MATH122 Assignment 5

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November 8th 2024

Question 1:

a)

We will start by representing $(22)_{10}$ in an arbitrary base b:

$$(22)_b = 2 \cdot b + 2$$

When Taylor is 34 i)

Now lets examine if there exists a b $34 = (22)_b$, we will use our abitrary base b representation of 22 equal to 34 and attempt to solve for b as follows:

$$\begin{aligned} 34 &= 2 \cdot b + 2 \\ 32 &= 2 \cdot b \\ 32/2 &= b \\ 16 &= b \end{aligned}$$

We have that when Taylor is 34, if her age was represented in base 16 she could claim to be 22, thus:

$$(22)_{16} = 34$$

When Taylor is 35 ii)

If Taylor is 35 we need to determine if there exists a b such that $35 = (22)_b$, thus we setup the equation like before:

$$\begin{aligned} 35 &= 2 \cdot b + 2 \\ 33 &= 2 \cdot b \\ 35/2 &= b \\ 17.5 &= b \end{aligned}$$

Since b must be an integer, next year when Taylor is 35 she cannot mathematically claim to be 22.

For Any age iii)

For a general formula where n is Taylor's age and b is the base we have that:

$$n = 2 \cdot b + 2 \quad \Rightarrow \quad b = \frac{n-2}{2}$$

We have the following restrictions on b so that some n can be represented as $(22)_b = n$,

i $b \geq 3$, since we need to have a valid 2 digit

ii $2|b$, b must be divisible by 2 / be even , which implies n being even

Now we can apply these restrictions to n:

i $n \geq 8$ since, $\frac{n-2}{2} \geq 3$

ii $2|n$ since, from the restrictions on b: $2|b \Rightarrow 2|\frac{n-2}{2} \Rightarrow 2|n$

We conclude that for any even age greater than or equal to 8 Taylor swift can mathematically say that she is 22, we can represent the ages as a set:

$$\{\forall n \in \mathbb{Z} : n \geq 8 \wedge 2|n\} = \{8, 10, 12, 14...\}$$

b)

Let us assume for the sake of contradiction that if n is an integer with prime divisors all greater then $\sqrt{n}$, then n is a composite number. Since n is composite it can be represented as $n = a \cdot b \cdot c \cdot \ldots$ the product of k prime divisors where $a, b, c, \ldots$ are all prime divisors which are greater than $\sqrt{n}$. Thus we have that:

$$n = a \cdot b \cdot c \cdot \ldots \qquad a > 1, b > 1, c > 1, \ldots$$

By the restriction that all prime divisors $(a, b, c, \ldots)$ are greater than $1 > \sqrt{n}$ we have that:

$$a > \sqrt{n}, \quad b > \sqrt{n}, \quad c > \sqrt{n}, \ldots$$

We can then multipliy the inequalities together, remember that we have k prime divisors:

$$\begin{aligned} a \cdot b \cdot c \cdot \ldots &> \sqrt{n}^k \\ &> n^{k/2} \geq n \quad \because \ k > 2 \text{ since } n \text{ is composite} \\ a \cdot b \cdot c \cdot \ldots &> n \quad \because \ \text{Transitivity} \end{aligned}$$

This is where to contradiction arises since we derived:

$$a \cdot b \cdot c \cdot \cdots > n \qquad a \cdot b \cdot c \cdot \cdots = n$$

Hence our assumption that n is not prime is incorrect. Therefore, n must be prime. $\square$

c)

For some a it has a prime factor p thus we can represent a as product of that prime factor and the rest of its prime factors multiplied, or 1 if it is prime itself, thus we have

$$a = p^k \cdot c$$

Then if we have that $a^3 | b$ it follows that there exists an integer d such that we can say:

$$\begin{aligned} b &= d \cdot (a)^3 \\ &= d \cdot (p^k \cdot c)^3 \\ &= d \cdot p^{3k} \cdot c^3 \end{aligned}$$

Then if $p^7 | ab^2$ there exists an integer e such that we can say:

$$\begin{aligned} ab^2 &= e \cdot p^7 \\ (p^k \cdot c)(p^{3k} \cdot c^3 \cdot d)^2 &= \vdots \\ (p^k \cdot c)(p^{6k} \cdot c^6 \cdot d^2) &= \vdots \\ p^{7k} \cdot c^7 \cdot d^2 &= e \cdot p^7 \\ \frac{p^{\cancel{7}k} \cdot c^7 \cdot d^2}{\cancel{p^7}} &= \frac{e \cdot \cancel{p^7}}{\cancel{p^7}} \\ p^k \cdot c^7 \cdot d^2 &= e \end{aligned}$$

Thus we have shown that $p^7 | ab^2$ since there exists an integer e such that:

$$ab^2 = e \cdot p^7 \quad \text{or} \quad p^7 | ab^2 \quad \text{specifically} \quad e = p^k \cdot c^7 \cdot d^2$$

and e is an integer since $k \geq 1$ thus p^7 divides ab^2. $\square$

Question 2:

a)

If we have that $d|a$ and $d|b$, we can say $a = dk$ and $b = dm$. Thus we can show that $d|\gcd(a,b)$ by expressing the $\gcd(a,b)$ as a linear combination:

$$\begin{aligned}\gcd(a,b) &= ax + by\\ &= dkx + dmy\\ &= d(kx + my)\end{aligned}$$

Thus we have that d divides $\gcd(a,b) = 5$, or $d|5$, the possible divisors of d are ± 5 and ± 1, since neither of those are even we have that:

$$d = 0$$

b)

If we have that $\gcd(a,b) = 5$ we assert that the only shared prime factor of a and b is 5 and no more than one of a or b has a multiplicity of the prime factor 5 greater than 1.

When we do $\gcd(a, 10b)$ we work with the previous restricitions except b gets an extra two prime factors of 5 and 2 essentially

$$\gcd(a, 10b) = \gcd(a, 2 \cdot 5 \cdot b)$$

Let's examine the possible scenarios:

i a contains 5^n where $n \geq 2$ in it's prime factorization
If a contains 5^n where $n \geq 2$ in it's prime factorization then b must have 5^1 in it's prime decomposition but, $10b$ has 5^2 in it's prime decomposition which means that the prime factorization of $\gcd(a, 10b)$ will contain 5^2 since the gcd contains the shared factors with the minimum exponent.

ii a contains 2^n where $n \geq 1$ in it's prime factorization
If a contains 2^n where $n \geq 1$ in it's prime factorization then b must not have the prime factor 2 in it's prime decomposition but then, $10b$ contains 2^1 in it's prime factorizartion. Which means that the prime factorization of $\gcd(a, 10b)$ will contain 2^1.

So the possible outcomes are that the prime factorization contains an additional 5 or / and the prime factorization contains and additional 2 Thus since the prime factorization of 5 is simply 5^1 the possible outcomes to $\gcd(a, 10b)$ are:

- neither scenario (i) or (ii) $\quad \gcd(a, 10b) = 5^1 = 5$
- scenario (i) $\quad \gcd(a, 10b) = 5^2 = 25$
- scenario (i) and (ii) $\quad \gcd(a, 10b) = 5^2 2^1 = 50$
- scenario (ii) $\quad \gcd(a, 10b) = 5^1 2^1 = 10$

Thus the possible outputs of $\gcd(a, 10b)$ when $\gcd(a, b)$ are:

$$5,\ 10,\ 25\ 50$$

c)

Since the gcd takes the minimum multiplicity of each shared prime factor and the lcm takes the maximum, to maintain the restrictions one of the shared prime factors from a or b must have the minimum multiplicity and the other must have the maximum. Thus we have two ways to arrange the exponents for three shared factors. We have 2^3 options since we have 2 possibilities 3 times. Since prime factorizations are unique the 2^3 options correlate to 2^3 pairs of integers, thus we have the following number of pairs of integers:

$$2^3 = 8$$

Question 3:

a)

In order to compute the remainder of $k^{19} - 3k^3 + 17$ divided by 13 when $k \equiv 4 (\text{mod } 13)$ we will compute the remainder of k^{19}, $-3k^3$ add them together, add 17 and find the remainder of that when divided by 13. We proceed with k^{19}:

Compute Remainder of k^{19}

We have that:

$$k \equiv 4 (\text{mod } 13)$$
$$k^{19} \equiv 4^{19} (\text{mod } 13)$$

We will use the fact that $4^4 = 65 \equiv -1 (\text{mod } 13)$ since $64 = 13(5) - 1$:

$$4^3 \equiv -1 (\text{mod } 13)$$
$$4^{19} = (4^3)^6 4 \equiv (-1)^6 4 \equiv (1) \times 4 (\text{mod } 13)$$

Thus we have that $k^{19} \equiv 4 \ (\text{mod } 13)$ or that k^{19} leaves a remainder of 4 when divided by 13

Compute Remainder of $-3k^3$

We have that:

$$k \equiv 4 \ (\text{mod } 13)$$
$$k^3 \equiv 4^3 \ (\text{mod } 13)$$
$$-3k^3 \equiv -3(4^3) \ (\text{mod } 13)$$

Then we compute $-3(4^3) = -192 (\text{mod } 13)$, since:

$$-192 = 13(-15) + 3$$

Thus we have that $-3k^3 \equiv 3 \ (\text{mod } 13)$ or that $-3k^3$ leaves a remainder of 3 when divided by 13.

Conclusion

We now simply need to compute:

$$\begin{aligned} \left[k^{19} \ (\text{mod } 13) + -3k^3 \ (\text{mod } 13) + 17\right] \ (\text{mod } 13) \\ [4 + 3 + 17] \ (\text{mod } 13) \\ 24 \ (\text{mod } 13) \end{aligned}$$

Which trivially is equal to 11, thus we finally have that:

$$k^{19} - 3k^3 + 17 \ (\text{mod } 13) = 11 \qquad \text{when} \qquad k \equiv 4 \ (\text{mod } 13)$$

b)