Math 211 Ass 3.

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October 2024

Question 1:

a)

We prove linearity by proving that our function satisfies *additive linearity* and *scalar linearity*:

Additive Linearity:
Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ then we form and simplify as follows,

$$\begin{array}{rcll} L(\vec{x}+\vec{y}) & = & ((\vec{x}+\vec{y}) \times \vec{v}) + ((\vec{x}+\vec{y}) \cdot \vec{v})\vec{v} & \text{Definition of } L \\ & = & (-\vec{v} \times (\vec{x}+\vec{y})) + (\vec{v} \cdot (\vec{x}+\vec{y}))\vec{v} & \text{Anti Commutativity \& Commutativity} \\ & = & ((-\vec{v} \times \vec{x}) + (-\vec{v} \times \vec{y})) + ((\vec{v} \cdot \vec{x}) + (\vec{v} \cdot \vec{y}))\vec{v} & \text{Distributivity of Cross Product and Distributivity of Dot Product} \\ & = & ((-\vec{v} \times \vec{x}) + (-\vec{v} \times \vec{y})) + (\vec{v} \cdot \vec{x})\vec{v} + (\vec{v} \cdot \vec{y})\vec{v} & \text{Multiplicative Distributivity} \\ & = & (-\vec{v} \times \vec{x}) + (\vec{v} \cdot \vec{x})\vec{v} + (-\vec{v} \times \vec{y}) + (\vec{v} \cdot \vec{y})\vec{v} & \text{Associativity \& Commutativity} \\ & = & (\vec{x} \times \vec{v}) + (\vec{x} \cdot \vec{v})\vec{v} + (\vec{y} \times \vec{v}) + (\vec{y} \cdot \vec{v})\vec{v} & \text{Anti Commutativity } \times 2 \text{ \& Commutativity } \times 2 \\ & = & L(\vec{x}) + L(\vec{y}) & \text{Definition of } L \end{array}$$

Thus since $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$, the property of additive linearity is satisified.

Scalar Linearity:
Let $\vec{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, then we form and simplify as follows,

$$\begin{array}{rcll} L(t\vec{x}) & = & ((t\vec{x}) \times \vec{v}) + ((t\vec{x}) \cdot \vec{v})\vec{v} & \text{Definition of } L \\ & = & t(\vec{x} \times \vec{v}) + (t(\vec{x} \cdot \vec{v}))\vec{v} & \text{Associativity of Cross Product and Dot Product} \\ & = & t(\vec{x} \times \vec{v}) + t\vec{v}(\vec{x} \cdot \vec{v}) & \text{Distributivity} \\ & = & t(\vec{x} \times \vec{v}) + t(\vec{v}(\vec{x} \cdot \vec{v})) & \text{Associativity} \\ & = & t(\vec{x} \times \vec{v}) + t((\vec{x} \cdot \vec{v})\vec{v}) & \text{Commutativity} \\ & = & t((\vec{x} \times \vec{v}) + ((\vec{x} \cdot \vec{v})\vec{v})) & \text{Distributivity} \\ & = & t\ L(\vec{x}) & \text{Definition of } L \end{array}$$

Thus since $L(t\vec{x}) = t\ L(\vec{x})$, the property of scalar linearity is satisfied.

Since Additive Linearity and Scalar Linearity both hold, L is a linear operator.

b)

To compute $[L]$ we need to determine the result of L on the standard basis vectors of $\mathbb{R}^3$, which are $\vec{e_1}, \vec{e_2}$ and $\vec{e_3}$. Thus $[L]$ is given by:

$$[L] = \begin{bmatrix} | & | & | \\ L(\vec{e_1}) & L(\vec{e_2}) & L(\vec{e_3}) \\ | & | & | \end{bmatrix}$$

Thus we perform the following computations:

$\underline{L(\vec{e_1})}$

$\underline{\vec{e_1} \times \vec{v}}$:

$$= \begin{bmatrix} 0(v_3) - 0(v_2) \\ 0(v_1) - 1(v_3) \\ 1(v_2) - 0(v_1) \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-v_3 \\ v_2-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -v_3 \\ v_2 \end{bmatrix}$$

$\underline{\vec{e_1} \cdot \vec{v}}$:
$= 1v_1 + 0v_2 + 0v_3 = v_1$

$\underline{(\vec{e_1} \cdot \vec{v})\vec{v}}$:

$$= (v_1)\vec{v} = v_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1^2 \\ v_1v_2 \\ v_1v_3 \end{bmatrix}$$

$\underline{L(\vec{e_1})}$
$= (\vec{e_1} \times \vec{v}) + (\vec{e_1} \cdot \vec{v})\vec{v}$

$$= \begin{bmatrix} 0 \\ -v_3 \\ v_2 \end{bmatrix} + \begin{bmatrix} v_1^2 \\ v_1v_2 \\ v_1v_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1^2 \\ v_1v_2 - v_3 \\ v_1v_3 + v_2 \end{bmatrix}$$

$\underline{L(\vec{e_2})}$

$\underline{\vec{e_2} \times \vec{v}}$:

$$= \begin{bmatrix} 1(v_3) - 0(v_2) \\ 0(v_1) - 0(v_3) \\ 0(v_2) - 1(v_1) \end{bmatrix} = \begin{bmatrix} v_3-0 \\ 0-0 \\ 0-v_1 \end{bmatrix} = \begin{bmatrix} v_3 \\ 0 \\ -v_1 \end{bmatrix}$$

$\underline{\vec{e_2} \cdot \vec{v}}$:
$= 0v_1 + 1v_2 + 0v_3 = v_2$

$\underline{(\vec{e_2} \cdot \vec{v})\vec{v}}$:

$$= (v_2)\vec{v} = v_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2v_1 \\ v_2^2 \\ v_2v_3 \end{bmatrix}$$

$\underline{L(\vec{e_2})}$
$= (\vec{e_2} \times \vec{v}) + (\vec{e_2} \cdot \vec{v})\vec{v}$

$$= \begin{bmatrix} v_3 \\ 0 \\ -v_1 \end{bmatrix} + \begin{bmatrix} v_2v_1 \\ v_2^2 \\ v_2v_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_2v_1 + v_3 \\ v_2^2 \\ v_2v_3 - v_1 \end{bmatrix}$$

$\underline{L(\vec{e_3})}$

$\underline{\vec{e_3} \times \vec{v}}$:

$$= \begin{bmatrix} 0(v_3) - 1(v_2) \\ 1(v_1) - 0(v_3) \\ 0(v_2) - 0(v_1) \end{bmatrix} = \begin{bmatrix} 0-v_2 \\ v_1-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \\ 0 \end{bmatrix}$$

$\underline{\vec{e_3} \cdot \vec{v}}$:
$= 0v_1 + 0v_2 + 1v_3 = v_3$

$\underline{(\vec{e_3} \cdot \vec{v})\vec{v}}$:

$$= (v_3)\vec{v} = v_3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3v_1 \\ v_3v_2 \\ v_3^2 \end{bmatrix}$$

$\underline{L(\vec{e_3})}$
$= (\vec{e_3} \times \vec{v}) + (\vec{e_3} \cdot \vec{v})\vec{v}$

$$= \begin{bmatrix} -v_2 \\ v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} v_3v_1 \\ v_3v_2 \\ v_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} v_3v_1 - v_2 \\ v_3v_2 + v_1 \\ v_3^2 \end{bmatrix}$$

Since we have computed $L(\vec{e_1}), L(\vec{e_2})$ and $L(\vec{e_3})$, $[L]$ is given by:

$$[L] = \begin{bmatrix} | & | & | \\ L(\vec{e_1}) & L(\vec{e_2}) & L(\vec{e_3}) \\ | & | & | \end{bmatrix} = \begin{bmatrix} v_1^2 & v_2v_1 + v_3 & v_3v_1 - v_2 \\ v_1v_2 - v_3 & v_2^2 & v_3v_2 + v_1 \\ v_1v_3 + v_2 & v_2v_3 - v_1 & v_3^2 \end{bmatrix}$$

Question 2:

Let us first find the individual transformation matrices, then we will multiply them:

Reflects about the plane $2x_1 - x^2 + 2x_3$:

We need to find $[\text{refl}_{\vec{n}}]$ for $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ which is given by:

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} | & | & | \\ \text{refl}_{\vec{n}}(\vec{e_1}) & \text{refl}_{\vec{n}}(\vec{e_2}) & \text{refl}_{\vec{n}}(\vec{e_3}) \\ | & | & | \end{bmatrix}$$

Thus we need to compute $\text{refl}_{\vec{n}}$ for the three basis vectors:

$\text{refl}_{\vec{n}}(\vec{e_1})$:

textitproj$_{\vec{n}}(\vec{e_1})$:

$$\text{proj}_{\vec{n}}(\vec{e_1}) = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{2}{4+1+4} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{2}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\text{proj}_{\vec{n}}(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/9 \\ -4/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 4/9 \\ -8/9 \end{bmatrix}$$

$\text{refl}_{\vec{n}}(\vec{e_2})$:

$\text{proj}_{\vec{n}}(\vec{e_2})$:

$$\text{proj}_{\vec{n}}(\vec{e_2}) = \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{-1}{4+1+4} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{-1}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/9 \\ 1/9 \\ -2/9 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}(\vec{e_2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\text{proj}_{\vec{n}}(\vec{e_2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -2/9 \\ 1/9 \\ -2/9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -4/9 \\ 2/9 \\ -4/9 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 7/9 \\ 4/9 \end{bmatrix}$$

$\text{refl}_{\vec{n}}(\vec{e_3})$:

$\text{proj}_{\vec{n}}(\vec{e_3})$:

$$\text{proj}_{\vec{n}}(\vec{e_3}) = \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{2}{4+1+4} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{2}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}(\vec{e_3}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\text{proj}_{\vec{n}}(\vec{e_3}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/9 \\ -4/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} -8/9 \\ 4/9 \\ 1/9 \end{bmatrix}$$

Then we have:

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} | & | & | \\ \text{refl}_{\vec{n}}(\vec{e_1}) & \text{refl}_{\vec{n}}(\vec{e_2}) & \text{refl}_{\vec{n}}(\vec{e_3}) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/9 & 4/9 & -8/9 \\ 4/9 & 7/9 & 4/9 \\ -8/9 & 4/9 & 1/9 \end{bmatrix}$$

Rotates by an angle of $5\pi/6$ about the y axis:

The rotation matrix that rotates a vector about the y-axis by an angle of $5\pi/6$ is given by:

$$[R_{5\pi/6,y}] = \begin{bmatrix} \cos(5\pi/6) & 0 & \sin(5\pi/6) \\ 0 & 1 & 0 \\ -\sin(5\pi/6) & 0 & \cos(5\pi/6) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 \end{bmatrix}$$

Reflects about the yz-plane:

The rotation matrix that reflects a vector about the yz-plane is given by:

$$[R_{y,z}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After computing all three transformation matrices, we multiply them in reverse order to maintain order through matrix multiplication:

$$[L] = ([R_{y,z}][R_{5\pi/6,y}])[\text{refl}_{\vec{n}}]$$

First:

$$[R_{y,z}][R_{5\pi/6,y}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-\frac{\sqrt{3}}{2}) + (0)(0) + (0)(-\frac{1}{2}) & (-1)(0) + (0)(1) + (0)(0) & (-1)(\frac{1}{2}) + (0)(0) + (0)(-\frac{\sqrt{3}}{2}) \\ (0)(-\frac{\sqrt{3}}{2}) + (1)(0) + (0)(-\frac{1}{2}) & (0)(0) + (1)(1) + (0)(0) & (0)(\frac{1}{2}) + (1)(0) + (0)(-\frac{\sqrt{3}}{2}) \\ (0)(-\frac{\sqrt{3}}{2}) + (0)(0) + (1)(-\frac{1}{2}) & (0)(0) + (0)(1) + (1)(0) & (0)(\frac{1}{2}) + (0)(0) + (1)(-\frac{\sqrt{3}}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} + 0 + 0 & 0 + 0 + 0 & -\frac{1}{2} + 0 + 0 \\ 0 + 0 + 0 & 0 + 1 + 0 & 0 + 0 + 0 \\ 0 + 0 + -\frac{1}{2} & 0 + 0 + 0 & 0 + 0 + -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

Then we compute: $[L] = ([R_{y,z}][R_{5\pi/6,y}])[\text{refl}_{\vec{n}}]$

$$[L] = ([R_{y,z}][R_{5\pi/6,y}])[\text{refl}_{\vec{n}}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1/9 & 4/9 & -8/9 \\ 4/9 & 7/9 & 4/9 \\ -8/9 & 4/9 & 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} (-\frac{\sqrt{3}}{2})(\frac{1}{9}) + (0)(\frac{4}{9}) + (-\frac{1}{2})(-\frac{8}{9}) & (-\frac{\sqrt{3}}{2})(\frac{4}{9}) + (0)(\frac{7}{9}) + ((-\frac{1}{2})(\frac{4}{9}) & (-\frac{\sqrt{3}}{2})(-\frac{8}{9}) + (0)(\frac{4}{9}) + (-\frac{1}{2})(\frac{1}{9}) \\ (0)(\frac{1}{9}) + (1)(\frac{4}{9}) + (0)(-\frac{8}{9}) & (0)(\frac{4}{9}) + (1)(\frac{7}{9}) + (0)(\frac{4}{9}) & (0)(-\frac{8}{9}) + (1)(\frac{4}{9}) + (0)(\frac{1}{9}) \\ (-\frac{1}{2})(\frac{1}{9}) + (0)(\frac{4}{9}) + (-\frac{\sqrt{3}}{2})(-\frac{8}{9}) & (-\frac{1}{2})(\frac{4}{9}) + (0)(\frac{7}{9}) + (-\frac{\sqrt{3}}{2})(\frac{4}{9}) & (-\frac{1}{2})(-\frac{8}{9}) + (0)(\frac{4}{9}) + (-\frac{\sqrt{3}}{2})(\frac{1}{9}) \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{\sqrt{3}}{18} + 0 + \frac{8}{18} & -\frac{4\sqrt{3}}{18} + 0 + -\frac{4}{18} & \frac{8\sqrt{3}}{18} + 0 + -\frac{1}{18} \\ 0 + \frac{4}{9} + 0 & 0 + \frac{7}{9} + 0 & 0 + \frac{4}{9} + 0 \\ -\frac{1}{18} + 0 + \frac{8\sqrt{3}}{18} & -\frac{4}{18} + 0 + -\frac{4\sqrt{3}}{18} & \frac{8}{18} + 0 + -\frac{\sqrt{3}}{18} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}+8}{18} & \frac{-4\sqrt{3}-4}{18} & \frac{8\sqrt{3}-1}{18} \\ \frac{4}{9} & \frac{7}{9} & \frac{4}{9} \\ \frac{-1+8\sqrt{3}}{18} & \frac{-4-4\sqrt{3}}{18} & \frac{8-\sqrt{3}}{18} \end{bmatrix}$$

Finally we have that:

$$[L] = \begin{bmatrix} \frac{-\sqrt{3}+8}{18} & \frac{-4\sqrt{3}-4}{18} & \frac{8\sqrt{3}-1}{18} \\ 4/9 & 7/9 & 4/9 \\ \frac{-1+8\sqrt{3}}{18} & \frac{-4-4\sqrt{3}}{18} & \frac{8-\sqrt{3}}{18} \end{bmatrix}$$

Question 3:

a)

Let us first find a matrix interpretation for L and row reduce it to RREF:

$$\begin{bmatrix} x_2 + 2x_2 + 3x_3 + 4x_4 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 \\ 11x_1 + 14x_2 + 17x_3 + 20x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 11 & 14 & 17 & 20 \end{bmatrix}$$

Then we will row reduce it:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 11 & 14 & 17 & 20 \end{bmatrix} \overset{R_2 + -5R_1}{\underset{R_3 + -11R_1}{\Rightarrow}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \overset{R_2 \div -4}{\Rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \end{bmatrix} \overset{R_3 + 8R_2}{\Rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i. *Col*(A)

By observing the row reduced matrix, we see that columns 1 & 2 contain the leading ones so the columns 1 & 2 of the original matrix form the basis for the column-space of L:

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 14 \end{bmatrix} \right\}$$

ii. *Null*(A)

The null-space of A is given by its solution to $A\vec{x} = 0$ as follows, we can set our row reduced matrix equal to $\vec{0}$ as it has the same solution as the original:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

Then for the general solution since there is 2 free variables we represent x_1 and x_2 in terms of x_3 and x_4, first we find x_2: $x_2 = -2x_3 - 3x_4$

Then we can substitute x_2 in R_1 and expand and simplify as follows:
$x_1 + 2(-2x_3 - 3x_4) + 3x_3 + 4x_4 = 0$
$x_1 + -4x_3 - 6x_4 + 3x_3 + 4x_4 = 0$
$x_1 + -x_3 - 2x_4 = 0$
$x_1 = x_3 + 2x_4$

Now we can represent $x_3 = a$ and $x_4 = b$ to represent our free variables where $a, b \in \mathbb{R}$:

$$\begin{cases} x_1 = a + 2b \\ x_2 = -2a - 3b \\ x_3 = a \\ x_4 = b \end{cases} \Rightarrow \quad \vec{x} = \begin{bmatrix} a + 2b \\ -2a - 3b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Since a and b are free variables we can assert that the following is a basis for the null-space of A

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

iii. *Col*(A^T)

The column-space of the transpose of A also known as the row-space of A is given by the non zero rows of the row reduced matrix:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

iv. *Null*(A^T)

To find the left null-space or the null-space of the transpose we must first find the transpose of A

$$A^T = \begin{bmatrix} 1 & 5 & 11 \\ 2 & 6 & 14 \\ 3 & 7 & 17 \\ 4 & 8 & 20 \end{bmatrix}$$

Then we row reduce the transposed matrix:

$$\begin{bmatrix} 1 & 5 & 11 \\ 2 & 6 & 14 \\ 3 & 7 & 17 \\ 4 & 8 & 20 \end{bmatrix} \begin{matrix} R_2 + -2R_1 \\ \Rightarrow \\ R_3 + -3R_1 \\ R_4 + -4R_1 \end{matrix} \begin{bmatrix} 1 & 5 & 11 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \\ 0 & -12 & -24 \end{bmatrix} \begin{matrix} R_2(-1/4) \\ \Rightarrow \\ R_3(-1/8) \\ R_4(-1/12) \end{matrix} \begin{bmatrix} 1 & 5 & 11 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_3 + -2R_2 \\ \Rightarrow \\ R_4 + -3R_2 \end{matrix} \begin{bmatrix} 1 & 5 & 11 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we set the matrix equal to $\vec{0}$ and we have that:

$$\begin{cases} x_1 + 5x_2 + 11x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

We let $x_3 = t$ where $t \in \mathbb{R}$ and the represent x_1 and x_2 in terms of t, first we find x_2:
$x_2 = -2x_3 = -2t$

Then we use the above to make a substitution in R_1 and expand and simplify as follows:
$x_1 + 5x_2 + 11x_3 = 0$
$x_1 + 5(-2x_3) + 11x_3 = 0$
$x_1 + -10x_3 + 11x_3 = 0$
$x_1 + x_3 = 0$
$x_1 = -x_3 = -t$

Then we have that:

$$\begin{cases} x_1 = -t \\ x_2 = -2t \\ x_3 = t \\ x_4 = 0 \end{cases} \Rightarrow \begin{bmatrix} -t \\ -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Since t is a free variable we can assert that the following is a basis for Null(A^T):

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Finally we have that

$$\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 14 \end{bmatrix} \right\} \qquad \text{Null}(A) = \text{Span}$$

$$\text{Col}(A^T) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\} \qquad \text{Null}(A^T) = \text{Span} \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b)

By the rank nullity theorem the sum of the rank of a matrix and the nullity of the same matrix is equal to the number of columns in the respective matrix, thus we have that:

$$\text{Rank}(A) = 2 \quad \text{Nullity}(A) = 2 \quad \text{Rank}(A) + \text{Nullity}(A) = 4$$

Meaning that the matrix A has 4 columns which is true.

We also have that:

$$\text{Rank}(A^T) = 2 \quad \text{Nullity}(A^T) = 1 \quad \text{Rank}(A^T) + \text{Nullity}(A^T) = 3$$

Meaning that the matrix A^T has 3 columns which is true.

Since the rank is 2 it means that the function L collapses the input in $\mathbb{R}^4$ to some plane in $\mathbb{R}^3$

Question 4 :

a)

To find the LU factorization of A we will first row reduce A, while keeping track of the elementary matrices respective to each row operation:

$$\begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 1 \\ 3 & -7 & 2 \end{bmatrix} \overset{R_2+2R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 3 & -7 & 2 \end{bmatrix} \overset{R_3-3R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 5 \end{bmatrix} \overset{R_3+4R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Then we find the respective inverted elementary matrices since $A = (E_1^{-1}E_2^{-1}E_3^{-1})U$ and $L = (E_1^{-1}E_2^{-1}E_3^{-1})$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

Now to find L we need only multiply $E_1^{-1}(E_2^{-1}E_3^{-1})$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

Thus we have that:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

To solve the system $A\vec{x} = \vec{b}$, we will first make the substitution $\vec{y} = U\vec{x}$:

$$L\vec{y} = \vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} y_1 = 1 \\ -2(y_1) + y_2 = -1 \\ 3(y_1) - 4(y_2) + y_3 = 1 \end{cases}$$

Then we can use forward-substitution to solve for $\vec{y}$
R_1: $y_1 = 1$

$R_2 : -2(y_1) + y_2 = -1$
$-2 + y_2 = -1$
$y_2 = 1$

$R_3 : 3(y_1) - 4(y_2) + y_3 = 1$
$3(1) - 4(1) + y_3 = 1$
$-1 + y_3 = 1$
$y_3 = 2$

We now have that:

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Now having $\vec{y}$ we can solve the system $U\vec{x} = \vec{y}$ and the computed $\vec{x}$ will be the solution to the matrix $[A|\vec{x}]$:

$$U\vec{x} = \vec{y} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 - x_2 - x_3 = 1 \\ x_2 - x_3 = 1 \\ x_3 = 2 \end{cases}$$

Then we can use back-substitution to solve for $\vec{x}$
R_3: $x_3 = 2$

$R_2 : x_2 - x_3 = 1$
$x_2 - 2 = 1$
$x_2 = 3$

R_1 :
$x_1 - x_2 - x_3 = 1$
$x_1 - 3 - 2 = 1$
$x_1 = 6$

We now have that:

$$\vec{x} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

Which is the solution to:

$$[A|\vec{b}] = [LU|\vec{b}] = \vec{x} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

As desired.