Model-Free Quantiles and Market Returns

Conall O'Sullivan and Yan Wang

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Abstract

Rare disaster risk embedded in the short-term option contracts is priced in the long-term equity risk premium (ERP). A novel formula is proposed to identify the risk-neutral return quantiles from European option prices in a model-free manner. Moreover, we show the symmetric quantile difference (QD) at extreme probability levels is a highly sensitive gauge of the latent stochastic disaster probability under the model of Wachter (2013), and we use QDs to forecast the ERP. The QD at 5/95% is highly persistent and significantly forecasts the ERP of more than one year, which suggests that QD is a proxy of the long-term ERP constituent compensating for the rare disaster risk. In contrast, the variance risk premium (VRP) of Bollerslev, Tauchen and Zhou (2009) which is a proxy of the stochastic volatility of volatility risk, is significant in forecasting the ERP of less than one year.

Conall O'Sullivan: Michael Smurfit Graduate Business School, University College Dublin, Carysfort Avenue, Blackrock, Co. Dublin, Ireland, conall.osulivan@ucd.ie. Yan Wang is on the job market. yanwang.eir@gmail.ie.
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1. Introduction

Fixed-income markets facilitate the pricing of elementary contingent claims along the time dimension, whereas options markets augment the pricing of elementary contingent claims along the state dimension offering more refined price information. We study the content of this refined information using quantiles inferred from the option prices. The contribution of our study is three fold. First, we show the risk-neutral return quantiles can be estimated in a model-free manner using European options. The model-free quantiles complement the existing model-free measures which are now widely used for inference of the risk-neutral distributions that contain ex-ante information. Second, we study the information content of the risk-neutral quantiles estimated from a market index. Quantiles decompose the price information of options along the state dimension, which is an alternative of the study that focuses on the decomposition of return processes according to the continuous and discontinuous components, hence can provide potentially richer information on the shape of the pricing kernel. Third, the risk-neutral quantile estimated using real-time market prices of options is a natural reference of the risk measure called economic value-at-risk (VaR). Traditional VaR calculations do not take into account the economic valuation when the loss occurs which can change dramatically with the market condition. The risk-neutral quantile enhances the traditional VaR in this regard by weighting the loss outcomes by the marginal utility of a representative agent.

The risk-neutral quantile estimator we propose utilises the objective function of regression quantiles (Koenker (2005)), which turns the estimation into a convex optimisation problem, bypassing any of the parametric assumptions required on the data generating process. Hence it belongs to the family of model-free measures, e.g. Carr and Madan (2001), Bakshi et al. (2003), Britten-Jones and Neuberger (2000), Carr and Wu (2009), Du and Kapadia (2011), Martin (2017), Todorov (2019). The model-free measures are calculated using a continuum of options at different strikes. These measures have gained increased popularity over the years in the academic research, e.g. Han (2008), Conrad et al. (2013), Bollerslev et al. (2009), Andersen et al. (2015). And the methodology has been widely accepted in the financial industry to construct one of the most popular indexes, i.e. VIX as a fear gauge (Demeterfi et al. (1999)).

However, inference of the optioned asset prices through the means of model-free measures remains a challenging task, given the rich dynamics they have. For instance, the presence of jumps in price would distort the convergence of model-free integrated variance. Jiang and Tian (2005), Carr and Wu (2009) show that the model-free integrated variance is a close approximate of the risk-neutral quadratic variation up to the second order of jump size. Du and Kapadia (2012) point out when the optioned asset price follows a Merton (1976) jump-diffusion process, the model-free variance of Bakshi, Kapadia and Madan (2003) is in effect equivalent to the risk-neutral quadratic variation. And they devise a model-free jump measure under this assumption. Bondarenko (2014) shows that the model-free integrated variance closely resembles the risk-neutral realised variance computed from daily returns, which underlies the settlement price of most volatility instruments. More recently, Todorov (2019) proposes a model-free spot volatility estimator based on the conditional characteristic function and develops its central limit theorem.

A second challenge to the model-free measures is their finite sample performance. That is, their rigorous equivalence to the risk-neutral measures we intend to estimate hinges on a continuum of options that have strikes spanning all possible states of the optioned asset. Early study on this topic utilises mostly the approach of simulation and the conclusions are genuinely model-dependent, e.g. Dennis and Mayhew (2002), Jiang and Tian (2005). But the formal analysis is still sparse up to date, e.g. Andersen et al. (2015), Martin (2017). Estimation of the model-free measures is therefore almost always accompanied by nonparametric estimators to augment the sample size and to retain the most flexibility from available market prices.

We follow the approach of Figlewski (2008) using a quartic spline to fit the implied volatilities of option prices at a fixed maturity, and the fitted curve passes through the bid-ask spreads. This approach has been used in the study by Birru and Figlewski (2012), Linn, Shiv and Shumway (2017) to estimate the time-series of state-price densities, which evidences its robustness. The model-free quantiles are then estimated through the prices of put-call pairs at different strikes. This makes them different from other potential solutions that are based on the seminal paper of Breeden and Litzenberger (1978). For instance, one could numerically integrate the risk-neutral density function or differentiate the cumulative function and find the quantile estimates. But both of these approaches require knowledge on the risk-free discounting and using the first approach one has to make extra assumptions on the tail decay.

We estimate the time-series of return quantiles for the S&P 500 index from January 1996 to June 2019. The 5 percentile and 95 percentile exhibit the most stylish pattern. They vary mildly in a symmetric manner during the tranquil periods of the market but sharply deviate during financial crises with the 5 percentile declines much more than the increase of the 95 percentile. Based on this observation, we hypothesise the difference between 5 and 95 percentiles is a proxy of the rare disaster risk embedded in the option prices.

More formally, we solve for the risk-neutral quantiles under the stochastic disaster risk model of Wachter (2013). The choice of Wachter model is because of three reasons. First, the model generates time-varying risk-neutral quantiles, as opposed to being static (Backus et al. (2011)). Second, the model uses a Poisson process with stochastic jump intensity to emphasise the role of stochastic disaster risk, which is essential for reproducing the variation of extreme quantile differences we have observed. Third, the model is exhaustive and solves the equity risk premium (ERP), dividend price ratio (DP) and realised variance (RV) simultaneously. Hence the model can provide a reference to our following results.

After showing the difference between 5 and 95 percentile is a highly sensitive gauge of the stochastic disaster probability, we test whether it can forecast the future ERP. The quantile difference at 5/95% is highly persistent with an AR(1) coefficient equal to 0.79. Because an increase in the disaster risk increases the mean of ERP based on Wachter's study and further if our disaster risk proxy is persistent, this is expected to turn into the long-term predictability of the ERP (Cochrane (2009)). We find that the quantile differences possess strong predictive power of the future ERP. The predictive power is especially strong, in univariate predictive regressions, for quantile differences from at 10/90% to at 20/80% and for future ERP between two to five years. The extreme quantile difference at 5/95% is less significant but because of the larger standard deviation, which is consistent with what Wachter's model implies. The extreme quantile difference also has a high correlation coefficient with the dividend price ratio at 35%, consistent with the finding of Campbell and Shiller (1989) that the variation of the dividend price ratio is driven mainly by the variation of future discount rates.

The rest of the paper is organised as follows. Section 2 explains the model-free quantiles. Section 3 describes the estimation of quantiles and presents the estimates for the S&P 500 index. Section 4 discusses the risk-neutral quantiles under Wachter model. Section 5 reports the predictive regression results. The final sections concludes.

2. Model-Free Quantiles

The model-free quantile estimator is based on the following proposition. It is a generalisation of the objective function used to estimate the regression quantiles in Koenker (2005) to the context of option pricing. As no assumption is made regarding the data generating process[1], except for the implicit assumption of no-arbitrage and consequently, the existence of a risk-neutral measure (Duffie (2001)), the estimator is 'model-free'.

Proposition 1. The τth quantile of a risk-neutral distribution is the solution to the following minimisation problem

$$\min_{K \in \mathbb{K}} \; (1-\tau)\, Put(K,T) + \tau\, Call(K,T)\,,$$

where K denotes the strike, T denotes the expiry date, and $\mathbb{K}$ denotes the space of strikes.

Proof 1. The objective function is

$$(1-\tau)\, e^{-r_f T} \int_0^K (K - S_T)\, dQ(S_T) + \tau\, e^{-r_f T} \int_K^\infty (S_T - K)\, dQ(S_T)\,,$$

where r_f denotes the risk-free rate, $Q(S)$ denotes the risk-neutral distribution function of the optioned asset price. Because the objective function is convex by no-arbitrage condition, the first order condition with respective to K is a global solution,

$$0 = (1-\tau) \int_0^K dQ(S) - \tau \int_K^\infty dQ(S) = Q(K) - \tau\,.$$

[1] More rigorously, the fundamental theorem of asset pricing states that the existence of an equivalent martingale measure for the data generating process of asset returns is essentially equivalent to the absence of arbitrage opportunities (Dybvig and Ross (1987)), and the process should be a semimartingale (Delbaen and Schachermayer (1994)).

3. Risk-Neutral Quantiles of the S&P 500 Index

3.1 Data

The options data we use is from OptionMetrics. The sample period is from January 1996 to June 2019. We focus on the contant 30-day risk-neutral quantiles which can be estimated using the most liquid options, and our choice of the S&P 500 index options sample follows the VIX methodology[2]. Before October 6, 2014, a near-term (at least one week) and a next-term SPX option chains are included in the sample. Since then, the sample also includes Friday settled Weeklys[3], and the sample contains two option chains with more than 23 days but no more than 37 days to maturity. We also make adjustments to maturity calculations to distinguish between AM and PM settlements. AM settled SPX options are counted one day less than PM settled SPXW options. For SPX options, we use the settlement date as opposed to the expiration date to calculate maturity. The sample is further cleaned by eliminating any zero bid quotes, excluding any more out-of-the-money options once two zero bid quotes are observed, where an at-the-money option is defined by its strike price that equals the forward price[4].

[2]See the VIX white paper and the Information Circular IC14-075 from CBOE for further information.

[3]A number of measures are taken to ensure a smooth transition to Weekly options including taking into account CBOE expiry adjustments due to holidays to identify Friday settled Weeklys using the SPXW symbol prefix in OptionMetrics. Further detail is available from the authors upon request.

[4]We use the forward price provided by OptionMetrics directly which is different from the CBOE methodology. Occasionally, the OptionMetrics forward price field has missing records for some of the option strikes. We backfill these missing records using valid forward price records which match the date and expiration fields. Further detail on the difference between OptionMetrics and CBOE forward prices can be found in the OptionMetrics manual and the VIX white paper.

3.2 Nonparametric smoothing

We follow Figlewski (2008) to augment the strike space by a quartic spline that smoothly passes through the bid-ask spreads of options market quotes. The robustness of the approach has been evidenced by the study of Birru and Figlewski (2012) and, more recently, Linn, Shiv and Shumway (2017).

For each date within the sample, for a fixed tenor, we first discard very deep out-of-the-money options using a minimum bid cutoff of 50 cents. We combine put and call samples by first retaining the out-of-the-money and at-the-money options. The bid, ask and mid-point quotes are converted into implied volatilities (IV) using the Black-Scholes formula. A weighted average of these IVs is then taken in a region ±20 of the forward price, according to

$$\mathrm{IV}_{\mathrm{blend}}(K) = w\,\mathrm{IV}_{\mathrm{put}}(K) + (1-w)\,\mathrm{IV}_{\mathrm{call}}(K)\,,$$

where

$$w = \frac{K_{\mathrm{high}} - K}{K_{\mathrm{high}} - K_{\mathrm{low}}}\,.$$

K_{high} and K_{low} are respectively the highest and lowest strikes within the region. Afterwards, a quartic spline with a single break placed at the forward price is fitted to the mid-point IVs by weighted least square. The weights are specified using a Gaussian kernel

$$w(\mathrm{IV}) = \begin{cases} G\left(\dfrac{\mathrm{IV} - \mathrm{IV}_{\mathrm{ask}}}{h}\right) & \text{if } \mathrm{IV} \geq \mathrm{IV}_{\mathrm{mid}} \\ G\left(\dfrac{\mathrm{IV}_{\mathrm{bid}} - \mathrm{IV}}{h}\right) & \text{if } \mathrm{IV} < \mathrm{IV}_{\mathrm{mid}}\,. \end{cases}$$

Following Figlewski (2008), we set $h = 0.001$, which gives full penalty to deviations outside the bid-ask spreads but little to those within.

3.3 A demo

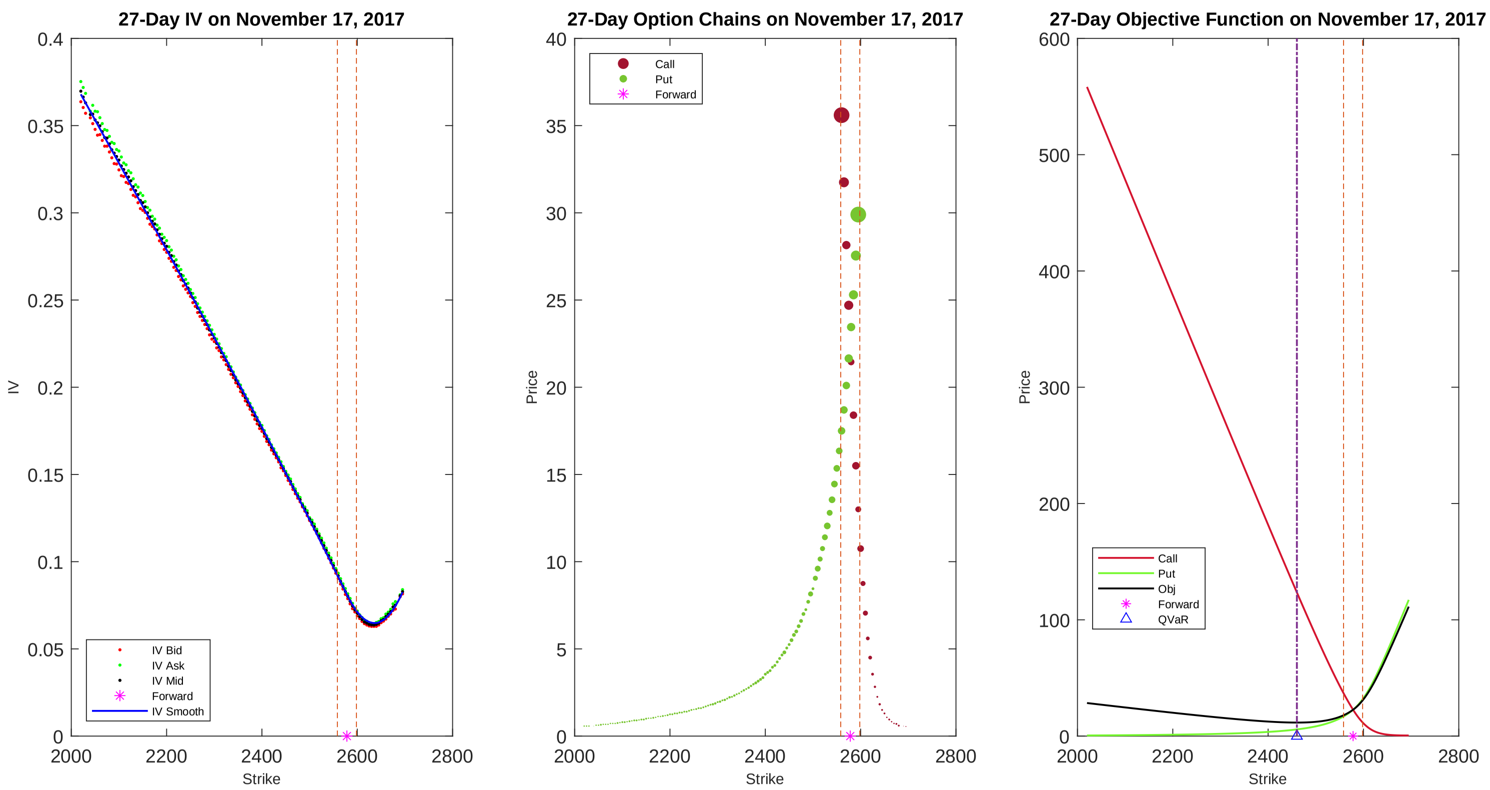
Figure 1 shows the steps that are taken to estimate the 5 percentile of the S&P 500 index levels on November 17, 2017. The options belong to the near-term chain with 27 days to expiry. The sample consists of out-of-the-money puts and calls and a mixture of both in the region between the dashed lines (± 20 of the forward price). The left panel depicts the Black-Scholes implied volatilities (IV) that correspond to the bid, ask and mid-point prices, and a smooth IV curve extracted using the approach of Figlewski (2008) is overlaid. The middle panel depicts the mid-point prices of the options, and the marker size indicates the magnitude of the bid-ask spread. The right panel depicts the put and call prices that correspond to the smooth IV curve, the objective function of the quantile estimator at the 5% probability level, and a vertical line that indicates the solution.

We repeat this estimation on each date, for near- and next-term option chains separately from January 1996 to June 2019. To obtain constant 30-day quantile estimates, we linearly interpolate two quantile estimates based on the near-term and next-term option chains.

3.4 Time-series of the S&P 500 index quantiles

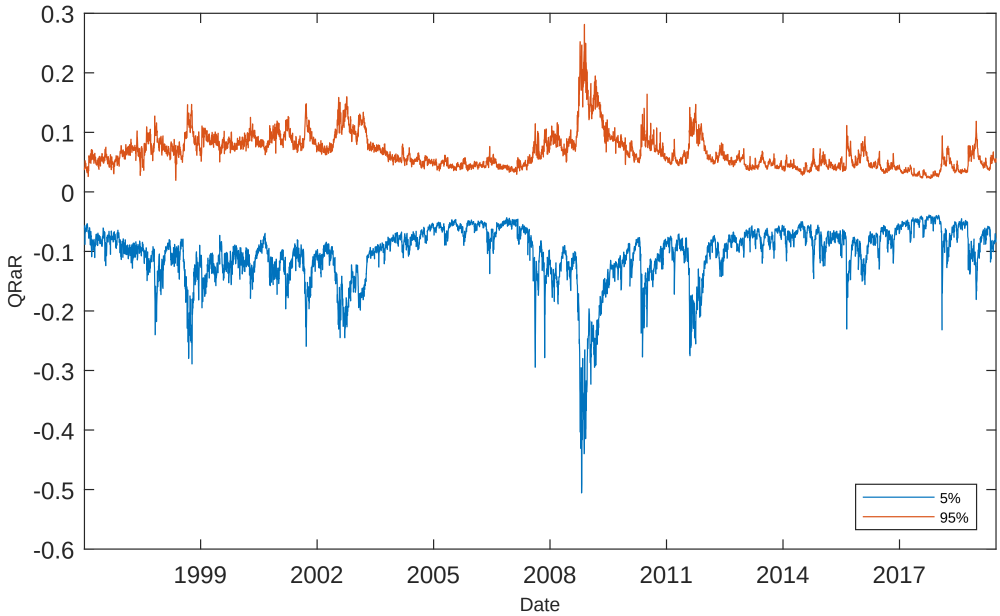
Denote by QVaR the solution to Proposition 1. We define QRaR as $\log \mathrm{QVaR} - \log S$. Figure 2 depicts the time series of 30-day $\mathrm{QRaR}_{5\%}$ and $\mathrm{QRaR}_{95\%}$ for the S&P 500 index from January 1996 to June 2019 on daily basis. On average, $\mathrm{QRaR}_{5\%}$ is larger in absolute magnitude than $\mathrm{QRaR}_{95\%}$. This is as expected, because the pricing kernel is downward sloping which reflects higher marginal utility in the disastrous states of the economy. The evolution of $\mathrm{QRaR}_{5\%}$ and $\mathrm{QRaR}_{95\%}$ is mild and symmetric about half the time within the sample period, which is driven by the diffusion of the return process. But during the remained half of the sample period which corresponds roughly to the shape declines of the S&P 500 index, or periods of the financial crises, these two quantiles deviate remotely from each other, and $\mathrm{QRaR}_{5\%}$ increases much more in absolute amount than $\mathrm{QRaR}_{95\%}$. These strong asymmetries cannot be explained by the volatility component alone as the Girsanov theorem states.

Figure 1: Estimation of Risk-Neutral Quantiles



The option sample consists of out-of-the-money S&P 500 index puts and calls and a mixture of both for the region between the dashed lines. The left panel depicts Black-Scholes implied volatilities (IV) corresponding to option bid, ask and mid-point prices. The smoothed IV curve extracted using the method of Figlewski (2008) is overlaid. The middle panel depicts the mid-point prices of the option sample. The size of the marker indicates the magnitudes of the bid-ask spreads. The right panel depicts put and call prices corresponding to the smoothed IV curve, and the objective function of QVaR estimator.

Figure 2: Risk-Neutral Quantiles of the S&P 500 Index



Daily 30-day risk-neutral return quantiles at 5% and 95% probability levels of the S&P 500 index from January 1996 to June 2019. The estimates are based on linear interpolants of the near- and next-term quantile estimates.

4. Risk-Neutral Quantiles under Wachter Model

Wachter (2013) proposes a stochastic disaster risk model that generates realistic values for the ERP using a reasonable risk aversion parameter. The model results in excess stock market volatility, and the dividend-price ratio positively predicting the ERP. All of these stylised features of the markets are reproduced with a model that is calibrated to international data on consumption declines without reference to market data itself. In this model stochastic disaster risk is positively associated with the time-varying ERP. However, stochastic disaster risk is a latent process and very difficult to measure using time series data. In this section we show that in the context of the Watcher model, the 5/95% quantile difference is very sensitive to, and positively associated with, this latent stochastic disaster risk. This then motivates the subsequent empirical section where we forecast the ERP with the 5/95% quantile difference.

4.1 Change of measure

Wachter (2013) models the aggregate consumption (the endowment) as a stochastic jump process

$$dC_t = \mu C_{t^-} dt + \sigma C_{t^-} dB_t + (e^{Z_t} - 1)C_{t^-} dN_t,$$

where B_t is a standard Brownian motion and N_t is a Poisson process with time-varying intensity λ_t. The intensity follows the process

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t},$$

where $B_{\lambda,t}$ is also a Brownian motion, and B_t, $B_{\lambda,t}$, and N_t are assumed to be independent. Z_t has distribution ν, which is time-invariant and independent of B_t, $B_{\lambda,t}$, and N_t.

Seo and Wachter (2018) argue, the aggregate dividend claim in equilibrium under Wachter (2013) model can be represented as follows, using a log-linear approximation of the price-dividend ratio

$$\log \frac{S_T}{S_t} = \phi \log \frac{C_T}{C_t} + b_\phi^*(\lambda_T - \lambda_t), \tag{1}$$

where S is the aggregate dividend claim price, C is the aggregate consumption, ϕ is the leverage (see Abel (1999)), b_ϕ^* is fixed for a given λ^*, it is a weighted average of the coefficients $b_\phi(\tau)$, where the average is over τ, and the weights are proportional to $\exp\{a_\phi(\tau) + b_\phi(\tau)\lambda^*\}$. Given this representation, $\log S$ under the subjective $\mathbb{P}$ measure behaves as follows,

$$d \log S_t = \left[\phi(\mu - \frac{1}{2}\sigma^2) + b_\phi^* \kappa(\bar{\lambda} - \lambda_t)\right] dt + \phi\sigma dB_t + b_\phi^* \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t} + \phi Z_t dN_t \tag{2}$$

where we have used the fact that by Ito's Lemma (see Protter (2005)),

$$d \log C_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t + Z_t dN_t$$

under $\mathbb{P}$ measure.

The state-price deflator of Wachter (2013) model is given in Seo and Wachter (2018) as

$$\pi_t = \exp\left(\eta t - \beta b \int_0^t \lambda_s ds\right) \beta^\gamma C_t^{-\gamma} e^{a+b\lambda_t},$$

where

$$
\begin{aligned}
a &= \frac{1-\gamma}{\beta}\left(\mu - \frac{1}{2}\gamma\sigma^2\right) + (1-\gamma)\log\beta + b\frac{\kappa\bar{\lambda}}{\beta}, \\
b &= \frac{\kappa+\beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa+\beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]}{\sigma_\lambda^2}}, \\
\eta &= \beta(1-\gamma)\log\beta - \beta a - \beta,
\end{aligned}
$$

E_ν denotes the expectation with respect to the distribution ν of Z_t, γ is relative risk aversion, and β is the rate of time preference. By definition of the state-price deflator, and denote by r the risk-free rate,

$$
\xi_t = \exp\left(\int_0^t r_s ds\right)\frac{\pi_t}{\pi_0}
$$

is a martingale which defines the density process or Radon-Nikodým process of an equivalent measure $\mathbb{Q}$ of the subjective measure $\mathbb{P}$, see Duffie (2010) or Shreve (2004) for full detail. In particular, for any $\mathscr{F}_T$-measurable random variable W, where $T > t$, and such that $E^{\mathbb{Q}}(|W|) < \infty$,

$$
E_t^{\mathbb{Q}}(W) = \frac{E_t^{\mathbb{P}}(\xi_T W)}{\xi_t}.
$$

Furthermore, $\mathbb{Q}$ is a martingale measure for the discounted process $\exp(-\int_0^T r_s ds)W_T$. Hence $\mathbb{Q}$ is also commonly referred as the equivalent martingale measure. In Wachter (2013), the risk-free rate is

$$
r_t = \beta + \mu - \gamma\sigma^2 + \lambda_t E_\nu\left[e^{-\gamma Z}\left(e^Z - 1\right)\right].
$$

Hence the density process is

$$
\xi_t = \exp\left\{\int_0^t (\eta + r_s - \beta b\lambda_s)\, ds\right\}\exp\left\{\gamma\log C_0 - b\lambda_0 - \gamma\log C_t + b\lambda_t\right\}.
$$

According to the Proposition 5 of Duffie et al. (2000), the state vector $[\log C, \lambda]$ should satisfy the following stochastic differential equations under $\mathbb{Q}$,

$$
\begin{aligned}
d\log C_t &= (\mu - (\frac{1}{2} + \gamma)\sigma^2)dt + \sigma dB_t^{\mathbb{Q}} + Z_t^{\mathbb{Q}} dN_t^{\mathbb{Q}}, \\
d\lambda_t &= (\kappa\bar{\lambda} + (b\sigma_\lambda^2 - \kappa)\lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t}^{\mathbb{Q}},
\end{aligned}
$$

where $B_t^{\mathbb{Q}}$ and $dB_{\lambda,t}^{\mathbb{Q}}$ are independent Brownian motions. $Z_t^{\mathbb{Q}}$ has a fixed probability distribution $\nu^{\mathbb{Q}}$, and $N_t^{\mathbb{Q}}$ is a Poisson process with intensity $\lambda_t^{\mathbb{Q}} = \exp(-\mu_j\gamma + \frac{1}{2}\sigma_j^2\gamma^2)\lambda_t$. For

$\nu^{\mathbb{P}} \sim \mathcal{N}\left(\mu_j, \sigma_j^2\right)$, $\theta^{\mathbb{Q}}(c) = \int_{\mathbb{R}} \exp(cz) d\nu^{\mathbb{Q}}(z) = \exp(\mu_j c + \frac{1}{2}\sigma_j^2 c^2 - c\gamma\sigma_j^2)$, which says $\nu^{\mathbb{Q}} \sim \mathcal{N}\left(\mu_j - \gamma\sigma_j^2, \sigma_j^2\right)$.

Therefore, according to (1), $\log S$ under the risk-neutral measure $\mathbb{Q}$ behaves as follows,

$$\begin{aligned} d\log S_t &= \mu_{\log S}^{\mathbb{Q}} dt + \phi\sigma dB_t^{\mathbb{Q}} + b_\phi^* \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}^{\mathbb{Q}} + \phi Z_t^{\mathbb{Q}} dN_t^{\mathbb{Q}}, \\ d\lambda_t &= (\kappa\bar{\lambda} + (b\sigma_\lambda^2 - \kappa)\lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}^{\mathbb{Q}}, \end{aligned}$$

where $\mu_{\log S}^{\mathbb{Q}} = \phi\mu - (\frac{1}{2} + \gamma)\phi\sigma^2 + b_\phi^* \kappa\bar{\lambda} + (b\sigma_\lambda^2 - \kappa) b_\phi^* \lambda_t$.

4.2 Fourier transform

The Fourier inversion theorem states,

$$F_X(x) = P(X \leq x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}\phi_X(u)}{iu} du,$$

where F_X is the cumulative distribution function of X, and $\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} dF_X(x)$ defines the characteristic function of X. Taking the derivative of $F_X(x)$ and using the symmetry of characteristic functions one can obtain the probability density function,

$$f_X(x) = \frac{1}{\pi} \int_0^{\infty} \Re\left[e^{-iux}\phi_X(u)\right] du.$$

Once the solution of $\phi_X(u)$ is obtained, F_X can be solved via numerical integration of f_X which does not have a singularity. Subsequently, the quantile function can be obtained. Note $\lambda_t^{\mathbb{Q}} = \exp(-\mu_j\gamma + \frac{1}{2}\sigma_j^2\gamma^2)\lambda_t$. This also allows us to bridge from risk-neutral quantiles to ERP, DP and RV, where the former is defined under $\mathbb{Q}$ measure and subject to $\mathbb{Q}$ shocks (indirectly to $\mathbb{P}$ shocks), but the latter are subject to $\mathbb{P}$ shocks.

The Fourier transform of $\log S_T$ under $\mathbb{Q}$ corresponds to transform ψ^χ in Duffie et al. (2000), with $X_t = \begin{bmatrix} \log S_t \\ \lambda_t \end{bmatrix}$, $u = \begin{bmatrix} iu \\ 0 \end{bmatrix}$. The characteristic is $\chi = (K, H, l, \theta, \rho)$, with $K_0 = \begin{bmatrix} \mu_c^{\mathbb{Q}} \\ \kappa\bar{\lambda} \end{bmatrix}$, $\mu_c^{\mathbb{Q}} = \phi\mu - (\frac{1}{2} + \gamma)\phi\sigma^2 + b_\phi^*\kappa\bar{\lambda}$, $K_1 = \begin{bmatrix} 0 & (b\sigma_\lambda^2 - \kappa)b_\phi^* \\ 0 & (b\sigma_\lambda^2 - \kappa) \end{bmatrix}$, $\rho_0 = 0$, $\rho_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H_0 = \begin{bmatrix} \phi^2\sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$, $H_1 = \begin{bmatrix} (0, {b_\phi^*}^2\sigma_\lambda^2) & (0, b_\phi^*\sigma_\lambda^2) \\ (0, b_\phi^*\sigma_\lambda^2) & (0, \sigma_\lambda^2) \end{bmatrix}$, $l_0 = 0$, $l_1 = \begin{bmatrix} 0 \\ \exp(-\mu_j\gamma + \frac{1}{2}\sigma_j^2\gamma^2) \end{bmatrix}$, $\theta^{\mathbb{Q}}(c) = \int_{\mathbb{R}^2} \exp(c \cdot z) d\nu^{\mathbb{Q}}(z) = \exp\left(\left(\mu_j - \gamma\sigma_j^2\right)\phi c_1 + \frac{1}{2}\phi^2\sigma_j^2 c_1^2\right)$, with $c \in \mathbb{C}^2$.

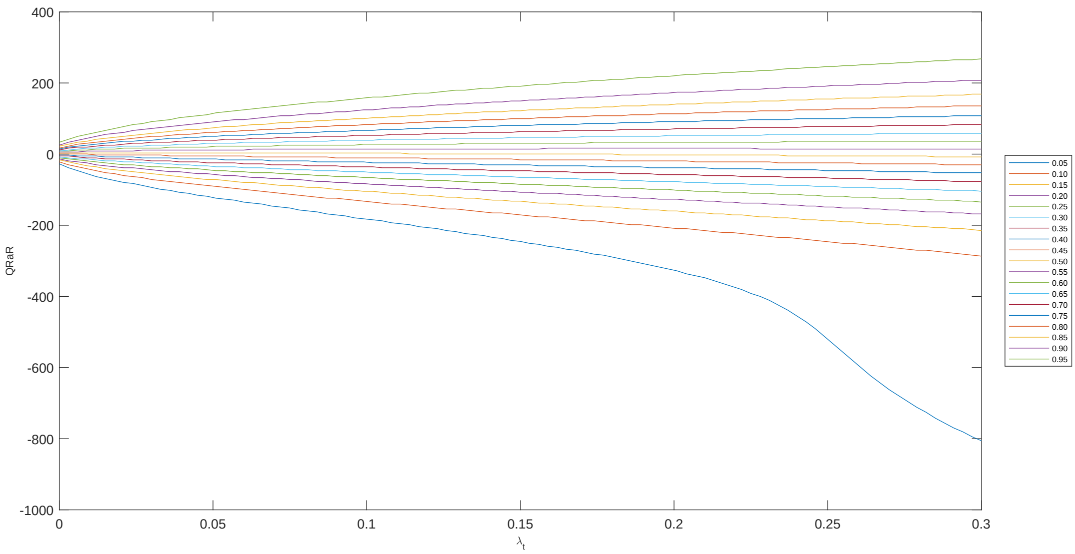
By their Proposition 1, $\psi^{\chi}(u, X_t, t, T) = E^{\chi}\left(e^{iu\log S_T}|\mathscr{F}_t^{\mathbb{Q}}\right) = e^{\alpha(t)+\beta(t)\cdot X_t}$, where

$$\dot{\beta}(t) = \rho_1 - K_1^T\beta(t) - \frac{1}{2}\beta(t)^T H_1\beta(t) - l_1(\theta(\beta(t)) - 1),$$
$$\dot{\alpha}(t) = \rho_0 - K_0\cdot\beta(t) - \frac{1}{2}\beta(t)^T H_0\beta(t) - l_0(\theta(\beta(t)) - 1).$$

with $\beta(T) = \begin{bmatrix} iu \\ 0 \end{bmatrix}$, $\alpha(T) = 0$. Proof of sufficiency is by showing that when α and β satisfy the above partial differential equations, $\Psi_t = \exp\left(-\int_0^t R(X_s)ds\right)e^{\alpha(t)+\beta(t)\cdot X_t}$ is a martingale. The solution of $\beta(t)$ involves a Riccati equation that can not be solved in close-form. We thus use Runge-Kutta to solve the PDEs numerically at a fine grid of u and invert the resulting characteristic function by nonuniform fast Fourier transform.

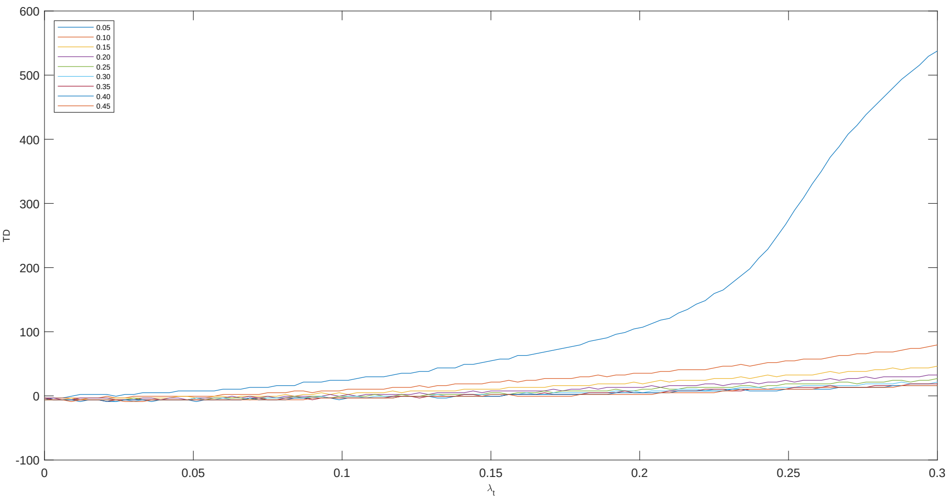
4.3 The solution

Figure 3: Risk-Neutral Quantiles and λ_t



The 30-day risk-neutral return quantiles at various λ_t levels under the model of Wachter (2013). The parameter values are based on Wachter's calibration, and the jump size of log consumption growth follows $\mathcal{N}(-0.3, 0.0225)$.

Figure 4: Symmetric Risk-Neutral Quantile Differences and λ_t



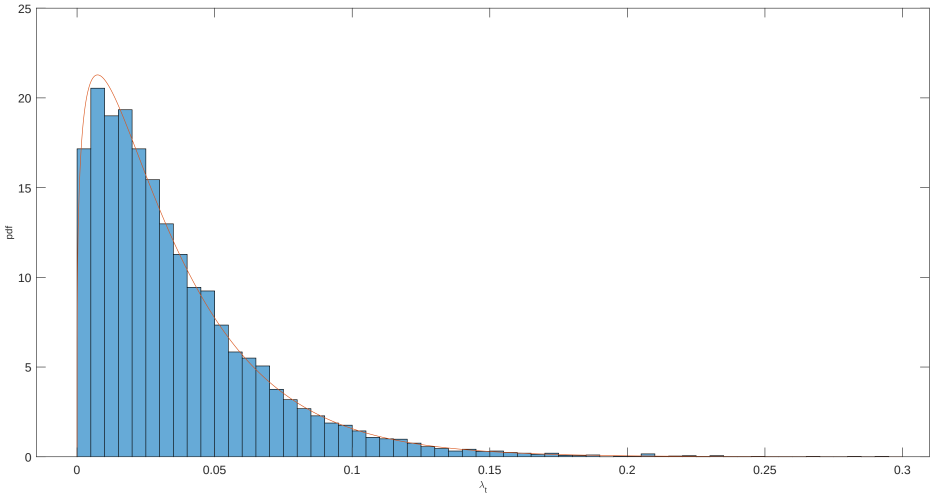
The 30-day symmetric risk-neutral return quantile differences at various λ_t levels under the model of Wachter (2013). The parameter values are based on Wachter's calibration, and the jump size of log consumption growth follows $\mathcal{N}(-0.3, 0.0225)$.

4.4 Supplement

We carry out a simulation study to assess whether the 5%/95% quantile difference predicts the ERP in the Wachter model. We simulate λ_t using the stationary distribution of λ under the physical measure $\mathbb{P}$ which is Gamma with shape parameter $2\kappa\bar{\lambda}/\sigma_\lambda^2$ and scale parameter $\sigma_\lambda^2/(2\kappa)$ (Cox et al. (1985)). To study the distribution of QDs and the relationship between QD, ERP, and DP, we simulate an i.i.d. sample of λ_t from its stationary distribution using a sample size of 10^4, see Figure 5. The QDs at different probability levels as a finction of the simulated λ_t are depicted in Figure 6.

Table 1 reports the results from this simulation study of the Wachter model. The first panel reports the mean and standard deviation of the 30-day risk-neutral return quantile differences at various probability levels. The second panel reports ordinary least square regression results with the instantaneous mean of ERP as the dependent variable and QD as the independent variable. The third panel reports the correlation coefficients of DP and QD along with their p-values. All variables are annualised ($\times 12$) and multiplied by 100. The table shows that in the Wachter model $\text{QD}_{5\%}$ is a significant predictor of the ERP with a beta coefficient of 0.322 and with an adjusted R^2 of 61%. In fact, the less extreme quantile difference of $\text{QD}_{10\%}$ is associated with a higher adjusted R^2 and a larger beta coefficient. The reason for this is a result of the lower standard deviation of $\text{QD}_{10\%}$ regressor relative to $\text{QD}_{5\%}$. The correlation coefficient of QD and the DP ratio in the Wachter model is high at 78% for $QD_{5\%}$ and 91% for $QD_{10\%}$. In the next section we will assess whether a similar relationship holds between QD, ERP and DP using real data.

Figure 5: Histogram of λ_t Sample



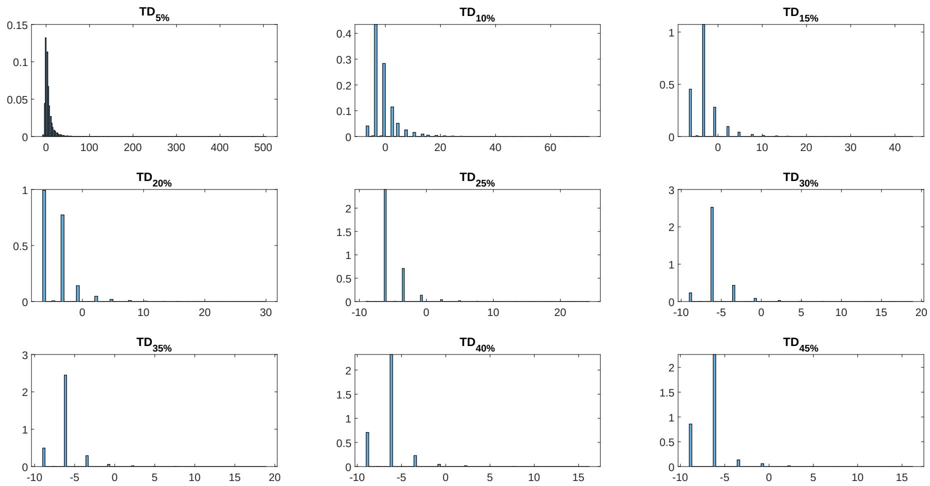
The histogram of 10^4 i.i.d. λ_t drawn from $\Gamma(1.2653, 0.0281)$. Overlaid is the pdf of $\Gamma(1.2653, 0.0281)$.

Table 1: Model-Implied Statistics of Wachter (2013)

	$QD_{5\%}$	$QD_{10\%}$	$QD_{15\%}$	$QD_{20\%}$	$QD_{25\%}$	$QD_{30\%}$	$QD_{35\%}$	$QD_{40\%}$	$QD_{45\%}$
Mean	5.620498	-0.48082	-2.8167	-4.19201	-5.07957	-5.67055	-6.09439	-6.36448	-6.55908
Std. Dev.	13.9330	4.9754	3.3909	2.6762	2.1415	1.9302	1.8516	1.8282	1.8119
					ERP				
β	0.321988	1.054381	1.475237	1.760275	2.156164	2.217812	2.09801	1.975867	1.989838
t-stat	125.2004	224.8271	177.511	143.5197	135.3008	111.875	91.87343	80.93448	80.68074
Adj.R^2	0.610527	0.834851	0.759108	0.673191	0.646732	0.555877	0.457717	0.395772	0.394271
					DP				
Cor	0.781476	0.91452	0.872195	0.821513	0.805376	0.746877	0.677866	0.630425	0.629263
p-value	0	0	0	0	0	0	0	0	0

The first panel reports the mean and standard deviation of the 30-day risk-neutral return quantile differences at various probability levels. The second panel reports ordinary least square regression results with the instantaneous mean of ERP as the dependent variable and QD as the independent variable. The third panel reports the correlation coefficients of DP and QD, and their p-values. All variables are at the annual scale and multiplied by 100.

Figure 6: Histogram of QDs



The sample histogram of 30-day risk-neutral return quantile differences under the model of Wachter (2013) at various probability levels. The x-axis is the QD value. The y-axis is the sample pdf. The sample size is equal to 10^4.

5. Forecasting Stock Market Returns

The results in the previous section suggest that the symmetric quantile difference at a very low probability level is a highly sensitive gauge of the stochastic disaster risk under the model of Wachter (2013).

Assume the quantile difference is zero, which holds when the return distribution is symmetric and investors are risk-neutral. The intercept of the predictive regression is the mean of the excess return so it should be significantly different from zero. On the other hand, if the return distribution is symmetric, the effect of pricing kernel should converge to symmetry around the mean, based on Jackwerth (2000), or close to symmetry based on Linn et al. (2017). In either case, the coefficient in front of the quantile difference should be insignificant and close to zero. And the pattern we should observe is that the intercept becomes more significant and closer to the mean of the excess return as we shrink the quantile difference. What if the physical distribution is asymmetric?

To test the hypothesis, we run predictive regressions of the following type

$$r_{m,t+h} - r_{f,t} = \alpha + \beta \times TD_{x\%,t} + \epsilon_{t+h},$$

Table 2: Summary Statistics

	Sample Period		obs #	Mean	Std. Dev.	Skewness	Kurtosis	Min	Max	AR(1)
ERP	Jan-1996	May-2020	293	6.42	55.21	-0.87	4.61	-226.93	153.54	0.07
TD	Jan-1996	Jun-2019	282	44.37	25.15	2.27	10.81	7.45	187.73	0.79
DP	Jan-1996	Dec-2019	288	-401.14	20.14	-0.11	4.26	-452.36	-328.1	0.98
DY	Jan-1996	Dec-2019	288	-400.56	20.13	-0.23	4.11	-453.09	-329.48	0.97
EP	Jan-1996	Dec-2019	288	-315.06	36.56	-2.17	9.6	-483.65	-256.56	0.98
DE	Jan-1996	Dec-2019	288	-86.08	41.78	3.39	16.37	-124.42	137.95	0.98
RV	Jan-1996	Dec-2019	288	3.47	5.94	6.73	64.11	0.18	69.71	0.7
BM	Jan-1996	Dec-2019	288	26.57	7.01	-0.17	2.26	12.05	44.11	0.96
NtIs	Jan-1996	Dec-2019	288	-0.01	1.92	-0.64	2.91	-5.77	3.11	0.98
TBl	Jan-1996	Dec-2019	288	2.18	2.03	0.45	1.62	0.01	6.17	0.99
LTY	Jan-1996	Dec-2018	276	4.44	1.43	0	1.94	1.75	7.26	0.98
LTR	Jan-1996	Dec-2019	288	0.58	3.01	0.12	5.18	-11.24	14.43	0
TmS	Jan-1996	Dec-2018	276	2.25	1.27	-0.06	2.05	-0.59	4.53	0.97
DfY	Jan-1996	Dec-2019	288	0.99	0.41	3.08	15.52	0.55	3.38	0.96
DfR	Jan-1996	Dec-2019	288	0.04	1.76	-0.46	9.02	-9.75	7.37	0.02
Infl	Jan-1996	Dec-2019	288	0.18	0.35	-0.89	7.68	-1.92	1.22	0.47
LJV	Jan-1996	May-2017	257	5.34	5.05	3.91	24.25	0.18	44.51	0.66
VRP	Jan-1996	Dec-2019	288	1.84	2.59	-3.61	51.57	-26.23	13.9	0.26

All variables are at the monthly frequency, annualised and scaled by 100. ERP is the excess return on the market, where the market return is the value-weighted return of all US firms incorporated in the CRSP and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. The data is from Kenneth French's website. TD is the difference between 30-day negative 5% RaR and 95% RaR of the S&P 500 index. The variables DP to Infl are monthly market predictors assessed in Goyal and Welch (2007)with the data avilable on Amit Goyal's website. LJV is the left risk-neutral jump tail variation studied by Bollerslev, Todorov and Xu (2015). The data is from Lai Xu's website. VRP is the variance risk premium studied by Bollerslev, Tauchen and Zhou (2009). The data is from Hao Zhou's website.

where the left hand side of the equation is the equity risk premium (ERP) observed at $t+h$ for the past h periods. Specifically, ERP is the excess return on the market, where the market return is the value-weighted return of all US firms incorporated in the CRSP and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. The data is from Kenneth French's website. $TD_{x\%,t}$ is the difference between 30-day negative $x\%$ RaR and $(100-x)\%$ RaR of the S&P 500 index that can be observed at t.

Table 4 reports the results of the predictive regressions for return horizons between 1 month and 5 years, using $TD_{5\%}$ as the predictor. All variables are annualised and scaled by 100. We report t-statistics based on three standard error estimates: Hansen and Hodrick (1980) corrected for heteroscedasticity, Hodrick (1992), and Newey-West that are depicted inside, respectively, parenthese, square brackets and curly brackets. The intercept is insignificant across all return horizons, and the coefficients of TD are strongly significant for return horizons from 2 to 5 years. Discuss Ross's argument on R square and Sharpe ratio. TD has a standard deviation of 25.15 during our sample period from January 1996 to June 2019. The sample

Table 3: Correlation Matrix

	ERP	TD	DP	DY	EP	DE	RV	BM	NtIs	TBl	LTY	LTR	TmS	DfY	DfR	Infl	LJV	VRP
ERP	1.00																	
TD	-0.31***	1.00																
DP	-0.08	0.35***	1.00															
DY	0.13*	0.28***	0.98***	1.00														
EP	-0.03	-0.29***	-0.00	-0.01	1.00													
DE	-0.01	0.42***	0.48***	0.48***	-0.88***	1.00												
RV	-0.39***	0.76***	0.27***	0.18**	-0.28***	0.37***	1.00											
BM	-0.06	0.17**	0.68***	0.66***	0.40***	-0.02	0.07	1.00										
NtIs	0.08	-0.13*	-0.49***	-0.47***	0.07	-0.30***	-0.22***	-0.23***	1.00									
TBl	-0.06	-0.19**	-0.58***	-0.58***	-0.03	-0.25***	-0.03	-0.69***	0.29***	1.00								
LTY	-0.03	-0.09	-0.54***	-0.54***	-0.21***	-0.08	0.04	-0.68***	0.56***	0.80***	1.00							
LTR	-0.24***	0.23***	0.03	-0.02	0.05	-0.03	0.19**	0.05	0.04	0.02	-0.05	1.00						
TmS	0.06	0.21***	0.35***	0.35***	-0.19**	0.33***	0.10	0.38***	0.15*	-0.74***	-0.18**	-0.10	1.00					
DfY	-0.12	0.62***	0.58***	0.55***	-0.50***	0.72***	0.59***	0.34***	-0.47***	-0.39***	-0.25***	0.03	0.36***	1.00				
DfR	0.46***	-0.16**	0.01	0.11	-0.18**	0.17**	-0.26***	-0.03	0.02	-0.09	-0.02	-0.48***	0.12*	0.11	1.00			
Infl	0.02	-0.25***	-0.15*	-0.14*	0.07	-0.13*	-0.22***	-0.08	0.04	0.12*	0.15*	-0.10	-0.03	-0.26***	-0.07	1.00		
LJV	-0.28***	0.74***	0.32***	0.26***	-0.35***	0.46***	0.70***	0.14*	-0.11	-0.19**	-0.10	0.12*	0.21***	0.54***	-0.12*	-0.25***	1.00	
VRP	0.03	0.00	-0.15**	-0.15*	-0.24***	0.13*	-0.38***	-0.19**	0.36***	0.13*	0.21***	0.06	0.02	-0.07	0.06	0.04	-0.08	1.00

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

period of the ERP is from January 1996 to May 2020 and its sample mean is 6.42. Hence, a one standard deviation increase in TD is associated with an approximately 3% annual increase in the ERP in the following 2 to 5 years.

In Table 5, we repeat the analysis in Table 4 but for $TD_{x\%,t}$ with x equal to 5 to 45. To save space, we drop the results on the intercepts and report only the t-statistics (inside paranthese) based on the standard error estimates of Hodrick (1992). Several patterns emerge from Table 5. TDs are significant in forecasting market returns in the long-run, starting from the 2-year horizon. The largest Adj.R^2 is observed on $TD_{20\%}$. The Adj.R^2 is equal to 31.76% for 2-year ERP and climbs up to 54.72% for 5-year ERP. However, the predictive power of TD starts diminishing from at 20% and wears off completely for $TD_{35\%}$, generating a hump-shaped pattern. The lack of predictability of TD at very low probability levels could be a consequence of the noisy deep OTM option prices or it could be caused by the finite range of available option strikes. In the second case, if the quantile estimator is assigned a probability level that it cannot reach, as there are no options retained by our filters at this level, the estimator will return the lowest (highest) available strike price as the result.

Table 4: Equity Risk Premium Predictive Regressions — Univariate (TD)

Horizon (monthly)	1	6	12	24	36	48	60
Constant	2.32	-0.59	1.02	-0.6	0.46	0.84	0.03
	(0.3)	(-0.14)	(0.22)	(-0.13)	(0.13)	(0.28)	(0.01)
	[0.31]	[-0.1]	[0.23]	[-0.16]	[0.13]	[0.22]	[0.01]
	{0.3}	{-0.15}	{0.27}	{-0.14}	{0.12}	{0.27}	{0.01}
TD	0.09	0.16	0.12	0.15	0.11	0.1	0.11
	(0.51)	(1.94)	(1.58)	.	(3.36)	(2.22)	(3.37)
	[0.53]	[1.18]	[1.25]	[2.03]	[1.93]	[1.92]	[2.49]
	{0.51}	{1.86}	{1.74}	{3.51}	{2.48}	{2.1}	{3.35}
Adj.R^2 (%)	-0.53	2.13	2.26	7.82	7.68	8.2	15.45

Robust t-statistics based on the standard error estimates of Hansen and Hodrick (1980) corrected for heteroscedasticity, Hodrick (1992), and Newey-West are reported, respectively, in parenthese, square brackets and curly brackets.

Table 5: Equity Risk Premium Predictive Regressions — Univariate (TDs)

Horizon (monthly)	1	6	12	24	36	48	60
$TD_{5\%}$	0.09	0.16	0.12	0.15	0.11	0.1	0.11
	(0.53)	(1.18)	(1.25)	(2.03)	(1.93)	(1.92)	(2.49)
Adj.R^2 (%)	-0.53	2.13	2.26	7.82	7.68	8.2	15.45
$TD_{10\%}$	0.24	0.33	0.28	0.37	0.32	0.27	0.28
	(0.69)	(1.2)	(1.39)	(2.39)	(2.49)	(2.47)	(3.07)
Adj.R^2 (%)	-0.4	2.55	3.62	13.56	16.29	18.06	28.66
$TD_{15\%}$	0.5	0.67	0.69	0.91	0.83	0.7	0.66
	(0.83)	(1.29)	(1.63)	(2.62)	(2.72)	(2.64)	(3.1)
Adj.R^2 (%)	-0.28	3.42	7.26	25.78	34.74	38.42	50.03
$TD_{20\%}$	0.64	0.83	1.1	1.38	1.3	1.11	0.94
	(0.92)	(1.25)	(1.79)	(2.41)	(2.46)	(2.39)	(2.57)
Adj.R^2 (%)	-0.34	2.62	10.28	31.76	46.76	52.26	54.72
$TD_{25\%}$	0.31	0.45	0.81	0.93	0.92	0.8	0.6
	(0.55)	(0.91)	(1.65)	(1.94)	(2.04)	(2.01)	(1.92)
Adj.R^2 (%)	-0.6	0.52	6.89	18	28.73	32.54	26.51
$TD_{30\%}$	0.13	0.2	0.52	0.54	0.55	0.48	0.31
	(0.27)	(0.54)	(1.42)	(1.53)	(1.69)	(1.68)	(1.41)
Adj.R^2 (%)	-0.69	-0.33	4.02	8.68	14.94	16.89	10.36
$TD_{35\%}$	0.01	0.09	0.37	0.35	0.36	0.32	0.18
	(0.03)	(0.28)	(1.23)	(1.25)	(1.43)	(1.43)	(1.05)
Adj.R^2 (%)	-0.72	-0.62	2.56	4.66	8.57	9.67	4.25
$TD_{40\%}$	-0.01	0.05	0.3	0.27	0.28	0.24	0.12
	(-0.02)	(0.17)	(1.1)	(1.07)	(1.25)	(1.26)	(0.8)
Adj.R^2 (%)	-0.72	-0.68	1.85	2.9	5.68	6.29	1.82
$TD_{45\%}$	-0.02	0.04	0.27	0.23	0.25	0.21	0.1
	(-0.05)	(0.12)	(1.03)	(0.97)	(1.16)	(1.17)	(0.69)
Adj.R^2 (%)	-0.72	-0.7	1.5	2.2	4.58	5.01	0.99

$TD_{x\%}$ is the difference between 30-day negative $x\%$ RaR and $(100 - x)\%$ RaR of the S&P 500 index. Robust t-statistics based on the standard error of Hodrick (1992) are reported in parentheses.

6. Forecasting Stock Market Returns – Cont.

In this section, we compare the predictive power of the quantile difference on future market returns with other well known market predictors. These include Goyal and Welch (2007) factors, the variance risk premium of Bollerslev, Tauchen and Zhou (2009), and the left risk-neutral jump tail variation of Bollerslev, Todorov and Xu (2015). Table 2 reports the summary statistics of all the variables. Note that TD has a high AR(1) coefficient equal to 0.79 and this is not surprising. Other successful long-run market predictors like the dividend price ratio (DP), the term spread between long and short term bonds (TmS), the default spread (DfY), and the level of the T-bill rate (TBl) all have AR(1) coefficients close to 1. Cochrane (2011) made the following argument, *If daily returns are very slightly predictable by a slow-moving variable, that predictability adds up over long horizons ... Thus, a central fact driving the predictability of returns is that the dividend price ratio is very persistent ...*

Table 3 contains the pairwise correlation coefficients of all variables. TD has a significantly negative contemporaneous correlation coefficient with the ERP at -0.31. This is in line with Figure 2 where the 5% quantile increases much more in magnitude than the 95% quantile when the market drops. The dividend price ratio (DP) is significantly correlated with TD at 0.35. Campbell and Shiller (1989) show that the variation of the dividend price ratio is driven mainly by the variation of future discount rates, or the ERP for the market. Hence, if TD predicts future market returns, it is not surprising to find this large correlation coefficient, and the insignificant contemporaneous correlation between DP and ERP. The correlation coefficient between TD and LJV is especially high and significant at 0.74. The LJV of Bollerslev, Todorov and Xu (2015) measures the risk-neutral large left jump variation. The information content of TD is expected to be very similar to LJV for jump diffusion processes where the diffusion component is cancelled by subtraction. However, for semimartingales TD also captures the asymmetry induced by the continuous part. The realised variance (RV) has a correlation coefficient with ERP equal to -0.39, which is known as the leverage effect, and its correlation coefficient with TD is 0.76. Unlike traditional market predictors, TD and LJV have negative correlation coefficients with ERP. It is necessary to test whether the predictive power comes from reversal of the return. Or a deeper question is, if TD and LJV increase each time the market crashes, as long as the market still exists as of today, we should always expect TD and LJV to positively predict the future market returns?!

Table 6: Equity Risk Premium Predictive Regressions — Univariate

	One-month Horizon			Six-month Horizon			One-year Horizon			Two-year Horizon			Three-year Horizon			Four-year Horizon			Five-year Horizon		
	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2	Coeff.	t-stat.	Adj.R^2
TD	2.35	0.53	-0.53	3.96	1.18	2.13	2.98	1.25	2.26	3.75	2.03	7.82	2.97	1.93	7.68	2.52	1.92	8.2	2.85	2.49	15.45
DP	7.09	1.61	1.07	7.71	1.92	10.16	8.11	2.2	21.39	8.18	2.39	40.02	7.08	2.23	47.44	6.19	2.19	53.4	5.4	2.28	57.55
DY	7.76	1.85	1.42	7.84	2.01	10.53	8.31	2.28	22.49	8.25	2.42	40.67	7.04	2.22	46.87	6.12	2.18	52.23	5.25	2.23	54.42
EP	2.73	0.7	-0.44	0.68	0.22	-0.62	1.1	0.39	-0.31	-0.01	0	-0.75	0.62	0.32	-0.41	0.16	0.1	-0.79	-0.42	-0.33	-0.51
DE	1.03	0.23	-0.66	3.12	0.91	1.08	2.94	1.02	2.2	3.94	1.71	8.68	2.85	1.62	7.03	2.81	2.03	10.33	2.91	2.8	16.07
RV	-7.66	-1.27	1.36	-0.34	-0.09	-0.68	1.09	0.43	-0.32	1.34	0.77	0.35	1.01	0.76	0.19	1.18	1.17	1.16	1.79	2.32	5.56
BM	4.05	1.09	-0.13	7.18	2.13	8.73	7.19	2.23	16.65	6.27	1.99	23.18	5.27	1.81	25.97	4.76	1.72	31.29	4.55	1.92	40.7
NtIs	4.25	1.02	-0.07	5.18	1.25	4.2	3.79	0.97	4.11	0.77	0.25	-0.39	-0.43	-0.17	-0.61	-1.03	-0.43	0.67	-1.97	-0.92	6.95
TBl	-3.28	-1.01	-0.32	-4.13	-1.27	2.42	-5.33	-1.67	8.82	-7.03	-2.24	29.34	-6.64	-2.35	41.61	-5.86	-2.26	47.77	-5.2	-2.53	53.37
LTY	-4.23	-1.42	-0.11	-4.24	-1.43	2.48	-3.87	-1.33	4.24	-4.39	-1.6	10.98	-4.23	-1.8	16.41	-4.06	-1.73	22.55	-4.54	-2.32	40.53
LTR	2.82	0.75	-0.42	1.18	0.92	-0.45	0.36	0.53	-0.68	0.21	0.5	-0.72	0.45	1.88	-0.6	0.13	0.74	-0.8	0.09	0.62	-0.85
TmS	0.69	0.21	-0.72	2.19	0.66	0.13	4.44	1.45	5.81	6.72	2.46	26.72	6.44	2.47	39.06	5.45	2.44	41.16	3.97	2.5	30.8
DfY	-2.98	-0.6	-0.39	1.05	0.25	-0.5	2.52	0.78	1.42	3.65	1.43	7.38	3.07	1.49	8.3	3.56	1.98	17.13	4.08	2.66	32.45
DfR	4.23	0.73	-0.07	1.67	1.04	-0.19	1.39	1.07	-0.07	0.9	1.09	-0.26	0.72	1.33	-0.28	0.61	1.43	-0.3	0.55	1.61	-0.25
Infl	3.13	0.79	-0.36	-3.86	-1.66	2.01	-2.9	-2.61	2.11	-1.9	-2.82	1.45	-1.3	-2.34	0.83	-1.15	-2.31	1.03	-1.09	-2.65	1.51
LJV	0.96	0.19	-0.76	5.52	1.48	4.4	4.47	1.9	5.48	3.54	1.94	6.53	1.93	1.32	2.79	1.84	1.62	3.99	2.33	2.39	9.97
VRP	12.07	2.29	4.43	5.37	2.41	4.56	2.08	1.18	0.73	1.23	0.87	0.17	0.36	0.29	-0.66	-0.13	-0.11	-0.8	-0.23	-0.2	-0.76

All variables are at monthly frequency, annualised and scaled by 100. ERP is the excess return on the market, where the market return is the value-weighted return of all US firms incorporated in the CRSP and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. TD is the difference between 30-day negative 5% RaR and 95% RaR of the S&P 500 index. The variables DP to Infl are monthly market predictors used in Goyal and Welch (2007). LJV is the left risk-neutral jump tail variation introduced in Bollerslev, Todorov and Xu (2015). VRP is variance risk premium of Bollerslev, Tauchen and Zhou (2009). t-statistics are based on the standard errors of Hodrick (1992). For comparison, reported coefficients are scaled by the standard deviation of the regressors.

Table 7: Equity Risk Premium Predictive Regressions — Bivariate (Up to One Year)

	One-month Horizon					Six-month Horizon					One-year Horizon				
	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2
DP	-0.17	-0.04	7.19	1.66	0.69	1.38	0.43	7.34	1.84	10.39	0.15	0.06	8.06	2.1	21.11
DY	0.16	0.04	7.75	1.88	1.03	1.85	0.58	7.45	1.93	11.07	0.67	0.29	8.12	2.19	22.35
EP	3.42	0.75	3.72	0.93	-0.44	4.53	1.35	2	0.64	2.44	3.58	1.54	2.12	0.76	3.3
DE	2.33	0.51	0.06	0.01	-0.89	3.2	1.01	1.82	0.55	2.27	2.11	0.98	2.06	0.73	3.08
RV	19.14	2.42	-22.2	-2.18	6.45	9.99	2.86	-7.98	-1.63	6.73	5.03	1.78	-2.72	-0.87	2.97
BM	1.69	0.39	3.9	1.09	-0.37	2.83	0.86	6.69	2.01	9.69	1.82	0.76	6.88	2.11	17.43
NtIs	2.97	0.67	4.78	1.14	-0.1	4.7	1.44	5.74	1.4	7.68	3.53	1.58	4.25	1.1	7.89
TBl	1.8	0.41	-2.94	-0.92	-0.6	3.29	0.99	-3.59	-1.11	4.04	2.06	0.86	-4.94	-1.53	9.87
LTY	1.85	0.41	-4.07	-1.38	-0.36	3.67	1.08	-3.93	-1.33	4.51	2.64	1.09	-3.65	-1.24	6.19
LTR	1.77	0.38	2.48	0.6	-0.69	3.91	1.14	0.18	0.15	1.78	3.06	1.24	-0.36	-0.48	1.95
TmS	2.15	0.48	0.24	0.07	-0.93	3.71	1.11	1.42	0.43	2.11	2.11	0.91	4	1.32	6.89
DfY	6.73	1.37	-7.1	-1.24	0.2	5.38	1.78	-2.31	-0.52	2.37	2.29	1.01	1.11	0.31	2.16
DfR	3.12	0.69	4.79	0.81	-0.11	4.34	1.29	2.4	1.57	2.8	3.28	1.39	1.91	1.55	3.11
Infl	3.33	0.76	3.97	1.02	-0.37	3.18	0.97	-3.18	-1.51	3.5	2.41	0.98	-2.31	-1.96	3.59
LJV	3.77	0.61	-1.84	-0.27	-0.93	0.03	0.01	5.5	1.32	4.02	-0.66	-0.18	4.95	1.43	5.16
VRP	2.31	0.52	12.35	2.31	4.42	3.94	1.17	5.35	2.37	6.99	2.97	1.25	2.07	1.17	3.35

All variables are at monthly frequency, annualised and scaled by 100. ERP is the excess return on the market, where the market return is the value-weighted return of all US firms incorporated in the CRSP and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. TD is the difference between 30-day negative 5% RaR and 95% RaR of the S&P 500 index. The variables DP to Infl are monthly market predictors used in Goyal and Welch (2007). LJV is the left risk-neutral jump tail variation introduced by Bollerslev, Todorov and Xu (2015). VRP is variance risk premium of Bollerslev, Tauchen and Zhou (2009). t-statistics are based on the standard error of Hodrick (1992). For comparison, reported coefficients are scaled by the standard deviation of the regressors.

Table 6 compares univariate predictive regressions at return horizons from 1 month to five years where the single regressor used to predict the ERP includes: $TD_{5\%}$, a number of well known monthly market predictors used in Goyal and Welch (2007), the left risk-neutral jump tail variation (LJV) of Bollerslev, Todorov and Xu (2015), and the variance risk premium (VRP) of Bollerslev, Tauchen and Zhou (2009). All variables are annualised and scaled by 100. We report t-statistics based on the standard error estimates of Hodrick (1992). As previously reported in 4, TD becomes significant at return horizons of two years or longer. Given that TD is most closely associated to the left risk-neutral jump tail variation (LJV) of Bollerslev, Todorov and Xu (2015), as both capture tail risk aversion and both predictors are implied from the prices of traded options, we focus in particular on comparing the results from using $TD_{5\%}$ and LJV as the single regressor. We find that TD has higher t-statistics and is associated with higher Adj.R^2 values than LJV at return horizons of 2 to 5 years. Comparing TD to the dividend price ratio, DP, and the dividend yield, DY, we find that TD has lower t-statistics and lower Adj.R^2 at all horizons, however, TD captures different aspects of return dynamics than DP and DY.

In Table 7 we report bivariate predictive regressions where TD and another regressor are used to predict the ERP. Results on return horizons up to one year are reported in this table. In each horizon panel, the TD coefficient and t-statistic are reported is the first two columns and the second regressor and its associated t-statistic are reported in the third and fourth column. At the one-month horizon, TD is insignificant in all bivariate regressions with the exception of the regression that includes RV is a second regressor. Including RV as a second regressors results in the TD coefficient being highly significant and positive and the RV coefficient being highly significant and negative with an Adj.R^2 value of 6.45%. However, there is a significant negative correlation between TD and RV so this result could be as a result of multicollinearity. A similar result on the bivariate regression using TD and RV is observed at the six month and one year return horizons.

Table 8 reports bivariate predictive regressions where TD and another regressor are used to predict the ERP at the longer return horizons of two to five years. We find that TD does not remain significant at longer horizons once DP or DY are included as a second regressor. TD is robust to the inclusion of EP as a second regressor. Similar to the shorter horizon return results in Table 7, we find that TD is robust to the inclusion of RV as a second regressor, with the TD coefficient being highly significant and positive and the RV coefficient being highly significant and negative at all longer return horizons from two- to five-years. TD is robust to

Table 8: Equity Risk Premium Predictive Regressions — Bivariate (More than One Year)

	Two-year Horizon					Three-year Horizon					Four-year Horizon					Five-year Horizon				
	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2	TD Coeff.	TD t-stat.	Coeff.	t-stat.	Adj.R^2
DP	0.97	0.5	7.84	2.15	40.3	0.43	0.24	6.92	1.98	47.39	0.18	0.11	6.12	1.91	53.25	0.94	0.64	5.04	1.88	58.89
DY	1.48	0.8	7.82	2.22	41.68	0.92	0.55	6.76	2.01	47.4	0.63	0.41	5.92	1.94	52.54	1.32	0.99	4.83	1.9	57.37
EP	4.09	2.28	1.16	0.5	8.23	3.44	2.4	1.63	0.86	9.65	2.82	2.26	1	0.61	9.1	2.99	2.69	0.47	0.38	15.49
DE	2.55	1.6	2.86	1.31	11.6	2.14	1.74	1.93	1.25	10.26	1.6	1.46	2.1	1.77	12.88	1.95	1.93	2.05	2.39	21.88
RV	6.37	2.96	-3.46	-1.78	10.62	5.1	2.6	-2.83	-1.81	10.67	3.76	1.99	-1.65	-1.18	9.49	3.49	2.02	-0.85	-0.74	15.71
BM	2.81	1.48	5.81	1.81	27.58	2.19	1.37	4.93	1.65	30.2	1.79	1.26	4.47	1.57	35.41	2.19	1.81	4.21	1.73	49.83
NtIs	3.96	2.24	1.38	0.44	8.61	2.99	1.92	0.11	0.04	7.33	2.42	1.68	-0.5	-0.2	8.15	2.54	1.99	-1.42	-0.63	18.96
TBl	2.52	1.34	-6.55	-2.05	32.8	1.67	1.06	-6.3	-2.17	43.95	1.24	0.93	-5.57	-2.1	49.63	1.77	1.55	-4.8	-2.29	59.16
LTY	3.36	1.8	-4.06	-1.47	17.48	2.42	1.53	-3.89	-1.62	21.61	1.83	1.33	-3.72	-1.54	26.85	2.1	1.83	-4.16	-2.1	48.86
LTR	3.92	2.04	-0.71	-1.43	7.76	3.03	1.9	-0.26	-0.83	7.38	2.64	1.92	-0.49	-1.83	8.14	2.99	2.51	-0.6	-2.51	15.77
TmS	2.49	1.37	6.21	2.26	30.07	1.82	1.23	6.09	2.34	41.89	1.56	1.27	5.15	2.33	44.27	2.18	1.98	3.56	2.26	39.76
DfY	2.41	1.28	2.16	0.73	9.22	1.73	0.98	2.01	0.81	9.74	0.56	0.32	3.22	1.38	17.06	0.57	0.35	3.73	1.8	32.57
DfR	3.99	2.13	1.52	1.83	8.86	3.14	1.98	1.19	1.88	8.65	2.67	1.95	0.99	1.87	9.18	2.98	2.51	0.96	2.16	16.91
Infl	3.5	1.83	-1.04	-1.54	8.1	2.82	1.79	-0.57	-1.11	7.61	2.39	1.79	-0.52	-1.23	8.18	2.75	2.41	-0.39	-1.32	15.37
LJV	2.91	1.26	1.38	0.63	8.39	3.42	2.41	-0.61	-0.5	7.48	2.59	2.23	-0.09	-0.15	7.82	2.53	2.47	0.43	0.89	15.24
VRP	3.75	2.02	1.23	0.87	8.41	2.97	1.93	0.4	0.31	7.47	2.52	1.9	-0.08	-0.07	7.83	2.85	2.48	-0.2	-0.17	15.16

All variables are at monthly frequency, annualised and scaled by 100. ERP is the excess return on the market, where the market return is the value-weighted return of all US firms incorporated in the CRSP and listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. TD is the difference between 30-day negative 5% RaR and 95% RaR of the S&P 500 index. The variables DP to Infl are monthly market predictors used in Goyal and Welch (2007). LJV is the left risk-neutral jump tail variation introduced by Bollerslev, Todorov and Xu (2015). VRP is variance risk premium of Bollerslev, Tauchen and Zhou (2009). t-statistics are based on the standard error of Hodrick (1992). For comparison, reported coefficients are scaled by the standard deviation of the regressors.

the inclusion of LJV as a second regressors and TD remains significant at the three-, four- and five-year return horizons whereas LJV is insignificant at all horizons. Finally, TD is robust to the inclusion of the VRP at longer return horizons, with TD remaining significant whereas the VRP is insignificant at these longer horizons. This is a result typical to the literature, see Bollerslev, Todorov and Xu (2015) among others, who find the VRP is a strong return predictor at short horizons but not at long horizons.

The paper is still a work in process ... We apologise for missing a conclusion, the poor quality of discussion on the regression results, an incomplete reference list, and the random thoughts that appear in the body of the text with a different colour ... Thank you for reading to this point!

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