Math 211 Ass 2.

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Question 1,

Let $\vec{x}, \vec{y} \in Y$, $\vec{v_1}$,$\vec{v_2}$ be vectors in $\mathbb{R}^n$ and $t \in \mathbb{R}$. Since $\vec{x}, \vec{y} \in Y$ then,

$$\begin{cases} x_1v_1 + x_2v_2 = 0 \\ y_1v_1 + y_2v_2 = 0 \end{cases}$$

<u>Non Empty</u>:

$$\begin{array}{ll} \text{Since} & 0\vec{v_1} + 0\vec{v_2} \\ & = \vec{0} + \vec{0} \\ & = \vec{0} = \vec{0} \end{array}$$

Then $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in Y$

<u>Closure Under Addition</u>:

Form $\vec{w} = \vec{x} + \vec{y}$. We have,

$$\begin{array}{rcll} w_1\vec{v_1} + w_2\vec{v_2} & = & (x_1 + y_1)\vec{v_1} + (x_2 + y_2)\vec{v_2} & \\ & = & x_1\vec{v_1} + y_1\vec{v_1} + x_2\vec{v_2} + y_2\vec{v_2} & \text{Distributivity} \\ & = & (x_1v_1 + x_2v_2) + (y_1v_1 + y_2v_2) & \text{Associativity} \\ & = & (0) + (0) & \because \vec{x}, \vec{y} \in Y \\ & = & 0 & \end{array}$$

Thus since $w_1\vec{v_1} + w_2\vec{v_2} = 0$ then, $\vec{w} = \vec{x} + \vec{y} \in Y$

<u>Closure Under Scalar-Multiplication</u>:

Form $\vec{w} = t\vec{x}$. We have,

$$\begin{array}{rcll} w_1\vec{v_1} + w_2\vec{v_2} & = & (tx_1)\vec{v_1} + (tx_2)\vec{v_2} & \\ & = & tx_1\vec{v_1} + tx_2\vec{v_2} & \text{Associativity} \\ & = & t(x_1\vec{v_1} + x_2\vec{v_2}) & \text{Distributivity} \\ & = & t(0) & \because \vec{x} \in Y \\ & = & 0 & \end{array}$$

Thus since, $w_1\vec{v_1} + w_2\vec{v_2} = 0$ then $\vec{w} = t\vec{x} \in Y$

$\therefore$ As Y satisfies being non-empty, closure under vector addition, closure under scalar multiplication then Y is a subspace by definition. $\square$

Question 2,

a)

Let $\vec{x} \in W$ and $\vec{t} \in \mathbb{R}$,

W is not a subspace since it fails closure under scalar multiplication, since our scalar $t \in \mathbb{R}$ need not be an integer, when we perform scalar multiplication on $\vec{x}$ the result may not lead to x_1, x_2 and x_3 being integers, for example:

Let $t = \frac{1}{2}$ and let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, thus $\vec{x}$ satisfies the condition of x_1, x_2 and x_3 being integers, under scalar multiplication, form $\vec{w} = t\vec{x}$:

$$\vec{w} = t\vec{x} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2})1 \\ (\frac{1}{2})2 \\ (\frac{1}{2})3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

$\vec{w} = t\vec{x}$ does not satisfy the condition of the subspace W that w_1, w_2 and w_3 are all integers. Thus W is not a subspace since it fails under scalar multiplication.

b)

Let $\vec{x}, \vec{y} \in X$,
X is not a subspace since it fails closure under vector addition, while maintaining the property that $x_1 = x_2$ or $x_1 = x_3$ and $y_1 = y_2$ or $y_1 = y_3$ it is not necessarily true that that the result of $\vec{x} + \vec{y}$ will yield a result that satisfies this condition, for example:
Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ then form $\vec{w} = \vec{x} + \vec{y}$,

$$\vec{w} = \vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

We see that in $\vec{w}$ that nor $w_1 = w_2$ or $w_1 = w_3$ is true, thus X fails closure under scalar addition proving that X is not a subspace by definition.

Question 3,

Let $\vec{x}, \vec{y} \in W$, $\vec{v}$ be a vector in $\mathbb{R}^n$ and $t \in \mathbb{R}$, then:

$$\begin{cases} \vec{x} \cdot \vec{v} = 0 \\ \vec{y} \cdot \vec{v} = 0 \end{cases}$$

Non-Empty:
Let $\vec{x} = \vec{0}$ we have:

$$\begin{array}{rcll} \vec{x} \cdot \vec{v} & = & x_1v_1 + x_2v_2 + \cdots + x_nv_n & \text{Dot Product Definition} \\ & = & 0v_1 + 0v_2 + \cdots + 0v_3 & \\ & = & 0 + 0 + \cdots + 0 & \\ & = & 0 & \end{array}$$

Thus since $\vec{x} \cdot \vec{v} = 0$ when $\vec{x} = \vec{0}$, then $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in W$

Closure Under Vector Addition:
Form $\vec{w} = \vec{x} + \vec{y}$, we have:

$$\begin{array}{rcll} \vec{w} \cdot \vec{v} & = & (\vec{x} + \vec{y}) \cdot \vec{v} & \\ & = & (\vec{x} \cdot \vec{v}) + (\vec{y} \cdot \vec{v}) & \text{Distributivity} \\ & = & 0 + 0 & \because \vec{x}, \vec{y} \in W \\ & = & 0 & \end{array}$$

Thus since $\vec{w} \cdot v = 0$ then $\vec{w} = \vec{x} + \vec{y} \in W$

Closure Under Scalar Multiplication:
Form $\vec{w} = t\vec{x}$, we have:

$$\begin{array}{rcll} \vec{w} \cdot \vec{v} & = & (t\vec{x}) \cdot \vec{v} & \\ & = & t(\vec{x} \cdot \vec{v}) & \text{Associativity} \\ & = & t(0) & \because \vec{x} \in W \\ & = & 0 & \end{array}$$

Thus since $\vec{w} \cdot \vec{v} = 0$ then $\vec{w} = t\vec{x} \in W$

$\therefore$ As W satisfies being non-empty, closure under vector addition, closure under scalar multiplication then W is a subspace by definition. $\square$

Question 4,

a)

Using properties of the dot product and given conditions we determine the value:

$$
\begin{array}{rcll}
\vec{z}\cdot(2\vec{y}+\vec{z}) & = & (2\vec{y}\cdot z)+(\vec{z}\cdot\vec{z}) & \text{Distributivity} \\
 & = & (2\vec{y}\cdot 3\vec{x})+(3\vec{x}\cdot 3\vec{x}) & \text{Given Condition (i)} \\
 & = & 2(\vec{y}\cdot 3\vec{x})+3(\vec{x}\cdot 3\vec{x}) & \text{Associativity } \times 2 \\
 & = & 6(\vec{y}\cdot\vec{x})+9(\vec{x}\cdot\vec{x}) & \text{Associativity } \times 2 \\
 & = & 6(4)+9(\vec{x}\cdot\vec{x}) & \text{Given Condition (ii)} \\
 & = & 6(4)+9(\|\vec{x}\|^2) & \text{Relation between length and dot product} \\
 & = & 6(4)+9(5^2) & \text{Given Condition (iv)} \\
 & = & 24+9(25) & \\
 & = & 24+225 & \\
 & = & 249 &
\end{array}
$$

b)

Using properties of the cross product and given conditions we determine the value:

$$
\begin{array}{rcll}
\|(\vec{x}+2\vec{y})\times\vec{z}\| & = & \|-\vec{z}\times(\vec{x}+2\vec{y})\| & \text{Anti-Commutativity} \\
 & = & \|(-\vec{z}\times\vec{x})+(-\vec{z}\times 2\vec{y})\| & \text{Distributivity} \\
 & = & \|(-3\vec{x}\times\vec{x})+(-\vec{z}\times 2\vec{y})\| & \text{Given Condition (i)} \\
 & = & \|-3(\vec{x}\times\vec{x})+2(-\vec{z}\times\vec{y})\| & \text{Associativity } \times 2 \\
 & = & \|-3(\vec{x}\times\vec{x})+-2(\vec{z}\times\vec{y})\| & \text{Associativity} \\
 & = & \|-3(\vec{0})+-2(\vec{z}\times\vec{y})\| & \text{Self-Degenerate} \\
 & = & \|\vec{0}+-2(\vec{z}\times\vec{y})\| & \text{Properties of } \vec{0} \\
 & = & \|-2(4\hat{u})\| & \text{Given Condition (iii)} \\
 & = & \|-8\hat{u}\| & \text{Associativity} \\
 & = & |-8|\|\hat{u}\| & \text{Common Factor} \\
 & = & 8(1) & \text{Absolute Value \& Property of the Unit Vector} \\
 & = & 8 &
\end{array}
$$