line integral over the boundary.
- Surface integral over closed surface: use the Divergence Theorem to convert it to a volume integral over the manifold enclosed by the surface.