CSC-225 Assignment 5

Dryden Bryson

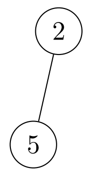
March 27, 2025

Question 1:

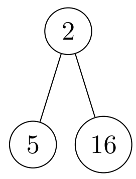
Insertions:

(1) Insert 2:

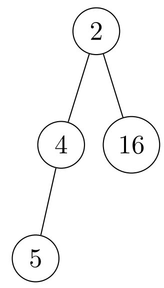
(2) Insert 5:



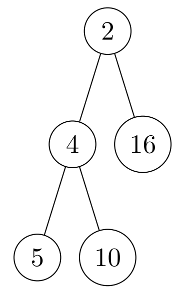
(3) Insert 16:



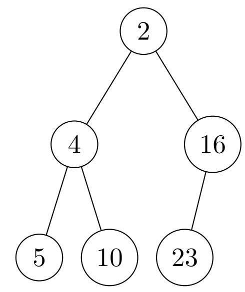
(4) Insert 4:



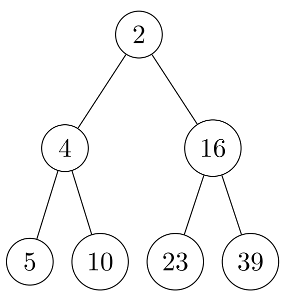
(5) Insert 10:



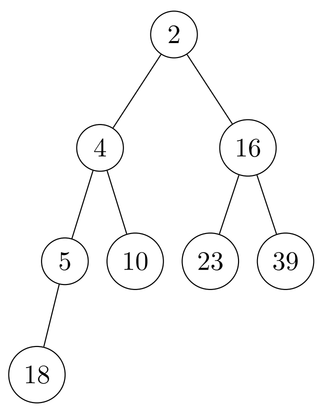
(6) Insert 23:



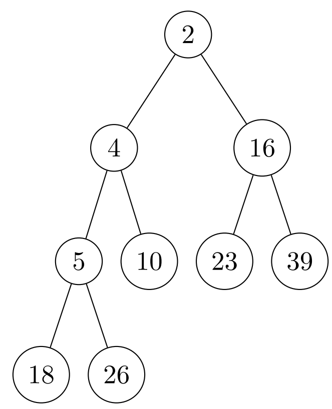
(7) Insert 39:



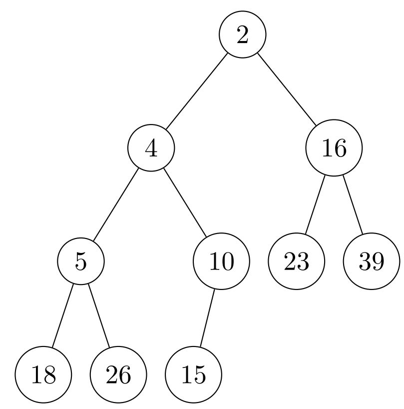
(8) Insert 18:



(9) Insert 26:

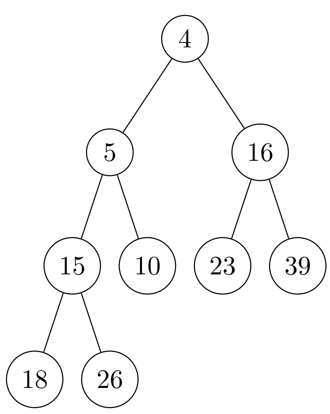


(10) Insert 15:

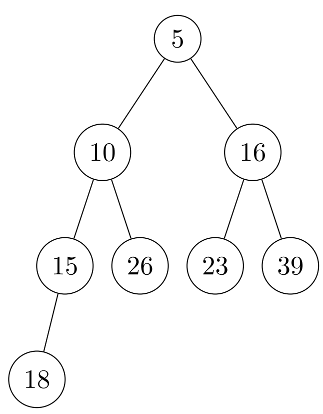


Removals

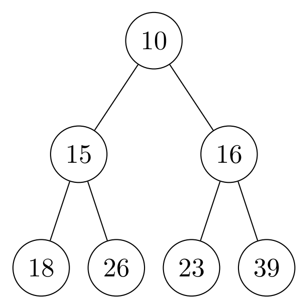
(11) Remove 2:



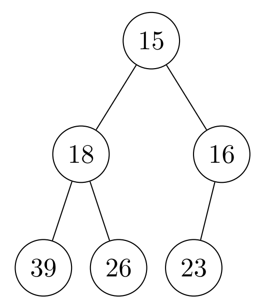
(12) Remove 4:



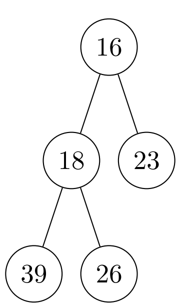
(13) Remove 5:



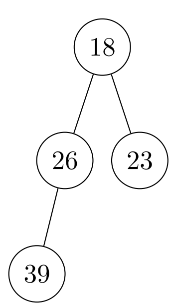
(14) Remove 10:



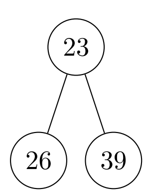
(15) Remove 15:



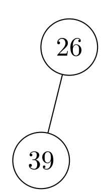
(16) Remove 16:



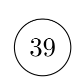
(17) Remove 18:



(18) Remove 23:



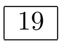
(19) Remove 26:



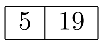
(20) Remove 39:

Question 2:

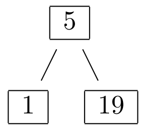
(1) Insert 19:



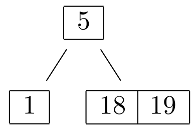
(2) Insert 5:



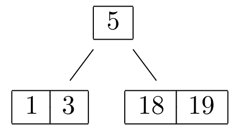
(3) Insert 1:



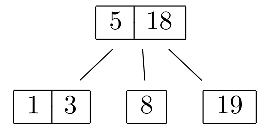
(4) Insert 18:



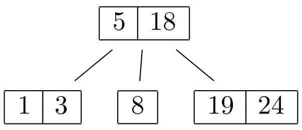
(5) Insert 3:



(6) Insert 8:



(7) Insert 24:



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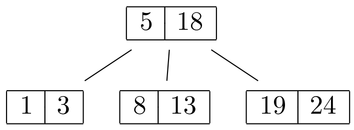

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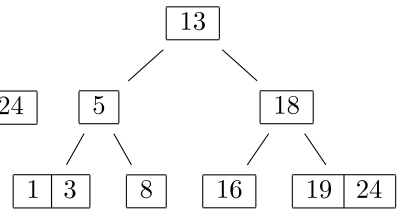
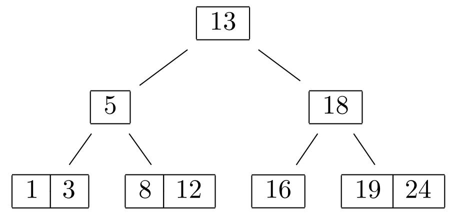
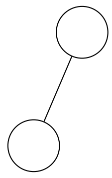


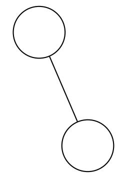
(10) Insert 12:



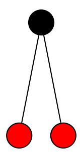
Question 3:

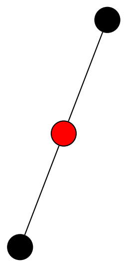
$n = 2$

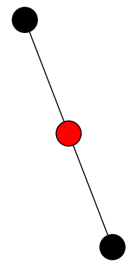




$n = 3$







Question 4:

To prove by induction on n that $2m \leq n^2 - n$ for all $n \geq 1$ for any simple undirected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ we will proceed with induction on n:

Base Case:

For $n = 1$ the graph is simply a single vertex with no edges thus $m = 0$. We have then that $2m = 2 \cdot 0 = 0$ and $n^2 - n = 1 - 1 = 0$. Therefore $0 \leq 0$, which holds true.

Inducitve Hypothesis:

We assume that for any simple undirected graph with k vertices and m edges the following inequality holds true:

$$2m \leq k^2 - k$$

Inductive Step:

We want to show that for any simple undirected graph with $k+1$ vertices satisfies $2m' \leq (k+1)^2 - (k+1)$. We will being constructing G' which represents a graph with $k+1$ vertices and m' edges. We will define m' as follows:

$$m' = m + c$$

Where c is the number of edges which connect to the $k+1$'th vertex in G' we bound $c \leq k$ since the vertex could be connected to all other vertices adding at most m edges.

We then have a simple undirected graph $G' = (V', E')$ where $|V'| = k + 1$ and $|E'| = m + c$, we will proceed with the original equation from the inductive hypothesis and make the following substitutions:

$$\begin{aligned}
2m &\leq k^2 - k \\
2(m' - c) &\leq k^2 - k \\
2m' - 2c &\leq k^2 - k \\
2m' &\leq k^2 - k + 2c \\
2m' &\leq k^2 - k + 2k && \text{Since } c \leq k \text{ thus } 2c \leq 2k \\
2m' &\leq k^2 + k \\
2m' &\leq (k+1)^2 - (k+1) && \text{Since } (k+1)^2 - (k+1) = k^2 + 2k + 1 - k - 1 = k^2 + k
\end{aligned}$$

Thus we have $2m' \leq (k+1)^2 - (k+1)$ where m' is the number of edges in a simple undirected graph with $k+1$ edges. This shows if the statement holds for $n = k$ it must hold for $n = k+1$.

Conclusion:

By induction, the base case $n = 1$ holds and if the statement holds for $n = k$ it must hold for $n = k+1$, therefore for all integers $n \geq 1$ any simple undirected graphs with n vertices and m edges:

$$2m \leq n^2 - n$$

$\square$

Question 5:

a)

The graph has 9 total edges and a spanning subgraph is a graph which includes all vertices with any subset of edges thus we want to calculate the cardinality of the power set of the set of all edges where each edge either exists or doesn't which is:

$$2^9 = 512$$

b)

The graph has contains one 3-cycle which we can remove one edge from and all vertices will still remain accesible, otherwise all other edges are neccesary and must be kept, this means we can either remove 0 edges or any 1 of the 3 edges in the 3-cycle giving us a total of

$$4$$

connected spanning subgraphs.

c)

To count the number of spanning subgraphs which have 0 as an isolated subgraph we can imagine the graph as if 0 where not connected and compute the amount of spanning subgraphs normally, this works since for 0 to be isolated it must not have any edges connecting to this so we assume the 3 edges that connect to it are not part of the powerset of the set of all edges, thus there exists:

$$2^5 = 32$$

spanning subgraphs where the vertex 0 is isolated.