Math 211 Assignment 5

Dryden Bryson

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Question 1:

We will first start by performing row operations to reduce row 1 to a row containing only x:

$$\begin{vmatrix} x & 3 & 4 \\ x^2 & 9 & 16 \\ 7 & x+4 & x+3 \end{vmatrix} \xrightarrow{C_3 - \frac{4}{3}C_2} \begin{vmatrix} x & 3 & 0 \\ x^2 & 9 & 4 \\ 7 & x+4 & \frac{-x-7}{3} \end{vmatrix} \xrightarrow{C_2 - \frac{3}{x}C_1} \begin{vmatrix} x & 0 & 0 \\ x^2 & 9-3x & 4 \\ 7 & x+4-\frac{21}{x} & \frac{-x-7}{3} \end{vmatrix}$$

Thus by the cofactor expansion formula we have that:

$$p(x) = x \cdot \begin{vmatrix} 9-3x & 4 \\ x+4-\frac{21}{x} & \frac{-x-7}{3} \end{vmatrix}$$

Then by the formula for the determinant of a 2×2 matrix we have that:

$$\begin{aligned} p(x) &= x \cdot \begin{vmatrix} 9-3x & 4 \\ x+4-\frac{21}{x} & \frac{-x-7}{3} \end{vmatrix} = x \cdot ((9-3x)(\tfrac{-x-7}{3}) - 4(x+4-\tfrac{21}{x})) \\ &= x \cdot (\tfrac{(9-3x)(-x-7)}{3} - (4x+16-\tfrac{84}{x})) \\ &= x \cdot (\tfrac{3x^2+12x-63}{3} - (4x+16-\tfrac{84}{x})) \\ &= x \cdot ((x^2+4x-21) - (4x+16-\tfrac{84}{x})) \\ &= x \cdot (x^2+4x-21-4x-16+\tfrac{84}{x}) \\ &= x \cdot (x^2-37+\tfrac{84}{x}) \\ &= x^3-37x+84x \end{aligned}$$

Now it suffices to find the roots of:

$$p(x) = x^3 - 37x + 84x$$

Through the use of the rational root theorem we can identify 3 as one of the roots, thus we can perform synthetic division to form a quadratic which we can solve using the quadratic formula, we need to perform:

$$(x^3 - 37x + 84) \div (x-3)$$

The following synthetic division table:

$$\begin{array}{c|cccc} 3 & 1 & 0 & -37 & 84 \\ & & 3 & 9 & -84 \\ \hline & 1 & 3 & -28 & 0 \end{array}$$

We have that the factor expression is:

$$(x-3)(x^2+3x-28)$$

By using the quadratic formula on $(x^2+3x-28)$ we have that the remaining roots are, 4 and -7 which gives us:

$$p(x) = \begin{vmatrix} x & 3 & 4 \\ x^2 & 9 & 16 \\ 7 & x+4 & x+3 \end{vmatrix} = x^3 - 37x + 84x = (x-3)(x-4)(x+7)$$

Thus we have that the roots of $p(x)$ are 3,4 and -7

Question 2:

a)

Since we know that if two matrices have the same characteristic polynomial then they have the same eigen values. Thus it suffices to show that A and A^T have the same characteristic polynomial. The characteristic polynomial of $A^T$is $(A^T - \lambda I)$

We will first demonstrate that: $(\lambda I)^T = (\lambda I)$ we can observe this visually:

$$(\lambda I) = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \qquad (\lambda I)^T = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Which makes sense since the matrices are diagonally symmtrical, thus clearly: $(\lambda I)^T = (\lambda I)$. We can proceed with the property of "Determinant of Transpose" where:

$$\left|A^T\right| = |A|$$

Thus we have that

$$|A - \lambda I| = \left|(A - \lambda I)^T\right|$$

And then by the transpose property of matrix addition we have that:

$$|A - \lambda I| = \left|A^T - (\lambda I)^T\right|$$

and since we have shown that $(\lambda I)^T = (\lambda I)$ we have that:

$$|A - \lambda I| = \left|A^T - \lambda I\right|$$

Which demonstrates that the characteristic polynomial of a matrix is the same as the characteristic polynomial for the transposed matrix, thus the matrices A and A^T will have the same eigen values. $\square$

b)

We have that $A\vec{v} = \lambda\vec{v}$ and we want to show that $(A - kI)\vec{v} = (\lambda - k)\vec{v}$, thus we will being by observing the matrix $A - kI$ acting on $\vec{v}$:

$$\begin{aligned} (A - kI)\vec{v} &= A\vec{v} - kI\vec{v} && \text{Right Distributivity of Matrix Multiplication} \\ &= A\vec{v} - k\vec{v} && \text{Multiplication of identity matrix } (IM = M) \\ &= \lambda\vec{v} - k\vec{v} && \text{Substitution from given information } (A\vec{v} = \lambda\vec{v}) \\ &= (\lambda - k)\vec{v} && \text{Right Distributivity of Matrix Multiplication} \end{aligned}$$

We have shown that: $(A - kI)\vec{v} = (\lambda - k)\vec{v}$ or that $\vec{v}$ is an eigen vector of the matrix $A - kI$ with and eigen value of $\lambda - k$. $\square$

c)

Since A is idempotent we have that:

$$A = A^2 \Rightarrow A\vec{v} = A^2\vec{v}$$

Next we can represent A^2 as follows:

$$\begin{aligned} A^2\vec{v} &= A(A\vec{v}) && \text{Definition of exponentiation} \\ &= A(\lambda\vec{v}) && \text{Subsitution from eigenvalue equation } A\vec{v} = \lambda\vec{v} \\ &= \lambda(A\vec{v}) && \text{Associativity of Scalars} \\ &= \lambda(\lambda\vec{v}) && \text{Subsitution from eigenvalue equation } A\vec{v} = \lambda\vec{v} \\ &= \lambda^2\vec{v} && \text{Definition of exponentiation} \end{aligned}$$

We can connect the two equations which yeilds:

$$A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$$

Then we can form and simplify the following eigenvalue equation:

$$\begin{aligned} A\vec{v} &= \lambda^2\vec{v} && \text{From above equation} \\ \lambda\vec{v} &= \lambda^2\vec{v} && \text{Substitution from eigenvalue equation} \\ \lambda\vec{v} - \lambda^2\vec{v} &= 0 && \text{Subtract } \lambda^2\vec{v} \text{ from both sides} \\ (\lambda - \lambda^2)\vec{v} &= 0 && \text{Distributivity of Scalar Addition} \\ (\lambda - \lambda^2) &= 0 && \text{Divide both sides by } \vec{v} \\ \lambda(1 - \lambda) &= 0 && \text{Distributivity} \end{aligned}$$

Then trivially we can see how the only possible values for λ are 0 and 1, since $\lambda(1 - \lambda) = 0$ thus $\lambda_1 = 0, \lambda_2 = 1$. $\square$

Question 3:

a)

We will begin with the cofactor decomposition of $A - \lambda I$ along the first row, which is as follows:

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 1-\lambda \begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3 \begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

Let us then compute the sub-determinants:

i)
$$\begin{aligned} \begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} &= (1-\lambda)(-5-\lambda) - (3 \cdot -3) \\ &= \lambda^2 + 4\lambda - 5 + 9 \\ &= \lambda^2 + 4\lambda + 4 \\ &= (\lambda+2)(\lambda+2) \end{aligned}$$

ii)
$$\begin{aligned} \begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} &= -3(1-\lambda) - (3 \cdot -3) \\ &= -3 + 3\lambda + 9 \\ &= 3\lambda + 6 \end{aligned}$$

iii)
$$\begin{aligned} \begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix} &= (3 \cdot -3) - 3(-5-\lambda) \\ &= -9 + 15 + 3\lambda \\ &= 3\lambda + 6 \end{aligned}$$

Which gives us the entire equation:

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 1-\lambda(\lambda+2)(\lambda+2) \cancel{-3(3\lambda+6)} \cancel{+3(3\lambda+6)}$$

Thus we have that the simplified characteristic polynomial is:

$$(1-\lambda)(\lambda+2)(\lambda+2)$$

Which gives us the following eigen values: $\lambda_1 = \lambda_2 = -2, \lambda_3 = 1$

b)

We need to setup the homogenous equations $(A - \lambda I)\vec{v} = 0$ for each λ

Case i) $\lambda = -2$

We have that:

$$A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

Thus we form the homogenous system and reduce it:

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \underset{\rightarrow}{\overset{R_2 + R_1}{R_3 - R_1}} \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \cdot \frac{1}{3}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have that $v_1 + v_2 + v_3 = 0$, let $v_2 = s, v_3 = t$ thus we have the system:

$$\begin{cases} v_1 = -s - t \\ v_2 = s \\ v_3 = t \end{cases}$$

And so:

$$\vec{v} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{thus:} \quad \vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case ii) $\lambda = 1$

We have that:

$$A - (1)I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

Thus we form the homogenous system and reduce it:

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \underset{R_2 + R_3}{\overset{R_2 + R_1}{\rightarrow}} \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \overset{R_3 - R_1}{\rightarrow} \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \end{array}\right] \underset{R_3 \leftrightarrow R_1}{\overset{R_1 \leftrightarrow R_2}{\rightarrow}} \left[\begin{array}{ccc|c} 3 & 0 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \cdot 1/3}{\overset{R_1 \cdot 1/3}{\rightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have that: $v_1 - v_3 = 0$ and $v_2 + v_3 = 0$. Let $v_3 = t$ thus we have the system:

$$\begin{cases} v_1 = t \\ v_2 = -t \\ v_3 = t \end{cases}$$

And so:

$$\vec{v} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{thus:} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus we have the following eigenpairs:

$$\lambda_1 = \lambda_2 = -2, \vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \lambda_3 = 1, \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

A is Diagonalizable:

Per the diagonalization theorem, our 3×3 matrix A can be diagonalized if and only if our eigenvectors form a basis for $\mathbb{R}^3$. We know they form a basis if the system is not defficient and since our matrix has a geometric and algebreaic multiplicity of 3, A is not defficient and therefore the eigenvectors are linearly independent and form a basis for $\mathbb{R}^3$. As a result the matrix is diagonalizable.

c)

We aim to construct $A = PDP^{-1}$ and we have that:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

and thus it suffice to find P^{-1} in order to find the diagonalization $A = PDP^{-1}$. To compute P^{-1} we set up the augmented matrix $[P|I]$ and reduce as follows:

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$