Math 122 Assignment 4

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Question 1:

a)

Basis

We prove that the statement is true for the base case where $n = 1$:

LHS	**RHS**
$(1+x)^1$	$1+(1)x$
$1+x$	$1+x$

We can see that $LHS = RHS$ thus the inequality holds for the base case.

Inductive Hypothesis

We assume that the inequality holds for all values of $n \geq 1$:

$$(1+x)^n \geq 1+nx$$

Inductive Step

We have that:

$$\begin{aligned} (1+x)^{n+1} &= (1+x)(1+x)^n \\ &\geq (1+x)(1+nx) && \text{From the inductive hypothesis} \\ &\geq 1+(n+1)x+nx^2 \end{aligned}$$

Thus so far we have that:

$$(1+x)^{n+1} \geq 1+(n+1)x+nx^2$$

And since $x^2 \geq 0$ we know $nx^2 \geq 0$ We have that:

$$1+(n+1)x+nx^2 \geq 1+(n+1)x$$

Since we previously established that $(1+x)^{n+1} \geq 1+(n+1)x+nx^2$ we have that:

$$(1+x)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x$$

And consequentialy:

$$(1+x)^{n+1} \geq 1+(n+1)x$$

Conclusion

We have shown that $(1+x)^n \geq 1+nx$ for all $n \geq 1$. Which completes the proof $\square$

b)

We will start by making the following substitutions:

$$y = \frac{1}{2}(1+x) \qquad 1-y = \frac{1}{2}(1-x) \qquad y^n = \frac{1}{2^n}(1+x)^n \qquad (1-y)^m = \frac{1}{2^m}(1-x^m)$$

Then we can substitue them into the equality:

$$\begin{aligned} \frac{2^n y^n}{n} + \frac{2^m(1-y)^m}{m} &\geq \frac{1}{n} + \frac{1}{m} \\ \frac{2^n \frac{1}{2^n}(1+x)^n}{n} + \frac{2^m \frac{1}{2^m}(1-x^m)}{m} &\geq \frac{1}{n} + \frac{1}{m} \\ \frac{(1+x)^n}{n} + \frac{(1-x^m)}{m} &\geq \frac{1}{n} + \frac{1}{m} \end{aligned}$$

Now that we have the re-written inequality we can prove it using the information form part (a), we have:

$$(1+x)^n \geq 1+nx \qquad (1-x)^m \geq 1-mx$$

Then we divide them both by n and m respectively:

$$\frac{(1+x)^n}{n} \geq \frac{1}{n} + x \qquad \frac{(1-x)^m}{m} \geq \frac{1}{m} - x$$

And then we combine them:

$$\frac{(1+x)^n}{n} + \frac{(1-x)^m}{m} \geq \frac{1}{n} + \cancel{x} \frac{1}{m} \cancel{-x}$$

Finally we have:

$$\frac{(1+x)^n}{n} + \frac{(1-x)^m}{m} \geq \frac{1}{n} + \frac{1}{m}$$

Which completes the proof $\square$

Question 2:

We begin by checking small values of n:

$n = 1$:

$$(1+1)! = 2! = 2 \qquad \text{and} \qquad (1+3)^3 = 4^3 = 64$$

Here $2 < 64$ thus the inequality does not hold.

$n = 2$:

$$(2+1)! = 3! = 6 \qquad \text{and} \qquad (2+3)^3 = 5^3 = 125$$

Here $6 < 125$ thus the inequality does not hold.

$n = 3$:

$$(3+1)! = 4! = 24 \qquad \text{and} \qquad (3+3)^3 = 6^3 = 216$$

Here $24 < 216$ thus the inequality does not hold.

$n = 4$:

$$(4+1)! = 5! = 120 \qquad \text{and} \qquad (4+3)^7 = 4^3 = 343$$

Here $120 < 343$ thus the inequality does not hold.

$n = 5$:

$$(5+1)! = 6! = 720 \qquad \text{and} \qquad (5+3)^3 = 8^3 = 512$$

Here $720 > 512$ thus the inequality is satisfied.
Now it suffices to prove that for all $n \geq 5$ that

$$(n+1)! \geq (n+3)^3$$

We will proceed by applying induction on n

Basis

The basis has already been proved above, the equality holds for $n = 5$

Inductive Hypothesis

We assume that the following is true for all $n \geq 5$

$$(n+1)! \geq (n+3)^3$$

Inductive Step

We want to show that the inequality holds for all values of $n+1$ where $n \geq 5$:

$$\begin{array}{rcll} ((n+1)+1)! & \geq & ((n+1)+3)^3 & \\ (n+2)! & \geq & (n+4)^3 & \\ (n+2)(n+1)! & \geq & (n+4)^3 & \\ (n+2)(n+3)^3 & \geq & (n+4)^3 & \qquad \text{From the inductive hypothesis} \end{array}$$

We can see that if we expanded the LHS of the equality it would have a leading term of n^4 and the RHS would have a leading term of n^3 thus we can say that the equality is satisifed for $n+1$

Conclusion

We have shown that $(n+1)! \geq (n+3)^3$ is true for all $n \geq 5$ which completes the proof $\square$

Question 3:

We prove the definition by applying inducito n on n

Base Case

We prove that the relation is true for $n = 1$

LHS	RHS
f_1^2	$f_1 f_2$
1^2	$1 \cdot 1$
1	1

We can see that the LHS = RHS thus the definition holds for the base case of $n = 1$

Inductive Hypothesis

We assume that the definition:

$$f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$$

holds for all values of $n \geq 1$

Inductive Step

We need to prove that the statement holds for $n + 1$, we start with the LHS:

$$f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2$$
$$= f_k f_{k+1} + f_{k+1}^2 \text{ From the inductive hypothesis}$$
$$= f_{k+1}(f_k + f_{k+1})$$
$$= f_{k+1}(f_{k+2}) \text{ From the definition of the fibbonaci sequance}$$

We have shown that the definition holds by using the inductive hypothesis and the definition of the fibbonaci sequence.

Conclusion

We have shown that the definition:

$$f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$$

is true for all values of $n \geq 1$

Question 4:

a)

To prove that:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

We apply induction on n

Basis:

We prove that the formula holds for $n=1$

LHS	**RHS**
$\frac{1}{1\cdot 2}$	$\frac{1}{(1)+1}$
$\frac{1}{2}$	$\frac{1}{2}$

We see that the LHS = RHS thus the formula holds for the base case

Inductive Hypothesis

We assume that for any $n \geq 1$ that:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

Inductive Step:

We show that the formula is true for $n+1$ or that:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}=\frac{n+1}{n+2}$$

We start with the LHS:

$$\begin{aligned}
\tfrac{1}{1\cdot 2}+\tfrac{1}{2\cdot 3}+\cdots+\tfrac{1}{n\cdot(n+1)}+\tfrac{1}{(n+1)\cdot(n+2)} &= \tfrac{n}{n+1}+\tfrac{1}{(n+1)\cdot(n+2)} && \text{From the inductive hypothesis}\\
&= \tfrac{n(n+2)+1}{(n+1)(n+2)}\\
&= \tfrac{n^2+2n+1}{(n+1)(n+2)}\\
&= \tfrac{(n+1)^2}{(n+1)(n+2)}\\
&= \tfrac{n+1}{n+2}
\end{aligned}$$

We have shown that:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}=\frac{n+1}{n+2}$$

Conclusion

We have now shown that for all values of $n \geq 1$ that:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

b)

i)

Let us first compute the first few values:

1. $t_0 = 1$
2. $t_1 = 1$
3. $t_2 = t_1 - \frac{t_0}{2} = 1 - \frac{1}{2} = \frac{1}{2}$
4. $t_3 = t_2 - \frac{t_1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$
5. $t_4 = t_3 - \frac{t_2}{4} = \frac{1}{6} - \frac{1/2}{4} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

Based on the pattern web observed, we conjecture that:

$$t_n = \frac{1}{n!}$$

For $n \geq 0$

ii)

We will prove the formula by applying induction on n

Base Cases

For $n = 0$

$$t_0 = 1 = \frac{1}{0!}$$

For $n = 1$

$$t_1 = 1 = \frac{1}{1!}$$

Which both match our formula thus the formula holds for the base cases of $n = 0$ and $n = 1$

Inductive Hypothesis

We assume that for all $n \geq 2$ that:

$$t_n = \frac{1}{n!} \qquad \text{and} \qquad t_{n-1} = \frac{1}{(n-1)!}$$

Inductive Step

We aim to show that the formula holds for $n + 1$ or that:

$$t_{n+1} = \frac{1}{(n+1)!}$$

We start with the given recursive definition

$$t_{n+1} = t_n - \frac{t_{n-1}}{n+1}$$

Then we make substitutions from the inductive hypothesis

$$\begin{aligned}
t_n - \tfrac{t_{n-1}}{n+1} &= \tfrac{1}{n!} - \tfrac{\frac{1}{(n-1)!}}{n+1} \\
&= \tfrac{1}{n!} - \tfrac{1}{n+1}\tfrac{1}{(n-1)!} \\
&= \tfrac{1}{n(n-1)!} - \tfrac{1}{(n+1)(n-1)!} \\
&= \tfrac{(n+1)}{n(n-1)!(n+1)} - \tfrac{n}{n(n+1)(n-1)!} \\
&= \tfrac{n+1-n}{n(n-1)!(n+1)} \\
&= \tfrac{1}{n!(n+1)} \\
&= \tfrac{1}{(n+1)!}
\end{aligned}$$

Thus we have shown that $t_{n+1} = \frac{1}{(n+1)!}$ using the recursive definition and the inductive hypothesis

Conclusion

We have shown that for all $n \geq 0$ that:

$$t_{n+1} = \frac{1}{(n+1)!}$$

Question 5:

a)

Since $k!$ is the product of all positive integers from 1 to k, $k!$ is divisible by each integer d in $\{2, 3, \ldots, k\}$ because each d is a factor of $k!$.

If we consider $k! - 1$ Since $k!$ is divisible by d it leaves a remained of 0 when divided by any d thus, $k! - 1$ will leave a remainder of -1, since the convention is not to use negative numbers as a remainder the remainder will be $d - 1$

b)

If we have $k = p$ a prime number and examine $p! - 1$, by part a $p! - 1$ is not divisible by and integer d in $\{2, 3, \ldots, p\}$. Since $p! - 1$ is an integer greater than 1, it must have a prime factor, for example we say it has a prime factor q, since it is not divisible by any integer in d we know q is not in d and must thus be larger than p. By definition q is prime since we declared it was a prime factor of $p! - 1$

c)

Suppose for contradition that there is a largest prime number, call it p. According to part (b), for any prime p, there exsts a prime $q > p$. This contradictions the assumption that p is the largest prime, because part (b) guarantees the existence of a prime number larger than p

d)

Since part c shows that there is no largest prime number, it follows that the primes do not stop at any finite value. For any prime p, there will always be a prime number larger than p, thus the sequence of primes continues infinitely.