MATH 122 Ass. 1

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September 2024

Question 1,

A)

p	q	r	If p then q else r
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

B)

We know that when p is false r determines the value of "if p then q else r" and that when p is true q determines the value. Thus we create an or statement with an and on either side with p and $\neg p$ in each and. When p is true the and with $\neg p$ will be automatically false leaving the other side of the $p \wedge$ to determine if the compound statement will be true or false and vice versa, we can write this L.E as:

$$(p \wedge q) \vee (\neg p \wedge r)$$

To satisfy the condition that we only use the symbols $(p, q, r, \neg, \rightarrow, \wedge)$

$$\begin{aligned}
(p \wedge q) \vee (\neg p \wedge r) \quad &\Leftrightarrow \quad \neg\big(\neg(p \wedge q)\big) \vee \neg\big(\neg(\neg p \wedge r)\big) && \text{Double Negation } \times 2 \\
&\Leftrightarrow \quad \neg\big(\neg(p \wedge q) \wedge \neg(\neg p \wedge r)\big) && \text{DeMorgan's}
\end{aligned}$$

Now we have as desired, a compound statement logically equivalent to if p then q else r using only $p, q, r, \neg, \rightarrow, \wedge$

$$\neg\big(\neg(p \wedge q) \wedge \neg(\neg p \wedge r)\big)$$

C)

Using mostly DeMorgan's Law and the Known L.E $(p \rightarrow q \Leftrightarrow \neg p \vee q)$ we find a logically equivalent compound statement without the use of $\wedge$ and $\vee$

$$\begin{aligned}
\neg\big(\neg(p \wedge q) \wedge \neg(\neg p \wedge r)\big) \quad &\Leftrightarrow \quad \neg\Big(\neg\big(\neg(\neg p) \wedge \neg(\neg q)\big) \wedge \neg\big(\neg p \wedge \neg(\neg r)\big)\Big) && \text{Double Negation } \times 3 \\
&\Leftrightarrow \quad \neg\Big(\neg\neg\big(\neg p \vee \neg q\big) \wedge \neg\neg\big(p \vee \neg r\big)\Big) && \text{DeMorgan's } \times 2 \\
&\Leftrightarrow \quad \neg\Big(\big(\neg p \vee \neg q\big) \wedge \big(\neg(\neg p) \vee \neg r\big)\Big) && \text{Double Negation} \\
&\Leftrightarrow \quad \neg\Big(\big(p \rightarrow \neg q\big) \wedge \big(\neg p \rightarrow \neg r\big)\Big) && \text{Known L.E } \times 2 \\
&\Leftrightarrow \quad \neg\Big(\neg\neg\big(p \rightarrow \neg q\big) \wedge \neg\neg\big(\neg p \rightarrow \neg r\big)\Big) && \text{Double Negation } \times 2 \\
&\Leftrightarrow \quad \neg\big(p \rightarrow \neg q\big) \vee \neg\big(\neg p \rightarrow \neg r\big) && \text{DeMorgan's} \\
&\Leftrightarrow \quad (p \rightarrow \neg q) \rightarrow \neg(\neg p \rightarrow \neg r) && \text{Known L.E}
\end{aligned}$$

Question 2,

A)

Let: s, b, i be equivalent to the statements:

- $s =$ you are sitting still
- $b =$ you are behaving
- $i =$ you can have ice cream

The first statement: "If you sit still and behave, then you can have ice cream" becomes:

$$(s \land b) \rightarrow i$$

The second statement: "If you don't sit still or you don't behave, then you can't have ice cream" becomes:

$$(\neg s \lor \neg b) \rightarrow \neg i$$

These two statements are not logically equivalent, we can prove this with their respective truth tables:

s	b	i	$(s \land b) \rightarrow i$	$(\neg s \lor \neg b) \rightarrow \neg i$
1	1	1	1	1
1	1	0	0	1
1	0	1	1	0
1	0	0	1	1
0	1	1	1	0
0	1	0	1	1
0	0	1	1	0
0	0	0	1	1

B)

Let: q, t, u be equivalent to the statements:

- $q =$ She qualifies for the Olympic 100m final
- $t =$ She finishes top 2 in the semifinal
- $u =$ She finishes in under 11 seconds

The first statement: "To qualify for the Olympic 100m final, she needs to finish top 2 in the semifinal or finish in under 11 seconds:

$$(t \lor u) \rightarrow q$$

The second statement: If she does not finish in the top 2 in the semifinal and she does not finish in under 11 seconds, then she will not qualify for the Olympic 100m final" becomes:

$$(\neg t \land \neg u) \rightarrow \neg q$$

These two statements are not logically equivalent, we can prove this with their respective truth tables:

q	t	u	$(t \lor u) \rightarrow q$	$(\neg t \land \neg u) \rightarrow \neg q$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	0
1	0	0	1	1
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	1	1

C)

Let s, p be equivalent to the statements:

- $s =$ he studies
- $p =$ he passes the quiz

Then the first statement: "He will not pass the quiz if he doesn't study" becomes:

$$\neg s \rightarrow \neg p$$

And the second statement: "It is not true that he will pass and not study" becomes:

$$\neg(p \wedge \neg s)$$

We can prove the statements are logically equivalent using logical equivalences and laws of logic:

$$\begin{array}{rcll} \neg s \rightarrow \neg p & \Leftrightarrow & s \vee \neg p & \text{Known L.E} \\ & \Leftrightarrow & \neg(\neg s) \vee \neg p & \text{Double Negation} \\ & \Leftrightarrow & \neg(\neg s \wedge p) & \text{DeMorgan's} \\ & \Leftrightarrow & \neg(p \wedge \neg s) & \text{Commutativity} \end{array}$$

As desired we demonstrated that the first statement is logically equivalent to the second statement, thus:

$$\neg s \rightarrow \neg p \Leftrightarrow \neg(p \wedge \neg s)$$

$$\text{and} \quad \text{"He will not pass the quiz if he doesn't study"} \Leftrightarrow \text{"It is not true that he will pass and not study"}$$

Question 3,

a)

We prove that $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$:

$$\begin{array}{rcll}
p \wedge (q \vee r) & \Leftrightarrow & \neg\big(\neg p \vee \neg(q \vee r)\big) & \text{DeMorgan's} \\
& \Leftrightarrow & \neg\big(\neg p \vee (\neg q \wedge \neg r)\big) & \text{DeMorgan's} \\
& \Leftrightarrow & \neg\big((\neg p \vee \neg q) \wedge (\neg p \vee \neg r)\big) & \text{Distributive D1} \\
& \Leftrightarrow & \neg\big(\neg(p \wedge q) \wedge \neg(p \wedge r)\big) & \text{DeMorgan's} \times 2 \\
& \Leftrightarrow & (p \wedge q) \wedge (p \wedge r) & \text{DeMorgan's, as desired}
\end{array}$$

b)

First we find an statement for all rows with $s = 1$ and find an equation only true for the truth values:

p	q	r	s		
0	0	0	0		
0	0	1	0		
0	1	0	0	$\Longrightarrow$ Row 3.	$\neg p \wedge q \wedge r$
0	1	1	1	$\Longrightarrow$ Row 7.	$p \wedge q \wedge \neg r$
1	0	0	0	$\Longrightarrow$ Row 8.	$p \wedge q \wedge r$
1	0	1	0		
1	1	0	1		
1	1	1	1		

Now combining these into disjunctive normal form gives us an equation valid for the given truth table:

$$(\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$$

c)

We prove that:

$$q \wedge (\neg r \rightarrow p) \Leftrightarrow (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$$

$$\begin{array}{rcll}
(\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) & \Leftrightarrow & (\neg p \wedge (q \wedge r)) \vee (p \wedge (q \wedge r)) \vee ((p \wedge \neg r) \wedge q) & \text{Associative} \times 3\text{, Commutative} \times 3 \\
& \Leftrightarrow & ((q \wedge r) \wedge (p \vee \neg p)) \vee ((p \wedge \neg r) \wedge q) & \text{Distributive} \\
& \Leftrightarrow & ((q \wedge r) \wedge (1)) \vee ((p \wedge \neg r) \wedge q) & \text{Known Tautology} \\
& \Leftrightarrow & (q \wedge r) \vee ((p \wedge \neg r) \wedge q) & \text{Identity} \\
& \Leftrightarrow & q \wedge (r \vee (p \wedge \neg r)) & \text{Distributive} \\
& \Leftrightarrow & q \wedge ((r \vee p) \wedge (r \vee \neg r)) & \text{Distributive} \\
& \Leftrightarrow & q \wedge ((r \vee p) \wedge (1)) & \text{Known Tautology} \\
& \Leftrightarrow & q \wedge (r \vee p) & \text{Identity} \\
& \Leftrightarrow & q \wedge (\neg(\neg r) \vee p) & \text{Double Negation} \\
& \Leftrightarrow & q \wedge (\neg r \rightarrow p) & \text{Known Identity, as desired}
\end{array}$$

Question 4,

a)

i) In the statement $p \wedge (p \vee q)$ if p is true, the truth value of q does not matter in the statement $(p \vee q)$ as if there is 1 side of the $\vee$ that is true the entire statement is true. This means our statement becomes $p \wedge p$ which if of course true if p is true.

ii) In the statement $p \wedge (p \vee q)$ if p is false, we can immediately tell the entire statement is false because if one side of an $\wedge$ statement is false, in this case p, we can disregard the other side of the $\wedge$, and by definition the entire statement is false.

We showed that for the entire universe of p the statement is equal to p, and demonstrated that the truth value of q was irrelevant, thus proving the logical equivalence.

b)

We show that $(r \rightarrow q) \wedge (q \wedge \neg p)$ is equivalent to $\neg(\neg q \vee p)$ by using logical equivalences and the absorption law:

$$\begin{array}{rcll} (r \rightarrow q) \wedge (q \wedge \neg p) & \Leftrightarrow & (\neg r \vee q) \wedge (q \wedge \neg p) & \text{Known L.E} \\ & \Leftrightarrow & ((\neg r \vee q) \wedge q) \wedge \neg p & \text{Associative} \\ & \Leftrightarrow & q \wedge \neg p & \text{Absorption} \\ & \Leftrightarrow & \neg(\neg q \vee p) & \text{DeMorgan's, as desired} \end{array}$$

Question 5,

a)

Let the following statements, e, c, w, s & d be equal to:

- $e =$ I ate tomatoes
- $c =$ I consumed enough vitamin C
- $w =$ The tomato plant got enough water
- $s =$ The tomato plant got enough sunlight
- $d =$ The tomato plant died

Writing it in symbolic form:

$$\begin{array}{r} \neg e \rightarrow \neg c \\ (\neg w \vee \neg s) \rightarrow (d \wedge \neg e) \\ c \\ \hline \therefore \quad s \end{array}$$

Now using inference rules to prove the implication:

1.	$\neg e \rightarrow \neg c$	Given Premise
2.	c	Given Premise
3.	$(\neg w \vee \neg s) \rightarrow (d \wedge \neg e)$	Given Premise
4.	$c \rightarrow e$	Contrapositive (1)
5.	e	Modus Ponens (2,4)
6.	$\neg(d \wedge \neg e) \rightarrow \neg(\neg w \vee \neg s)$	Contrapositive (3)
7.	$(\neg d \vee e) \rightarrow (w \wedge s)$	DeMorgan's $\times 2$
8.	$\neg d \vee e$	Disjunctive Amplification (5,7)
9.	$w \wedge s$	Modus Ponens (7,8)
10.	s	Conjunctive Simplification (9)

From the premises we have inferred that s is true thus proving the implication.

b)

- $w =$ I walk to work
- $c =$ I cycle to work
- $m =$ I listen to music
- $g =$ I chew gum
- $a =$ I am alone

Writing it in symbolic form

$$\begin{array}{r} w \vee c \\ \neg(w \wedge c) \\ w \rightarrow ((m \wedge g) \leftrightarrow a) \\ c \rightarrow \neg g \\ \hline \therefore \quad g \rightarrow m \end{array}$$

Using a counter example we prove the argument is invalid, let:

- $w = 1$ I walked to work
- $c = 0$ I did not cycle to work
- $g = 1$ I chewed gum
- $m = 0$ I did not listen to music
- $a = 0$ I am not alone

We see that all the premises are true but the conclusion is false:

Prem 1. $w \vee c$	Prem 2. $\neg(w \wedge c)$	Prem 3. $w \rightarrow ((m \wedge g) \leftrightarrow a)$	Prem 4. $c \rightarrow \neg g$	Conclusion $g \rightarrow m$
$1 \vee 0$	$\neg(1 \wedge 0)$	$1 \rightarrow ((0 \wedge 1) \leftrightarrow 0)$	$0 \rightarrow \neg 1$	$1 \rightarrow 0$
$\vdots$	$\neg(0)$	$1 \rightarrow (0 \leftrightarrow 0)$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$1 \rightarrow 1$	$\vdots$	$\vdots$
1	1	1	1	0

Using the above truth values we have that all the premises are true yet the conclusion is false, thus the conclusion does not follow the premises and it is an **invalid argument**.

Question 6 (Bonus),

The boxes we need to check are:

- **Vanilla Sponge Box**, to ensure that all icing is either raspberry or chocolate.
- **Lemon Icing Box**, to ensure that there is no vanilla sponge with lemon icing.
- **Elderflower Icing Box**, to ensure that there is no vanilla sponge with elder flower icing.

The other boxes we do not need to check since:

- **Red Velvet Sponge Box**, since our rule only applies to vanilla sponge.
- **Raspberry Icing**, this box would already comply with the rule, if by chance it contained vanilla sponge.
- **Chocolate Icing**, this box would already comply with the rule, if by chance it contained vanilla sponge.

Thus only 3 boxes need to be opened in order to verify that the head bakers rule is being followed.